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ON THE DOMAIN GEOMETRY DEPENDENCE OF THE LBB CONDITION*,**

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Abstract. The LBB condition is well-known to guarantee the stability of a finite element (FE) velocity pressure pair in incompressible flow calculations To ensure the condition to be satisfied a certain constant should be positive and mesh-independent The paper studies the dependence of the LBB condition on the domain geometry For model domains such as strips and rings the substantial dependence of this constant on geometry aspect ratios is observed In domains with highly anisotropic substructures this may require special care with numerics to avoid failures similar to those when the LBB condition is violated In the core of the paper we prove that for any FE velocity pressure pair satisfying usual approximation hypotheses the mesh-independent limit in the LBB condition is not greater than its continuous counterpart, the constant from the Nečas inequality For the latter the explicit and asymptotically accurate estimates are proved The analytic results are illustrated by several numerical experiments

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INTRODUCTION

Consider the Stokes problem in a bounded domain $\Omega \subset R^n$, $n = 2, 3$

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{1}$$

Equations (1) describe the slow motion of a viscous incompressible fluid driven by external forces $\mathbf{f}(\mathbf{x})$ The unknowns are the vector function $\mathbf{u}(\mathbf{x})$ (velocity) and the scalar function $p(\mathbf{x})$ (pressure) subject to the integral condition $\int_\Omega p(\mathbf{x})\mathrm{d}\mathbf{x} = 0$ Problem (1) serves also as a model or auxiliary problem in many CFD applications Let $\mathbf{U}_h$ and P_h be some FE approximations of the velocity space

$$\mathbf{U} = \mathrm{H}_0^1(\Omega)^n \quad \text{with} \quad ||\mathbf{u}||_1 = ||\nabla \mathbf{u}||_{L_2(\Omega)^n}$$

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and the pressure space

$$P = \{p : p \in L_2(\Omega),\ (p,1) = 0\} \quad \text{with} \quad ||p||_0 = ||p||_{L_2(\Omega)}.$$

Assume for a moment $\mathbf{U}_h \subset \mathbf{U}$ and $P_h \subset P$. The discrete counterpart of (1) reads: find $\{\mathbf{u}_h, p_h\}$ from $\{\mathbf{U}_h, P_h\}$ such that for any $\{\mathbf{v}_h, q_h\}$ from $\{\mathbf{U}_h, P_h\}$

$$\begin{array}{rcl} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) & = & (\mathbf{f}, \mathbf{v}_h), \\ (\operatorname{div} \mathbf{u}_h, q_h) & = & 0. \end{array} \tag{2}$$

It is well-known that for the well-posedness of (2) and stability of $\{\mathbf{u}_h, p_h\}$ the following inequality should be valid (see [2, 7]):

$$\inf_{q_h \in P_h} \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q_h, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1 \, ||q_h||_0} = \gamma_h \geq \gamma(\Omega) > 0 \tag{3}$$

with some positive constant $\gamma(\Omega)$ independent of the mesh parameter h. Throughout the paper, we assume that $\sup_x$ and $\inf_x$ are taken for $x \neq 0$ if $||x||$ appears in the denominator.

Condition (3) is commonly referred as LBB (*Ladyzhenskaya - Babuška - Brezzi*) or *inf-sup* condition and is not satisfied by an arbitrary pair of FE spaces $\mathbf{U}_h$ and P_h. One example when (3) fails are piecewise-linear continuous elements ($P_1^n \times P_1$ pair), if the same triangulation is used for both pressure and velocity grids. For discussions and historical remarks see, *e.g.*, [14]. Condition (3) is also crucial in proving estimates and convergence for discrete solution. It is classic (see, *e.g.*, [8]) to have

$$||p_h||_0 \leq 2\,\gamma_h^{-1}\, ||\mathbf{f}||_{-1}, \quad ||\mathbf{u}_h||_1 \leq ||\mathbf{f}||_{-1} \quad \text{with} \quad ||\mathbf{f}||_{-1} = \sup_{\mathbf{v} \in \mathbf{U}} \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{||\mathbf{v}||_1} \tag{4}$$

and

$$||\mathbf{u}_h - \mathbf{u}||_1 + ||p_h - p||_0 \leq 3(1 + \gamma_h^{-1}) \left(\inf_{\mathbf{v}_h \in \mathbf{U}_h} ||\mathbf{u} - \mathbf{v}_h||_1 + \inf_{q_h \in P_h} ||p - q_h||_0 \right).$$

As shown in [3], condition (3), together with the so called ellipticity in the kernel, is also a *necessary* condition for (4). Moreover, the convergence rate of many iterative methods to solve (2) depends essentially on γ_h (see, *e.g.*, [6, 9, 17, 23]). For example the Uzawa - CG algorithm for (2), which is generally believed to be one of the most efficient, has the asymptotic convergence rate

$$\kappa = \frac{1 - \gamma_h}{1 + \gamma_h}. \tag{5}$$

Therefore for small γ_h one may expect poor algebraic properties of (2).

There are a lot of papers (see the overview in [14]), in which the LBB condition is checked for particular FE pairs. The main address of these papers is commonly the mesh dependence of γ_h, rather than the domain dependence. On the other hand, the degradation of (3) for some configurations, *e.g.* for flows in channels, is a phenomena well known for practitioners. In this paper it is proved that at least for a certain type of domains, such as strips and rings, $\gamma(\Omega)$ tends to zero if the domain becomes in some sense anisotropic. Moreover, estimates involving a measure of anisotropy are given. This *a priori* information can be quite important for the prediction of a numerical solution quality and the solvers behaviour in domains with anisotropic substructures and for domain-decomposition methods.

The remainder of the paper is organised as follows. In Section 1 the theorem is proved that for any FE pair, which possesses usual approximation properties, one has

$$\gamma(\Omega) \leq \mu(\Omega),$$

where $\mu(\Omega)$ is the optimal constant from the Nečas inequality:

$$\mu(\Omega)||p||_0 \le ||\nabla p||_{-1} \qquad \forall p \in P. \tag{6}$$

In Section 2 constant $\mu(\Omega)$ is linked with the minimal eigenvalue of a certain operator associated with the Stokes problem (1). This enables us to show that in

$$\Omega = \{(x_1, x_2) : 0 < x_i < L_i,\ i = 1, 2\}\,, \quad \ell = \max(L_1/L_2, L_2/L_1)$$

the following estimates hold:

$$\frac{1}{2\sqrt{15}}\ell^{-1} \le \mu(\Omega) \le \frac{\pi}{2\sqrt{3}}\ell^{-1}. \tag{7}$$

In Section 3, we show for the ring

$$\Omega = \{\mathbf{x} = (x_1, x_2) : 0 < R_1 < |\mathbf{x}| < R_2\}\,, \quad R_2/R_1 = 1 + \delta,\ \delta > 0$$

that for $\delta \in (0, 1]$

$$\mu(\Omega) \le \sqrt{\frac{7}{6}\frac{\delta}{2}}\,. \tag{8}$$

These results imply that for thin strips, long channels, or rings the constant $\gamma(\Omega)$ from the LBB condition tends to zero at least with the linear dependence on the domain anisotropy parameter (ℓ^{-1} or δ). In Section 4, results of numerical experiments with two conforming and one non-conforming FE pair are presented. They support the theory and indicate that this dependence is indeed linear.

