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THE MORTAR FINITE ELEMENT METHOD FOR BINGHAM FLUIDS

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Abstract. This paper deals with the flow problem of a viscous plastic fluid in a cylindrical pipe. In order to approximate this problem governed by a variational inequality, we apply the nonconforming mortar finite element method. By using appropriate techniques, we are able to prove the convergence of the method and to obtain the same convergence rate as in the conforming case.

Résumé. On considère le problème de l'écoulement d'un fluide visqueux plastique dans une conduite cylindrique. Afin d'approcher ce problème régi par une inéquation variationnelle, nous appliquons la méthode non conforme des éléments finis avec joints. En utilisant des techniques appropriées, on devient en mesure de prouver la convergence de la méthode avec un taux de convergence identique au cas conforme.

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1. Introduction

The nonconforming mortar domain decomposition method allows the coupling of different approximation methods (*e.g.* finite elements, spectral elements, wavelets) and also the efficient handling of independent discretizations of the subdomains. The setting of the method as well as the first analyses have been performed in [6, 8]. Then the mortar procedure has been studied and extended to numerous areas and especially in fluid and solid mechanics.

In the fluid mechanics context on which we will focus, the mortar finite element approach has been considered from a theoretical or numerical point of view in [1, 3, 12], for the Stokes and the Navier-Stokes equations.

From a mathematical point of view, the mortar method was originally studied for problems governed by variational equalities and the first extension of the method to variational inequalities was achieved in [5, 16] for the two-dimensional unilateral contact problem in elasticity when using finite elements.

Our purpose in this paper is to consider a variational inequality arising in fluid mechanics and modeling the flow of a viscous plastic fluid (also called Bingham fluid) in a cylindrical pipe. As for unilateral contact, our aim is to prove that the mortar finite element method leads to a convergence rate which is similar to the rate obtained when using conforming finite elements (see [15]). Let us mention that significant differences will occur in the convergence analysis in comparison with unilateral contact. In the latter case, the inequality of the problem is "concentrated" on the boundary whereas in the context of the Bingham fluid the inequality problem

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holds on the entire domain. This fact leads to the use of new techniques in the error estimates, particularly in the consistency error estimate due to the nonconformity of the method.

The outline of the paper is as follows. The variational formulation of the viscous-plastic medium is given in the next section and in Section 3 the well posed finite element approximation of order one is stated. Section 4 deals with the convergence analysis of the method and begins with an adapted version of Falk's lemma (see [14]) to our problem. The main characteristic of this tool is to measure an important term of the consistency error in the $W^{1,1}$-norm that will lead us to a global convergence rate of order $h^{\frac{1}{2}}$ as in the conforming case (see [15]).

Notations. Let Ω be an open bounded subset of $\mathbb{R}^2$ whose generic point is denoted $\boldsymbol{x} = (x_1, x_2)$ and denote by $L^p(\Omega)$, $1 \leq p < \infty$ the set of real-valued Lebesgue measurable functions ψ such that $|\psi|^p$ is integrable. The Banach space $L^p(\Omega)$ is endowed with the norm

$$\|\psi\|_{L^p(\Omega)} = \Big(\int_\Omega |\psi(\boldsymbol{x})|^p \,\mathrm{d}\Omega\Big)^{\frac{1}{p}}.$$

When $p = 2$, $L^2(\Omega)$ is the Hilbert space associated with the inner product

$$(\varphi, \psi) = \int_\Omega \varphi(\boldsymbol{x})\psi(\boldsymbol{x}) \,\mathrm{d}\Omega.$$

Let $m \in \mathbb{N}$ and $p \geq 1$. Define the Sobolev spaces

$$W^{m,p}(\Omega) = \Big\{\psi \in L^p(\Omega), D^\alpha\psi \in L^p(\Omega),\ |\alpha| \leq m\Big\},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi–index in $\mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$. The notation D^α denotes the partial derivative $\dfrac{\partial^{\alpha_1}\partial^{\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. The convention $W^{0,p}(\Omega) = L^p(\Omega)$ is adopted. The Banach spaces $W^{m,p}(\Omega)$ are equipped with the norm

$$\|\psi\|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha\psi\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}}.$$

We shall denote by $W_0^{m,p}(\Omega)$ the closure of $\mathscr{D}(\Omega)$ (*i.e.* the space of indefinitely differentiable functions with compact support in Ω) in $W^{m,p}(\Omega)$. When $p = 2$, the spaces $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$ are denoted by $H^m(\Omega)$ and $H_0^m(\Omega)$ respectively which are Hilbert spaces.

Let γ be a connected portion of the boundary of Ω. For any $\tau \in \mathbb{R}_+ \setminus \mathbb{N}$, the Hilbert space $H^\tau(\gamma)$ is assigned with the norm

$$\|\psi\|_{H^\tau(\gamma)} = \left(\|\psi\|^2_{H^m(\gamma)} + \int_\gamma\int_\gamma \frac{(D^m\psi(\boldsymbol{x}) - D^m\psi(\boldsymbol{y}))^2}{|\boldsymbol{x}-\boldsymbol{y}|^{1+2\theta}} \,\mathrm{d}\gamma \,\mathrm{d}\gamma\right)^{\frac{1}{2}},$$

where m is the integer part of τ and θ its decimal part (see [2]). In the previous integral, $D^m\psi$ stands for the m–order derivative of ψ along γ and $\mathrm{d}\gamma$ denotes the linear measure on γ.

