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Modélisation Mathématique et Analyse Numérique

A CLASS OF TIME DISCRETE SCHEMES FOR A PHASE–FIELD SYSTEM OF PENROSE–FIFE TYPE

Olaf Klein[1]

Abstract. In this paper, a phase field system of Penrose–Fife type with non–conserved order parameter is considered. A class of time–discrete schemes for an initial–boundary value problem for this phase–field system is presented. In three space dimensions, convergence is proved and an error estimate linear with respect to the time–step size is derived.

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1. Introduction

In [29], Penrose and Fife derived a phase–field system modeling the dynamics of diffusive phase transitions. In the case of a non–conserved order parameter, their approach leads to the following system:

$$c_0\theta_t + \lambda'(\chi)\chi_t + \kappa\Delta\left(\frac{1}{\theta}\right) = g, \tag{1.1}$$

$$\eta\chi_t - \varepsilon\Delta\chi + \beta(\chi) - \sigma'(\chi) \ni -\frac{\lambda'(\chi)}{\theta}\,. \tag{1.2}$$

This system of an energy balance (1.1) coupled with an evolution equation (1.2) for the order parameter determines the evolution of the absolute temperature θ and the order parameter χ. Here, c_0 and κ denote the specific heat and thermal conductivity respectively, which are supposed to be positive constants. The datum g represents heat sources or sinks, and η stands for a positive space–dependent relaxation coefficient. Choosing this coefficient in a particular way, an anisotropic growth can be simulated.

The positive constant ε is a relaxation coefficient and β denotes the subdifferential of the convex but non–smooth part of a potential on $\mathbb{R}$, while $-\sigma$ corresponds to the non–convex but differentiable part of the potential. The latent heat of the phase transition is represented by $\lambda'(\chi)$.

In the context of solid–liquid phase transitions, one typically has a quadratic or linear function λ and

$$\sigma(s) = \frac{\lambda(s)}{\theta_C} + \rho s^2, \quad \forall\, s \in \mathbb{R}, \tag{1.3}$$

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where θ_C denotes some critical temperature and ρ some positive constant. For $\beta(s) = 2\rho s^3$, we see that $\beta(s) - \sigma'(s) + \theta_C^{-1}\lambda'(s)$ is the derivative of the *double well potential* $\frac{\rho}{2}(s-1)^2(s+1)^2$. If β is the subdifferential of the indicator function $I_{[-1,1]}$ of the interval $[-1,1]$, we see that $\beta(s) - \sigma'(s) + \theta_C^{-1}\lambda'(s)$ corresponds to the "derivative" of the *double obstacle potential* $I_{[-1,1]}(s) + \rho(1-s^2)$, which has been introduced for the standard phase–field system by Blowey and Elliott (see [3]).

In the mean–field theory of the Ising ferromagnet as in Section 4 of [29], one has quadratic functions σ and λ, $D(\beta) = (0,1)$, and

$$\beta(s) = \rho^* \frac{\partial}{\partial s}\left(s\ln s + (1-s)\ln(1-s) - \ln\left(\frac{1}{2}\right)\right) = \rho^* \ln\left(\frac{s}{1-s}\right), \quad \forall\, s \in D(\beta),$$

where ρ^* is some positive constant.

In [13], Horn considers a time–discrete scheme in one space dimension for the Penrose–Fife system with a double well potential and quadratic λ and σ. He derives an error estimate of order $\sqrt{h}$, where h denotes the time–step size.

In previous works [16,17] of the author, a time discrete scheme for a Penrose–Fife system with σ linear or quadratic and special choices for β has been considered and an error estimate of order $\sqrt{h}$ has been shown. These results hold in three space dimensions, but are restricted to the situation, when $\lambda'(\chi)$ is some constant, such that some cancellations of the coupling terms can be used in the derivation of *a priori* estimates and of error estimates.

If $\lambda'(\chi)$ is a function of χ, as in the original Penrose–Fife system, these cancellations do not appear. Hence, in this case a more sophisticated argument for dealing with the coupling terms is needed to prove the convergence of the numerical scheme in three space dimensions.

Therefore, the first main novelty of this work is the derivation of a class of such schemes for Penrose–Fife systems with quite general λ, which covers all the situations discussed above.

The other main novelty is the error estimate for the schemes which is linear with respect to h, while in [13,16,17] only one of order $\sqrt{h}$ is derived. The linear order of the error estimate is proved by using arguments similar to Nochetto, Savaré, and Verdi in [28]. Moreover, in view of the results for Euler schemes for linear parabolic problems (*cf.* Sect. 3.1 of Chap. II in [11]), this result is optimal.

Using the time–discrete scheme, the existence of a unique solution to the Penrose–Fife system is proved. This result is a minor novelty of this paper, because of the weakened regularity assumption used for λ and σ. These functions are supposed to be C^1–functions on $\mathbb{R}$ with λ' and σ' locally Lipschitz continuous such that the Lipschitz constants fulfill some growth conditions.

Until now, in papers concerning existence, uniqueness, and regularity of similar Penrose–Fife systems, these functions are supposed to be at least C^2–functions with λ'' bounded (see, *e.g.* [12,15,20,22,30] or C^1–functions with λ convex (see [10]) or λ' globally Lipschitz continuous (see [18,19]).

The same holds for papers like [5,6,9,23], where more general heat flux laws are considered.

The layout of this paper is as follows: In Section 2, a precise formulation of the considered phase–field system is given, the class of time–discrete schemes is introduced, and the existence and approximation results are presented. The remaining sections are devoted to the proof of these results, and they are briefly discussed at the end of Section 2.

2. The Penrose–Fife system and the time–discrete schemes

In this section, a precise formulation of the considered phase–field system of Penrose–Fife type is given. Moreover, existence results and approximation results for a class of time–discrete schemes are presented.

2.1. The phase–field system

In the sequel, $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ denotes a bounded, open domain with smooth boundary Γ and $T > 0$ stands for a final time. Let $\Omega_T := \Omega \times (0,T)$ and $\Gamma_T := \Gamma \times (0,T)$. We consider the following Penrose–Fife system:

(PF) Find a quadruple (θ, u, χ, ξ) fulfilling

$$\theta \in H^1(0,T;L^2(\Omega)), \quad u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{2.1a}$$

$$\chi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.1b}$$

$$\xi \in L^\infty(0,T;L^2(\Omega)), \tag{2.1c}$$

$$\theta > 0, \quad u = \frac{1}{\theta}, \quad \chi \in D(\beta), \quad \xi \in \beta(\chi) \quad \text{a.e. in } \Omega_T, \tag{2.1d}$$

$$c_0\theta_t + \lambda'(\chi)\chi_t + \kappa\Delta u = g \quad \text{a.e. in } \Omega_T, \tag{2.1e}$$

$$\eta\chi_t - \varepsilon\Delta\chi + \xi - \sigma'(\chi) = -\lambda'(\chi)u \quad \text{a.e. in } \Omega_T, \tag{2.1f}$$

$$\kappa\frac{\partial u}{\partial n} + \gamma u = \zeta, \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{a.e. in } \Gamma_T, \tag{2.1g}$$

$$\theta(\cdot,0) = \theta^0, \quad \chi(\cdot,0) = \chi^0 \quad \text{a.e. in } \Omega. \tag{2.1h}$$

For dealing with this system, the following assumptions will be used:

(A1) Let β be a maximal monotone graph on $\mathbb{R}$ and $\phi : \mathbb{R} \to [0,\infty]$ a convex, lower semicontinuous function $\phi : \mathbb{R} \to [0,\infty]$ satisfying

$$\beta = \partial\phi, \quad 0 \in D(\beta), \quad 0 \in \beta(0), \quad \text{int } D(\beta) \neq \emptyset.$$

(A2) There are positive constants C_1^*, p, q such that

$$\lambda \in W^{2,\infty}_{loc}(\mathbb{R}), \quad \sigma \in W^{2,\infty}_{loc}(\mathbb{R}), \quad p < 1, \quad q < 4,$$
$$-\lambda(s) \le C_1^*(\phi(s)+1), \quad (\sigma'(s))^2 \le C_1^*(\phi(s)+1), \quad \forall\, s \in D(\beta),$$
$$|\lambda''(s)| \le C_1^*\left(|s|^p + 1\right), \quad |\sigma''(s)| \le C_1^*\left(|s|^q + 1\right) \quad \text{for a.e. } s \in D(\beta).$$

(A3) We have positive constants c_η, c_γ, and c_ζ such that

$$g \in H^1(0,T;L^\infty(\Omega)), \quad \eta \in L^\infty(\Omega), \quad \eta \ge c_\eta \quad \text{a.e. in } \Omega,$$
$$\gamma \in L^\infty(0,T;C^1(\Gamma)), \quad \gamma_t \in L^\infty(\Gamma_T), \quad \gamma \ge c_\gamma \quad \text{a.e. in } \Gamma_T,$$
$$\zeta \in H^1(0,T;L^2(\Gamma)) \cap L^\infty(\Gamma_T) \cap L^\infty(0,T;H^{\frac{1}{2}}(\Gamma)), \quad \zeta \ge c_\zeta \quad \text{a.e. in } \Gamma_T.$$

(A4) We consider initial data $\theta^0, \chi^0, u^0, \xi^0$ such that

$$\theta^0, u^0 \in H^1(\Omega) \cap L^\infty(\Omega), \quad \chi^0 \in H^2(\Omega), \quad \xi^0 \in L^2(\Omega), \quad \phi(\chi^0) \in L^1(\Omega),$$
$$\theta^0 > 0, \quad u^0 = \frac{1}{\theta^0}, \quad \chi^0 \in D(\beta), \quad \xi^0 \in \beta(\chi^0) \quad \text{a.e. in } \Omega, \quad \frac{\partial \chi^0}{\partial n} = 0 \quad \text{a.e. in } \Gamma.$$

2.2. The class of time discrete schemes

To allow for variable time–steps, we consider decompositions of $(0,T)$ that do not need to be uniform, but satisfy the following assumption, where c_{down} and c_{up} are fixed positive constants such that $c_{\text{down}} \le 1 \le c_{\text{up}}$.

(A5) The decomposition $Z = \{t_0, t_1, \dots, t_K\}$ with $0 = t_0 < t_1 < \cdots < t_K = T$ and $h_m := t_m - t_{m-1}$, for $1 \le m \le K$, fulfills

$$c_{\mathrm{down}} h_{m-1} \le h_m \le c_{\mathrm{up}} h_{m-1}, \quad \forall\, 1 < m \le K.$$

We define the *width* $|Z|$ of the decomposition by $|Z| := \max_{1 \le m \le K} h_m$, and, for $1 \le m \le K$,

$$g_m(x) := \frac{1}{h_m} \int_{t_{m-1}}^{t_m} g(x,t)\,\mathrm{d}t\,, \qquad \forall x \in \Omega, \tag{2.2a}$$

$$\gamma_m(\sigma) := \frac{1}{h_m} \int_{t_{m-1}}^{t_m} \gamma(\sigma,t)\,\mathrm{d}t\,, \quad \zeta_m(\sigma) := \frac{1}{h_m} \int_{t_{m-1}}^{t_m} \zeta(\sigma,t)\,\mathrm{d}t\,, \qquad \forall \sigma \in \Gamma. \tag{2.2b}$$

Now, the following time–discrete scheme $(\mathbf{D_Z})$ for the Penrose–Fife system is considered

$(\mathbf{D_Z})$ **For $1 \le m \le K$, find**

$$\theta_m \in L^2(\Omega), \qquad u_m, \chi_m \in H^2(\Omega), \qquad \xi_m \in L^2(\Omega) \tag{2.3a}$$

such that

$$0 < u_m, \quad \theta_m = \frac{1}{u_m}, \quad \chi_m \in D(\beta), \quad \xi_m \in \beta(\chi_m) \quad \text{a.e. in } \Omega, \tag{2.3b}$$

$$c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + \lambda_d'(\chi_m, \chi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} + \kappa \Delta u_m = g_m \quad \text{a.e. in } \Omega, \tag{2.3c}$$

$$\eta \frac{\chi_m - \chi_{m-1}}{h_m} - \varepsilon \Delta \chi_m + \xi_m - \sigma_d'(\chi_m, \chi_{m-1}) = -\lambda_d'(\chi_m, \chi_{m-1})\, u_m \quad \text{a.e. in } \Omega, \tag{2.3d}$$

$$-\kappa \frac{\partial u_m}{\partial n} = \gamma_m u_m - \zeta_m, \quad \frac{\partial \chi_m}{\partial n} = 0 \quad \text{a.e. in } \Gamma, \tag{2.3e}$$

with

$$\theta_0 := \theta^0, \quad u_0 := u^0, \quad \chi_0 := \chi^0, \quad \xi_0 := \xi^0. \tag{2.3f}$$

Here, approximations λ_d' and σ_d' for λ' and σ' are used such that the following assumption is satisfied:

(A6) Let $\lambda_d', \sigma_d' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, and let C_2^*, p, q be positive constants with $p < 1$, $q < 4$ such that, for all $r, s, r', s' \in D(\beta)$,

$$\begin{gathered}
\lambda_d'(s,s) = \lambda'(s), \quad \sigma_d'(s,s) = \sigma'(s), \quad (\sigma_d'(r,s))^2 \le C_2^*(\phi(r) + \phi(s) + 1), \\
|\lambda_d'(r,r') - \lambda_d'(s,s')| \le C_2^* \left(|r-s| + |r'-s'|\right)\left(|r|^p + |r'|^p + |s|^p + |s'|^p + 1\right), \\
|\sigma_d'(r,r') - \sigma_d'(s,s')| \le C_2^* \left(|r-s| + |r'-s'|\right)\left(|r|^q + |r'|^q + |s|^q + |s'|^q + 1\right), \\
-\lambda_d'(r,s)(r-s) \le -\lambda(r) + \lambda(s) + C_2^*(r-s)^2.
\end{gathered} \tag{2.4}$$

Remark 2.1. The time–discrete scheme $(\mathbf{D_Z})$ is an Euler scheme in time for the Penrose–Fife system **(PF)**, which is fully implicit, except for the treatment of the nonlinearities λ' and σ'. The time–discrete scheme $(\mathbf{D_Z})$, especially the approximation used for the coupling terms, is chosen in such a way that one can use discrete versions of the *a priori* estimates derived by Sprekels and Zheng (*cf.* [30]).