Finally, we recall the result from [15], which states that for bounded simply-connected 2D domains the following relation between $\mu(\Omega)$ and the optimal constant from the 2nd Korn's inequality, defined below by $\eta(\Omega)$, holds:

$$\eta(\Omega) = 2\mu(\Omega)^{-1}. \tag{9}$$

Therefore, as a by-product of our analysis new estimates in rectangular domains for $\eta(\Omega)$ are obtained, which could be useful in elasticity. See also Remark 3.5 in Section 3 for the case of a ring.

1. LBB condition and Nečas inequality

Further we consider both conforming and non-conforming finite elements for velocity. First assume that FE subspaces $\mathbf{U}_h$ and P_h are such that $\mathbf{U}_h \subset \mathbf{U}$ and $P_h \subset P$ for each $h > 0$. In this case the only assumption we need is the following standard approximation hypothesis for P_h :

- A1. For each $q \in P \cap \mathrm{H}^1(\Omega)$ there exists a function $q_h \in P_h$ such that

$$||q - q_h||_0 \le C\,h\,||q||_{\mathrm{H}^1(\Omega)} \tag{10}$$

  with C independent of q and h.

In the non-conforming case ($\mathbf{U}_h \not\subset \mathbf{U}$) assume $\mathbf{u}_h$ to be a polynomial on every element τ of the subdivision T of Ω. Then it is standard to define $(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = \sum_{\tau\in T}(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_\tau$. Naturally $||\mathbf{u}_h||_1 = (\nabla \mathbf{u}_h, \nabla \mathbf{u}_h)^{\frac{1}{2}}$ is the mesh-depended norm, scalar product $(p_h, \operatorname{div} \mathbf{u}_h)$ is defined similarly. In the non-conforming case we need additionally two assumptions. These are the full elliptic regularity for the solution to the Poisson problem and the convergence assumption for a discrete solution of the Poisson problem:

- A2. For any $\mathbf{f} \in L_2(\Omega)^n$ and the solution $\boldsymbol{\phi}$ to

$$\Delta \boldsymbol{\phi} = \mathbf{f} \quad \text{in } \Omega, \quad \boldsymbol{\phi} = 0 \quad \text{on } \partial\Omega \tag{11}$$

  one gets $\boldsymbol{\phi} \in \mathbf{U} \cap \mathrm{H}^2(\Omega)^n$ and $||\boldsymbol{\phi}||_{\mathrm{H}^2(\Omega)^n} \le c||\mathbf{f}||_{L_2(\Omega)^n}$.
- A3. Let $\boldsymbol{\phi} \in \mathbf{U} \cap \mathrm{H}^2(\Omega)$ be a solution to (11) with $\mathbf{f} = -\nabla q$ and $\boldsymbol{\phi}_h \in \mathbf{U}_h$ is a solution to the problem

$$(\nabla \boldsymbol{\phi}_h, \nabla \mathbf{v}_h) = (q, \operatorname{div} \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{U}_h,$$

  then

$$||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_1 \le \omega(h)||\boldsymbol{\phi}||_{\mathrm{H}^2(\Omega)} \tag{12}$$

  with $\omega(h) \to 0$, if $h \to 0$.

The former assumption imposes some restriction on Ω (*cf.* [13]). As an example it is valid for bounded domains with a piecewise-smooth boundary with no entering corners. The second assumption is usually the consequence of approximation and consistency properties due to Stang's lemma [25]. The consistency follows from by the standard arguments (see, *e.g.* [5]) if the functions from $\mathbf{U}_h$ has continuous fluxes on the edges of elements, since in this case

$$\sum_{\tau \in T} \sum_{e \in \partial\tau} \int_e (\mathbf{v}_h \cdot \mathbf{n}) q \, \mathrm{d}s = 0$$

for any $\mathbf{v}_h \in \mathbf{U}_h$ and $q \in \mathrm{H}^1(\Omega)$, which implies $(q, \operatorname{div} \mathbf{v}_h) = -(\nabla q, \mathbf{v}_h)$. Examples are the Crouzeix-Raviart element [8] or $(\tilde{Q}_2)^n \times Q^0$ quadrilateral elements from [22].

Consider $\mu(\Omega)$ from (6). By definition one has

$$||\nabla p||_{-1} = \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(p, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1}.$$

And it is clear that we can set

$$\mu(\Omega) = \inf_{q \in P} \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1 \, ||q||_0}. \tag{13}$$

Lemma 1.1. *For $\mu(\Omega)$ from (13) one has*

$$\mu(\Omega) = \inf_{q \in P \cap \mathrm{H}^1(\Omega)} \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1 \, ||q||_0}.$$

Proof. The inequality

$$\mu(\Omega) \le \inf_{q \in P \cap \mathrm{H}^1(\Omega)} \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1 \, ||q||_0} \tag{14}$$

is evident since $P \cap \mathrm{H}^1(\Omega) \subset P$.

On the other hand

$$\mu(\Omega) \ge \inf_{q \in P \cap \mathrm{H}^1(\Omega)} \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1 \, ||q||_0}, \tag{15}$$

easily follows from the density of $P \cap \mathrm{H}^1(\Omega)$ in P [16]. $\square$

Now we are in a position to prove the following theorem.

