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SEMI-GLOBAL C^1 SOLUTION AND EXACT BOUNDARY CONTROLLABILITY FOR REDUCIBLE QUASILINEAR HYPERBOLIC SYSTEMS *

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Abstract. By means of a result on the semi global C^1 solution, we establish the exact boundary controllability for the reducible quasilinear hyperbolic system if the C^1 norm of initial data and final state is small enough

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1 INTRODUCTION

We consider the exact boundary control for the following reducible quasilinear hyperbolic system

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r,s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \mu(r,s)\dfrac{\partial s}{\partial x} = 0 \end{cases} \tag{1 1}$$

with the initial data

$$r(0,x) = r_0(x), \qquad s(0,x) = s_0(x), \quad 0 \le x \le 1 \tag{1 2}$$

and the nonlinear boundary feedback controls

$$\begin{cases} s = g(t,r) + v(t) & \text{at} \quad x = 0, \\ r = f(t,s) + u(t) & \text{at} \quad x = 1 \end{cases} \tag{1 3}$$

Without loss of generality, we may assume that

$$f(t,0) \equiv g(t,0) \equiv 0 \tag{1 4}$$

Keywords and phrases reducible quasilinear hyperbolic system, semi-global C^1 solution, exact boundary controllability

* *Dedicated to Roger Temam for his 60th birthday*

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Since we consider only C^1 solution, we assume that the coefficients λ, μ and the nonlinear feedback laws f, g are all C^1 functions on the domain under consideration. Moreover, we suppose that the system is strictly hyperbolic and

$$\lambda(r,s) < 0 < \mu(r,s). \tag{1.5}$$

Also we assume that the initial data (r_0, s_0) and the input control (u, v) are all C^1 functions satisfying the compatibility conditions

$$\begin{cases} s_0(0) = g(0, r_0(0)) + v(0), \\ r_0(1) = f(0, s_0(1)) + u(0) \end{cases} \tag{1.6}$$

and

$$\begin{cases} \mu(r_0(0), s_0(0))s_0'(0) = -\dfrac{\partial g}{\partial t}(0, r_0(0)) + \dfrac{\partial g}{\partial r}(0, r_0(0))\lambda(r_0(0), s_0(0))r_0'(0) - v'(0), \\ \lambda(r_0(1), s_0(1))r_0'(1) = -\dfrac{\partial f}{\partial t}(0, s_0(1)) + \dfrac{\partial f}{\partial s}(0, s_0(1))\mu(r_0(1), s_0(1))s_0'(1) - u'(0). \end{cases} \tag{1.7}$$

As in [5], we will consider the following exact boundary controllability:

Given initial data $r_0, s_0 \in C^1[0,1]$ and finial data $r_T, s_T \in C^1[0,1]$, can we find a time $T > 0$ and boundary input controls $u, v \in C^1[0,T]$ such that the mixed initial-boundary value problem (1.1)–(1.3) admits a unique C^1 solution $(r(t,x), s(t,x))$ verifying the final condition

$$r(T,x) = r_T(x), \qquad s(T,x) = s_T(x), \quad \forall 0 \le x \le 1? \tag{1.8}$$

First of all, because of the finite speed of the wave propagation, the exact boundary controllability of hyperbolic system requires that the controllability time T must be greater than a given constant. On the other hand, following the local existence theorem of C^1 solution (see [6]), there exists a constant $\delta > 0$ such that the mixed initial-boundary value problem (1.1)–(1.3) admits a unique C^1 solution $(r(t,x), s(t,x))$ on the domain

$$D_\delta = \{(t,x) |\ 0 \le t \le \delta, \quad 0 \le x \le 1\}. \tag{1.9}$$

But this C^1 solution may blow up in a finite time (see Ref. [4]). So, the mixed initial-boundary value problem (1.1)–(1.3) has no global C^1 solution in general. We even don't know if the life span of the C^1 solution could be greater than a given $T > 0$. In order to avoid this difficulty, in [5] the authors considered the linearly degenerate case:

$$\lambda(r,s) \equiv \lambda(s), \qquad \mu(r,s) \equiv \mu(r). \tag{1.10}$$

In that case, the global existence of C^1 solution $(r(t,x), s(t,x))$ and the global exact boundary controllability for the system (1.1)–(1.3) were actually proved.