In order to define the space $H_{00}^{\frac{1}{2}}(\gamma)$, let us introduce the map ρ as the distance to the extreme points p_1 and p_2 of γ:

$$\rho(\boldsymbol{x}) = \mathrm{dist}\ (\boldsymbol{x}, \{p_1, p_2\}), \qquad \forall \boldsymbol{x} \in \gamma.$$

The space $H_{00}^{\frac{1}{2}}(\gamma)$ is then endowed with the norm

$$\|\psi\|_{H_{00}^{\frac{1}{2}}(\gamma)} = \left(\|\psi\|^2_{H^{\frac{1}{2}}(\gamma)} + \int_\gamma \frac{\psi(\boldsymbol{x})^2}{\rho(\boldsymbol{x})} \,\mathrm{d}\gamma\right)^{\frac{1}{2}}.$$

2. The variational formulation of the problem

Let us consider the laminar stationary flow of a Bingham fluid in a cylindrical pipe of cross-section $\Omega \subset \mathbb{R}^2$. According to Duvaut and Lions [13], the problem consists of finding the velocity field u defined in Ω and solution of the variational inequality

$$u \in V, \qquad \mu \int_\Omega \nabla u.(\nabla v - \nabla u) \, \mathrm{d}\Omega + g \int_\Omega (|\nabla v| - |\nabla u|) \, \mathrm{d}\Omega \geq \int_\Omega f(v-u) \, \mathrm{d}\Omega, \qquad \forall v \in V, \tag{2.1}$$

where $V = H_0^1(\Omega)$. The notation $\mu > 0$ stands for the viscosity of the fluid and $g > 0$ denotes the yield limit of the fluid. Such a fluid starts to flow only when the applied forces locally exceed g. The following notations have been used for any $v \in V$:

$$\nabla v = \Big(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\Big) \quad \text{ and } \quad |\nabla v| = \sqrt{\Big(\frac{\partial v}{\partial x_1}\Big)^2 + \Big(\frac{\partial v}{\partial x_2}\Big)^2}.$$

Finally, f represents the decay of the pressure in the pipe. Henceforward we assume that $f \in L^2(\Omega)$.

The existence and uniqueness statement for the variational inequality (2.1) follows directly from Lions-Stampacchia's theorem [17]. We recall this result (see [15]):

Proposition 2.1. *Problem (2.1) admits a unique solution $u \in V$ satisfying the stability property $\|u\|_{H^1(\Omega)} \leq (C/\mu)\|f\|_{L^2(\Omega)}$ where the positive constant C is independent of f.*

Concerning the regularity of the solution u, the result of [10] is as follows:

Proposition 2.2. *The solution u of (2.1) satisfies $u \in H^2(\Omega) \cap V$. Moreover, if Ω is a convex set, there exists a positive constant C independent of f such that $\|u\|_{H^2(\Omega)} \leq (C/\mu)\|f\|_{L^2(\Omega)}$.*

A significant investigation on the qualitative properties of the solution u has been accomplished in the references [18–20]. In particular, the authors proved that there always exists at least one region of Ω where the fluid behaves like a rigid medium (*i.e.* $\nabla u(x) = 0$) and looked for the shape of such zones. The research of stagnant regions (*i.e.* $u(x) = 0$) as well as their shape was also carried out.

In the case where Ω is a circular domain and if the function f is constant in Ω then an exact solution can be exhibited (see [15]). This solution depends then only on the variable $\sqrt{x_1^2 + x_2^2}$. In that case, the velocity field lies in $V \cap W^{2,\infty}(\Omega) \cap H^s(\Omega)$ for any $s < \frac{5}{2}$, (see [15]).

Remark 2.1. Let u be the solution of problem (2.1). Then u is solution of the minimization problem

$$u \in V, \qquad J(u) = \min_{v \in V} J(v),$$

where

$$J(v) = \frac{\mu}{2} \int_\Omega \nabla v.\nabla v \, \mathrm{d}\Omega + g \int_\Omega |\nabla v| \, \mathrm{d}\Omega - \int_\Omega f v \, \mathrm{d}\Omega.$$

Moreover u is characterized by the existence of p satisfying:

$$\begin{aligned} &u \in V, \qquad \mu \int_\Omega \nabla u.\nabla v \, \mathrm{d}\Omega + g \int_\Omega p.\nabla v \, \mathrm{d}\Omega = \int_\Omega f v \, \mathrm{d}\Omega, \qquad \forall v \in V, \\ &p \in \Lambda, \qquad p.\nabla u = |\nabla u| \quad \text{a.e. in } \Omega, \end{aligned}$$

where

$$\Lambda = \big\{q, \quad q \in (L^2(\Omega))^2, \quad |q(\boldsymbol{x})| \leq 1 \quad \text{a.e. in } \Omega\big\}.$$