By introducing the general approximations $\lambda'_d(\chi_m, \chi_{m-1})$ and $\sigma'_d(\chi_m, \chi_{m-1})$ in $(\mathbf{D_Z})$, the same formulation can be used to investigate a bunch of different time–discrete schemes. A fully implicit scheme corresponds to the choices $\lambda'_d(r,s) = \lambda'(r)$ and $\sigma'_d(r,s) = \sigma'(r)$. A fully explicit treatment of nonlinearities λ' and σ' corresponds to $\lambda'_d(r,s) = \lambda'(s)$ and $\sigma'_d(r,s) = \sigma'(s)$.

For the time–discrete scheme there holds:

Theorem 2.1. *Assume that* **(A1–A6)** *hold. The scheme has a unique solution, if* $|Z|$ *is sufficiently small.*

Appropriate choices for the approximations λ'_d and σ'_d are discussed in the following remark.

Remark 2.2. If σ is quadratic, the implicit approximation will be linear in χ_m and should be used, *cf.* Remark 2.4. Only if σ' is not linear, more general approximations can be really useful. In this case, one would like to use approximations which are still linear in the implicit part, *e.g.* $\sigma'_d(r,s) = \sigma_0(r-s) + \sigma'(s)$ with $\sigma_0 \in \mathbb{R}$ fixed or $\sigma'_d(r,s) = \sigma''(s)(r-s) + \sigma'(s)$, if $\sigma \in C^2(D(\beta))$.

If the explicit approximation for λ' is used, $\lambda'_d(\chi_m, \chi_{m-1})$ does not depend on χ_m, and the coupling between the two equations (2.3c, 2.3d) becomes a linear one. For any other choice for λ'_d, the coupling term in the discrete energy balance (2.3c) is nonlinear, and the $\lambda'_d(\chi_m, \chi_{m-1})u_m$–term in the discrete order parameter equation depends on χ_m and u_m, such that it becomes more complicated to solve this system numerically.

The following choices for σ'_d and λ'_d fulfill **(A6)**, if **(A2)** is satisfied for λ and σ, see Lemma 3.1:

(a) Any convex combination of $\sigma'(\chi_m)$ and $\sigma'(\chi_{m-1})$ can be used for $\sigma'_d(\chi_m, \chi_{m-1})$.

(b) One particular choice for λ'_d is the following approximation for a derivative, which has been used by Niezgódka and Sprekels in equation (2.3) of [27]:

$$\lambda'_*(r,s) := \begin{cases} \dfrac{\lambda(r) - \lambda(s)}{r-s}, & \text{if} \quad r \neq s, \\ \lambda'(r), & \text{if} \quad r = s. \end{cases} \tag{2.5}$$

If one chooses this function as λ'_d, the approximation for $\lambda'(\chi)\chi_t$ used in the discrete energy balance (2.3c) will coincide with the discrete difference quotient arising in the approximation of $(\lambda(\chi))_t$.

(c) If we have a uniform upper and a uniform lower bound for λ'' a.e. on $D(\beta)$, we can use every convex combination of $\lambda'(\chi_m)$ and $\lambda'(\chi_{m-1})$ for $\lambda'_d(\chi_m, \chi_{m-1})$.

If we have a uniform upper bound for λ'' a.e. on $D(\beta)$, we can use the explicit approximation $\lambda'_d(\chi_m, \chi_{m-1}) = \lambda'(\chi_{m-1})$. If we have a uniform lower bound for λ'' a.e. on $D(\beta)$, we can use the implicit approximation $\lambda'_d(\chi_m, \chi_{m-1}) = \lambda'(\chi_m)$.

The following corollary and remark yield conditions to ensure the existence of a unique solution to the scheme in concrete situations.

Corollary 2.1. *Assume that* **(A1–A6)** *hold. There exists a solution to* $(\mathbf{D_Z})$, *if* $|Z| \leq h^*$, *where* h^* *and* C_3^* *are positive constants with*

$$h^* \left(2\,(\sigma'_d(r,s))^2 - C_3^*(\phi(s)+1)\right) \leq c_\eta \phi(r), \quad \forall\, r,s \in D(\beta). \tag{2.6}$$

The solution to the scheme is unique, if, in addition,

$$\lambda'_d(r,s) = \lambda'(s), \quad 2\,|Z|\,|\sigma'_d(r,s) - \sigma'_d(r',s)| \leq c_\eta\,|r - r'|\,, \quad \forall\, r, r', s \in D(\beta). \tag{2.7}$$

Remark 2.3. Assume that **(A1–A6)** hold. If $D(\beta)$ is bounded, Corollary 2.1 yields that the scheme has a solution for any time step, since one may always choose $C_3^* > 0$ such that $(2\,(\sigma'_d(r,s))^2 - C_3^*(\phi(s)+1)) \leq 0$, for all $r, s \in D(\beta)$.

If $D(\beta)$ is unbounded, we obtain from **(A6)** that (2.6) is satisfied for $h^* = c_\eta/2C_2^*$ and $C_3^* = 2C_2^*$, but this value for the upper bound h^* does not need to be the optimal one.

For σ'_d explicit, *i.e.* $\sigma'_d(r,s) = \sigma'(s)$, we do not get any restriction for the time–step size from (2.6) or (2.7).

If λ' is approximated explicitly and σ'_d is globally Lipschitz continuous in the first variable on $D(\beta) \times D(\beta)$, the conditions (2.6, 2.7) lead to a computable upper bound for the time–step size to ensure the existence of a unique solution.

In order to illustrate the use of Corollary 2.1, we consider an example: Let $c_\eta = 1 < T$, $\phi(r) = r^4$, $\sigma'(r) = r^2$, and $\sigma'_d(r,s) = 2rs - s^2$, for $r, s \in D(\beta) = \mathbb{R}$. Using **(A6)**, we see that (2.6) holds for $h^* = c_\eta/2C_2^* < \frac{1}{2}T$, but applying Young's inequality yields that (2.6) is also satisfied for $h_* = T$ and $C_3^* := 32\,(T/c_\eta) + 6\,(T/c_\eta)^{1/3} + 2$. Since $|\sigma'_d(r,s) - \sigma'_d(r',s)| = |s|\,|r-s|$, (2.7) does not hold for any decomposition Z. Hence, for this example, Corollary 2.1 yields that the scheme has a solution for any time step, but the corollary cannot be used to ensure the uniqueness of the solution.

2.3. Existence and approximation results

We use the solution to $(\mathbf{D_Z})$ to construct an approximate solution $(\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\chi}^Z, \overline{\xi}^Z)$ in $\left(L^\infty(0,T;L^2(\Omega))\right)^4$ to the Penrose–Fife system **(PF)**. The function $\widehat{\theta}^Z$ is defined to be linear in time on $[t_{m-1}, t_m]$ for $m = 1, \ldots, K$ such that $\widehat{\theta}^Z(t_k) = \theta_k$ holds for $k = 0, \ldots, K$. The functions $\widehat{u}^Z$ and $\widehat{\chi}^Z$ are defined analogously. We define $\overline{\xi}^Z$ piecewise constant in time by $\overline{\xi}^Z(t) = \xi_k$ for $t \in (t_{k-1}, t_k]$ and $k = 1, \ldots, K$.

Theorem 2.2. *Assume that* **(A1–A4)** *hold. Then there is a unique solution* (θ, u, χ, ξ) *to the Penrose-Fife system* **(PF)**. *For this solution it holds that*

$$\theta \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T) \cap W^{1,\infty}(0,T;H^1(\Omega)^*), \tag{2.8}$$

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(\Omega_T), \tag{2.9}$$

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). \tag{2.10}$$

Assume that **(A6)** *is satisfied. As, for decompositions* Z *with* **(A5)**, $|Z|$ *tends to* 0, *we have,*

$$\widehat{\theta}^Z \longrightarrow \theta \quad \textit{weakly in} \quad H^1(0,T;L^2(\Omega)), \tag{2.11}$$

$$\textit{weakly-star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{2.12}$$

$$\textit{weakly-star in} \quad W^{1,\infty}(0,T;H^1(\Omega)^*), \tag{2.13}$$

$$\widehat{u}^Z \longrightarrow u \quad \textit{weakly in} \quad H^1(0,T;L^2(\Omega)), \tag{2.14}$$

$$\textit{weakly-star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{2.15}$$

$$\textit{weakly in} \quad L^2(t_*,T;H^2(\Omega)),\ \forall\, 0 < t_* < T, \tag{2.16}$$

$$\widehat{\chi}^Z \longrightarrow \chi \quad \textit{weakly in} \quad H^1(0,T;H^1(\Omega)), \tag{2.17}$$

$$\textit{weakly-star in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.18}$$

$$\overline{\xi}^Z \longrightarrow \xi \quad \textit{weakly-star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.19}$$

The following error–estimate is the main result of this work.

Theorem 2.3. *Assume that* **(A1–A6)** *hold and that* $|Z|$ *is sufficiently small. Let* (θ, u, χ, ξ) *be the solution to the Penrose–Fife system* **(PF)**. *We have a positive constant* C*, independent of* Z*, such that*

$$\left\|\widehat{\theta}^Z - \theta\right\|_{L^2(0,T;L^2(\Omega))\,\cap\, C([0,T];H^1(\Omega)^*)} + \left\|\widehat{u}^Z - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\chi}^Z - \chi\right\|_{C([0,T];L^2(\Omega))\,\cap\, L^2(0,T;H^1(\Omega))} \leq C\,|Z|. \quad (2.20)$$

Remark 2.4 (Numerical implementation). In a lot of physically relevant situations, see [29], the considered functions λ and σ are quadratic and ϕ is bounded from below by a quadratic function, *i.e.* we have positive constants C_4^*, C_5^* with

$$\phi(s) + C_4^* \geq C_5^* s^2, \quad \forall\, s \in D(\beta). \quad (2.21)$$

Hence, **(A2)** holds, and **(A6)** is satisfied for

$$\sigma_d'(r,s) := \sigma'(r), \quad \lambda_d'(r,s) := \lambda'(s), \quad \forall\, r,s \in \mathbb{R},$$

which are the most promising choices for the numerical computations, because of the following properties: A careful inspection of the use of the assumption **(A5)** in the proof yields that the lower bound $c_{\text{down}} h_{m-1}$ for h_m in **(A5)** can be omitted, if the implicit approximation for σ' is used. This approximation is linear in χ_m, since σ is quadratic. Moreover, the use of the explicit approximation for λ' is the only choice for λ_d', such that the two equations (2.3c, d) are linearly coupled, *cf.* Remark 2.1.

If $\sigma''(0) = 0$ or otherwise $|Z| \leq \min(c_\eta C_5^*/4\,|\sigma''(0)|^2,\, c_\eta/2\,|\sigma''(0)|)$ holds, Corollary 2.1 yields that the scheme has a unique solution. Theorem 2.3 yields a convergence linear with respect to the time–step size. Moreover, a finite element discretization and a nonlinear Gauss–Seidel scheme similar to the one used in Section 10 of [16] can be employed to find approximative solutions to $(\mathbf{D_Z})$.

Remark 2.5. If the regularity assumption for g in **(A3)** is weakened to $g \in L^\infty(\Omega_T)$, all results of this work still holds, except for the error estimates in Theorem 2.3.

The layout of the proof is as follows: In Section 3, estimates concerning the approximation of the data are derived, and, by using a fixed point argument, the existence of a solution to the scheme is shown under the additional assumption that the domain $D(\beta)$ is bounded.

In Section 4, uniform *a priori* estimates are derived. The first *a priori* estimate in Lemma 4.2 is an energy estimate, where the coupling terms cancel each other, thanks to the chosen approximations of $\lambda'(\chi)$. The coupling terms do not cancel each other completely in the second *a priori* estimate in Lemma 4.3. Hence, the discrete version of $u\,(\lambda'(\chi))_t$ has to be estimated by using the first *a priori* estimate.

Afterwards, an *a priori* estimate for the $H^2(\Omega)$–norm of χ_m is derived in Lemma 4.4. The main results of Lemmas 4.5–4.8 are the uniform bounds for θ_m and $1/\theta_m = u_m$ in (4.39).

Based on the results of Section 3 and Section 4, the existence of a unique solution to the scheme is proved in Section 5, to finish the proofs of Theorem 2.1 and Corollary 2.1. This is done by considering the time–discrete scheme with β replaced by $\beta + \partial I_{[-C,C]}$, where $I_{[-C,C]}$ denotes the indicator function of the interval $[-C, C]$ for some sufficiently large $C > 0$.

In Section 6, the proofs of Theorem 2.2 and Theorem 2.3 are completed, *i.e.* the existence of a unique solution to the Penrose–Fife system is proved and the error estimate is shown. The first error estimate is derived in Lemma 6.3. Here, and even more extensively in Lemma 6.2, we apply estimates similar to those used by Nochetto, Savaré, and Verdi in [28], to improve the order of the error estimate from $\sqrt{|Z|}$ as in [16,17] to $|Z|$.

The first error estimate is used to prove the uniqueness of the solution to the Penrose–Fife system and is afterwards improved in Section 6.3 to derive the error estimate (2.20).