Theorem 1.2. *Under the above assumptions on* $\mathbf{U}_h$ *and* P_h *we have*

$$\gamma(\Omega) \le \mu(\Omega). \tag{16}$$

Proof. Thanks to Lemma 1.1 it suffices to check that

$$\gamma(\Omega) \le \inf_{q \in P \cap \mathrm{H}^1(\Omega)} \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1\, ||q||_0}. \tag{17}$$

Consider arbitrary $q \in P \cap \mathrm{H}^1(\Omega)$ and $\varepsilon \in (0,1)$. For sufficiently small h we have

$$\max(C\,h, \omega(h))||q||_{\mathrm{H}^1(\Omega)} \le \varepsilon ||q||_0 \tag{18}$$

with constant C from (10) and $\omega(h)$ from (12). Hence, owing to approximation hypothesis A1 we can choose $q_h \in P_h$ such that

$$||q - q_h||_0 \le \varepsilon ||q||_0.$$

Thus we have

$$\begin{aligned}\gamma(\Omega) \le \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q_h, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1\, ||q_h||_0} &\le \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1\, ||q_h||_0} + \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q_h - q, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1\, ||q_h||_0} \\ &\le \frac{1}{(1-\varepsilon)} \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1\, ||q||_0} + \frac{\varepsilon}{1-\varepsilon}\end{aligned} \tag{19}$$

In the case of conforming FE, thanks to $\mathbf{U}_h \subset \mathbf{U}$ and the arbitrary choice of $\varepsilon \in (0,1)$ we get from (19)

$$\gamma(\Omega) \le \sup_{\mathbf{v} \in \mathbf{U}} \frac{|(q, \operatorname{div} \mathbf{v})|}{||\mathbf{v}||_1\, ||q||_0},$$

which leads to (17).

In the case of non-conforming velocity elements we get from (19)

$$\gamma(\Omega) \le \frac{1}{(1-\varepsilon)} \sup_{\mathbf{v}_h \in \mathbf{U}_h} \frac{|(q, \operatorname{div} \mathbf{v}_h)|}{||\mathbf{v}_h||_1\, ||q||_0} + \frac{\varepsilon}{1-\varepsilon} \tag{20}$$

It is straightforward to check (*cf.* (23)–(26) below) that for a given q the supremum in (20) is attained for the $\hat{\mathbf{v}}_h$ which solves the problem

$$(\nabla \hat{\mathbf{v}}_h, \nabla \mathbf{v}_h) = (q, \operatorname{div} \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{U}_h.$$

Together with $\hat{\mathbf{u}}_h$ consider $\hat{\mathbf{u}}$ from $\mathbf{U}$, which solves

$$(\nabla \hat{\mathbf{v}}, \nabla \mathbf{v}) = (q, \operatorname{div} \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathbf{U}.$$

Assumption A2 implies $\hat{\mathbf{v}} \in \mathbf{U} \cap \mathrm{H}^2(\Omega)^n$. Therefore, thanks to (12), (18), (6), and (26) we get

$$||\hat{\mathbf{v}} - \hat{\mathbf{v}}_h||_1 \le \omega(h)||\hat{\mathbf{v}}||_{\mathrm{H}^2(\Omega)^n} \le c\omega(h)||\nabla q||_{L_2(\Omega)^n} \le c\varepsilon||q||_0 \le c\mu(\Omega)^{-1}\varepsilon||\nabla q||_{-1} = c_1\varepsilon||\hat{\mathbf{v}}||_1.$$

Now the following estimates hold:

$$
\begin{aligned}
\gamma(\Omega) &\le \frac{1}{(1-\varepsilon)} \sup_{\mathbf{v}_h\in\mathbf{U}_h} \frac{|(q,\operatorname{div}\mathbf{v}_h)|}{||\mathbf{v}_h||_1\,||q||_0} + \frac{\varepsilon}{1-\varepsilon} = \frac{1}{(1-\varepsilon)} \frac{|(q,\operatorname{div}\hat{\mathbf{v}}_h)|}{||\hat{\mathbf{v}}_h||_1\,||q||_0} + \frac{\varepsilon}{1-\varepsilon} \\
&\le \frac{1}{(1-\varepsilon)} \frac{|(q,\operatorname{div}\hat{\mathbf{v}})|}{(1-c_1\varepsilon)||\hat{\mathbf{v}}||_1\,||q||_0} + \frac{1}{(1-\varepsilon)} \frac{|(q,\operatorname{div}(\hat{\mathbf{v}}_h-\hat{\mathbf{v}}))|}{(1-c_1\varepsilon)||\hat{\mathbf{v}}||_1\,||q||_0} + \frac{\varepsilon}{1-\varepsilon} \\
&\le \frac{1}{(1-\varepsilon)(1-c_1\varepsilon)} \sup_{\mathbf{v}\in\mathbf{U}} \frac{|(q,\operatorname{div}\mathbf{v})|}{||\mathbf{v}||_1\,||q||_0} + \frac{c_1\varepsilon}{1-c_1\varepsilon} + \frac{\varepsilon}{1-\varepsilon}\cdot
\end{aligned}
\tag{21}
$$

Since the choice of $q \in P \cap \mathrm{H}^1(\Omega)$ and ε was arbitrary, we have proved (17). □

Remark 1.3. It is well known that γ_h^2 equals the minimal eigenvalue of a certain eigenvalue problem associated with the discrete system (2) (*cf.* Sect. 4). Hence, from the observation of the next section, where $\mu(\Omega)$ is linked with the minimal eigenvalue of a certain continuous operator, and approximation properties one could conclude that

$$\lim_{h\to 0} \gamma_h = \mu(\Omega). \tag{22}$$

However, as shown in [3], besides the approximation properties the *necessary* condition for the convergence of eigenvalues of mixed problems is the existence of a certain projection operator from $\mathbf{U}$ to $\mathbf{U}_h$. The existence of such an operator does not follow from the LBB and ellipticity conditions (see also [4]). So to establish (22) one has to check the existence of such an operator for every particular FE pair of interest, the latter is a non-standard task.

2. Estimates for a strip

One can rewrite the Stokes problem (1) as follows:

$$
\begin{aligned}
A_0 p &= \operatorname{div}\Delta_0^{-1}\mathbf{f}, \\
\mathbf{u} &= \Delta_0^{-1}(\nabla p - \mathbf{f})
\end{aligned}
$$

with

$$A_0 = \operatorname{div}\Delta_0^{-1}\nabla,$$

where Δ_0^{-1} is the solution operator for the vector Poisson problem: Given a functional $\mathbf{g}$ on $\mathbf{U}$ find $\mathbf{v} \in \mathbf{U}$ such that $\Delta\mathbf{v} = \mathbf{g}$.

The operator $A_0 : P \to P$ is a Schur complement for problem (1). From the papers [18] and [10] it follows that in this continuous setting the operator A_0 is self-adjoint, positive definite, has a discrete spectrum, and possesses a complete orthonormal system of eigenfunctions in P.