3. Finite element approximation

The present section consists of building the spaces approximating $H^1_0(\Omega)$ in the mortar finite element context in order to set the approximation of problem (2.1). The framework of the mortar domain decomposition method consists of dividing Ω into K polygonal open subdomains. For the sake of simplicity, we assume that the polygonally shaped domain Ω is the union of two subdomains Ω^1 and Ω^2 with $\overline{\Omega^1} \cap \overline{\Omega^2} = \gamma$ where γ is the straight line segment $[p_1, p_2]$. We set

$$X(\Omega^\ell) = \Big\{ v^\ell \in H^1(\Omega^\ell), \quad v^\ell|_{\partial\Omega \cap \partial\Omega^\ell} = 0 \Big\}, \qquad \ell = 1, 2,$$

where $\partial\Omega, \partial\Omega^\ell$ denote the boundaries of Ω and Ω^ℓ respectively. Define

$$X = \Big\{ v \in L^2(\Omega), \quad \forall \ell, \quad v^\ell = v|_{\Omega^\ell} \in X(\Omega^\ell) \Big\} = \prod_{\ell=1}^{2} X(\Omega^\ell).$$

The norm on X, denoted $\|.\|$, is as follows

$$\|v\| = \Big(\|v^1\|^2_{H^1(\Omega^1)} + \|v^2\|^2_{H^1(\Omega^2)} \Big)^{\frac{1}{2}}, \qquad \forall v = (v^1, v^2) \in X.$$

The space V can be identified with the subspace of X containing the functions satisfying continuity conditions on γ:

$$V = \Big\{ v = (v^1, v^2) \in X, \quad v^1|_\gamma = v^2|_\gamma \Big\}.$$

Let us define the continuous bilinear form $a(.,.)$, the continuous functional $j(.)$ and the continuous linear form $L(.)$:

$$\begin{aligned} a(u, v) &= \sum_{\ell=1}^{2} \int_{\Omega^\ell} \nabla u^\ell . \nabla v^\ell \, \mathrm{d}\Omega^\ell, \qquad \forall u, v \in X, \\ j(v) &= \sum_{\ell=1}^{2} \int_{\Omega^\ell} |\nabla v^\ell| \, \mathrm{d}\Omega^\ell, \qquad \forall v \in X, \\ L(v) &= \sum_{\ell=1}^{2} \int_{\Omega^\ell} f^\ell v^\ell \, \mathrm{d}\Omega^\ell, \qquad \forall v \in X. \end{aligned}$$

With each subdomain Ω^ℓ is associated a regular family of discretizations $\mathscr{T}^\ell_h$ (see [11]) of triangles κ of diameter h_κ so that

$$h_\ell = \max_{\kappa \in \mathscr{T}^\ell_h} h_\kappa$$

represents the discretization parameter on Ω^ℓ and we set

$$h = \max(h_1, h_2).$$

Let $\mathbb{P}_q(\kappa)$ denote the space of polynomial functions whose degree is $\leq q$ on κ. Define

$$V_h(\Omega^\ell) = \Big\{ v^\ell_h \in \mathscr{C}(\overline{\Omega}^\ell), \quad \forall \kappa \in \mathscr{T}^\ell_h, \quad v^\ell_h|_\kappa \in \mathbb{P}_1(\kappa), \quad v^\ell_h|_{\partial\Omega \cap \partial\Omega^\ell} = 0 \Big\}.$$

Let $\mathcal{I}_h^\ell$ denote the Lagrange interpolation operator of order one on $\mathscr{T}_h^\ell$. The following error estimate is obtained from [11] by Hilbertian interpolation: for any pair of real numbers $(\eta,\nu) \in [0,1]\times]1,2]$, there exists a constant $C = C(\eta,\nu)$ verifying:

$$\|v^\ell - \mathcal{I}_h^\ell v^\ell\|_{H^\eta(\Omega^\ell)} \leq C(\eta,\nu) h_\ell^{\nu-\eta} \|v^\ell\|_{H^\nu(\Omega^\ell)}, \qquad \forall v^\ell \in H^\nu(\Omega^\ell). \tag{3.1}$$

The trace space of $V_h(\Omega^\ell)$ on γ is given by

$$W_h^\ell(\gamma) = \Big\{ v_h^\ell|_\gamma,\ v_h^\ell \in V_h(\Omega^\ell) \Big\},$$

and corresponds to the continuous functions on γ, piecewise linear on the trace $\mathcal{T}_h^\ell$ of the triangulation $\mathscr{T}_h^\ell$ on γ and vanishing at p_1 and p_2. Notice that (3.1) remains true when Ω^ℓ is replaced by γ and when $\mathcal{I}_h^\ell$ is replaced by the Lagrange interpolation operator of order one on $\mathcal{T}_h^\ell$. As $\mathscr{T}_h^1$ and $\mathscr{T}_h^2$ are generated independently, it follows that the meshes of both subdomains do not coincide on the interface γ and therefore $W_h^1(\gamma) \neq W_h^2(\gamma)$. In order to use inverse inequalities, we suppose that both families of one-dimensional triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are uniformly regular. We then consider the spaces $M_h^\ell(\gamma)$ defined as follows

$$M_h^\ell(\gamma) = \Big\{ q_h^\ell \in \mathscr{C}(\overline{\gamma}), \quad \forall T \in \mathcal{T}_h^\ell, \quad q_h^\ell|_T \in \mathbb{P}_1(T) \ \text{ and } \ q_h^\ell|_T \in \mathbb{P}_0(T) \text{ if } p_1 \text{ or } p_2 \in T \Big\}.$$

The space approximating $V = H_0^1(\Omega)$ becomes (see [8]):

$$V_h = \Big\{ v_h = (v_h^1, v_h^2) \in V_h(\Omega^1) \times V_h(\Omega^2), \ \int_\gamma (v_h^1 - v_h^2) q_h \,\mathrm{d}\gamma = 0, \ \forall q_h \in M_h(\gamma) \Big\}, \tag{3.2}$$

where $M_h(\gamma) = M_h^1(\gamma)$ or $M_h(\gamma) = M_h^2(\gamma)$.