In the special situation $\lambda'(\chi) \equiv L \equiv \lambda'(\chi_m, \chi_{m-1})$ for some constant L which has been considered in the previous works [16,17], the coupling terms in the proof of the second *a priori* estimate cancelled each other. Hence, this estimate could be derived directly and led to uniform *a priori* estimates also if ε and/or η tend to zero, which is not the case for the *a priori* estimates derived in the present work. Moreover, in the error estimate in this work the coupling terms do not cancel each other as in [16,17], such that additional terms have to be estimated.

3. Some properties of the approximation of the data and a special existence result

To prepare the proof of the theorems and the corollary in the last section, some notations will be fixed and some properties for the approximation of the data will be proved. Moreover, the existence of a unique solution will be shown, under the additional condition that $D(\beta)$ is bounded.

In the sequel, we use the notation $\|\cdot\|_p$ for the $L^p(\Omega)$–norm, for all $p \in [1, \infty]$. Moreover, $\|\cdot\|_2$ will also be used for the $(L^2(\Omega))^2$ resp. $(L^2(\Omega))^3$ norm.

3.1. Properties of the data and their approximations

In the following lemma it is shown that those approximations discussed in Remark 2.2 fulfill the condition **(A6)**.

Lemma 3.1. *Assume that* **(A2)** *holds. Let $\omega \in [0,1]$ be given and define $\sigma_d' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by*

$$\sigma_d'(r,s) = \omega\sigma'(r) + (1-\omega)\sigma'(s), \quad \forall r,s \in \mathbb{R}. \tag{3.1}$$

(a) *If $\lambda_d' = \lambda_*'$ (cf. (2.5)), we have* **(A6)** *and*

$$\lambda_*'(r,s)(r-s) = \lambda(r) - \lambda(s), \quad \forall r,s \in \mathbb{R}. \tag{3.2}$$

(b) *Let*

$$\lambda_d'(r,s) = \omega^*\lambda'(r) + (1-\omega^*)\lambda'(s), \quad \forall r,s \in \mathbb{R}, \tag{3.3}$$

with some $\omega^* \in [0,1]$.

If we have positive constants C_1, C_2 such that $-C_1 \le \lambda''(s) \le C_2$ for a.e. $s \in D(\beta)$, the assumption **(A6)** *holds.*

If $\omega^* = 0$ and we have a positive constant C_3 with $\lambda''(s) \le C_3$ for a.e. $s \in D(\beta)$, the assumption **(A6)** *is satisfied.*

If $\omega^* = 1$ and we have a positive constant C_4 with $-C_4 \le \lambda''(s)$ for a.e. $s \in D(\beta)$, the assumption **(A6)** *holds.*

Proof. First, we consider part (a) of the lemma. Thanks to (2.5), we have $\lambda_*'(r,s) = \int_0^1 \lambda'(s + \tau(r-s))\,\mathrm{d}\tau$ and (3.2). Hence, for $\lambda_d' = \lambda_*'$, we can use (3.1), Schwarz's inequality, and **(A2)**, to show that **(A6)** is satisfied. This yields part (a) of the Lemma.

To prove part (b) of the lemma, we need only to show that the last estimate in **(A6)**, *i.e.* (2.4), is satisfied, since the remaining assumptions in **(A6)** follow by an argumentation similar to the one above. For $r, s \in D(\beta)$, applying Taylor's formula and (3.3) gives $\mu \in D(\beta)$ between r and s such that

$$-\lambda_d'(r,s)(r-s) + \lambda(r) - \lambda(s) = \omega^*(r-s)\int_r^\mu \lambda''(\tau)\,\mathrm{d}\tau + (1-\omega^*)(r-s)\int_s^\mu \lambda''(\tau)\,\mathrm{d}\tau\,.$$

Now, we see immediately that (2.4) holds under the considered assumptions. □

Lemma 3.2. *Assume that* **(A3)** *holds. Then there exist positive constants* $C_1, C_2, \ldots, C_6$*, such that, for all decompositions* Z *with* **(A5)**, *the functions* g_m, γ_m, *and* ζ_m *defined in* (2.2) *fulfill, for* $1 \le m \le K$,

$$C_1 \|v\|^2_{H^1(\Omega)} \le \kappa \|\nabla v\|_2^2 + \int_\Gamma \gamma_m v^2 \,\mathrm{d}\sigma \le C_2 \|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

$$\gamma_m v \in H^{\frac12}(\Gamma), \qquad \|\gamma_m v\|_{H^{\frac12}(\Gamma)} \le C_3 \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

$$c_\zeta \le \zeta_m \quad a.e.\ in\ \Gamma, \qquad \left|\int_\Gamma \zeta_m v \,\mathrm{d}\sigma\right| + \left|\int_\Omega g_m v \,\mathrm{d}x\right| \le C_4 \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

$$\|g_m\|_\infty + \|\gamma_m\|_{C^1(\Gamma)} + \|\zeta_m\|_{L^\infty(\Gamma)} + \|\zeta_m\|_{H^{\frac12}(\Gamma)} \le C_5,$$

and

$$\max_{1\le m\le K-1} \left\| \frac{\gamma_{m+1} - \gamma_m}{h_m} \right\|_{L^\infty(\Gamma)} + \sum_{m=1}^{K-1} h_m \left\| \frac{\zeta_{m+1} - \zeta_m}{h_m} \right\|^2_{L^2(\Gamma)} \le C_6,$$

where the positive constants c_ζ, c_γ *are specified in* ***(A3)***.

Proof. This lemma follows from **(A1, A5)**, the trace–mapping from $H^1(\Omega)$ to $H^{\frac12}(\Gamma)$, and the interpolation of $H^{\frac12}(\Gamma)$ by $H^1(\Gamma)$ and $L^2(\Gamma)$. □

3.2. The existence proof for $D(\beta)$ bounded

Lemma 3.3. *Assume that* **(A1–A6)** *hold and that* $D(\beta)$ *is bounded. Then there exists a solution to* $(\mathbf{D_Z})$.

Proof. From (2.3f), we get $\theta_0, u_0, \chi_0, \xi_0$. Now, we assume that $\theta_{m-1} \in L^2(\Omega), \chi_{m-1} \in H^2(\Omega)$ for some $m \in \{1, \ldots, K\}$ are given. To show that there exists a solution to the system in $(\mathbf{D_Z})$, *i.e.* to (2.3a–2.3e), we will first consider the discrete energy balance equation (2.3c) and the discrete equation (2.3d) for the order parameter separately. Afterwards, we will rewrite the system as a fixed point problem and apply Schauder's fixed point theorem.

Lemma 3.4. *For every* $\chi \in L^\infty(\Omega)$, *there is a unique* $\tilde u \in H^2(\Omega)$ *such that*

$$0 < \tilde u \quad a.e.\ in\ \Omega, \qquad \frac{1}{\tilde u} \in L^2(\Omega), \quad -\kappa \frac{\partial \tilde u}{\partial n} = \gamma_m \tilde u - \zeta_m \quad a.e.\ in\ \Gamma, \tag{3.4}$$

$$-\frac{c_0}{\tilde u} - h_m \kappa \Delta \tilde u = -c_0 \theta_{m-1} - h_m g_m + \lambda_d'(\chi, \chi_{m-1})(\chi - \chi_{m-1}) \quad a.e.\ in\ \Omega. \tag{3.5}$$

Proof. Let $\chi \in L^\infty(\Omega)$ be given. Thanks to **(A6)** and $\chi_{m-1} \in C(\bar\Omega)$, we have

$$\lambda_d'(\chi, \chi_{m-1})(\chi - \chi_{m-1}) \in L^2(\Omega).$$

By translating the proof of Corollary 13 of [4], we see that the operator corresponding to (3.4) and the left–hand side of (3.5) is maximal monotone. By showing that this operator is also coercive, we obtain that the operator is also surjective. The injectivity follows by estimating the difference between two given solutions. Details can be found in Lemma 5.1 of [16]. □

Lemma 3.5. *For every $\chi \in L^\infty(\Omega)$, $\tilde{u} \in L^2(\Omega)$ there exists a unique $\tilde{\chi}$ such that*

$$\tilde{\chi} \in H^2(\Omega), \qquad \tilde{\chi} \in D(\beta) \quad a.e.\ in\ \Omega, \qquad \frac{\partial \tilde{\chi}}{\partial n} = 0 \quad a.e.\ in\ \Gamma, \tag{3.6}$$

$$\eta \frac{\tilde{\chi} - \chi_{m-1}}{h_m} - \varepsilon \Delta \tilde{\chi} + \beta(\tilde{\chi}) \ni \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1})\,\tilde{u} \quad a.e.\ in\ \Omega, \tag{3.7}$$

$$-\eta \frac{\tilde{\chi} - \chi_{m-1}}{h_m} + \varepsilon \Delta \tilde{\chi} + \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1})\,\tilde{u} \ \in\ L^2(\Omega). \tag{3.8}$$

Proof. By **(A1, A3)**, we can rewrite (3.6–3.8) as

$$\frac{c_\eta}{h_m}\tilde{\chi} + B\tilde{\chi} \ni \sigma_d'(\chi, \chi_{m-1}) - \lambda_d'(\chi, \chi_{m-1})\,\tilde{u} + \frac{\eta}{h_m}\chi_{m-1}, \tag{3.9}$$

where $B : L^2(\Omega) \to \{W \subseteq L^2(\Omega)\}$ is a nonlinear operator. Using Corollary 13 of [4], we see that this operator is maximal monotone. Details can be found in (5.7, 5.8) and Lemma 5.5 of [16].

Because of **(A6, A3)**, $\chi \in L^\infty(\Omega)$, $\chi_{m-1} \in H^2(\Omega) \subset C(\overline{\Omega})$, we see that the right–hand side of (3.9) is in $L^2(\Omega)$. Hence, Theorem 2 of [4] yields that there is a unique solution $\tilde{\chi}$ to (3.6–3.8). □

In this proof, C_i, for $i \in \mathbb{N}$, will always denote generic positive constants, independent of $\chi \in \mathcal{M}$ with

$$\mathcal{M} := \left\{\chi \in L^2(\Omega) : \chi \in \overline{D(\beta)} \quad \text{a.e. in } \Omega\right\}. \tag{3.10}$$

This is a closed and convex subset of $L^2(\Omega)$.

We have:

Lemma 3.6. *The functions $r \mapsto \sigma_d'(r, \chi_{m-1}(x))$ and $r \mapsto \lambda_d'(r, \chi_{m-1}(x))$ are Lipschitz continuous on $\overline{D(\beta)}$ for every $x \in \overline{\Omega}$, with a Lipschitz constant independent of x. There is a positive constant C_1 such that, for all $\chi \in \mathcal{M}$,*

$$\|\lambda_d'(\chi, \chi_{m-1})\|_\infty + \|\sigma_d'(\chi, \chi_{m-1})\|_\infty + \|\chi\|_\infty + \|\chi_{m-1}\|_\infty \le C_1. \tag{3.11}$$

Proof. Since $\overline{D(\beta)}$ is bounded and $\chi_{m-1} \in H^2(\Omega) \subset C(\overline{\Omega})$, **(A6)** yields that the assertions of this lemma hold. □

Combining Lemma 3.4 and Lemma 3.5, we see that for every $\chi \in \mathcal{M}$ there is a unique $\tilde{u} \in H^2(\Omega)$ and a unique $\Psi(\chi) := \tilde{\chi} \in H^2(\Omega)$ such that (3.4–3.5) and (3.6–3.8) hold.

This defines a mapping $\Psi : \mathcal{M} \to \mathcal{M}$ and any fixed point of Ψ leads to a solution to the system in $(\mathbf{D_Z})$, *i.e.* to (2.3a–2.3e). Therefore, it is sufficient to prove that Ψ has a fixed point.

We test (3.5) by $h_m\tilde{u}$, apply Green's formula, Lemma 3.2, Hölder's inequality, (3.4, 3.11), and Young's inequality to conclude that

$$\begin{aligned} C_2\,\|\tilde{u}\|^2_{H^1(\Omega)} &\le c_0\,|\Omega| + h_m \int_\Gamma \zeta_m \tilde{u}\,\mathrm{d}\sigma + \int_\Omega \left(-c_0\theta_{m-1} - h_m g_m + \lambda_d'(\chi, \chi_{m-1})\,(\chi - \chi_{m-1})\right)\tilde{u}\,\mathrm{d}x \\ &\le C_3 + \frac{C_2}{2}\,\|\tilde{u}\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.12}$$

Owing to **(A1)**, we have $ws \ge 0$ for all $s \in D(\beta)$, $w \in \beta(s)$. Therefore, by testing (3.7) by $\tilde{\chi}$ and applying **(A3)**, Green's formula, (3.6, 3.11), Hölder's inequality, (3.12), and Young's inequality, we get

$$C_4\,\|\tilde{\chi}\|^2_{H^1(\Omega)} \le \left\|\eta\frac{\chi_{m-1}}{h_m} + \sigma_d'(\chi, \chi_{m-1}) - \lambda'(\chi, \chi_{m-1})\tilde{u}\right\|_2 \|\tilde{\chi}\|_2 \le C_5 + \frac{C_4}{2}\,\|\tilde{\chi}\|_2^2.$$

Hence, we see that $\tilde{\chi} \in \mathcal{M}_1$ with

$$\mathcal{M}_1 := \left\{ \tilde{\chi} \in \mathcal{M} : \|\tilde{\chi}\|^2_{H^1(\Omega)} \leq 2\frac{C_5}{C_4} \right\}.$$

Therefore, we observe that $\mathcal{M}_1$ is a nonempty, convex, compact set in $L^2(\Omega)$ and, by construction, that Ψ maps $\mathcal{M}_1$ into itself. Thanks to Lemma 3.7 below, Ψ is on $\mathcal{M}_1$ continuous. Now, Schauder's fixed point theorem yields the existence of a fixed point of Ψ in $\mathcal{M}_1$. □