Below we give a link between the minimal eigenvalue of A_0 and the constant $\mu(\Omega)$ from (13). To this end consider the following equalities for arbitrary function $q \in \mathrm{H}^1(\Omega) \cap P$:

$$
\begin{aligned}
(A_0 q, q) &= (\operatorname{div}\Delta_0^{-1}\nabla q, q) = -(\Delta_0^{-1}\nabla q, \nabla q) && (23)\\
&= -(\Delta_0^{-1}\nabla q, \Delta\Delta_0^{-1}\nabla q) = (-\Delta\mathbf{w}, \mathbf{w})|_{\mathbf{w}\,=\,\Delta_0^{-1}\nabla q} && (24)\\
&= ||\mathbf{w}||_1^2 = \sup_{\mathbf{v}\in\mathbf{U}} \frac{(-\Delta\mathbf{w},\mathbf{v})^2}{||\mathbf{v}||_1^2} = \sup_{\mathbf{v}\in\mathbf{U}} \frac{(\nabla q,\mathbf{v})^2}{||\mathbf{v}||_1^2} = \sup_{\mathbf{v}\in\mathbf{U}} \frac{(q,\operatorname{div}\mathbf{v})^2}{||\mathbf{v}||_1^2}\cdot && (25)
\end{aligned}
$$

Once again we use the fact that $\mathrm{H}^1(\Omega) \cap P$ is dense in P and pass to the limit in the equalities

$$(A_0 q, q) = ||\mathbf{w}||_1^2 = \sup_{\mathbf{v} \in \mathbf{U}} \frac{(q, \operatorname{div} \mathbf{v})^2}{||\mathbf{v}||_1^2} . \tag{26}$$

Thus relations (26) are valid for arbitrary $q \in P$ and $\mathbf{w} = \Delta_0^{-1} \nabla q \in \mathbf{U}$. In particular from (13) and (26) follows

$$\lambda_{\min}(A_0) = \inf_{q \in P} \frac{(A_0 q, q)}{||q||_0^2} = \inf_{q \in P} \sup_{\mathbf{v} \in \mathbf{U}} \frac{(q, \operatorname{div} \mathbf{v})^2}{||\mathbf{v}||_1^2 \, ||q||_0^2} = \mu(\Omega)^2,$$

where $\lambda_{\min}(A_0)$ is a minimal eigenvalue of the operator A_0.

We also define operators A_p and A_m. Similar to A_0 these operators are Schur complements for the Stokes problem, however they involve another boundary condition for the velocity. We assume that

$$\Omega = \{(x_1, x_2) \,|\, 0 < x_i < L_i, \ i = 1, 2\} . \tag{27}$$

By A_p denote the operator $A_p \cdot P \to P$ defined as

$$A_p = \operatorname{div} \Delta_p^{-1} \nabla,$$

here Δ_p^{-1} is the solution operator to the problem:

$$\begin{aligned} \Delta \mathbf{u} &= \mathbf{g} \quad \text{in} \quad \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \frac{\partial (\mathbf{u} \cdot \tau)}{\partial \mathbf{n}} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\mathbf{n}$ and τ are the normal and tangent vectors to $\partial\Omega$. In [19] it was shown that A_p is the identity operator on P.

In the same fashion we define the operator $A_m : P \to P$ as

$$A_m = \operatorname{div} \Delta_m^{-1} \nabla,$$

where Δ_m^{-1} is the solution operator to the problem:

$$\Delta \mathbf{u} = \mathbf{g} \quad \text{in} \quad \Omega$$

with boundary conditions

$$\begin{aligned} u_1 = 0, \quad \frac{\partial u_2}{\partial x_1} = 0 \quad &\text{for} \quad x_1 = 0, \, L_1, \\ u_1 = 0, \quad u_2 = 0 \quad &\text{for} \quad x_2 = 0, \, L_2. \end{aligned} \tag{28}$$

The weak solution of this problem belongs to

$$\mathbf{U}_m = \left\{ \mathbf{u} \in \left(\mathrm{H}^1(\Omega)\right)^2 : \, u_1 = 0 \text{ and } (0, u_2) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

For arbitrary $q \in P$ we have

$$(A_m q, q) = \sup_{\mathbf{v} \in \mathbf{U}_m} \frac{(q, \operatorname{div} \mathbf{v})^2}{||\mathbf{v}||_1^2} . \tag{29}$$

The estimates (7) follow from the estimates for the minimal eigenvalues of the operators A_i ($i = 0, p, m$). We state them below with ℓ as defined in the introduction:

$$\frac{1}{60}\frac{1}{\ell^2}\lambda_{\min}(A_p) \leq \lambda_{\min}(A_0) \leq \lambda_{\min}(A_m) \leq \frac{\pi^2}{12}\frac{1}{\ell^2} \tag{30}$$

Recall that all $\lambda(A_p) = 1$ and $\mu(\Omega)^2 = \lambda_{\min}(A_0)$.

The upper estimate from (30) is proved in Appendix A. The prove of the lower bound is rather technical and can be found in [20]. The inequality

$$\lambda_{\min}(A_0) \leq \lambda_{\min}(A_m)$$

follows from the embedding $\mathbf{U} \subset \mathbf{U}_m$ and thanks to (26), (29) and Rayleigh's rule:

$$\lambda_{\min}(A_0) = \inf_{q\in P}\sup_{\mathbf{v}\in\mathbf{U}} \frac{(q, \operatorname{div}\mathbf{v})^2}{||\mathbf{v}||_1^2\,||q||_0^2}, \qquad \lambda_{\min}(A_m) = \inf_{q\in P}\sup_{\mathbf{v}\in\mathbf{U}_m} \frac{(q, \operatorname{div}\mathbf{v})^2}{||\mathbf{v}||_1^2\,||q||_0^2}.$$

3. Estimate for a ring

In this section we assume that Ω is the ring

$$\Omega = \{\mathbf{x} = (x_1, x_2) : 0 < R_1 < |\mathbf{x}| < R_2\}, \qquad R_2/R_1 = 1 + \delta, \quad \delta > 0.$$

Since the relation $\mu(\Omega)^2 = \lambda_{\min}(A_0)$ obtained from (26) is still true, we consider the eigenvalue problem $A_0\, p = \lambda\, p$. The following theorem is valid.