In this paper, we consider the general case that system (1.1) is not necessary to be linearly degenerate. We first give suitable conditions on the initial data (r_0, s_0) and the input control (u, v) such that for a given $T > 0$, the mixed initial-boundary value problem (1.1)–(1.3) admits a unique C^1 solution $(r(t,x), s(t,x))$ on the domain

$$D_T = \{(t,x) |\ 0 \le t \le T, \quad 0 \le x \le 1\}. \tag{1.11}$$

We will refer to this solution as a semi-global C^1 solution.

Let $(r(t,x), s(t,x))$ be a local C^1 solution to the mixed initial-boundary value problem (1.1)–(1.3) on the domain D_δ with $0 < \delta \le T$. In order to extend this local C^1 solution up to the domain D_T, it suffices to establish the following uniform *a priori* estimate: for any given δ with $0 < \delta \le T$,

$$\|(r(t,\cdot), s(t,\cdot))\|_{C^1[0,1]} \le C(T), \qquad \forall 0 \le t \le \delta, \tag{1.12}$$

where C is a positive constant depending on T, but independent of δ.

In Section 2, we will prove the existence and uniqueness of semi-global C^1 solution to the mixed initial-boundary value problem (1.1)–(1.3), provided that the C^1 norm of initial data (r_0, s_0) and the boundary control (u, v) is small enough. In Section 3, using a similar approach as that in [5], we will establish the local exact boundary controllability for the system (1.1)–(1.3).

There is a number of publications concerning the exact controllability and uniform stabilization for linear hyperbolic systems (see [7, 8, 9] and the references therein). Furthermore, the exact boundary controllability for semilinear wave and plate equations were also established in [10] and [3]. However, only a little is known concerning quasilinear hyperbolic systems. We mention that M. Cirinà [1, 2] considered the local exact boundary controllability for quasilinear hyperbolic systems with linear boundary controls. In order to obtain the semi-global solution, the author of [1, 2] needed very strong conditions on the coefficients of the system (globally bounded and globally Lipschitz continuous). This is a grave restriction to the application. Since except [2] there is little results on the semi-global C^1 solution to quasilinear hyperbolic systems in the literature, we hope that the discussion in this paper on the semi-global C^1 solution to quasilinear hyperbolic systems would also promote a systematic investigation in this area.

2. Existence and uniqueness of semi-global C^1 solution

In this section, we will give the existence and uniqueness of semi-global C^1 solution to the mixed initial-boundary value problem (1.1)–(1.3). The main result is the following

Theorem 2.1. *For a given $T > 0$, the mixed initial-boundary value problem (1.1)–(1.3) admits a unique semi-global C^1 solution $(r(t,x), s(t,x))$ on the domain D_T defined in (1.11), provided that the C^1 norms $\|(r_0, s_0)\|_{C^1[0,1]}$ and $\|(u,v)\|_{C^1[0,T]}$ are small enough. Moreover the C^1 norm of the solution $(r(t,x), s(t,x))$ can be arbitrarily small.*

Proof. Following the local existence theorem of C^1 solution (see [6]), the mixed initial-boundary value problem (1.1)–(1.3) admits a unique local C^1 solution $(r(t,x), s(t,x))$ on the domain D_δ. In order to obtain the semi-global C^1 solution on the domain D_T, it is sufficient to prove that, for any given δ with $0 < \delta \le T$, the local C^1 solution $(r(t,x), s(t,x))$ on D_δ satisfies the following uniform *a priori* estimate

$$\|(r(t,\cdot), s(t,\cdot))\|_{C^1[0,1]} \le C(T), \qquad \forall 0 \le t \le \delta, \tag{2.1}$$

where C is a positive constant independent of δ.