Lemma 3.7. *$\Psi : \mathcal{M} \to \mathcal{M}$ is $L^2(\Omega)$–continuous.*

Proof. Let χ_1^*, χ_2^* in $\mathcal{M}$ be arbitrary, and

$$\tilde{\chi}_1 := \Psi(\chi_1^*), \quad \tilde{\chi}_2 := \Psi(\chi_2^*), \quad \chi^* := \chi_1^* - \chi_2^*, \quad \tilde{\chi} := \tilde{\chi}_1 - \tilde{\chi}_2.$$

Combining (3.4, 3.5), (3.6–3.8), and the definition of Ψ, we find $\tilde{u}_1, \tilde{u}_2 \in H^2(\Omega)$, $\tilde{\xi}_1, \tilde{\xi}_2 \in L^2(\Omega)$ such that

$$\tilde{u}_1 > 0, \quad \tilde{u}_2 > 0, \quad \tilde{\xi}_1 \in \beta(\tilde{\chi}_1), \quad \tilde{\xi}_2 \in \beta(\tilde{\chi}_2) \quad \text{a.e. in } \Omega, \tag{3.13}$$

$$-c_0\left(\frac{1}{\tilde{u}_1} - \frac{1}{\tilde{u}_2}\right) - h_m\kappa\Delta\left(\tilde{u}_1 - \tilde{u}_2\right) = \lambda_d'(\chi_1^*, \chi_{m-1})(\chi_1^* - \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})(\chi_2^* - \chi_{m-1}) \quad \text{a.e. in } \Omega, \tag{3.14}$$

$$\eta\frac{\tilde{\chi}}{h_m} - \varepsilon\Delta\tilde{\chi} + \tilde{\xi}_1 - \tilde{\xi}_2 = -\lambda_d'(\chi_1^*, \chi_{m-1})\tilde{u}_1 + \lambda_d'(\chi_2^*, \chi_{m-1})\tilde{u}_2 + \sigma_d'(\chi_1^*, \chi_{m-1}) - \sigma_d'(\chi_2^*, \chi_{m-1}) \quad \text{a.e. in } \Omega, \tag{3.15}$$

$$-\kappa\frac{\partial\left(\tilde{u}_1 - \tilde{u}_2\right)}{\partial n} = \gamma_m\left(\tilde{u}_1 - \tilde{u}_2\right), \quad \frac{\partial\tilde{\chi}}{\partial n} = 0 \quad \text{a.e. in } \Gamma. \tag{3.16}$$

Testing (3.14) by $\tilde{u} := \tilde{u}_1 - \tilde{u}_2$, integrating by parts, and using (3.16, 3.13), Lemma 3.2, Hölder's inequality, Lemma 3.6, and Young's inequality, we deduce

$$\begin{aligned} c_0\left\|\frac{\tilde{u}}{\sqrt{\tilde{u}_1\tilde{u}_2}}\right\|_2^2 + C_6\|\tilde{u}\|^2_{H^1(\Omega)} &\leq \int_\Omega \left(\lambda_d'(\chi_1^*, \chi_{m-1})\chi^* + \left(\lambda_d'(\chi_1^*, \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})\right)\left(\chi_2^* - \chi_{m-1}\right)\right)\tilde{u}\,\mathrm{d}x \\ &\leq C_7\|\chi^*\|_2\|\tilde{u}\|_2 \leq C_8\|\chi^*\|_2^2 + \frac{C_6}{2}\|\tilde{u}\|_2^2. \end{aligned} \tag{3.17}$$

We test (3.15) by $\tilde{\chi}$ and use (3.13), the monotonicity of β, **(A3)**, (3.16), and the generalized Hölder's inequality (see Lemma AP.2) to derive

$$\begin{aligned} C_9\|\tilde{\chi}\|^2_{H^1(\Omega)} \leq{}& \|\lambda_d'(\chi_1^*, \chi_{m-1})\|_{\frac{3}{2}}\|\tilde{u}\|_6\|\tilde{\chi}\|_6 + \|\lambda_d'(\chi_1^*, \chi_{m-1}) - \lambda_d'(\chi_2^*, \chi_{m-1})\|_{\frac{3}{2}}\|\tilde{u}_2\|_6\|\tilde{\chi}\|_6 \\ &+ \|\sigma_d'(\chi_1^*, \chi_{m-1}) - \sigma_d'(\chi_2^*, \chi_{m-1})\|_2\|\tilde{\chi}\|_2. \end{aligned}$$

Because of Lemma 3.6, (AP.1), (3.17), and (3.12), we see

$$C_9\|\tilde{\chi}\|^2_{H^1(\Omega)} \leq C_{10}\|\chi^*\|_2\|\tilde{\chi}\|_{H^1(\Omega)}.$$

Hence, thanks to Young's inequality, we have shown that Ψ is $L^2(\Omega)$–continuous. □

4. Uniform estimates

In this section, uniform estimates for the solutions to the time–discrete scheme are derived.

Assume that **(A1–A6)** hold and that $|Z| \leq h^*$, where h^* and C_3^* are positive constants such that (2.6) is satisfied.

Let $\beta^* := \partial\phi^*$ and $\phi^* : \mathbb{R} \to [0,\infty]$ be either ϕ or the function defined by

$$\phi^*(s) = \begin{cases} \phi(s), & \text{if } |s| \leq B, \\ \infty, & \text{otherwise}, \end{cases} \tag{4.1}$$

for some $B > \|\chi^0\|_\infty$. In the light of **(A1)**, we see that ϕ^* is a convex, lower semicontinuous function with

$$0 \leq \phi \leq \phi^* \text{ on } \mathbb{R}, \quad 0 \in D(\beta^*), \quad \operatorname{int} D(\beta^*) \neq \emptyset, \quad 0 \in \beta^*(0), \quad \phi^*|_{D(\beta^*)} = \phi|_{D(\beta^*)}\,. \tag{4.2}$$

Now, a modified version of the time–discrete scheme is considered, where β in $(\mathbf{D_Z})$, *i.e.* in (2.3b), is replaced by β^*. Let any solution to this scheme be given.

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z, the considered choice of ϕ^*, and the solution itself.

Remark 4.1. Recalling (2.3a, b, e, f), **(A4)**, and the definition of ϕ^*, we see that

$$\begin{aligned} &0 < u_m = \frac{1}{\theta_m}, \quad \chi_m \in D(\beta^*) \subseteq D(\beta), \quad \xi_m \in \beta^*(\chi_m) = \partial\phi^*(\chi_m) \text{ a.e. in } \Omega, \\ &\chi_m \in H^2(\Omega), \quad \frac{\partial \chi_m}{\partial n} = 0 \quad \text{a.e. in } \Gamma, \quad \forall\, 0 \leq m \leq K. \end{aligned} \tag{4.3}$$

Applying (2.3c), Green's formula, and (2.3e), we deduce that

$$\begin{aligned} \int_\Omega \left(c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + \frac{\lambda_m - \lambda_{m-1}}{h_m} \right) v \, \mathrm{d}x - \kappa \int_\Omega \nabla u_m \bullet \nabla v \, \mathrm{d}x - \int_\Gamma \gamma_m u_m v \, \mathrm{d}\sigma = \\ \int_\Omega g_m v \, \mathrm{d}x - \int_\Gamma \zeta_m v \, \mathrm{d}\sigma\,, \quad \forall v \in H^1(\Omega), \quad 1 \leq m \leq K, \end{aligned} \tag{4.4}$$

with

$$\lambda_0 := \lambda(\chi_0), \ \lambda_m := \lambda_{m-1} + \lambda_d'(\chi_m, \chi_{m-1})(\chi_m - \chi_{m-1}) \text{ a.e. in } \Omega, \ \forall\, 1 \leq m \leq K. \tag{4.5}$$

The following Lemmas use ideas from [8, 13–16, 20, 21, 30].

Lemma 4.1. (a) *There is a positive constant C_1 such that*

$$\begin{aligned} \|\phi^*(\chi_0)\|_1 + \|\xi_0\|_2 + \|\lambda_0\|_6 + \|\lambda_d'(\chi_0, \chi_0)\|_2 + \|\sigma_d'(\chi_0, \chi_0)\|_2 + \|\chi_0\|_{H^2(\Omega)} \\ + \|\theta_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} + \|u_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} + \|\ln(\theta_0)\|_1 \leq C_1. \end{aligned} \tag{4.6}$$

(b) *Let $\chi_{-1} \in L^2(\Omega)$ be defined by*

$$\eta \frac{\chi_0 - \chi_{-1}}{h_0} - \varepsilon \Delta \chi_0 + \xi_0 - \sigma_d'(\chi_0, \chi_0) = -\lambda_d'(\chi_0, \chi_0) u_0 \quad \textit{a.e. in } \Omega, \tag{4.7}$$

with $h_0 := |Z|$. We have a positive constant C_2 such that

$$\left\|\sqrt{\eta}\,\left(\frac{\chi_0-\chi_{-1}}{h_0}\right)\right\|_2^2 \le C_2. \tag{4.8}$$

Proof. If $\phi^* = \phi$, we use the initial condition (2.3f), **(A2, A4)**, Sobolev's embedding Theorem, **(A6)**, and (4.5) to show that (4.6) is satisfied. If $\phi^* \neq \phi$, in addition, (4.1) and $B > \|\chi^0\|_\infty$ are applied. Combining (4.7, 4.6), and **(A3)** leads to (4.8). □

Lemma 4.2. *There are two positive constants C_3, C_4 such that*

$$\max_{0\le m\le K}\left(\|\theta_m\|_1 + \|\ln(\theta_m)\|_1 + \|\chi_m\|_{H^1(\Omega)}^2 + \|\phi(\chi_m)\|_1\right) + \sum_{m=1}^K h_m\|u_m\|_{H^1(\Omega)}^2 + \sum_{m=1}^K h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2 + \sum_{m=1}^K\|\chi_m-\chi_{m-1}\|_{H^1(\Omega)}^2 \le C_3, \tag{4.9}$$

$$\max_{1\le m\le K}\|\sigma_d'(\chi_m,\chi_{m-1})\|_2 \le C_4. \tag{4.10}$$

Proof. Testing (2.3d) by $(\chi_m - \chi_{m-1})$, taking the sum from $m=1$ to $m=k$, and using **(A3)**, Green's formula, (4.3, AP.5, 4.6, 4.2, 4.5), Schwarz's inequality, and Young's inequality, we deduce

$$\frac{c_\eta}{2}\sum_{m=1}^k h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2 + \frac{\varepsilon}{2}\|\nabla\chi_k\|_2^2 + \frac{\varepsilon}{2}\sum_{m=1}^k\|\nabla\chi_m-\nabla\chi_{m-1}\|_2^2 + \|\phi(\chi_k)\|_1 \le C_5 - \sum_{m=1}^k\int_\Omega(\lambda_m-\lambda_{m-1})\,u_m\,\mathrm{d}x + \frac{1}{2c_\eta}\sum_{m=1}^k h_m\|\sigma_d'(\chi_m,\chi_{m-1})\|_2^2. \tag{4.11}$$

Let $\alpha := \min\left(1/2C_1^*, c_\eta/6C_2^*T\right)$, with C_1^*, C_2^* as in **(A2, A6)**. For $1 \le m \le K$, we insert $v = h_m\alpha - h_m u_m$ in (4.4), use (4.3) and that $-1/s$ is the derivative of the convex function $-\ln(s)$, take the sum from $m=1$ to $m=k$, and apply Lemma 3.2, (4.6), and Young's inequality, to show that

$$c_0\int_\Omega(-\ln(\theta_k))\,\mathrm{d}x + \alpha c_0\|\theta_k\|_1 + C_6\sum_{m=1}^k h_m\|u_m\|_{H^1(\Omega)}^2 \le C_7 + \sum_{m=1}^k\int_\Omega(\lambda_m-\lambda_{m-1})(u_m-\alpha)\,\mathrm{d}x\,. \tag{4.12}$$

Because of (4.5), **(A6, A2)**, (4.6), Young's inequality, and the definition of α, we have

$$-\alpha\sum_{m=1}^k\int_\Omega(\lambda_m-\lambda_{m-1})\,\mathrm{d}x \le C_8 + \frac{1}{2}\int_\Omega\phi(\chi_k)\,\mathrm{d}x + \frac{c_\eta}{6}\sum_{m=1}^k h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2.$$

Hence, by using Lemma AP.8 and adding (4.12) to (4.11), we deduce

$$C_9\|\theta_k\|_1 + c_0\|\ln(\theta_k)\|_1 + C_6\sum_{m=1}^k h_m\|u_m\|_{H^1(\Omega)}^2 + \frac{c_\eta}{3}\sum_{m=1}^k h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2 + \frac{\varepsilon}{2}\|\nabla\chi_k\|_2^2 + \frac{\varepsilon}{2}\sum_{m=1}^k\|\nabla(\chi_m-\chi_{m-1})\|_2^2 + \frac{1}{2}\|\phi(\chi_k)\|_1 \le C_{10} + \frac{1}{2c_\eta}\sum_{m=1}^k h_m\|\sigma_d'(\chi_m,\chi_{m-1})\|_2^2. \tag{4.13}$$

Since **(A6)**, (2.6), and $|Z| \leq h^*$ yield

$$\|\sigma_d'(\chi_m, \chi_{m-1})\|_2^2 \leq C_2^* \left(\|\phi(\chi_m)\|_1 + \|\phi(\chi_{m-1})\|_1 + \int_\Omega 1 \, \mathrm{d}x \right), \quad \forall 1 \leq m \leq K, \tag{4.14}$$

$$\frac{h_k}{2c_\eta} \|\sigma_d'(\chi_k, \chi_{k-1})\|_2^2 \leq \frac{1}{4} \|\phi(\chi_k)\|_1 + C_{11} \left(h_k \|\phi(\chi_{k-1})\|_1 + 1 \right),$$

we obtain from (4.13), **(A5)**, the discrete version of Gronwall's lemma, and (4.6) that (4.9) is satisfied. Therefore, (4.10) holds because of (4.14). □

Lemma 4.3. *There exists a constant C_{12} such that*

$$\max_{0 \leq m \leq K} \left(\left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 + \|u_m\|_{H^1(\Omega)}^2 \right) + \sum_{m=1}^K h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 + \sum_{m=1}^K \left\| (\xi_m - \xi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1$$
$$+ \sum_{m=1}^K \left\| \frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_2^2 + \sum_{m=1}^K h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|_2^2 + \sum_{m=1}^K \|u_m - u_{m-1}\|_{H^1(\Omega)}^2 \leq C_{12}, \tag{4.15}$$

with χ_{-1}, h_0 as in Lemma 4.1.