The integral condition incorporated in (3.2) expresses a "weak continuity" relation across γ. It is easy to see that the finite element approximation is nonconforming ($V_h \not\subset V$) in the general case of nonmatching meshes on γ.

When the meshes fit together on γ, then the integral condition in (3.2) is equivalent to $v_h^1 = v_h^2$ on γ so that the inclusion $V_h \subset V$ holds. The latter case is considered in [15].

The discretized problem issued from (2.1) becomes: find u_h such that

$$u_h \in V_h, \qquad \mu a(u_h, v_h - u_h) + g j(v_h) - g j(u_h) \geq L(v_h - u_h), \qquad \forall v_h \in V_h. \tag{3.3}$$

We are now in a position to state the following existence and uniqueness result.

Proposition 3.1. *Problem (3.3) admits a unique solution $u_h \in V_h$.*

Proof. The bilinear form $a(.,.)$ is continuous on V_h and V_h-elliptic (see [8]) and the linear form $L(.)$ is continuous on V_h. Moreover, $j(.)$ is a convex continuous functional on V_h. The hypotheses of Lions-Stampacchia's theorem are then fulfilled. □

4. Error analysis

This section consists of obtaining *a priori* error estimates in the $\|.\|$-norm committed by the finite element approximation. Our purpose is to generalize the convergence results of the conforming finite element method to the more general case described here and to prove that the error decays at least like $h^{\frac{1}{2}}$ which is the error bound obtained in the conforming case (see [15]). The starting point is the next lemma: an adaptation of Falk's lemma (see [14]) to our problem.

Lemma 4.1. *Let $u \in H^2(\Omega) \cap V$ be the solution of (2.1) and let $u_h \in V_h$ be the solution of (3.3). Then the following estimate holds:*

$$\|u-u_h\|^2 \leq C\Big\{ \inf_{v_h \in V_h} \Big(\|u-v_h\|^2 + \|u-v_h\| \Big) + \inf_{v\in V} \Big(\sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)} \Big) + \Big| \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^1 - u_h^2)\, \mathrm{d}\gamma \Big| \Big\}. \tag{4.1}$$

where the constant C is independent of h.

Proof. Let α be the ellipticity constant of $a(.,.)$ on X. Then,

$$\alpha\mu\|u-u_h\|^2 \leq \mu a(u-u_h, u-u_h) = \mu a(u,u) - \mu a(u,u_h) - \mu a(u_h,u) + \mu a(u_h,u_h).$$

Using (2.1) and (3.3), we write:

$$\begin{aligned} \mu a(u,u) &\leq \mu a(u,v) - L(v-u) + gj(v) - gj(u), \qquad \forall v \in V, \\ \mu a(u_h,u_h) &\leq \mu a(u_h,v_h) - L(v_h-u_h) + gj(v_h) - gj(u_h), \qquad \forall v_h \in V_h. \end{aligned}$$

Hence, the following inequality

$$\begin{aligned} \alpha\mu\|u-u_h\|^2 \leq\ & \mu a(u_h-u, v_h-u) + \mu a(u, v_h-u) - L(v_h-u) + gj(v_h) - gj(u) \\ & + \mu a(u, v-u_h) - L(v-u_h) + gj(v) - gj(u_h). \end{aligned} \tag{4.2}$$

We now estimate separately the terms of (4.2). Denoting by M the norm of the continuous bilinear form $a(.,.)$ on X yields

$$\mu a(u_h-u, v_h-u) \leq \mu M\|u-u_h\|\|u-v_h\| \leq \frac{\mu\alpha}{2}\|u-u_h\|^2 + \frac{\mu M^2}{2\alpha}\|u-v_h\|^2. \tag{4.3}$$

Using again the continuity of $a(.,.)$ so as the boundedness of $\|u\|$ gives

$$\mu a(u, v_h-u) \leq \mu M\|u\|\|u-v_h\| \leq C\|f\|_{L^2(\Omega)}\|u-v_h\|. \tag{4.4}$$

Moreover, the following estimate holds

$$L(v_h-u) \leq \|f\|_{L^2(\Omega)}\|v_h-u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v_h-u\|. \tag{4.5}$$

Noting that $|\nabla v_h^\ell| - |\nabla u^\ell| \leq |\nabla(v_h^\ell - u^\ell)|$, we can write

$$\begin{aligned} gj(v_h) - gj(u) = g\sum_{\ell=1}^{2}\int_{\Omega^\ell} |\nabla v_h^\ell| - |\nabla u^\ell|\, \mathrm{d}\Omega^\ell &\leq g\sum_{\ell=1}^{2}\int_{\Omega^\ell} |\nabla(v_h^\ell - u^\ell)|\, \mathrm{d}\Omega^\ell \\ &\leq g\sum_{\ell=1}^{2}\sqrt{\mathrm{meas}(\Omega^\ell)}\|v_h^\ell - u^\ell\|_{H^1(\Omega^\ell)} \\ &\leq g\sqrt{2}\sqrt{\mathrm{meas}(\Omega)}\|v_h - u\|. \end{aligned} \tag{4.6}$$