We first assume that there exists a constant $M > 0$ such that

$$\|(r(t,\cdot), s(t,\cdot))\|_{C^0[0,1]} \le M, \qquad \forall 0 \le t \le \delta. \tag{2.2}$$

We will justify this assumption at the end of the proof. Let

$$T_1 = \min_{|r|,|s| \le M} \left\{ \frac{1}{\mu}, \quad \frac{1}{|\lambda|} \right\} > 0. \tag{2.3}$$

For any given point $(t_0, x_0) \in D_{T_1}$, we consider the λ-characteristic $x = x_1(t)$ passing through (t_0, x_0):

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \lambda(r(t, x_1(t)), s(t, x_1(t))), \qquad x_1(t_0) = x_0. \tag{2.4}$$

We distinguish two cases: (a) The λ-characteristic $x = x_1(t)$ intersects the x-axis at a point $(0, \alpha)$. Then, since r is the corresponding Riemann invariant, we have

$$|r(t_0, x_0)| = |r(0, \alpha)| = |r_0(\alpha)| \leq \|r_0\|_{C^0[0,1]}. \tag{2.5}$$

(b) The λ-characteristic $x = x_1(t)$ intersects the right boundary of D_{T_1} at a point $(\tilde{t}_0, 1)$. Then, we consider the μ-characteristic $x = x_2(t)$ passing through $(\tilde{t}_0, 1)$:

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \mu(r(t, x_2(t)), s(t, x_2(t))), \qquad x_2(\tilde{t}_0) = 1. \tag{2.6}$$

By virtue of the choice of T_1, the μ-characteristic $x = x_2(t)$ must intersect the x-axis at a point $(0, \beta)$. Since r and s are the corresponding Riemann invariants respectively, we have

$$r(t_0, x_0) = r(\tilde{t}_0, 1), \quad s(\tilde{t}_0, 1) = s_0(\beta). \tag{2.7}$$

On the other hand, using the boundary condition (1.3) we have

$$r(\tilde{t}_0, 1) = f(\tilde{t}_0, s_0(\beta)) + u(\tilde{t}_0). \tag{2.8}$$

Then, noting (1.4) we get

$$\begin{aligned} |r(t_0, x_0)| &\leq |f(\tilde{t}_0, s_0(\beta))| + |u(\tilde{t}_0)| \\ &\leq \max_{|s| \leq M} \left|\frac{\partial f}{\partial s}(\tilde{t}_0, s)\right| |s_0(\beta)| + |u(\tilde{t}_0)| \\ &\leq A_0 \|s_0\|_{C^0[0,1]} + \|u\|_{C^0[0,T]} = (A_0 + 1)a_0, \end{aligned} \tag{2.9}$$

where

$$\begin{cases} a_0 = \max\left\{\|(r_0, s_0)\|_{C^0[0,1]}, \quad \|(u, v)\|_{C^0[0,T]}\right\}, \\ A_0 = \max\limits_{0 \leq t \leq T,\ |r|,|s| \leq M} \left\{\left|\dfrac{\partial f}{\partial s}(t, s)\right|, \quad \left|\dfrac{\partial g}{\partial r}(t, r)\right|\right\}. \end{cases} \tag{2.10}$$

Similarly, we have

$$|s(t_0, x_0)| \leq (A_0 + 1)a_0. \tag{2.11}$$

Combining (2.9) and (2.11) we get

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)a_0, \qquad \forall 0 \leq t \leq T_1. \tag{2.12}$$

Repeating the previous procedure with the new initial data $(r(T_1, x), s(T_1, x))$ on $t = T_1$, we obtain

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^2 a_0, \quad \forall T_1 \leq t \leq 2T_1. \tag{2.13}$$

In this way, after at most $N \leq [T/T_1] + 1$ iterations, we arrive at the estimate

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^N a_0, \quad \forall (N-1)T_1 \leq t \leq \delta. \tag{2.14}$$

Then, collecting the estimates (2.12)–(2.14), we obtain

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^N a_0 \triangleq C_1(T)a_0, \qquad \forall 0 \leq t \leq \delta, \tag{2.15}$$

where $C_1(T)$ is independent of δ. Moreover, since a_0 can be taken to be small enough, (2.15) also verifies the validness of (2.2).