Proof. Inserting $v = -(u_m - u_{m-1})$ in (4.4), taking the sum from $m = 1$ to $m = k$, and applying (4.3, AP.5, AP.4), Lemma 3.2, (4.9, 4.6), the generalized Hölder's inequality, $h_m \leq c_{\mathrm{up}} h_{m-1}$, and Young's inequality, we deduce that

$$\frac{c_o}{2} \sum_{m=1}^k h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|_2^2 + C_{13} \|u_k\|_{H^1(\Omega)}^2 + C_{13} \sum_{m=1}^k \|u_m - u_{m-1}\|_{H^1(\Omega)}^2$$
$$\leq C_{14} + \sum_{m=1}^k \int_\Omega \frac{\lambda_m - \lambda_{m-1}}{h_m} (u_m - u_{m-1}) \, \mathrm{d}x. \tag{4.16}$$

For $2 \leq m \leq K$, we test the difference of (2.3d) for m and $m-1$ by $(\chi_m - \chi_{m-1})/h_m$. By applying **(A3)**, Green's formula, (4.3), the monotonicity of β, (4.5), and (AP.5), we obtain that

$$\frac{1}{2} \left\| \sqrt{\eta} \, \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 - \frac{1}{2} \left\| \sqrt{\eta} \, \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_2^2 + \frac{c_\eta}{2} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_2^2$$
$$+ \varepsilon h_m \left\| \nabla \left(\frac{\chi_m - \chi_{m-1}}{h_m} \right) \right\|_2^2 + \left\| (\xi_m - \xi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1$$
$$\leq - \int_\Omega \left(\frac{\lambda_m - \lambda_{m-1}}{h_m} u_m - \lambda_{d,m-1}' \frac{\chi_m - \chi_{m-1}}{h_m} u_{m-1} \right) \mathrm{d}x$$
$$+ \int_\Omega \left(\sigma_d'(\chi_m, \chi_{m-1}) - \sigma_{d,m-1}' \right) \frac{\chi_m - \chi_{m-1}}{h_m} \, \mathrm{d}x, \tag{4.17}$$

with

$$\lambda_{d,m-1}' := \lambda_d'(\chi_{m-1}, \chi_{m-2}), \quad \sigma_{d,m-1}' := \sigma_d'(\chi_{m-1}, \chi_{m-2}) \quad \text{a.e. in } \Omega. \tag{4.18}$$

Testing the difference of (2.3d) for $m = 1$ and (4.7) by $(\chi_1 - \chi_0)/h_1$ and using the same argumentation as above, we deduce that (4.17) holds also for $m = 1$ with

$$\lambda'_{d,0} := \lambda'_d(\chi_0, \chi_0), \quad \sigma'_{d,0} := \sigma'_d(\chi_0, \chi_0) \quad \text{a.e. in } \Omega. \tag{4.19}$$

Summing up (4.17) from $m = 1$ to $m = k$, adding the resulting estimate to (4.16), and using **(A3)**, (4.9, 4.8), we conclude that

$$\begin{aligned}
&\frac{c_\eta}{2}\left\|\frac{\chi_k - \chi_{k-1}}{h_k}\right\|_2^2 + \frac{c_\eta}{2}\sum_{m=1}^{k}\left\|\frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|_2^2 + C_{15}\sum_{m=1}^{k} h_m \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{H^1(\Omega)}^2 \\
&\quad + \sum_{m=1}^{k}\left\|(\xi_m - \xi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_1 + \frac{c_o}{2}\sum_{m=1}^{k} h_m \left\|\frac{u_m - u_{m-1}}{h_m\sqrt{u_m u_{m-1}}}\right\|_2^2 + C_{13}\left\|u_k\right\|_{H^1(\Omega)}^2 \\
&\quad + C_{13}\sum_{m=1}^{k}\left\|u_m - u_{m-1}\right\|_{H^1(\Omega)}^2 \le C_{16} + I_{1,k} + I_{2,k},
\end{aligned} \tag{4.20}$$

with

$$I_{1,k} := \sum_{m=1}^{k}\int_\Omega \left(\lambda'_{d,m-1}\frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\lambda_m - \lambda_{m-1}}{h_m}\right) u_{m-1}\, \mathrm{d}x\,, \tag{4.21}$$

$$I_{2,k} := \sum_{m=1}^{k}\int_\Omega \left(\sigma'_d(\chi_m, \chi_{m-1}) - \sigma'_{d,m-1}\right)\frac{\chi_m - \chi_{m-1}}{h_m}\, \mathrm{d}x\,. \tag{4.22}$$

Using (4.5), the generalized Hölder's inequality, and Schwarz's inequality, we deduce that

$$\begin{aligned}
I_{1,k} &\le \left(\max_{1\le m\le k}\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2\right)\sqrt{I_{3,k}}\sqrt{\sum_{m=1}^{k} h_{m-1}\left\|u_{m-1}\right\|_6^2}, \\
\text{with } I_{3,k} &:= \sum_{m=1}^{k}\frac{1}{h_{m-1}}\left\|\lambda'_{d,m-1} - \lambda'_d(\chi_m, \chi_{m-1})\right\|_3^2.
\end{aligned} \tag{4.23}$$

Now, owing to (AP.1, 4.9, 4.6), and Young's inequality, we observe that

$$I_{1,k} = \frac{c_\eta}{4}\max_{1\le m\le k}\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 + C_{17} I_{3,k}. \tag{4.24}$$

Since $1/3 = 1/p_1 + p/6$ holds for $p_1 := 6/(2-p)$, we obtain, by (4.18, 4.19), **(A6)**, the generalized Hölder's inequality, $h_m \le c_{\mathrm{up}} h_{m-1}$ (AP.1, 4.9), that

$$\begin{aligned}
I_{3,k} &\le C_{18}\sum_{m=2}^{k}\left(\frac{h_m^2}{h_{m-1}}\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{p_1}^2 + h_{m-1}\left\|\frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|_{p_1}^2\right) \\
&\quad \times \left(\left\||\chi_m|^p\right\|_{\frac{6}{p}} + \left\||\chi_{m-1}|^p\right\|_{\frac{6}{p}} + \left\||\chi_{m-2}|^p\right\|_{\frac{6}{p}} + 1\right)^2 + C_{19}\frac{h_1^2}{|Z|}\left\|\frac{\chi_1 - \chi_0}{h_1}\right\|_{p_1}^2 \left(\left\||\chi_1|^p\right\|_{\frac{6}{p}} + \left\||\chi_0|^p\right\|_{\frac{6}{p}} + 1\right)^2 \\
&\le C_{20}\sum_{m=1}^{k} h_m \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{\frac{6}{2-p}}^2.
\end{aligned}$$

Because of $p < 1$, we can use the Gagliardo–Nirenberg inequality (see Lemma AP.5) and Young's inequality to deduce

$$C_{17} I_{3,k} \le \frac{C_{15}}{4} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 + C_{21} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 . \tag{4.25}$$

Defining $q_1 := 12/(6-q)$, we have $1 = 1/q_1 + q/6 + 1/q_1$. It follows from (4.22, 4.18, 4.19), **(A6)**, and the generalized Hölder's inequality that

$$\begin{aligned} I_{2,k} \le & C_{22} \sum_{m=2}^{k} \left(h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{q_1} + h_{m-1} \left\| \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_{q_1} \right) \\ & \times \left(\| |\chi_m|^q \|_{\frac{6}{q}} + \| |\chi_{m-1}|^q \|_{\frac{6}{q}} + \| |\chi_{m-2}|^q \|_{\frac{6}{q}} + 1 \right) \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{q_1} \\ & + C_{23} h_1 \left\| \frac{\chi_1 - \chi_0}{h_1} \right\|_{q_1} \left(\| |\chi_1|^q \|_{\frac{6}{q}} + \| |\chi_0|^q \|_{\frac{6}{q}} + 1 \right) \left\| \frac{\chi_1 - \chi_0}{h_1} \right\|_{q_1} . \end{aligned}$$

Using (AP.1), (4.9), Young's inequality, **(A5)**, the Gagliardo–Nirenberg inequality, and $q < 4$, we obtain that

$$I_{2,k} \le \frac{C_{15}}{4} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 + C_{24} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 . \tag{4.26}$$

Combining (4.20, 4.24–4.26, 4.9), we conclude that

$$\begin{aligned} & \frac{c_\eta}{2} \left\| \frac{\chi_k - \chi_{k-1}}{h_k} \right\|_2^2 + \frac{c_\eta}{2} \sum_{m=1}^{k} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} - \frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}} \right\|_2^2 + \frac{C_{15}}{2} \sum_{m=1}^{k} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_{H^1(\Omega)}^2 \\ & + \sum_{m=1}^{k} \left\| (\xi_m - \xi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1 + \frac{c_o}{2} \sum_{m=1}^{k} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|_2^2 + C_{13} \|u_k\|_{H^1(\Omega)}^2 + C_{13} \sum_{m=1}^{k} \|u_m - u_{m-1}\|_{H^1(\Omega)}^2 \\ & \le C_{25} + \frac{c_\eta}{4} \max_{1 \le m \le k} \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 . \end{aligned} \tag{4.27}$$

By taking the maximum from $m = 1$ to $m = K$, we see that (4.15) holds, because of (4.6). □

Lemma 4.4. *There exists a positive constant C_{26} such that*

$$\max_{1 \le m \le K} \|\xi_m\|_2 + \max_{0 \le m \le K} \|\chi_m\|_{H^2(\Omega)} \le C_{26} . \tag{4.28}$$

Proof. Testing formally (2.3d) by ξ_m and using Green's formula, (2.3e, 4.3), and Young's inequality, we obtain

$$\|\xi_m\|_2 \le \left\| -\lambda_d' (\chi_m, \chi_{m-1}) u_m - \eta \frac{\chi_m - \chi_{m-1}}{h_m} + \sigma_d'(\chi_m, \chi_{m-1}) \right\|_2 . \tag{4.29}$$

For a precise derivation of this inequality, one has to replace β^* by the Yosida approximation $\beta^*_{1/n}$ of β^*, see p. 104 of [4], and test the modified version of (2.3d) by the approximations of χ_m and ξ_m. Here, one has to use that the approximation of χ_m is an element of $H^{1,6}(\Omega)$ such that the generalized chain rules hold, see Theorem 1 of [26] and Lemma 2.1 and Remark 2.1 of [25]. Now, a passage to the limit and using Prob. 1.1(iv) of Chap. II in [2] lead to (4.29).

Applying **(A6)**, the generalized Hölder's inequality, $p < 1$, (AP.1), (4.9, 4.15), we obtain

$$\begin{aligned}\|\lambda_d'(\chi_m,\chi_{m-1})u_m\|_2 \leq& |\lambda_d'(0,0)|\,\|u_m\|_2 + C_{27}\left(\|\chi_m - 0\|_6 + \|\chi_{m-1} - 0\|_6\right)\left(\||\chi_m|^p\|_6 + \||\chi|^p_{m-1}\|_6 + 1\right)\|u_m\|_6 \\ \leq& C_{28}. \end{aligned} \tag{4.30}$$

Combining this with (4.29, 4.10), **(A3)**, (4.15) leads to

$$\|\xi_m\|_2 \leq C_{29}. \tag{4.31}$$

Comparing the terms in (2.3d), and using **(A3)**, (4.15, 4.10, 4.31, 4.30), we see that

$$\|\varepsilon\Delta\chi_m\|_2 = \left\|\eta\frac{\chi_m - \chi_{m-1}}{h_m} + \xi_m - \sigma_d'(\chi_m,\chi_{m-1}) + \lambda_d'(\chi_m,\chi_{m-1})\,u_m\right\|_2 \leq C_{30}.$$

Now, using Lemma AP.4, (4.9, 4.3), we conclude $\|\chi_m\|_{H^2(\Omega)} \leq C_{31}$. Combining this with (4.31, 4.6), we see that (4.28) is satisfied. □

Lemma 4.5. *There exists a positive constant C_{32} such that*

$$\begin{aligned}\max_{1\leq m\leq K}\left(\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty + \left\|\frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_2 + \left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|_{H^1(\Omega)^*}\right)& \\ + \sum_{m=1}^K h_m\left\|\frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|^2_{H^1(\Omega)} + \max_{0\leq m\leq K}\|\lambda_m\|_{H^1(\Omega)} \leq C_{32}.&\end{aligned} \tag{4.32}$$

Proof. By looking at the terms in (4.4) and using (4.15) and Lemma 3.2, we see that

$$\max_{1\leq m\leq K}\left\|c_0\frac{\theta_m - \theta_{m-1}}{h_m} + \frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_{H^1(\Omega)^*} \leq C_{33}. \tag{4.33}$$

Thanks to (4.28), Sobolev's embedding Theorem, and **(A6)**, we have

$$\max_{0\leq m\leq K}\|\chi_m\|_{H^{1,6}(\Omega)} + \max_{1\leq m\leq K}\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty \leq C_{34}. \tag{4.34}$$