The term $\mu a(u, v - u_h)$ is handled by using Green's formula and the property $v^1 = v^2$ on γ. The notation $\partial u^\ell/\partial n^\ell$ stands for the outward normal derivative of u^ℓ on Ω^ℓ and we have $\partial u^1/\partial n^1 + \partial u^2/\partial n^2 = 0$ on γ.

$$\begin{aligned}
\mu a(u, v - u_h) &= \mu \sum_{\ell=1}^{2} \int_{\Omega^\ell} \nabla u^\ell . \nabla (v^\ell - u_h^\ell)\, \mathrm{d}\Omega^\ell \\
&= -\mu \sum_{\ell=1}^{2} \int_{\Omega^\ell} \Delta u^\ell (v^\ell - u_h^\ell)\, \mathrm{d}\Omega^\ell + \mu \sum_{\ell=1}^{2} \int_\gamma \frac{\partial u^\ell}{\partial n^\ell}(v^\ell - u_h^\ell)\, \mathrm{d}\gamma \\
&\leq \mu \|\Delta u\|_{L^2(\Omega)} \|v - u_h\|_{L^2(\Omega)} + \mu \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^2 - u_h^1)\, \mathrm{d}\gamma \\
&\leq \mu \|u\|_{H^2(\Omega)} \|v - u_h\|_{L^2(\Omega)} + \mu \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^2 - u_h^1)\, \mathrm{d}\gamma \\
&\leq C \sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)} + \mu \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^2 - u_h^1)\, \mathrm{d}\gamma
\end{aligned} \tag{4.7}$$

where the continuous embedding $W^{1,1}(\Omega^\ell) \hookrightarrow L^2(\Omega^\ell)$ has been used (see [2]). Notice that if Ω is a convex set then the constant C of (4.7) does not depend on u according to Proposition 2.2.

Using the same imbedding as previously yields

$$L(v - u_h) \leq \|f\|_{L^2(\Omega)} \|v - u_h\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)}. \tag{4.8}$$

The last term $gj(v) - gj(u_h)$ is evaluated as follows:

$$\begin{aligned}
gj(v) - gj(u_h) = g \sum_{\ell=1}^{2} \int_{\Omega^\ell} |\nabla v^\ell| - |\nabla u_h^\ell|\, \mathrm{d}\Omega^\ell &\leq g \sum_{\ell=1}^{2} \int_{\Omega^\ell} |\nabla (v^\ell - u_h^\ell)|\, \mathrm{d}\Omega^\ell \\
&\leq g \sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)}.
\end{aligned} \tag{4.9}$$

Putting the estimates obtained in (4.3)–(4.9) into (4.2), and taking both infimum on V and V_h, we conclude to the existence of a positive constant independent of h satisfying (4.1). That ends the proof of the lemma. □

The nonconformity of the method leads to two supplementary terms in (4.1) in comparison with the conforming case studied in [15]: the second infimum (on V) as well as the integral term. The estimate of the first infimum (*i.e.* the approximation error) is a standard result of the mortar finite element method proved by Bernardi, Maday and Patera in [8] which we recall hereafter to render the paper self-contained and also to introduce some useful tools. Afterwards, in order to simplify the notations, we will choose $M_h(\gamma) = M_h^1(\gamma)$ in the definition of the approximation space in (3.2). Of course the symmetrical definition is also possible.

Lemma 4.2. *Let $u \in H^2(\Omega) \cap V$ be the solution of (2.1). Then there exists $v_h \in V_h$ such that:*

$$\|u - v_h\| \leq Ch,$$

where the positive constant C is independent of h.

Proof. Denoting by $\mathcal{I}_h^\ell$ the Lagrange interpolation operator of order one on $\mathscr{T}_h^\ell$ and from the definition of the norm $\|.\|$, we get for any $v_h \in V_h$

$$\begin{aligned}
\|u - v_h\| &\leq \|u^1 - v_h^1\|_{H^1(\Omega^1)} + \|u^2 - v_h^2\|_{H^1(\Omega^2)} \\
&\leq \|u^1 - \mathcal{I}_h^1 u^1\|_{H^1(\Omega^1)} + \|\mathcal{I}_h^1 u^1 - v_h^1\|_{H^1(\Omega^1)} \\
&\quad + \|u^2 - \mathcal{I}_h^2 u^2\|_{H^1(\Omega^2)} + \|\mathcal{I}_h^2 u^2 - v_h^2\|_{H^1(\Omega^2)} \\
&\leq \|\mathcal{I}_h^1 u^1 - v_h^1\|_{H^1(\Omega^1)} + \|\mathcal{I}_h^2 u^2 - v_h^2\|_{H^1(\Omega^2)} + Ch, \qquad (4.10)
\end{aligned}$$

where the error bounds (3.1) committed by $\mathcal{I}_h^\ell, \ell = 1, 2$ have been used. Choosing

$$v_h^1 = \mathcal{I}_h^1 u^1 + \mathcal{R}_h^1(\pi_h^1(\mathcal{I}_h^2 u^2 - \mathcal{I}_h^1 u^1)) \quad \text{and} \quad v_h^2 = \mathcal{I}_h^2 u^2, \qquad (4.11)$$

where π_h^1 represents the projection operator on $W_h^1(\gamma)$ defined for any function $\varphi \in H_{00}^{\frac{1}{2}}(\gamma)$ by

$$\begin{aligned}
\pi_h^1 \varphi &\in W_h^1(\gamma), \\
\int_\gamma (\varphi - \pi_h^1 \varphi)\psi_h \, \mathrm{d}\gamma &= 0, \qquad \forall \psi_h \in M_h^1(\gamma). \qquad (4.12)
\end{aligned}$$