Theorem 3.1. *Define $s = R_2/R_1 > 1$, and let (r, φ) be the polar coordinates on R^2, then all the eigenvalues of the problem $A_0\, p = \lambda\, p$ belong to*

$$\{1\} \cup \mathcal{L}_1 \cup \mathcal{L}_2,$$

where

$$\mathcal{L}_1 = \left\{\frac{1}{2}\left(1 + \sqrt{\frac{s^2-1}{s^2+1}\frac{1}{\ln s}}\right)\right\} \bigcup \left\{\frac{1}{2}\left(1 + \frac{(s^{m+1} - s^{m-1})\sqrt{m^2-1}}{\sqrt{(s^{2(m+1)}-1)(s^{2(m-1)}-1)}}\right)\right\},$$

$$\mathcal{L}_2 = \left\{\frac{1}{2}\left(1 - \sqrt{\frac{s^2-1}{s^2+1}\frac{1}{\ln s}}\right)\right\} \bigcup \left\{\frac{1}{2}\left(1 - \frac{(s^{m+1} - s^{m-1})\sqrt{m^2-1}}{\sqrt{(s^{2(m+1)}-1)(s^{2(m-1)}-1)}}\right)\right\}$$

for $m = 2, 3, \ldots$. The eigenvalue $\lambda = 1$ is of infinite multiplicity and some corresponding eigenfunctions are

$$p_k(r, \varphi) = \frac{\pi\, k\, r}{R_2 - R_1}\cos \pi\, k\frac{r - R_1}{R_2 - R_1} - \frac{1}{r}\sin \pi\, k\frac{r - R_1}{R_2 - R_1} + C_k, \quad k = 1, 2, \ldots,$$

each eigenvalue $\lambda \neq 1$ is of double multiplicity and all corresponding eigenfunctions are

$$p^1(r, \varphi) = r\left(1 \mp \left[\frac{R_1}{r}\right]^2\sqrt{\frac{s^4-1}{4\ln s}}\right)\alpha_1(\varphi),$$

$$p^m(r, \varphi) = r^m\left(1 \mp \left[\frac{R_1}{r}\right]^{2m} s^{m-1}\sqrt{\frac{m-1}{m+1}\,\frac{s^{2(m+1)}-1}{s^{2(m-1)}-1}}\right)\alpha_m(\varphi), \quad m = 2, 3, \ldots$$

with $\alpha_m(\varphi) = \cos(m\varphi)$ or $\alpha_m(\varphi) = \sin(m\varphi)$.

Proof. The proof is based on the presentation

$$[u_1, u_2, p] = \sum_j \left[u_1^j(r), u_2^j(r), p^j(r)\right] \exp\{\mathbf{i}j\varphi\}$$

and the possibility to decouple $A_0 p = \lambda p$ into separate differential problems for $u_1^j(r)$, $u_2^j(r)$, and $p^j(r)$, which are solved explicitly. Details can be found in Appendix B. □

Remark 3.2. If R_2 is fixed and $R_1 \to 0$ (*i.e.* $s \to \infty$) we have $\lambda \to \frac{1}{2}$ for all $\lambda \neq 1$. Hence the result of Crouzeix [10] for a circle is recovered.

Corollary 3.3. *With the above assumptions on Ω we have for $\mu(\Omega)$ from (13)*

$$\mu(\Omega) \leq \sqrt{\frac{7}{6}\frac{\delta}{2}}, \ \delta \in (0, 1]$$

and

$$\mu(\Omega) \sim \frac{\delta}{2\sqrt{3}}, \ \delta \to 0.$$

Proof. Consider the eigenvalue

$$\bar{\lambda} = \frac{1}{2}\left(1 - \sqrt{\frac{s^2-1}{s^2+1}\frac{1}{\ln s}}\right).$$

Substituting $s = 1 + \delta$, we get for $\delta \in (0, 1]$

$$\sqrt{\frac{s^2-1}{s^2+1}\frac{1}{\ln s}} = \frac{2\delta + \delta^2}{2 + 2\delta + \delta^2} \frac{1}{\delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} \dots} \geq \frac{2+\delta}{2 + \delta + \frac{7}{6}\delta^2} \geq 1 - \frac{7}{12}\delta^2.$$

Therefore we have

$$\mu(\Omega)^2 = \lambda_{\min}(A_0) \leq \bar{\lambda} \leq \frac{1}{2}\left(1 - \sqrt{1 - \frac{7}{12}\delta^2}\right) \leq \frac{1}{2}\left(1 - (1 - \frac{7}{12}\delta^2)\right) = \frac{7}{24}\delta^2.$$

Thus the estimate for $\mu(\Omega)$ is proved.

To verify the asymptotic for $\mu(\Omega)$, when $\delta \to 0$ we substitute $s = 1 + \delta$ and calculate Taylor expansions w.r.t. δ for the functions from the definition of the eigenvalues. It is straightforward to obtain

$$\frac{2\delta + \delta^2}{2 + 2\delta + \delta^2} \frac{1}{\ln(1+\delta)} = 1 - \frac{1}{3}\delta^2 + \dots$$

and for $m = 2, 3, \dots$

$$\begin{aligned} \frac{((1+\delta)^{m+1} - (1+\delta)^{m-1})^2(m^2-1)}{((1+\delta)^{2(m+1)} - 1)((1+\delta)^{2(m-1)} - 1)} &= \frac{4 + 4(2m-1)\delta + (8m^2 - 12m + 5)\delta^2 + \dots}{4 + 4(2m-1)\delta + \frac{28m^2 - 36m + 15}{3}\delta^2 + \dots} \\ &= 1 - \frac{m^2}{3}\delta^2 + \dots . \end{aligned}$$

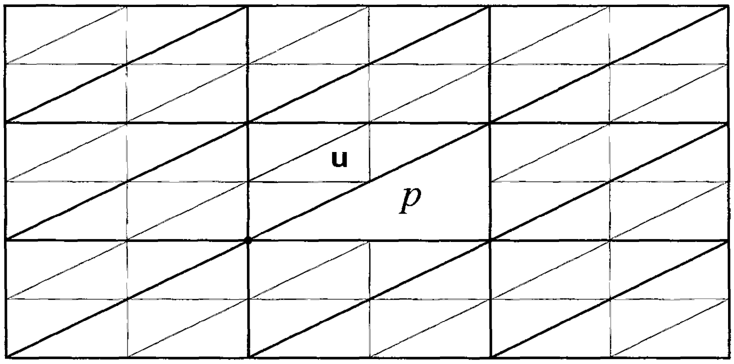

FIGURE 1. $(P^1_{h/2})^2 \times P^0$ velocity-pressure FE.

Using this expansions, we get for $\lambda_{\min}$

$$\lambda_{\min}(A_0) \sim \frac{1}{2}\left(1-\sqrt{1-\frac{1}{3}\delta^2}\right) \sim \frac{1}{2}\left(1-(1-\frac{1}{6}\delta^2)\right) = \frac{\delta^2}{12},\ \delta \to 0.$$

□

Remark 3.4. From the above expansions it is clearly seen that the asymptotic behaviour $\lambda \sim c\,\delta^2$ holds for all eigenvalues from the set $\mathcal{L}_2$ defined in Theorem 3.1.