Now we estimate the C^0 norm of $\frac{\partial r}{\partial x}$ and $\frac{\partial s}{\partial x}$. First let us recall the well-known Lax transformation [4]:

$$U = e^{h(r,s)}\frac{\partial r}{\partial x}, \quad V = e^{k(r,s)}\frac{\partial s}{\partial x}, \tag{2.16}$$

where the functions h, k are given by

$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{\lambda - \mu}\dfrac{\partial \lambda}{\partial s}, & h(0,0) = 0, \\ \dfrac{\partial k}{\partial r} = \dfrac{1}{\mu - \lambda}\dfrac{\partial \mu}{\partial r}, & k(0,0) = 0. \end{cases} \tag{2.17}$$

A straightforward computation shows that along the λ-characteristic $x = x_1(t)$, U satisfies the following Riccati's equation

$$\frac{\mathrm{d}U}{\mathrm{d}_1 t} = -\frac{\partial \lambda}{\partial r}e^{-h(r,s)}U^2, \tag{2.18}$$

where $\frac{\mathrm{d}}{\mathrm{d}_1 t} = \frac{\partial}{\partial t} + \lambda(r,s)\frac{\partial}{\partial x}$, and along the μ-characteristic $x = x_2(t)$, V satisfies the following Riccati's equation

$$\frac{\mathrm{d}V}{\mathrm{d}_2 t} = -\frac{\partial \mu}{\partial s}e^{-k(r,s)}V^2 \tag{2.19}$$

where $\frac{\mathrm{d}}{\mathrm{d}_2 t} = \frac{\partial}{\partial t} + \mu(r,s)\frac{\partial}{\partial x}$.

We next define some constants:

$$b_0 = \max\{\|(r_0', s_0')\|_{C^0[0,1]}, \|(u', v')\|_{C^0[0,T]}\}, \tag{2.20}$$

$$A_1 = \max_{0 \le t \le T, |r|, |s| \le M}\left\{\left|\frac{\partial f}{\partial t}(t,s)\right|, \quad \left|\frac{\partial g}{\partial t}(t,r)\right|\right\}, \tag{2.21}$$

$$M_1 = \max_{|r|, |s| \le M}\left\{e^{|h(r,s)|}, \quad e^{|k(r,s)|}\right\}, \tag{2.22}$$

$$M_2 = \max_{|r|, |s| \le M}\left\{\left|\frac{\partial \lambda}{\partial r}(r,s)\right|, \quad \left|\frac{\partial \mu}{\partial s}(r,s)\right|\right\}, \tag{2.23}$$

$$M_3 = \max_{|r|, |s| \le M}\{|\lambda(r,s)|, \quad \mu(r,s)\}, \tag{2.24}$$

$$M_4 = \max_{|r|, |s| \le M}\left\{\frac{1}{|\lambda(r,s)|}, \quad \frac{1}{\mu(r,s)}, \quad 1\right\}. \tag{2.25}$$

As in the previous stage, we distinguish two cases:

(a) The λ-characteristic $x = x_1(t)$ passing through $(t_0, x_0) \in D_{T_1}$ intersects the x-axis at a point $(0, \alpha)$. Then solving (2.18), we get

$$U(t_0, x_0) = \frac{U(0,\alpha)}{1 + U(0,\alpha)\int_0^{t_0}\frac{\partial \lambda}{\partial r}e^{-h(r(\tau, x_1(\tau)), s(\tau, x_1(\tau)))}\mathrm{d}\tau}. \tag{2.26}$$