Such an operator is stable in $L^2(\gamma)$, in $H_0^1(\gamma)$ and in $H_{00}^{\frac{1}{2}}(\gamma)$ (the proofs require the uniform regularity of the family of one-dimensional meshes $\mathcal{T}_h^1$, see [8]): let $Y = L^2(\gamma)$ or $H_0^1(\gamma)$ or $H_{00}^{\frac{1}{2}}(\gamma)$, then

$$\|\pi_h^1 v\|_Y \leq C\|v\|_Y, \quad \forall v \in Y. \qquad (4.13)$$

Moreover the following approximation property holds (see [4]): for any $\frac{1}{2} < \nu \leq 2$

$$\|v - \pi_h^1 v\|_{L^2(\gamma)} + h_1^{\frac{1}{2}} \|v - \pi_h^1 v\|_{H_{00}^{\frac{1}{2}}(\gamma)} \leq C h_1^\nu \|v\|_{H^\nu(\gamma)}, \quad \forall v \in H^\nu(\gamma) \cap H_{00}^{\frac{1}{2}}(\gamma). \qquad (4.14)$$

In (4.11), the notation $\mathcal{R}_h^1$ stands for a lifting operator from $W_h^1(\gamma) \cap H_0^1(\gamma)$ into $V_h(\Omega^1)$ satisfying $\|\mathcal{R}_h^1 \psi_h^1\|_{H^1(\Omega^1)} \leq C\|\psi_h^1\|_{H_{00}^{\frac{1}{2}}(\gamma)}$ for any $\psi_h^1 \in W_h^1(\gamma) \cap H_0^1(\gamma)$ (see [7,9]). Besides, it is straightforward that $v_h \in V_h$. The definition of v_h and of the lifting operator, the stability condition (4.13) and the trace theorem yield

$$\begin{aligned}
\|\mathcal{I}_h^1 u^1 - v_h^1\|_{H^1(\Omega^1)} &= \|\mathcal{R}_h^1(\pi_h^1(\mathcal{I}_h^2 u^2 - \mathcal{I}_h^1 u^1))\|_{H^1(\Omega^1)} \\
&\leq C\|\pi_h^1(\mathcal{I}_h^2 u^2 - \mathcal{I}_h^1 u^1)\|_{H_{00}^{\frac{1}{2}}(\gamma)} \\
&\leq C\|\mathcal{I}_h^2 u^2 - \mathcal{I}_h^1 u^1\|_{H_{00}^{\frac{1}{2}}(\gamma)} \\
&\leq C\Big(\|u^2 - \mathcal{I}_h^2 u^2\|_{H_{00}^{\frac{1}{2}}(\gamma)} + \|u^1 - \mathcal{I}_h^1 u^1\|_{H_{00}^{\frac{1}{2}}(\gamma)}\Big) \\
&\leq C\Big(\|u^2 - \mathcal{I}_h^2 u^2\|_{H^1(\Omega^1)} + \|u^1 - \mathcal{I}_h^1 u^1\|_{H^1(\Omega^2)}\Big) \\
&\leq Ch. \qquad (4.15)
\end{aligned}$$

Using estimate (4.15) with (4.10) and noticing that $\|\mathcal{I}_h^2 u^2 - v_h^2\|_{H^1(\Omega^2)} = 0$ leads to the estimate of the lemma. □

Next, we estimate the integral term of Lemma 4.1 which disappears in the conforming case of matching meshes (see [15]).

Lemma 4.3. *Let $u \in H^2(\Omega) \cap V$ be the solution of (2.1) and let $u_h \in V_h$ be the solution of (3.3). Then the following estimate holds*

$$\Big| \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^1 - u_h^2)\, \mathrm{d}\gamma \Big| \le C(h\|u-u_h\| + h^2),$$

where the positive constant C is independent of h.

Proof. As u_h belongs to V_h, we can write

$$\int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^1 - u_h^2)\, \mathrm{d}\gamma \quad = \quad \int_\gamma \Big(\frac{\partial u^1}{\partial n^1} - \psi_h\Big)(u_h^1 - u_h^2)\, \mathrm{d}\gamma,$$

for all $\psi_h \in M_h^1(\gamma)$. Denoting by $(H_{00}^{\frac12}(\gamma))'$ the topological dual space of $H_{00}^{\frac12}(\gamma)$, we get

$$\begin{aligned} \Big| \int_\gamma \frac{\partial u^1}{\partial n^1}(u_h^1 - u_h^2)\, \mathrm{d}\gamma \Big| \quad &\le \quad \inf_{\psi_h \in M_h^1(\gamma)} \Big\| \frac{\partial u^1}{\partial n^1} - \psi_h \Big\|_{(H_{00}^{\frac12}(\gamma))'} \|u_h^1 - u_h^2\|_{H_{00}^{\frac12}(\gamma)} \\ &\le \quad Ch \Big\| \frac{\partial u^1}{\partial n^1}\Big\|_{H^{\frac12}(\gamma)} \|u_h^1 - u_h^2\|_{H_{00}^{\frac12}(\gamma)} \\ &\le \quad Ch\|u_h^1 - u_h^2\|_{H_{00}^{\frac12}(\gamma)}. \end{aligned} \tag{4.16}$$

where the infimum is bounded as in ([8], Sect. 5.2) and the trace theorem has been used.