Combining this with **(A6)** and Theorem 1 of [26], we see that $\lambda_d'(\chi_m,\chi_{m-1}) \in H^{1,6}(\Omega)$ and

$$\max_{1\leq m\leq K}\|\nabla\lambda_d'(\chi_m,\chi_{m-1})\|_6 \leq C_{35}.$$

Therefore, owing to (4.5), Young's inequality, the generalized Hölder's inequality, (4.34, 4.15), and Sobolev's embedding Theorem, we have

$$\begin{aligned}\max_{1\leq m\leq K}\left\|\frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_2^2 + \sum_{m=1}^K h_m\left\|\nabla\left(\frac{\lambda_m - \lambda_{m-1}}{h_m}\right)\right\|_2^2 \leq& \max_{1\leq m\leq K}\left(\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2\right)^2 \\ &+ 2\sum_{m=1}^K h_m\|\lambda_d'(\chi_m,\chi_{m-1})\|_\infty^2\left\|\nabla\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)\right\|_2^2 \\ &+ 2\sum_{m=1}^K h_m\|\nabla\lambda_d'(\chi_m,\chi_{m-1})\|_6^2\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_3^2 \\ \leq& C_{36}.\end{aligned}$$

Combining this with (4.33, 4.6), we see that (4.32) is satisfied. $\square$

Lemma 4.6. *We have $\theta_m \in H^1(\Omega)$ for $0 \le m \le K$.*

Proof. We have $\theta_0 \in H^1(\Omega)$ by (2.3f) and **(A4)**. For $1 \le m \le K$ with $\theta_{m-1} \in H^1(\Omega)$, we define the approximation $\theta_{m,n} \in H^1(\Omega) \cap L^\infty(\Omega)$ for θ_m by

$$\theta_{m,n} := \left(u_m + \frac{1}{n}\right)^{-1} \quad \text{a.e. in } \Omega, \quad \forall n \in \mathbb{N}.$$

The Lebesgue dominated convergence theorem and $\theta_m \in L^2(\Omega)$ yield that

$$\theta_{m,n} \xrightarrow[n\to\infty]{} \theta_m \quad \text{strongly in} \quad L^2(\Omega). \tag{4.35}$$

By applying (4.4) with $v = \theta_{m,n}^3$ and using (4.3), Hölder's inequality, Lemma 3.2, (4.32), (AP.1), and Young's inequality, we see that this sequence is bounded in $H^1(\Omega)$. Combining this with (4.35), we conclude that $\theta_m \in H^1(\Omega)$. $\square$

Lemma 4.7. *There exists a constant C_{37} such that*

$$\max_{0\le m\le K} \|\theta_m\|_2 \le C_{37}. \tag{4.36}$$

Proof. We multiply (2.3c) by h_m and use (4.5). Summing up the resulting equation for $m = 1$ to $m = i$, we find

$$c_0\theta_i + \lambda_i + \kappa \sum_{m=1}^{i} h_m \Delta u_m = c_0\theta_0 + \lambda_0 + \sum_{m=1}^{i} h_m g_m \quad \text{a.e. in } \Omega. \tag{4.37}$$

We test (4.37) by $h_i \cdot \Delta u_i$, take the sum from $i = 1$ to $i = k$, and apply Green's formula, (2.3e, 4.3), $\theta_m \in H^1(\Omega)$, (AP.3, AP.2), Lemma 3.2, and Schwarz's inequality, to derive

$$\begin{aligned} c_0 \sum_{i=1}^{k} h_i \left\| \frac{\nabla u_i}{u_i} \right\|_2^2 &+ \frac{\kappa}{2} \left\| \sum_{i=1}^{k} h_i \Delta u_i \right\|_2^2 + \frac{\kappa}{2} \sum_{i=1}^{k} h_i^2 \|\Delta u_i\|_2^2 + \frac{c_0 c_\zeta}{\kappa} \sum_{i=1}^{k} h_i \|\theta_i\|_{L^1(\Gamma)} \\ &\le C_{38} + \int_\Omega \left(\left(c_0\theta_0 + \lambda_0 + \sum_{i=1}^{k} h_i g_i \right) \sum_{i=1}^{k} h_i \Delta u_i \right) \mathrm{d}x - \sum_{i=1}^{k-1} h_{i+1} \int_\Omega g_{i+1} \sum_{m=1}^{i} h_m \Delta u_m \,\mathrm{d}x \\ &\qquad + \sum_{i=1}^{k} h_i \int_\Omega \nabla\lambda_i \bullet \nabla u_i \,\mathrm{d}x + \sum_{i=1}^{k} h_i \frac{1}{\kappa} \int_\Gamma \lambda_i \left(\gamma_i u_i - \zeta_i\right) \mathrm{d}\sigma . \end{aligned}$$

Now, by utilizing Young's inequality, (4.6), Lemma 3.2, (4.15, 4.32), and $h_m \le c_{\mathrm{up}} h_{m-1}$, we derive

$$c_0 \sum_{m=1}^{k} h_m \left\| \frac{\nabla u_m}{u_m} \right\|_2^2 + \frac{\kappa}{4} \left\| \sum_{m=1}^{k} h_m \Delta u_m \right\|_2^2 + \frac{\kappa}{2} \sum_{m=1}^{k} h_m^2 \|\Delta u_m\|_2^2 \le C_{39} + C_{40} \sum_{m=1}^{k-1} h_m \left\| \sum_{i=1}^{m} h_i \Delta u_i \right\|_2^2 . \tag{4.38}$$

By applying the discrete version of Gronwall's lemma, we get a uniform upper bound for the left-hand side of (4.38). Looking at the terms in (4.37) and applying (4.32, 4.6), and Lemma 3.2, we see that (4.36) holds. $\square$

Lemma 4.8. *We have $\theta_m \in C(\bar{\Omega})$ for $0 \le m \le K$, and there are two positive constants C_{41}, C_{42} such that*

$$\max_{0\le m\le K}\left(\|u_m\|_{C(\bar\Omega)} + \|\theta_m\|_{C(\bar\Omega)\cap H^1(\Omega)}\right) + \sum_{m=1}^{K} h_m \left(\left\|\frac{u_m - u_{m-1}}{h_m}\right\|_2^2 + \left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|_2^2\right) \le C_{41}, \tag{4.39}$$

$$\sum_{m=1}^{K} h_m \|u_m\|_{H^2(\Omega)}^2 \le C_{42}. \tag{4.40}$$

Proof. We deduce, by Lemma 3.2, (4.32, AP.1), that

$$\sum_{m=1}^{k} h_m \left\|g_m - \frac{\lambda_m - \lambda_{m-1}}{h_m}\right\|_6^2 \le C_{43}.$$

Thanks to (4.3–4.6, 4.15, 4.32, 4.36), and Lemma 3.2, we can apply Moser's technique as in Lemma 6.11 and 6.12 of [16], for $\varepsilon > 0$ fixed, and derive, by using (4.6), that

$$\max_{0\le m\le K}\left(\|u_m\|_{L^\infty(\Omega)} + \|\theta_m\|_{L^\infty(\Omega)}\right) \le C_{44}.$$

Combining this with $u_m \in H^2(\Omega) \subset C(\bar\Omega)$, and (4.3), we see that u_m is a continuous function on $\overline{\Omega}$ which is bounded from above and below by positive constants. Combining this with (4.3, 4.15), and Hölder's inequality, we see that $\theta_m \in C(\bar\Omega)$ and (4.39) hold. Now, by looking at the terms in (2.3c), and using (4.5, 4.32), and Lemma 3.2, we see that

$$\sum_{m=1}^{K} h_m \|\Delta u_m\|_2^2 \le C_{45}.$$

Now, Lemma AP.4 yields that (4.40) is satisfied, because of (2.3e), Lemma 3.2, and (4.15). □

Lemma 4.9. *We have*

$$\|\lambda(\chi_k) - \lambda_k\|_3 \le C_{46}\,|Z|, \quad \forall\, 1 \le k \le K. \tag{4.41}$$

Proof. Applying (4.5), **(A2)**, the mean value theorem, **(A6)**, (4.28), and Sobolev's embedding Theorem, we deduce

$$|\lambda(\chi_k) - \lambda_k| \le C_{47} \sum_{m=1}^{k} h_m^2 \left|\frac{\chi_m - \chi_{m-1}}{h_m}\right|^2 \quad \text{a.e. in } \Omega. \tag{4.42}$$

Hence, recalling (AP.1, 4.15), we conclude

$$\|\lambda(\chi_k) - \lambda_k\|_3 \le C_{48} \sum_{m=1}^{k} h_m^2 \left\|\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)^2\right\|_3 \le C_{49} \sum_{m=1}^{k} h_m^2 \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{H^1(\Omega)}^2 \le C_{50}\,|Z|.$$

□

5. Proof of Theorem 2.1 and Corollary 2.1

We assume that **(A1–A6)** hold.

In the framework of Theorem 2.1, we obtain from **(A6)** that (2.6) is satisfied for $h^* = c_\eta/2C_2^*$ and $C_3^* = 2C_2^*$. We assume that $|Z| \le h^*$.

In the framework of Corollary 2.1, it is part of the assumptions that $|Z| \leq h^*$ where h^* and C_3^* are positive constants fulfilling (2.6).

Because of **(A4)** and Sobolev's embedding Theorem, we see that $\|\chi^0\|_\infty$ is finite.

For any $B > \|\chi^0\|_\infty$, we can consider ϕ^* as in (4.1), β^*, and the corresponding modified version of the time–discrete scheme as in the last section. Lemma 3.3 yields that there exists a solution $(\theta_m^B, u_m^B, \chi_m^B, \xi_m^B)_{m=0}^K$ to this modified version of the scheme. Since the assumptions used in the last section are satisfied, the estimates derived therein hold for this solution. Now, because of (4.28) and Sobolev's embedding Theorem, there is some positive constant C', independent of B, such that

$$\max_{0\leq m\leq K} \|\chi_m^B\|_{C(\overline{\Omega})} \leq C'. \tag{5.1}$$

Now, we consider $B := C' + \|\chi^0\|_\infty + 2$. Thanks to (4.1), $\beta^* = \partial\phi^*$, and $\beta = \partial\phi$, we have

$$\beta^*|_{[-C'-1,C'+1]} = \beta|_{[-C'-1,C'+1]}\,.$$

This yields, by (4.3, 5.1), that the solution to the modified version of scheme is also a solution to the unmodified version of the scheme $(\mathbf{D_Z})$.

It remains to show the uniqueness of the solution. Assume that we have two solutions $\left(\theta_m^{(1)}, u_m^{(1)}, \chi_m^{(1)}, \xi_m^{(1)}\right)_{m=0}^K$ and $\left(\theta_m^{(2)}, u_m^{(2)}, \chi_m^{(2)}, \xi_m^{(2)}\right)_{m=0}^K$ to the scheme $(\mathbf{D_Z})$. Hence, the estimates in the last section are valid for both solutions.

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z and the considered solutions.

Thanks to (2.3f), we have $\theta_0^{(1)} = \theta_0^{(2)}$, $u_0^{(1)} = u_0^{(2)}$, $\chi_0^{(1)} = \chi_0^{(2)}$, $\xi_0^{(1)} = \xi_0^{(2)}$ a.e. on Ω.

To prove by induction that the two solutions coincide, we now assume that $1 \leq m \leq K$ is given such that

$$\theta_{m-1}^{(1)} = \theta_{m-1}^{(2)}, \quad u_{m-1}^{(1)} = u_{m-1}^{(2)}, \quad \chi_{m-1}^{(1)} = \chi_{m-1}^{(2)} =: \chi^* \quad \text{a.e. in } \Omega. \tag{5.2}$$

Now, let $u_m := u_m^{(1)} - u_m^{(2)}$ and $\chi_m := \chi_m^{(1)} - \chi_m^{(2)}$.

Using (2.3b, c, e), Green's formula, and (5.2), we deduce

$$\theta_m^{(1)} - \theta_m^{(2)} = \frac{-u_m}{u_m^{(1)} u_m^{(2)}} \quad \text{a.e. in } \Omega, \tag{5.3}$$

$$\frac{1}{h_m}\int_\Omega \left(c_0 \frac{-u_m}{u_m^{(1)}u_m^{(2)}} + \lambda_d'(\chi_m^{(1)},\chi^*)\left(\chi_m^{(1)} - \chi^*\right) - \lambda_d'(\chi_m^{(2)},\chi^*)\left(\chi_m^{(2)} - \chi^*\right)\right) v \,\mathrm{d}x$$
$$- \kappa\int_\Omega \nabla u_m \bullet \nabla v \,\mathrm{d}x - \int_\Gamma \gamma_m u_m v \,\mathrm{d}x = 0, \quad \forall v \in H^1(\Omega).$$

This yields for $v = -h_m u_m$, by Lemma 3.2,

$$c_0 \left\| \frac{u_m}{\sqrt{u_m^{(1)} u_m^{(2)}}} \right\|_2^2 + h_m C_1 \left\| u_m \right\|_{H^1(\Omega)}^2 \\ \leq \int\limits_\Omega \lambda_d'(\chi_m^{(1)}, \chi^*) \chi_m u_m \, \mathrm{d}x + \int\limits_\Omega \left(\lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*) \right) (\chi_m^{(2)} - \chi^*) u_m \, \mathrm{d}x \, . \quad (5.4)$$

Recalling (2.3d, 5.2), we have

$$\eta \frac{\chi_m}{h_m} - \varepsilon \Delta \chi_m + \xi_m^{(1)} - \xi_m^{(2)} - \sigma_d'(\chi_m^{(1)}, \chi^*) + \sigma_d'(\chi_m^{(2)}, \chi^*) = \\ - \lambda_d' \left(\chi_m^{(1)}, \chi^* \right) u_m - \left(\lambda_d' \left(\chi_m^{(1)}, \chi^* \right) - \lambda_d' \left(\chi_m^{(2)}, \chi^* \right) \right) u_m^{(2)} \quad \text{a.e. in } \Omega. \quad (5.5)$$

Testing this equation by χ_m and using **(A3)**, Green's formula, (2.3e, 2.3b), and the monotonicity of β, and adding the resulting estimate to (5.4), we obtain, by (4.39),

$$C_2 \left\| u_m \right\|_2^2 + h_m C_1 \left\| u_m \right\|_{H^1(\Omega)}^2 + \frac{c_\eta}{h_m} \left\| \chi_m \right\|_2^2 + \varepsilon \left\| \nabla \chi_m \right\|_2^2 \leq I_1 + I_2, \quad (5.6)$$

with

$$I_1 := \int\limits_\Omega \left(\lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*) \right) \left((\chi_m^{(2)} - \chi^*) u_m - u_m^{(2)} \chi_m \right) \mathrm{d}x \, , \quad (5.7)$$

$$I_2 := \int\limits_\Omega \left(\sigma_d'(\chi_m^{(1)}, \chi^*) - \sigma_d'(\chi_m^{(2)}, \chi^*) \right) \chi_m \, \mathrm{d}x \, . \quad (5.8)$$

Now, we consider the framework of Corollary 2.1 and Theorem 2.1 separately.