From the definition (2.16) and noting (1.2), we have

$$U(0,\alpha) = e^{-h(r_0(\alpha), s_0(\alpha))}r_0'(\alpha).$$

Then

$$|U(0,\alpha)| \le M_1|r_0'(\alpha)| \le M_1 b_0. \tag{2.27}$$

Thus

$$1 + U(0,\alpha)\int_0^{t_0} \frac{\partial\lambda}{\partial r} e^{-h(r(\tau,x_1(\tau)),s(\tau,x_1(\tau)))}\mathrm{d}\tau \ge 1 - T_1M_1^2M_2b_0 \ge \frac{1}{2}, \tag{2.28}$$

provided that b_0 is small enough. It follows from (2.26)–(2.28) that

$$|U(t_0,x_0)| \le 2M_1b_0. \tag{2.29}$$

(b) The λ-characteristic $x = x_1(t)$ passing through $(t_0,x_0) \in D_{T_1}$ intersects the right boundary of D_{T_1} at a point $(\tilde{t}_0,1)$. Then by virtue of the choice of T_1, the μ-characteristic $x = x_2(t)$ passing through $(\tilde{t}_0,1)$ must intersect the x-axis at a point $(0,\beta)$. Therefore, solving (2.18) and (2.19) respectively, we get

$$V(\tilde{t}_0,1) = \frac{V(0,\beta)}{1 + V(0,\beta)\int_0^{\tilde{t}_0} \frac{\partial\mu}{\partial s} e^{-k(r(\tau,x_2(\tau)),s(\tau,x_2(\tau)))}\mathrm{d}\tau} \tag{2.30}$$

and

$$U(t_0,x_0) = \frac{U(\tilde{t}_0,1)}{1 + U(\tilde{t}_0,1)\int_{\tilde{t}_0}^{t_0} \frac{\partial\lambda}{\partial r} e^{-h(r(\tau,x_1(\tau)),s(\tau,x_1(\tau)))}\mathrm{d}\tau}. \tag{2.31}$$

Similarly to (2.29), we get

$$|V(\tilde{t}_0,1)| \le 2M_1b_0. \tag{2.32}$$

On the other hand, differentiating the boundary condition (1.3) at the end $x = 1$ with respect to t, we have

$$\frac{\partial r}{\partial x} = -\frac{1}{\lambda}\Big(\frac{\partial f}{\partial t} + u'\Big) + \frac{\mu}{\lambda}\frac{\partial f}{\partial s}\frac{\partial s}{\partial x} \quad \text{on} \quad x = 1. \tag{2.33}$$

Thus, noting (2.16), we obtain that

$$U(\tilde{t}_0,1) = -\frac{e^h}{\lambda}\Big(\frac{\partial f}{\partial t}(\tilde{t}_0, s(\tilde{t}_0,1)) + u'(\tilde{t}_0)\Big) + \frac{\mu}{\lambda}\frac{\partial f}{\partial s}(\tilde{t}_0, s(\tilde{t}_0,1))e^{h-k}V(\tilde{t}_0,1). \tag{2.34}$$

It follows from (2.32) and (2.34) that

$$|U(\tilde{t}_0,1)| \le M_1M_4(A_1 + b_0) + A_0M_1^2M_3M_4\,|V(\tilde{t}_0,1)| \le M_1M_4\big(A_1 + b_0 + 2A_0M_1^2M_3b_0\big). \tag{2.35}$$

Moreover, noting (1.4) and (2.15), it is easy to see that $A_1 \to 0$ as $a_0 \to 0$. Therefore, similarly to (2.28), we have

$$1 + U(\tilde{t}_0,1)\int_{\tilde{t}_0}^{t_0} \frac{\partial\lambda}{\partial r} e^{-h(r(\tau,x_1(\tau)),s(\tau,x_1(\tau)))}\mathrm{d}\tau \ge \frac{1}{2}, \tag{2.36}$$

provided that a_0, b_0 are small enough. Noting (2.35) and (2.36), it follows from (2.31) that

$$|U(t_0,x_0)| \le 2M_1M_4\big(A_1 + b_0 + 2A_0M_1^2M_3b_0\big), \tag{2.37}$$

provided that a_0, b_0 are small enough. Since $M_4 \geq 1$, the estimate (2.37) remains true for the two cases.