Remark 3.5. The value of $\mu(\Omega)$ compared with the value of Korns' constant for a ring [11], shows that (9) is not valid now. Thus the assumption on Ω to be simply-connected is necessary for (9).

4. Numerical examples

First we consider two examples of conforming FE pairs, which are known (*e.g.*, [14]) to satisfy the LBB condition. These are piecewise linear or bilinear velocity functions w.r.t. a subdivision of Ω into triangles or rectangles, respectively. In both cases the discrete velocity is continuous over $\overline{\Omega}$, and the discrete pressure is piecewise constant over Ω. We assume

$$\Omega = (0, \ell) \times (0, 1).$$

Scheme I. Consider the regular ("north-east") triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}$ into triangles. Divide each macro-triangle in $\mathcal{T}_h$ into four mini-triangles by joining the mid-sides. This defines a finer triangulation $\mathcal{T}_{h/2}$.

Define

$$\begin{aligned} \mathbf{U}_h &= \left\{\mathbf{v}|\mathbf{v} \in P^1(\triangle)^2, \triangle \in \mathcal{T}_{h/2}; \quad \mathbf{v} \in C^0(\overline{\Omega}); \quad \mathbf{v} = 0 \text{ on } \partial\Omega\right\}, \\ P_h &= \left\{q|q \in P^0(\triangle), \triangle \in \mathcal{T}_h; \quad \int_\Omega q\, d\Omega = 0\right\}. \end{aligned}$$

Here $P^r(\triangle)$ denotes the space of polynomials of degree not greater then r on an element of the triangulation $\triangle \subset R^2$. We illustrate Scheme I in Figure 1.

Scheme II. We start with subdivision $\mathcal{Q}_h$ of the domain $\overline{\Omega}$ into rectangles. Subsequently we divide each rectangle into four smaller rectangles by joining the opposite mid-sides. This defines another subdivision $\mathcal{Q}_{h/2}$ of $\overline{\Omega}$.

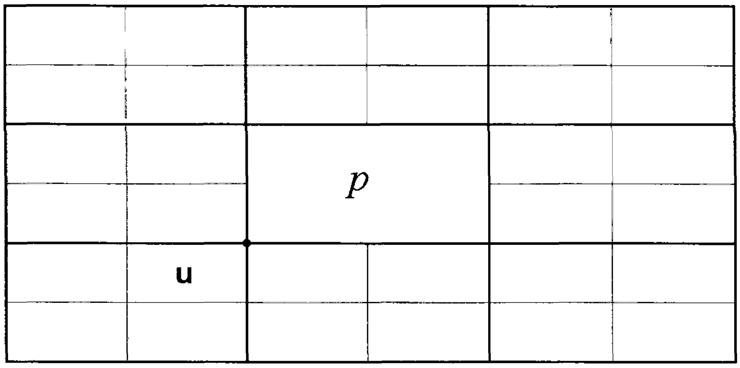

FIGURE 2. $(Q^1_{h/2})^2 \times Q^0$ velocity-pressure FE.

The FE velocity field consists of the piecewise bilinear functions w.r.t. the subdivision $\mathcal{Q}_{h/2}$, which are continuous over $\overline{\Omega}$ and vanish on $\partial\Omega$, *i.e.*

$$\mathbf{U}_h = \left\{ \mathbf{v} | \mathbf{v} \in Q^1(\Box)^2, \Box \in \mathcal{Q}_{h/2}; \quad \mathbf{v} \in C^0(\overline{\Omega}); \quad \mathbf{v} = 0 \text{ on } \partial\Omega \right\}.$$

The pressure FE space consists of piecewise constants w.r.t. the macro subdivision $\mathcal{Q}_h$ with zero mean over Ω, *i.e.*

$$P_h = \left\{ q | q \in Q^0(\Box), \Box \in \mathcal{Q}_h; \quad \int_\Omega q d\Omega = 0 \right\}.$$

See Figure 2.

For nodal functions associated with the FE functions we can define in the standard way the nodal Laplacian operator A, div-operator B, and pressure mass matrix M_p. Then, similar to the continuous case (*cf.* Sect. 2) the constant γ_h from (3) equals $\sqrt{\lambda^h_{\min}}$, where $\lambda^h_{\min}$ is the minimal eigenvalue of the eigenvalue problem

$$B A^{-1} B^T p = \lambda^h M_p p, \quad p \in \bar{P}_h,$$

where $\bar{P}_h$ is the space of the nodal functions associated with the pressure FE functions.

The minimal eigenvalue was determined by the subspace iteration algorithm (see [21]). The subroutine EA12 from the HARWELL Numerical Analysis Library was used. The process was considered as converged when the residual, normalised by the eigenvector, was less than 10^{-10} in the discrete L_2 norm.

In Table 1 we present the values of $\gamma_h(\Omega)$ for different ℓ with $h_1 = h_2 = 1/64$ and both FE schemes. In the bottom row the value of the upper bound

$$\bar{\mu} = \frac{\pi}{2\sqrt{3}} \ell^{-1}$$

is given for reference.

The data from Table 1 confirms the asymptotic behaviour

$$\gamma(\Omega) \approx O(\ell^{-1}) \quad \text{with} \quad \ell \to \infty,$$

that was predicted by the analysis of the paper.

Table 2 shows the calculated values of γ_h for different ℓ with $h_1 = \ell\, h_2$, $h_2 = 1/256$.

Compared with Table 1 numerical results from Table 2 show that the dependence of γ_h on the mesh aspect ratio h_1/h_2 is very weak. This agrees with the numerical experience in [26] for non-conforming velocity FE space.

TABLE 1. The dependence of γ_h on the domain aspect ratio.

	ℓ			
FE scheme	1	2	4	8
I	0.447424	0.387510	0.218469	0.112338
II	0.479624	0.388664	0.211852	0.112345
$\bar{\mu}(\Omega)$	0.9069	0.45345	0.226725	0.113363

TABLE 2. The dependence of γ_h on the domain aspect ratio and the mesh aspect ratio.

	ℓ			
FE scheme	1	2	4	8
I	0.440679	0.387417	0.218494	0.112432
II	0.460223	0.387812	0.218534	0.124900

TABLE 3. The dependence of γ_h on the mesh size.

	mesh size					
FE scheme	1/8	1/16	1/32	1/64	1/128	1/256
I	0.474990	0.461353	0.452987	0.447424	0.443527	0.440679
II	0.560231	0.521009	0.496087	0.479624	0.468308	0.460222

In Table 3 we present the values of γ_h with different $h_1 = h_2 = h$ for the unit square. These results illustrate that in general the LBB condition (3) holds for both FE pairs, *i.e.* the mesh-independent limit $\gamma(\Omega)$ exists for both FE schemes.