If we are in the framework of Corollary 2.1, the uniqueness needs only to be shown under the additional assumption that (2.7) holds. Therefore, we have $I_1 = 0$ and

$$I_2 \leq \frac{c_\eta}{2|Z|} \int\limits_\Omega (\chi_m)^2 \, \mathrm{d}x \leq \frac{c_\eta}{2h_m} \left\| \chi_m \right\|_2^2 .$$

Hence, (5.6, 5.3, 5.5) yield that

$$u_m = \chi_m = 0, \quad \theta_m^{(1)} = \theta_m^{(2)}, \quad \xi_m^{(1)} = \xi_m^{(2)} \quad \text{a.e. in } \Omega. \quad (5.9)$$

This finishes the proof of Corollary 2.1.

Now, we consider the framework of Theorem 2.1. **(A6)**, (4.28), and Sobolev's embedding Theorem yield that

$$\left| \lambda_d'(\chi_m^{(1)}, \chi^*) - \lambda_d'(\chi_m^{(2)}, \chi^*) \right| + \left| \sigma_d'(\chi_m^{(1)}, \chi^*) - \sigma_d'(\chi_m^{(2)}, \chi^*) \right| \leq C_3 \left| \chi_m \right| \quad \text{a.e. in } \Omega.$$

Hence, by applying the generalized Hölder's inequality, (4.28, 4.39), and Young's inequality, we deduce

$$\begin{aligned} I_1 + I_2 \leq & C_3 \left\|\chi_m\right\|_2 \left(\left\|\chi_m^{(2)} - \chi_{m-1}\right\|_\infty \left\|u_m\right\|_2 + \left\|u_m^{(2)}\right\|_\infty \left\|\chi_m\right\|_2\right) + C_3 \left\|\chi_m\right\|_2^2 \\ \leq & \frac{C_2}{2} \left\|u_m\right\|_2^2 + C_4 \left\|\chi_m\right\|_2^2 . \end{aligned}$$

Therefore, if we assume that $|Z| \leq c_\eta / 2C_4$, we obtain $I_1 + I_2 \leq (C_2/2) \left\|u_m\right\|_2 + (c_\eta / 2h_m) \left\|\chi_m\right\|_2$. Combining this with (5.6, 5.3, 5.5), we see that (5.9) is satisfied.

Since we have shown that the scheme has a unique solution, if $|Z|$ is sufficiently small, Theorem 2.1 is proved. □

6. Proof of Theorem 2.2 and Theorem 2.3

We assume that **(A1–A4, A6)** hold. Thanks to **(A6)**, we have positive constants h^* and C_5^* such that (2.6) is satisfied.

6.1. Properties of the approximations

In this section, we only consider decompositions Z with **(A5)** and $|Z|$ sufficiently small. Hence, Theorem 2.1 yields that there exists a unique solution to the time-discrete scheme $(\mathbf{D_Z})$. Let $\left(\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\chi}^Z, \overline{\xi}^Z\right)$ be the corresponding approximations derived from the solution to $(\mathbf{D_Z})$ as in Section 2.3.

For $(\lambda_m)_{m=0}^K$ as in (4.5), we define the piecewise linear function $\widehat{\lambda}^Z$ analogously to $\widehat{\chi}^Z$ The piecewise constant functions $\overline{\theta}^Z$, $\overline{u}^Z$, $\overline{\chi}^Z$, $\overline{\gamma}^Z$, $\overline{\zeta}^Z$, $\overline{g}^Z$, $\overline{\lambda}^Z$ are defined analogously to $\overline{\xi}^Z$, and $\underline{\chi}^Z \in L^\infty(0,T;H^2(\Omega))$ is defined by

$$\underline{\chi}^Z(t) = \chi_{m-1}, \quad \forall\, t \in (t_{m-1}, t_m),\ 1 \leq m \leq K. \tag{6.1}$$

Then, by the definition of the approximations, (2.3a–f, 4.5), we have

$$\widehat{\theta}^Z, \widehat{u}^Z, \widehat{\lambda}^Z \in H^1(0,T;H^1(\Omega)), \quad \overline{u}^Z \in L^2(0,T;H^2(\Omega)),\ \widehat{u}^Z \in L^2(|Z|,T;H^2(\Omega)), \tag{6.2a}$$

$$\widehat{\chi}^Z \in H^1(0,T;H^2(\Omega)), \quad \overline{\chi}^Z, \underline{\chi}^Z \in L^\infty(0,T;H^2(\Omega)), \tag{6.2b}$$

$$\overline{\xi}^Z \in L^\infty(0,T;L^2(\Omega)), \tag{6.2c}$$

$$0 < \widehat{u}^Z,\ 0 < \overline{u}^Z,\ \overline{\theta}^Z = \frac{1}{\overline{u}^Z}, \quad \overline{\chi}^Z, \widehat{\chi}^Z, \underline{\chi}^Z \in D(\beta), \quad \overline{\xi}^Z \in \beta\left(\overline{\chi}^Z\right) \quad \text{a.e. in } \Omega_T, \tag{6.2d}$$

$$c_0 \widehat{\theta}_t^Z + \widehat{\lambda}_t^Z + \kappa \Delta \overline{u}^Z = \overline{g}^Z \quad \text{a.e. in } \Omega_T, \tag{6.2e}$$

$$\eta \widehat{\chi}_t^Z - \varepsilon \Delta \overline{\chi}^Z + \overline{\xi}^Z - \sigma_d'(\overline{\chi}^Z, \underline{\chi}^Z) = -\lambda_d'(\overline{\chi}^Z, \underline{\chi}^Z) \overline{u}^Z \quad \text{a.e. in } \Omega_T, \tag{6.2f}$$

$$-\kappa \frac{\partial \overline{u}^Z}{\partial n} = \overline{\gamma}^Z \overline{u}^Z - \overline{\zeta}^Z, \quad \frac{\partial \widehat{\chi}^Z}{\partial n} = 0, \quad \frac{\partial \overline{\chi}^Z}{\partial n} = 0 \quad \text{a.e. in } \Gamma_T, \tag{6.2g}$$

$$\widehat{\theta}^Z(\cdot,0) = \theta^0, \quad \widehat{u}^Z(\cdot,0) = u^0, \quad \widehat{\chi}^Z(\cdot,0) = \chi^0, \quad \widehat{\lambda}^Z(\cdot,0) = \lambda(\chi^0) \quad \text{a.e. in } \Omega. \tag{6.2h}$$

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the decomposition Z.

We find, from (4.15, 4.28, 4.32, 4.39, 4.40):

$$\begin{aligned}\left\|\widehat{\theta}^Z\right\|_{W^{1,\infty}(0,T;H^1(\Omega)^*)\cap H^1(0,T;L^2(\Omega))\cap C(\overline{\Omega_T})\cap L^\infty(0,T;H^1(\Omega))} + \left\|\overline{\theta}^Z\right\|_{L^\infty(\Omega_T)\cap L^\infty(0,T;H^1(\Omega))} \\ + \left\|\widehat{u}^Z\right\|_{C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))\cap C(\overline{\Omega_T})} + \left\|\widehat{u}^Z\right\|_{L^2(|Z|,T;H^2(\Omega))} \\ + \left\|\overline{u}^Z\right\|_{L^\infty(0,T;H^1(\Omega))\cap L^\infty(\Omega_T)\cap L^2(0,T;H^2(\Omega))} \le C_1, \quad (6.3)\end{aligned}$$

$$\begin{aligned}\left\|\widehat{\chi}^Z\right\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega))\cap C([0,T];H^2(\Omega))} + \left\|\overline{\chi}^Z\right\|_{L^\infty(0,T;H^2(\Omega))} \\ + \left\|\underline{\chi}^Z\right\|_{L^\infty(0,T;H^2(\Omega))} + \left\|\overline{\xi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\widehat{\lambda}^Z\right\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega))} \le C_2. \quad (6.4)\end{aligned}$$

The difference between the piecewise linear and the piecewise constant approximations can be estimated, by using (4.15), **(A2)**, (4.28), Sobolev's embedding Theorem, (4.32, 4.39, 4.41):

$$\left\|\widehat{\theta}^Z - \overline{\theta}^Z\right\|_{L^2(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)^*)} + \left\|\widehat{u}^Z - \overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))} \le C_3\,|Z|, \quad (6.5)$$

$$\begin{aligned}\left\|\widehat{\chi}^Z - \overline{\chi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \left\|\widehat{\chi}^Z - \underline{\chi}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \\ + \left\|\lambda(\widehat{\chi}^Z) - \lambda(\overline{\chi}^Z)\right\|_{L^\infty(0,T;L^2(\Omega))} \le C_4\,|Z|, \quad (6.6)\end{aligned}$$

$$\left\|\widehat{\lambda}^Z - \overline{\lambda}^Z\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \left\|\lambda(\overline{\chi}^Z) - \overline{\lambda}^Z\right\|_{L^\infty(0,T;L^3(\Omega))} \le C_5\,|Z|, \quad (6.7)$$

$$\left\|\widehat{u}^Z - \overline{u}^Z\right\|_{L^2(0,T;H^1(\Omega))} \le C_6\sqrt{|Z|}. \quad (6.8)$$

For the approximation of the data, we have, by **(A3)**:

Lemma 6.1. *The functions* $\overline{g}^Z, \overline{\gamma}^Z, \overline{\zeta}^Z$ *fulfill*

$$\left\|\overline{g}^Z\right\|_{L^\infty(\Omega_T)} + \left\|\overline{\gamma}^Z\right\|_{L^\infty(0,T;C^1(\Gamma))} + \left\|\overline{\zeta}^Z\right\|_{L^\infty(\Gamma_T)\cap L^\infty(0,T;H^{\frac{1}{2}}(\Gamma))} \le C_7, \quad (6.9)$$

$$\left\|g - \overline{g}^Z\right\|_{L^2(0,T;L^\infty(\Omega))} + \left\|\gamma - \overline{\gamma}^Z\right\|_{L^\infty(\Gamma_T)} + \left\|\zeta - \overline{\zeta}^Z\right\|_{L^2(0,T;L^2(\Gamma))} \le C_8\,|Z|. \quad (6.10)$$

Now, estimates similar to [28] are used to prove the following lemma.

Lemma 6.2. *We have a positive constant* C_9 *such that*

$$-\int_0^s\!\!\int_\Omega \left(\xi - \overline{\xi}^Z\right)\left(\chi - \widehat{\chi}^Z\right)\,\mathrm{d}x\,\mathrm{d}t \le C_9\,|Z|^2, \quad \forall s \in [0,T], \quad (6.11)$$

for all $\chi, \xi \in L^2(0,T;L^2(\Omega))$ *with*

$$\chi \in D(\beta), \quad \xi \in \beta(\chi) \quad \textit{a.e. in } \Omega_T. \quad (6.12)$$

Proof. From (6.12, 6.2d), and $\beta = \partial\phi$, we get

$$A_1 := -\int_0^s\!\!\int_\Omega \left(\xi - \overline{\xi}^Z\right)\left(\chi - \widehat{\chi}^Z\right)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^s\!\!\int_\Omega \left(-\phi(\overline{\chi}^Z) + \overline{\xi}^Z\left(\overline{\chi}^Z - \widehat{\chi}^Z\right) + \phi(\widehat{\chi}^Z)\right)\mathrm{d}x\,\mathrm{d}t\,.$$

For $l^Z : (0,T] \to [0,1]$ defined by

$$l^Z(t) = \frac{t_m - t}{h_m}, \qquad \forall t \in (t_{m-1}, t_m],\ 1 \le m \le K, \tag{6.13}$$

holds

$$\widehat{\chi}^Z = \left(1 - l^Z\right) \overline{\chi}^Z + l^Z \underline{\chi}^Z = \overline{\chi}^Z + l^Z \left(\underline{\chi}^Z - \overline{\chi}^Z\right) \quad \text{a.e. in } \Omega_T.$$

We apply the convexity of ϕ, to show that

$$A_1 \le \int_0^s l^Z \int_\Omega \left(-\phi(\overline{\chi}^Z) + \phi(\underline{\chi}^Z) + \overline{\xi}^Z \left(\overline{\chi}^Z - \underline{\chi}^Z\right)\right) \mathrm{d}x \ \mathrm{d}t\,.$$

Since (6.2d) and $\beta = \partial\phi$ yield that the integrand is a.e. non–negative, we see, by (6.13, 2.3b,f), **(A4)**, and $\beta = \partial\phi$, that

$$\begin{aligned} A_1 &\le \sum_{m=1}^{K} \int_{t_{m-1}}^{t_m} \frac{t_m - t}{h_m} \,\mathrm{d}t \int_\Omega \left(-\phi(\chi_m) + \phi(\chi_{m-1}) + \xi_m \left(\chi_m - \chi_{m-1}\right)\right) \mathrm{d}x \\ &\le \frac{1}{2} |Z|^2 \sum_{m=1}^{K} \left\| (\xi_m - \xi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_1 . \end{aligned}$$

Hence, (6.11) holds because of (4.15). □

6.2. Error estimates

Now, we estimate the difference between the approximation and one exact solution. Here, ideas from [7, 8, 16, 28] are used.