In a similar way, we can prove that $|V(t_0, x_0)|$ satisfies also the estimate (2.37). Then, noting (2.16), we obtain

$$\Big\| \Big(\frac{\partial r}{\partial x}(t, \cdot), \frac{\partial s}{\partial x}(t, \cdot) \Big) \Big\|_{C^0[0,1]} \leq 2M_1^2 M_4 \big(A_1 + b_0 + 2A_0 M_1^2 M_3 b_0 \big) \tag{2.38}$$

for all t with $0 \leq t \leq T_1$. In particular, we see that the C^1 norm of $(r(T_1, x), s(T_1, x))$ can be sufficiently small as $a_0, b_0 \to 0$. Thus we can repeat the previous procedure with the new initial data $(r(T_1, x), s(T_1, x))$ on $t = T_1$. After at most $[T/T_1] + 1$ iterations, we obtain the following uniform *a priori* estimate

$$\Big\| \Big(\frac{\partial r}{\partial x}(t, \cdot), \frac{\partial s}{\partial x}(t, \cdot) \Big) \Big\|_{C^0[0,1]} \leq C_2(T; a_0, b_0), \qquad \forall 0 \leq t \leq \delta, \tag{2.39}$$

where $C_2(T; a_0, b_0)$ is a positive constant independent of δ and can be sufficiently small as $a_0, b_0 \to 0$. The combination of (2.15) and (2.39) gives the uniform *a priori* estimate (2.1). The proof is then completed.

3. Local exact boundary controllability

Now we can precise the framework of the exact boundary controllability. First of all, for a given constant $M > 0$, we put

$$\lambda_{\max} = \max_{|r|, |s| \leq M} \lambda(r, s), \qquad \mu_{\min} = \min_{|r|, |s| \leq M} \mu(r, s) \tag{3.1}$$

and we define the time T_0 by

$$T_0 = \max \Big\{ -\frac{1}{\lambda_{\max}}, \quad \frac{1}{\mu_{\min}} \Big\} \cdot \tag{3.2}$$

By Theorem 2.1, for any given $T \geq T_0$, the mixed initial-boundary value problem (1.1)–(1.3) admits a unique semi-global C^1 solution on the domain D_T, and the C^1 norm of the solution can be sufficiently small, provided that the C^1 norm of the initial data (r_0, s_0) and the boundary control (u, v) is small enough.

Using the same idea as in the proof of Theorem 2.1, we can get without any difficulty the following preliminary result.

Lemma 3.1. *The Cauchy problem (1.1)–(1.2) admits a unique global C^1 solution $(r(t,x), s(t,x))$ on the maximal determinate domain enclosed by the μ-characteristic passing through $(0,0)$, the λ-characteristic passing through $(0,1)$ and the x-axis, provided that the C^1 norm of the initial data $\|(r_0, s_0)\|_{C^1[0,1]}$ is small enough. Moreover, the C^1 norm of the solution $(r(t,x), s(t,x))$ can be arbitrarily small.*

Now we give the local exact boundary controllability for the system (1.1)–(1.3).

Theorem 3.1. *Let $T > T_0$. For any given initial data (r_0, s_0) and any given final state (r_T, s_T) with C^1 norm small enough, there exist two boundary controls u, v with C^1 norm small enough such that the mixed initial-boundary value problem (1.1)–(1.3) admits a unique semi-global C^1 solution $(r(t,x), s(t,x))$ on the domain D_T, which satisfies the final condition (1.8).*

Proof. First, thanks to Lemma 3.1, if the C^1 norm of the initial data (r_0, s_0) is small enough, the Cauchy problem (1.1)–(1.2) admits a unique global C^1 solution $(r_d(t,x), s_d(t,x))$ on the maximal determinate domain Ω_d enclosed by the x-axis, the λ-characteristic $x = x_1(t)$ passing through $A = (0,1)$ and the μ-characteristic