Since $u_h^1 = \pi_h^1 u_h^2$ where π_h^1 has been defined in (4.12), we obtain thanks to the stability (4.13), the approximation property (4.14) and the trace theorem:

$$\begin{aligned} \|u_h^1 - u_h^2\|_{H_{00}^{\frac12}(\gamma)} \quad &= \quad \|\pi_h^1 u_h^2 - u_h^2\|_{H_{00}^{\frac12}(\gamma)} \\ &\le \quad \|\pi_h^1(u_h^2 - u^2) - (u_h^2 - u^2)\|_{H_{00}^{\frac12}(\gamma)} + \|\pi_h^1 u^2 - u^2\|_{H_{00}^{\frac12}(\gamma)} \\ &\le \quad C\|u_h^2 - u^2\|_{H_{00}^{\frac12}(\gamma)} + Ch\|u^2\|_{H^{\frac32}(\gamma)} \\ &\le \quad C\|u - u_h\| + Ch, \end{aligned}$$

and combining the latter result with (4.16) ends the proof of the lemma. □

Having estimated the integral term, it remains to handle the second term of the consistency error which requires a quite specific treatment.

Lemma 4.4. *Let $u \in H^2(\Omega) \cap V$ be the solution of (2.1) and let $u_h \in V_h$ be the solution of (3.3). Then, there exists $v \in V$ such that:*

$$\sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)} \le C(h^{\frac12}\|u-u_h\| + h^{\frac32}),$$

where the positive constant C is independent of h.

Proof. **(i)** Let us choose $v^2 = u_h^2$ in Ω^2. In Ω^1, we define $\varphi^1 = u_h^1 + \mathcal{R}^1(u_h^2 - u_h^1)$ where $\mathcal{R}^1$ denotes a standard continuous lifting operator from $L^1(\gamma)$ into $W^{1,1}(\Omega^1)$ satisfying $\mathcal{R}^1(u_h^2 - u_h^1) = 0$ on $\partial\Omega \cap \partial\Omega^1$. Hence

$$\|\varphi^1 - u_h^1\|_{W^{1,1}(\Omega^1)} \quad = \quad \|\mathcal{R}^1(u_h^2 - u_h^1)\|_{W^{1,1}(\Omega^1)} \le C\|u_h^2 - u_h^1\|_{L^1(\gamma)} \le C'\|u_h^2 - u_h^1\|_{L^2(\gamma)}.$$

Denoting by i_h^2 the Lagrange interpolation operator of order one on $\mathcal{T}_h^2$ and noticing that $u_h^1 = \pi_h^1 u_h^2$ (definition of π_h^1 in (4.12)), it follows that

$$\begin{aligned}\|u_h^2 - u_h^1\|_{L^2(\gamma)} = \|u_h^2 - \pi_h^1 u_h^2\|_{L^2(\gamma)} \quad \leq \quad & \|u^2 - \pi_h^1 u^2\|_{L^2(\gamma)} \\ & + \|(i_h^2 u^2 - u^2) - \pi_h^1(i_h^2 u^2 - u^2)\|_{L^2(\gamma)} \\ & + \|(u_h^2 - i_h^2 u^2) - \pi_h^1(u_h^2 - i_h^2 u^2)\|_{L^2(\gamma)}. \end{aligned} \tag{4.17}$$

The first term of (4.17) is estimated with (4.14) so that

$$\|u^2 - \pi_h^1 u^2\|_{L^2(\gamma)} \leq Ch^{\frac{3}{2}} \|u^2\|_{H^{\frac{3}{2}}(\gamma)}. \tag{4.18}$$

The handling of the second term of (4.17) uses the $L^2(\gamma)$-norm stability (4.13) and the interpolation error estimate issued from (3.1):

$$\|(i_h^2 u^2 - u^2) - \pi_h^1(i_h^2 u^2 - u^2)\|_{L^2(\gamma)} \leq C\|i_h^2 u^2 - u^2\|_{L^2(\gamma)} \leq Ch^{\frac{3}{2}}\|u^2\|_{H^{\frac{3}{2}}(\gamma)}. \tag{4.19}$$

It remains to bound the third term of (4.17). Let $\ell = 1, 2$: the family of one-dimensional meshes $\mathcal{T}_h^\ell$ is supposed uniformly regular which means that there exists a constant C satisfying

$$\text{length}(T) \leq C \,\text{length}(T'), \qquad \forall T, T' \in \mathcal{T}_h^\ell.$$

We then denote by $\tilde{h}_1$ and $\tilde{h}_2$ the greatest length of the meshes belonging to $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ respectively. Set

$$\eta_h = \min\left(\frac{\tilde{h}_1}{2\tilde{h}_2}, \frac{\tilde{h}_2}{2\tilde{h}_1}\right). \tag{4.20}$$