In the next example we use non-conforming quadrilateral elements $(\tilde{Q}_h^2)^2 \times Q_h^0$ from [22], *i.e.* on every element the velocities are spanned by $\langle x^2 - y^2, x, y, 1\rangle$ and the pressure is constant. The domain and a coarse mesh are shown on Figure 3. The length of the channel equals 2.5 m and the hight 0.41 m, a cylinder of diameter 0.1 m is placed at 0.45 m from the inlet. For unsteady incompressible flows this is a benchmarking configuration (see details in [24]). Here the steady Stokes flow around cylinder was calculated. The Featflow software [27] was used.

The average convergence factors after 50 iterations of the Uzawa algorithm are presented in Table 4. The second line of the results shows the factors for the same configuration but with the outlet placed closer to the cylinder (1.5 m). We recall that the convergence of the Uzawa algorithm is ruled by γ_h (see (5)). The level number in Table 4 indicates the number of refining steps applied to the coarse mesh. At every step each element is divided to four finer elements by joining midpoints on opposite edges. As was expected the convergence is mesh-independed, however decreases while the channel becomes longer.

APPENDIX A

In this part of the appendix we prove the upper bound in (30) Assuming that Ω is defined as in (27), consider the eigenvalue problem $A_m p = \lambda p$. Introducing the auxiliary function $\mathbf{u} = \Delta_m^{-1}\nabla p$, we reformulate

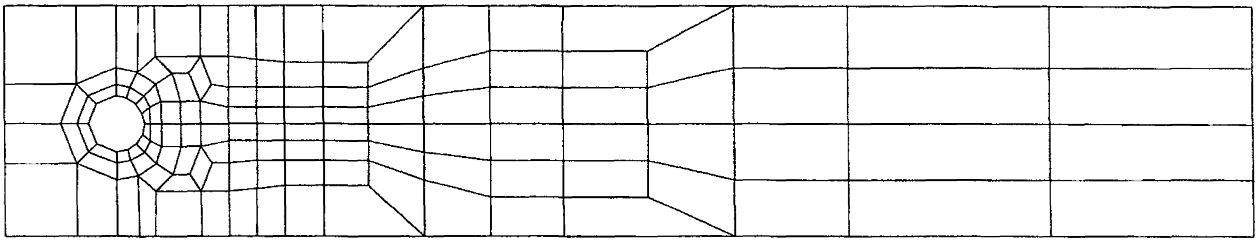

FIGURE 3. Coarse grid partition for non-conforming FE.

TABLE 4. The convergence rates of Uzawa algorithm.

	mesh levels			
Length	3	4	5	6
2.5	0.88	0.86	0.85	0.85
1.5	0.76	0.75	0.74	0.74

the problem: Find eigenvalues λ and eigenfunctions $p \in P$ which satisfy

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0, \\ \operatorname{div} \mathbf{u} = \lambda p \end{cases} \tag{31}$$

with some function $\mathbf{u} \in \mathbf{U}_m$ subject to the boundary conditions (28).

All solutions to (31), (28) can be found by the method of splitting the variables (*cf.* [1]). Here it suffices to note that the operator A_m has the eigenvalue

$$\bar{\lambda}(A_m) = \frac{1}{2}\left(1 - \frac{t}{\sinh t}\right) \quad \text{with } t = \pi \frac{L_2}{L_1}.$$

Indeed, consider the domain $\Omega = (0, L_1) \times (-b, b)$ with $b = L_1/2$. This shift of the original rectangle does not change the eigenvalues but simplifies the analysis. Further, setting $r = \pi/L_1$, by a straightforward substitution we check that the functions

$$u_1 = \frac{1}{2} \sin rx_1 \left[b \frac{\sinh rb}{\cosh rb} \cosh rx_2 - x_2 \sinh rx_2 \right],$$

$$u_2 = -\frac{1}{2} \cos rx_1 \left[b \frac{\cosh rb}{\sinh rb} \sinh rx_2 - x_2 \cosh rx_2 \right],$$

$$p = \cos rx_1 \cosh rx_2$$

satisfy (31), (28) together with the eigenvalue $\bar{\lambda}(A_m)$.

In Section 2 it was shown that $\lambda_{\min}(A_0) \le \lambda_{\min}(A_m)$. Thus one immediately gets

$$\lambda_{\min}(A_0) \le \lambda_{\min}(A_m) \le \bar{\lambda}(A_m) = \frac{1}{2}\left(1 - \frac{t}{\sinh t}\right)$$

with $t = \pi L_2/L_1 = \pi/\ell$. This proves the upper bound in (30) due to trivial calculations:

$$\frac{1}{2}\left(1 - \frac{t}{\sinh t}\right) = \frac{t^2}{2\cdot 3!}\,\frac{1 + \dfrac{t^2 3!}{5!} + \dfrac{t^4 3!}{7!} + \cdots}{1 + \dfrac{t^2}{3!} + \dfrac{t^4}{5!} + \cdots} \le \frac{\pi^2}{12}\,\frac{1}{\ell^2}.$$

Appendix B

In the appendix B we outline the proof of Theorem 3.1. Consider the eigenvalue problem $A_0 p = \lambda p$. Introducing the auxiliary function $\mathbf{u} = \Delta_0^{-1}\nabla p$, we reformulate the problem as: Find eigenvalues λ and eigenfunctions $p \in P$ which satisfy

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0, \\ \operatorname{div}\, \mathbf{u} = \lambda p \end{cases}$$

with some function $\mathbf{u} \in \mathbf{U}_0$.

Let us rewrite the eigenvalue problem in the polar coordinate system:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi} - \frac{\partial p}{\partial r} &= 0, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi} - \frac{1}{r}\frac{\partial p}{\partial \varphi} &= 0, \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,u) + \frac{\partial v}{\partial \varphi}\right] &= \lambda p, \end{aligned} \tag{32}$$

where we denote $u = u_1, v = u_2$.

For $\lambda = 1$ the theorem is checked by substituting functions

$$p_k = \frac{\partial u_k}{\partial r} + \frac{1}{r}u_k$$

and

$$u_k = \sin \pi\, k\frac{r - R_1}{R_2 - R_1}\,, \qquad v_k(r) = 0$$

to (32) for any integer k.