$x = x_2(t)$ passing through $O = (0,0)$. The C^1 norm of the solution $(r_d(t,x), s_d(t,x))$ can be arbitrarily small. Moreover, it is easy to see that the two characteristics intersect at a point $D = (t_d, x_d)$ with

$$t_d \le \frac{1}{\mu_{\min} - \lambda_{\max}} \cdot \tag{3.3}$$

Next, for the given final data (r_T, s_T) with C^1 norm small enough, the backward Cauchy problem for the system (1.1) with the final data (r_T, s_T) on $t = T$ admits a global C^1 solution $(r_u(t,x), s_u(t,x))$ on the maximal determinate domain Ω_u enclosed by the λ-characteristic $x = y_1(t)$ passing through $C = (T, 0)$, the μ-characteristic $x = y_2(t)$ passing through $B = (T, 1)$ and the segment BC. The two characteristics intersect at a point $U = (t_u, x_u)$ with

$$t_u \ge T - \frac{1}{\mu_{\min} - \lambda_{\max}} \cdot \tag{3.4}$$

Noting (3.2), it follows from (3.3) and (3.4) that

$$t_u - t_d \ge T - \frac{2}{\mu_{\min} - \lambda_{\max}} \ge T - T_0. \tag{3.5}$$

In particular, the subdomains Ω_d, Ω_u are disjointed provided that $T > T_0$.

Finally, let Ω_l be the subdomain enclosed by the characteristics $x = x_2(t), x = y_1(t)$ and the segments DU, OC, and Ω_r be the subdomain enclosed by the characteristics $x = x_1(t), x = y_2(t)$ and the segments DU, AB. The domain D_T is then divided into four subdomains $\Omega_d, \Omega_u, \Omega_l$ and Ω_r. Moreover, since $T > T_0$, we know (see Appendix in [5]) that the angle between the segment DU and the characteristic $x = x_1(t)$ (resp. $x = x_2(t)$, $x = y_1(t)$ and $x = y_2(t)$) is less than π. Thus we can consider the following mixed initial-boundary value problem on the subdomain Ω_l:

$$\begin{cases} \dfrac{\partial r}{\partial x} + \dfrac{1}{\lambda(r,s)} \dfrac{\partial r}{\partial t} = 0, \\ \dfrac{\partial s}{\partial x} + \dfrac{1}{\mu(r,s)} \dfrac{\partial s}{\partial t} = 0 \end{cases} \tag{3.6}$$

with the boundary conditions

$$\begin{cases} r = r_d(t, x_2(t)) & \text{on} \quad x = x_2(t), \quad 0 \le t \le t_d, \\ s = s_u(t, y_1(t)) & \text{on} \quad x = y_1(t), \quad t_u \le t \le T \end{cases} \tag{3.7}$$

and the initial data

$$r(t, x_3(t)) = r_m(t), \quad s(t, x_3(t)) = s_m(t), \quad t_d \le t \le t_u, \tag{3.8}$$

where $x = x_3(t)$ denotes the equation of the segment DU.

We notice that Theorem 2.1 applies well to problem (3.6)–(3.8). In fact, if the C^1 norm of the initial data (r_0, s_0) and the final data (r_T, s_T) is small enough, the C^1 norm of the boundary value $(r_d(t, x_2(t)), s_u(t, y_1(t)))$ is also small. In order to solve problem (3.6)–(3.8), the initial data $(r_m(t), s_m(t))$ should be small in C^1 norm and satisfy suitable compatibility conditions. Observing that $r_d(t,x)$ (resp. $s_d(t,x), r_u(t,x)$ and $s_u(t,x)$) is constant along the characteristic $x = x_1(t)$ (resp. $x = x_2(t), x = y_1(t)$ and $x = y_2(t)$), we get

$$\begin{cases} r_d(t_d, x_1(t_d)) = r_0(1), & s_d(t_d, x_2(t_d)) = s_0(0), \\ r_u(t_u, y_1(t_u)) = r_T(0), & s_u(t_u, y_2(t_u)) = s_T(1). \end{cases} \tag{3.9}$$

Then noting that the C^0 compatibility asks

$$\begin{cases} r_u(t_u, y_1(t_u)) = r_m(t_u), & s_d(t_d, x_2(t_d)) = s_m(t_d), \\ r_d(t_d, x_1(t_d)) = r_m(t_d), & s_u(t_u, y_2(t_u)) = s_m(t_u), \end{cases} \tag{3.10}$$

we deduce the C^0 compatibility conditions:

$$\begin{cases} r_m(t_u) = r_T(0), & s_m(t_d) = s_0(0), \\ r_m(t_d) = r_0(1), & s_m(t_u) = s_T(1). \end{cases} \tag{3.11}$$