Obviously $0 < \eta_h \leq \frac{1}{2}$. According to (4.14) and applying an inverse inequality gives

$$\begin{aligned}\|(u_h^2 - i_h^2 u^2) - \pi_h^1(u_h^2 - i_h^2 u^2)\|_{L^2(\gamma)} \quad & \leq \quad C\tilde{h}_1^{\frac{1}{2}+\eta_h} \|u_h^2 - i_h^2 u^2\|_{H^{\frac{1}{2}+\eta_h}(\gamma)} \\ & \leq \quad C\tilde{h}_1^{\frac{1}{2}} \left(\frac{\tilde{h}_1}{\tilde{h}_2}\right)^{\eta_h} \|u_h^2 - i_h^2 u^2\|_{H^{\frac{1}{2}}(\gamma)}.\end{aligned}$$

It is easy to check that

$$x^{(\min(\frac{x}{2}, \frac{1}{2x}))} \leq \mathrm{e}^{\frac{1}{2e}}, \qquad \forall x > 0.$$

Hence

$$\left(\frac{\tilde{h}_1}{\tilde{h}_2}\right)^{\eta_h} \leq \mathrm{e}^{\frac{1}{2e}}. \tag{4.21}$$

And consequently

$$\begin{aligned}\|(u_h^2 - i_h^2 u^2) - \pi_h^1(u_h^2 - i_h^2 u^2)\|_{L^2(\gamma)} \quad & \leq \quad Ch^{\frac{1}{2}} \|u_h^2 - i_h^2 u^2\|_{H^{\frac{1}{2}}(\gamma)} \\ & \leq \quad Ch^{\frac{1}{2}} (\|u_h^2 - u^2\|_{H^{\frac{1}{2}}(\gamma)} + \|u^2 - i_h^2 u^2\|_{H^{\frac{1}{2}}(\gamma)}) \\ & \leq \quad Ch^{\frac{1}{2}} (\|u_h - u\| + h).\end{aligned}$$

Putting together estimates (4.18), (4.19) and (4.22) in (4.17) leads to

$$\|\varphi^1 - u_h^1\|_{W^{1,1}(\Omega^1)} \quad \le \quad C(h^{\frac{1}{2}}\|u - u_h\| + h^{\frac{3}{2}}). \tag{4.22}$$

Since $\varphi^1 \in W^{1,1}(\Omega^1)$, the pair (φ^1, v^2) does not belong to V. The construction of an appropriate $(v^1, v^2) \in V$ is accomplished hereafter.

(ii) Next, we show that for every positive ε, there exists $v^1 \in X(\Omega^1)$ verifying $v^1 = u_h^2$ on γ and $\|v^1 - \varphi^1\|_{W^{1,1}(\Omega^1)} \le \varepsilon$. To do this, introduce $\psi^1 = u_h^1 + \mathcal{R}^{1*}(u_h^2 - u_h^1)$ where $\mathcal{R}^{1*}$ denotes a continuous lifting operator from $H_{00}^{\frac{1}{2}}(\gamma)$ into $H^1(\Omega^1)$ satisfying $\mathcal{R}^{1*}(u_h^2 - u_h^1) = 0$ on $\partial\Omega \cap \partial\Omega^1$. It follows that $\varphi^1 - \psi^1 \in W_0^{1,1}(\Omega^1)$.

Let then $\varepsilon > 0$ be given. A density argument implies that there exists $\chi^1 \in H_0^1(\Omega^1)$ verifying $\|\chi^1 - (\varphi^1 - \psi^1)\|_{W^{1,1}(\Omega^1)} \le \varepsilon$. Setting $v^1 = \chi^1 + \psi^1$, we deduce that

$$v^1 \in X(\Omega^1), \quad \|v^1 - \varphi^1\|_{W^{1,1}(\Omega^1)} \le \varepsilon \quad \text{and} \quad v = (v^1, v^2) \in V.$$

The latter estimate together with (4.22) gives

$$\begin{aligned}\sum_{\ell=1}^{2} \|v^\ell - u_h^\ell\|_{W^{1,1}(\Omega^\ell)} = \|v^1 - u_h^1\|_{W^{1,1}(\Omega^1)} &\le \|v^1 - \varphi^1\|_{W^{1,1}(\Omega^1)} + \|\varphi^1 - u_h^1\|_{W^{1,1}(\Omega^1)} \\ &\le \varepsilon + C(h^{\frac{1}{2}}\|u - u_h\| + h^{\frac{3}{2}}).\end{aligned}$$

Choosing $\varepsilon = h^{\frac{3}{2}}$ ends the proof of the lemma. □

Remark 4.1. The technique leading to the bound (4.21) by choosing η_h as in (4.20) avoids the introduction of the hypothesis "h_1/h_2 bounded as $h \to 0$" which should be used if η_h does not depend on h.

We are now in a position to exhibit an upper bound of the error committed by the mortar finite element approximation in the following theorem.

Theorem 4.5. *Let $u \in H^2(\Omega) \cap V$ be the solution of (2.1) and let $u_h \in V_h$ be the solution of (3.3). One has:*

$$\|u - u_h\| \le Ch^{\frac{1}{2}},$$

where the positive constant C is independent of h.

Proof. Putting together in (4.1) the estimates obtained in Lemmas 4.2, 4.3, and 4.4 yields the following bound

$$\|u - u_h\|^2 \le Ch + Ch^{\frac{1}{2}}\|u - u_h\|.$$

Writing $Ch^{\frac{1}{2}}\|u - u_h\| \le \frac{1}{2}\|u - u_h\|^2 + \frac{1}{2}C^2 h$ accomplishes the proof. □

Notice that the bound obtained in Theorem 4.5 is similar to that already known in the conforming case [15].

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