Further consider an eigenfunction p and auxiliary u, v, corresponding to some $\lambda \ne 1$. The periodical boundary conditions for φ imply the representation

$$[u, v, p] = \sum_{m=-\infty}^{+\infty} [u_m(r), v_m(r), p_m(r)]\, \exp\{im\varphi\}.$$

Since the system of exponents $\exp\{im\varphi\}$ is orthogonal in $L^2([0, 2\pi])$, from (32) follows

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u_m}{\partial r}\right) - \frac{m^2}{r^2}u_m - \frac{u_m}{r^2} - \frac{2im}{r^2}v_m - p_m' = 0, \tag{33}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v_m}{\partial r}\right) - \frac{m^2}{r^2}v_m - \frac{v_m}{r^2} + \frac{2im}{r^2}u_m - \frac{im}{r}p_m = 0, \tag{34}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\,u_m) + im\,v_m\right] = \lambda p_m \tag{35}$$

with boundary conditions

$$u_m(R_2) = v_m(R_2) = u_m(R_1) = v_m(R_1) = 0. \tag{36}$$

For the harmonic $m = 0$ it is straightforward to check (using (34) to get $v_0 = 0$) that the only possibility is $\lambda = 1$.

Thus we are interested in $m \neq 0$. Eliminating u_m and v_m we obtain from (33)–(35)

$$(\lambda - 1)\,\Delta_r p_m = 0.$$

Hence

$$p_m(r) = \frac{1}{2m}\left(C_+ r^m + C_- r^{-m}\right), \tag{37}$$

where C_+, C_- are arbitrary constants. From (33) and (34) we get

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u_m}{\partial r}\right) - \frac{m^2}{r^2}u_m - \frac{u_m}{r^2} - \frac{2im}{r^2}v_m &= \frac{1}{2}\left[C_+ r^{m-1} - C_- r^{-m-1}\right],\\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v_m}{\partial r}\right) - \frac{m^2}{r^2}v_m - \frac{v_m}{r^2} + \frac{2im}{r^2}u_m &= \frac{i}{2}\left[C_+ r^{m-1} + C_- r^{-m-1}\right].
\end{aligned}$$

Setting $w_\pm = u_m \pm i v_m$ we obtain a boundary value problem with parameter m

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial w_\pm}{\partial r}\right) - \frac{(m \mp 1)^2}{r^2}w_\pm = \pm C_\pm r^{\pm m-1},\\
&w_\pm(R_1) = w_\pm(R_2) = 0.
\end{aligned} \tag{38}$$

First, consider the case $m = 1$. We get

$$w_+ = A_+ \ln r + B_+ + \frac{C_+}{4}r^2, w_- = A_- r^2 + B_- r^{-2} + \frac{C_-}{4},$$

where

$$A_+ = \frac{C_+}{4\ln\dfrac{R_2}{R_1}}\left(R_1^2 - R_2^2\right), \quad B_+ = \frac{C_+}{4\ln\dfrac{R_2}{R_1}}\left(R_2^2 \ln R_1 - R_1^2 \ln R_2\right),$$

and

$$A_- = -\frac{C_-}{4}\frac{1}{R_1^2 + R_2^2}, \quad B_- = -\frac{C_-}{4}\frac{(R_1R_2)^2}{R_1^2 + R_2^2}.$$

Substituting $u_m = (w_+ + w_-)/2$, $iv_m = (w_+ - w_-)/2$ into (35) we obtain

$$\frac{C_+}{8}\left(\frac{R_1^2 - R_2^2}{\ln\frac{R_2}{R_1}} + 2\,r^2\right) - \frac{C_-}{4}\frac{1}{R_1^2 + R_2^2}\left(2\,r^2 - R_1^2 - R_2^2\right) = \frac{\lambda}{2}\left(C_+ r^2 + C_-\right).$$

Since the functions r^0 and r^2 are linearly independent, this yields the only possible values ($s = R_2/R_1 > 1$)

$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{\frac{s^2 - 1}{s^2 + 1}\frac{1}{\ln s}}\right). \tag{39}$$

We also find

$$p_1(r) = C\,r\left(1 \mp \left[\frac{R_1}{r}\right]^2 \sqrt{\frac{s^4-1}{4\ln s}}\right).$$

The same result is valid for the case $m = -1$. Therefore it follows that the λ from (39) are the eigenvalues corresponding to the eigenfunctions

$$\begin{aligned} p^1(r,\varphi) &= \frac{1}{2}p_1(r)(\exp{(\mathrm{i}\varphi)} + \exp{(-\mathrm{i}\varphi)}), \\ p^2(r,\varphi) &= \frac{1}{2\mathrm{i}}p_1(r)(\exp{(\mathrm{i}\varphi)} - \exp(-\mathrm{i}\varphi)). \end{aligned}$$

For $|m| > 1$ the solution to the problem (38) is

$$\begin{aligned} w_+ &= A_+\,r^{m-1} + B_+\,r^{-(m-1)} + \frac{C_+}{4m}r^{m+1}, \\ w_- &= A_-\,r^{m+1} + B_-\,r^{-(m+1)} + \frac{C_-}{4m}r^{-(m-1)}, \end{aligned}$$

where

$$\begin{aligned} A_+ &= -\frac{C_+}{4m}\frac{R_2^{2m} - R_1^{2m}}{R_2^{2(m-1)} - R_1^{2(m-1)}}, & B_+ &= -\frac{C_+}{4m}\frac{(R_1^2 - R_2^2)(R_1R_2)^{2(m-1)}}{R_2^{2(m-1)} - R_1^{2(m-1)}}, \\ A_- &= -\frac{C_-}{4m}\frac{R_2^2 - R_1^2}{R_2^{2(m+1)} - R_1^{2(m+1)}}, & B_- &= -\frac{C_-}{4m}\frac{(R_2^{2m} - R_1^{2m})(R_1R_2)^2}{R_2^{2(m+1)} - R_1^{2(m+1)}}. \end{aligned}$$

Similar considerations lead to the only possible values

$$\lambda = \frac{1}{2}\left(1 \pm \frac{(s^{m+1} - s^{m-1})\sqrt{m^2-1}}{\sqrt{(s^{2(m+1)}-1)(s^{2(m-1)}-1)}}\right)$$

and

$$p_m(r) = C\,r^k\left(1 \mp \left[\frac{R_1}{r}\right]^{2k} s^{k-1}\sqrt{\frac{k-1}{k+1}\,\frac{s^{2(k+1)}-1}{s^{2(k-1)}-1}}\right), \quad k = |m| > 1.$$

The theorem is proved.

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