Next, differentiating $r_m(t), s_m(t)$ with respect to t and noting that the segment DU is described by the equation $x = x_3(t)$

$$x_3(t) = x_u + \triangle(t - t_u), \qquad t_d \le t \le t_u, \tag{3.12}$$

where

$$\triangle = \frac{x_u - x_d}{t_u - t_d}, \tag{3.13}$$

we obtain that

$$\begin{cases} r'_m(t) = (\triangle - \lambda(r_m(t), s_m(t)))\dfrac{\partial r}{\partial x}(t, x_3(t)), \\ s'_m(t) = (\triangle - \mu(r_m(t), s_m(t)))\dfrac{\partial s}{\partial x}(t, x_3(t)). \end{cases} \tag{3.14}$$

Then, noting (3.11), we obtain the C^1 compatibility conditions:

$$\begin{cases} r'_m(t_d) = (\triangle - \lambda(r_0(1), s_0(0)))\dfrac{\partial r_d}{\partial x}(t_d, x_d), \\ s'_m(t_u) = (\triangle - \mu(r_T(0), s_T(1)))\dfrac{\partial s_u}{\partial x}(t_u, x_u). \end{cases} \tag{3.15}$$

Similarly, on the subdomain Ω_r, we consider the mixed initial-boundary value problem for system (3.6) with the boundary conditions

$$\begin{cases} s = s_d(t, x_1(t)) & \text{on} \quad x = x_1(t), \quad 0 \le t \le t_d, \\ r = r_u(t, y_2(t)) & \text{on} \quad x = y_2(t), \quad t_u \le t \le T \end{cases} \tag{3.16}$$

and the same initial data as in (3.8). This time, except the C^0 compatibility conditions (3.11), we need the following C^1 compatibility conditions:

$$\begin{cases} r'_m(t_u) = (\triangle - \lambda(r_T(0), s_T(1)))\dfrac{\partial r_u}{\partial x}(t_u, x_u), \\ s'_m(t_d) = (\triangle - \mu(r_0(1), s_0(0)))\dfrac{\partial s_d}{\partial x}(t_d, x_d). \end{cases} \tag{3.17}$$

Taking (3.11, 3.15, 3.17) into account, we can choose the initial data $(r_m(t), s_m(t))$ as the Hermite interpolation on the interval $[t_d, t_u]$, which is uniquely determined by the values $r_0(1), s_0(0), r_T(0), s_T(1)$ and the derivatives $\frac{\partial r_d}{\partial x}(t_d, x_d)$, $\frac{\partial s_d}{\partial x}(t_d, x_d)$, $\frac{\partial r_u}{\partial x}(t_u, x_u)$, $\frac{\partial s_u}{\partial x}(t_u, x_u)$. Since the C^1 norm of $(r_m(t), s_m(t))$ can be sufficiently small, applying Theorem 2.1 we can find $(r_l(t,x), s_l(t,x))$ and $(r_r(t,x), s_r(t,x))$ which solve the problem (3.6)–(3.8) and the problem (3.6, 3.8, 3.16) on the subdomains Ω_l, Ω_r respectively.

Finally, taking $(r(t,x), s(t,x))$ as the collection of the solutions on the four subdomains $\Omega_d, \Omega_u, \Omega_l, \Omega_r$, and defining the boundary controls u, v by

$$\begin{cases} v(t) = s_l(t,0) - g(t, r_l(t,0)) & \text{at} \quad x = 0, \\ u(t) = r_r(t,1) - f(t, s_r(t,1)) & \text{at} \quad x = 1, \end{cases} \tag{3.18}$$

we check easily that $(r(t,x), s(t,x))$ solves the problem (1.1)–(1.3) and satisfies the final condition (1.8). The proof is thus achieved.

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