Lemma 6.3. *For every solution (θ, u, χ, ξ) to the Penrose–Fife system* **(PF)** *there are positive constants C_{10}, C_{11} such that*

$$\begin{aligned} &\max_{0\le s\le T} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 + \max_{0\le s\le T} \left\| \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_\Gamma^2 + \left\| \frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}} \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad + \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}^2 + \left\| \theta - \overline{\theta}^Z \right\|_{L^2(0,T;L^1(\Omega))}^2 + \left\| \nabla \left(\chi - \overline{\chi}^Z\right) \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad + \left\| \chi - \widehat{\chi}^Z \right\|_{L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))}^2 \le C_{10} \left(|Z|^2 + |Z| \left\| u - \overline{u}^Z \right\|_{L^2(0,T;L^2(\Omega))}\right) \qquad (6.14) \\ &\quad \le C_{11} |Z|\,. \qquad (6.15) \end{aligned}$$

Proof. The generic constants may depend on the solution to the Penrose–Fife system.
Thanks to (2.1a,b), Sobolev's embedding Theorem, and **(A2)**, we have

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} + \|\chi\|_{L^\infty(\Omega_T)} + \|\lambda'(\chi)\|_{L^\infty(\Omega_T)} \le C_{12}. \tag{6.16}$$

First, we work on the equation for θ and u. Integrating the difference of (2.1e) and (6.2e) in time, and testing the corresponding equation by v, and using (2.1g, h, 6.2g, h), we obtain for all $v \in H^1(\Omega)$,

$$\int_\Omega \Big(c_0(\theta(t) - \widehat{\theta}^Z(t)) + \lambda(\chi(t)) - \widehat{\lambda}^Z(t)\Big)\, v \,\mathrm{d}x - \kappa \int_0^t \int_\Omega \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right) \bullet \nabla v \,\mathrm{d}x \,\mathrm{d}\tau$$
$$= \int_\Omega \int_0^t \left(g(\tau) - \overline{g}^Z(\tau)\right) \mathrm{d}\tau \, v \,\mathrm{d}x + \int_0^t \int_\Gamma \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right) v \,\mathrm{d}\sigma \,\mathrm{d}\tau$$
$$+ \int_0^t \int_\Gamma \Big(\left(\gamma(\tau) - \overline{\gamma}^Z(\tau)\right)\overline{u}^Z(\tau) - \Big(\zeta(\tau) - \overline{\zeta}^Z(\tau)\Big)\Big) v \,\mathrm{d}\sigma \,\mathrm{d}\tau, \quad \forall t \in (0,T). \tag{6.17}$$

For a.e. $t \in (0,T)$, this yields, with $v = -\left(u(t) - \overline{u}^Z(t)\right)$, by (2.1d, 6.2d),

$$\int_\Omega c_0 \left(\frac{\left(u - \overline{u}^Z\right)^2}{u\overline{u}^Z} - \left(\overline{\theta}^Z - \widehat{\theta}^Z\right)\left(u - \overline{u}^Z\right)\right) \mathrm{d}x - \int_\Omega \left(\lambda(\chi) - \widehat{\lambda}^Z\right)\left(u - \overline{u}^Z\right) \mathrm{d}x$$
$$= -\int_\Omega \int_0^t \left(g(\tau) - \overline{g}^Z(\tau)\right) \mathrm{d}\tau \,\left(u - \overline{u}^Z\right) \mathrm{d}x - \int_\Gamma \left(\int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right) \left(u - \overline{u}^Z\right) \mathrm{d}\sigma$$
$$- \int_\Gamma \int_0^t \Big(\left(\gamma(\tau) - \overline{\gamma}^Z(\tau)\right)\overline{u}^Z(\tau) \; - \Big(\zeta(\tau) - \overline{\zeta}^Z(\tau)\Big)\Big) \mathrm{d}\tau \,\left(u - \overline{u}^Z\right) \mathrm{d}\sigma$$
$$- \kappa \int_\Omega \int_0^t \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \bullet \nabla\left(u - \overline{u}^Z\right) \mathrm{d}x =: A_2 + A_3 + A_4 + A_5. \tag{6.18}$$

Owing to (6.2d, 2.1d), the generalized Hölder's inequality, (AP.1, 6.3, 6.16), we see that

$$\int_0^s \left(\left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2 + \left\|\theta - \overline{\theta}^Z\right\|_1^2 \right) \mathrm{d}t \le C_{13} \int_0^s \left\| \frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}} \right\|_2^2 \mathrm{d}t. \tag{6.19}$$

We have, by Hölder's inequality, (6.10), and Young's inequality,

$$A_2 \le C_{14} \left\|g - \overline{g}^Z\right\|_{L^2(0,T;L^\infty(\Omega))} \left\|u - \overline{u}^Z\right\|_{\frac{3}{2}} \le C_{15} \left|Z\right|^2 + \frac{c_0}{4C_{13}} \left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2, \tag{6.20}$$

$$A_3 = -\frac{1}{2}\partial_t \left\| \frac{1}{\sqrt{\gamma(t)}} \int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2$$
$$- \int_\Gamma \frac{\gamma_t(t)}{2(\gamma(t))^2} \left(\int_0^t \gamma(\tau)\left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right)^2 \mathrm{d}\sigma, \tag{6.21}$$

$$A_5 = -\frac{\kappa}{2}\partial_t \left\| \int_0^t \nabla\left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_2^2. \tag{6.22}$$

By integrating (6.18) from 0 to s and using (6.19–6.21), we obtain

$$\frac{c_0}{2C_{13}}\int_0^s\left(\frac{1}{2}\left\|u-\overline{u}^Z\right\|_{\frac{3}{2}}^2+\left\|\theta-\overline{\theta}^Z\right\|_1^2\right)\,\mathrm{d}t+\frac{c_0}{2}\int_0^s\left\|\frac{u-\overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|_2^2\,\mathrm{d}t+\frac{\kappa}{2}\left\|\nabla\int_0^s\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_2^2$$
$$+\frac{1}{2}\left\|\frac{1}{\sqrt{\gamma(s)}}\int_0^s\gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{L^2(\Gamma)}^2\leq\int_0^s\int_\Omega\left(\left(\lambda(\chi)-\lambda(\widehat{\chi}^Z)\right)+\left(\lambda(\widehat{\chi}^Z)-\widehat{\lambda}^Z\right)\right)\left(u-\overline{u}^Z\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$+c_0\int_0^s\int_\Omega\left(\overline{\theta}^Z-\widehat{\theta}^Z\right)\left(u-\overline{u}^Z\right)\,\mathrm{d}x\,\mathrm{d}t+\int_0^s\int_\Gamma\int_0^t\left(\zeta(\tau)-\overline{\zeta}^Z(\tau)-\left(\gamma(\tau)-\overline{\gamma}^Z(\tau)\right)\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\,\left(u-\overline{u}^Z\right)\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+TC_{15}\left|Z\right|^2-\int_0^s\int_\Gamma\frac{\gamma_t(t)}{2(\gamma(t))^2}\left(\int_0^t\gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right)^2\,\mathrm{d}\sigma\,\mathrm{d}t=:A_6+A_7+A_8+TC_{15}\left|Z\right|^2+A_9. \tag{6.23}$$

Applying Poincaré's inequality and Hölder's inequality, we get a positive constant C_{16} such that

$$C_{16}\left\|\int_0^s\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{H^1(\Omega)}^2\leq\frac{\kappa}{2}\left\|\nabla\int_0^s\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_2^2+\frac{c_0}{8C_{13}}\int_0^s\left\|u(\tau)-\overline{u}^Z(\tau)\right\|_{\frac{3}{2}}^2\,\mathrm{d}\tau. \tag{6.24}$$

Using **(A2)**, (6.16, 6.4), Sobolev's embedding Theorem, Hölder's inequality, (6.5–6.7), we derive

$$A_6+A_7\leq C_{17}\int_0^s\left\|\left(\chi-\widehat{\chi}^Z\right)\left(u-\overline{u}^Z\right)\right\|_1\,\mathrm{d}t+C_{18}\left|Z\right|\left\|u-\overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))}. \tag{6.25}$$

Partial integration with respect to time and applying Hölder's inequality result in

$$A_8\leq\left(\left\|\int_0^s\left(\zeta-\overline{\zeta}^Z\right)\,\mathrm{d}t\right\|_{L^2(\Gamma)}+\int_0^s\left\|\gamma-\overline{\gamma}^Z\right\|_{L^\infty(\Gamma)}\,\mathrm{d}t\,\left\|\overline{u}^Z\right\|_{L^\infty(0,T;L^2(\Gamma))}\right)\left\|\int_0^s\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{L^2(\Gamma)}$$
$$+\int_0^s\left(\left\|\zeta-\overline{\zeta}^Z\right\|_{L^2(\Gamma)}+\left\|\gamma-\overline{\gamma}^Z\right\|_{L^\infty(\Gamma)}\left\|\overline{u}^Z\right\|_{L^2(\Gamma)}\right)\left\|\int_0^t\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{L^2(\Gamma)}\,\mathrm{d}t.$$

Because of the trace theorem, (6.3, 6.10), and Young's inequality, we observe

$$A_8\leq\frac{C_{16}}{2}\left\|\int_0^s\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{H^1(\Omega)}^2+\frac{1}{2}\int_0^s\left\|\int_0^t\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{H^1(\Omega)}^2\,\mathrm{d}t+C_{19}\left|Z\right|^2. \tag{6.26}$$

In the light of Hölder's inequality and **(A3)**, we see

$$A_9\leq C_{20}\int_0^s\left\|\frac{1}{\sqrt{\gamma(t)}}\int_0^t\gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|_{L^2(\Gamma)}^2\,\mathrm{d}t. \tag{6.27}$$

Hence, we get, by using Hölder's inequality, (6.23–6.27), and Young's inequality,

$$
\begin{aligned}
&\frac{C_{16}}{2}\left\|\int_0^s \left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{H^1(\Omega)} + \frac{c_0}{2C_{13}}\int_0^s \left(\frac{1}{4}\left\|u-\overline{u}^Z\right\|^2_{\frac{3}{2}} + \left\|\theta-\overline{\theta}^Z\right\|^2_1\right)\,\mathrm{d}t\\
&+\frac{c_0}{2}\int_0^s \left\|\frac{u-\overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|_2^2\,\mathrm{d}t + \frac{1}{2}\left\|\frac{1}{\sqrt{\gamma(s)}}\int_0^s \gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)}\\
\leq &C_{17}\int_0^s \left\|\left(\chi-\widehat{\chi}^Z\right)\left(u-\overline{u}^Z\right)\right\|_1\,\mathrm{d}t + C_{18}\left|Z\right|\left\|u-\overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))}\\
&+\frac{1}{2}\int_0^s \left\|\int_0^t \left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{H^1(\Omega)}\,\mathrm{d}t\\
&+\left(C_{19}+TC_{15}\right)\left|Z\right|^2 + C_{20}\int_0^s \left\|\frac{1}{\sqrt{\gamma(t)}}\int_0^t \gamma(\tau)\left(u(\tau)-\overline{u}^Z(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)}\,\mathrm{d}t\,. \qquad (6.28)
\end{aligned}
$$

Now, estimates for χ will be derived. Subtracting (6.2f) from (2.1f), we obtain that

$$
\eta(\chi_t-\widehat{\chi}^Z_t) - \varepsilon\Delta\left(\chi-\overline{\chi}^Z\right) + \xi - \overline{\xi}^Z - \sigma'(\chi) + \sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z) = -\lambda'(\chi)u + \lambda_d'(\overline{\chi}^Z,\underline{\chi}^Z)\overline{u}^Z \quad \text{a.e. in } \Omega_T. \qquad (6.29)
$$

Testing this with $\chi-\widehat{\chi}^Z$ and recalling **(A3)**, (2.1g, 6.2g), we end up with

$$
\begin{aligned}
&\frac{1}{2}\partial_t\left\|\sqrt{\eta}\,\left(\chi-\widehat{\chi}^Z\right)\right\|_2^2 + \varepsilon\int_\Omega \nabla\left(\chi-\overline{\chi}^Z\right)\bullet\nabla\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x + \int_\Omega \left(\xi-\overline{\xi}^Z\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\
\leq &\int_\Omega \left(\sigma'(\chi)-\sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x - \int_\Omega \left(\lambda'(\chi)u-\lambda_d'(\overline{\chi}^Z,\underline{\chi}^Z)\overline{u}^Z\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\
=:&\ A_{10}+A_{11}. \qquad (6.30)
\end{aligned}
$$

We have

$$
\varepsilon\int_\Omega \nabla\left(\chi-\overline{\chi}^Z\right)\bullet\nabla\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x = \frac{\varepsilon}{2}\left\|\nabla\left(\chi-\overline{\chi}^Z\right)\right\|_2^2 + \frac{\varepsilon}{2}\left\|\nabla\left(\chi-\widehat{\chi}^Z\right)\right\|_2^2 - \frac{\varepsilon}{2}\left\|\nabla\left(\overline{\chi}^Z-\widehat{\chi}^Z\right)\right\|_2^2. \qquad (6.31)
$$

Using (6.30), **(A6, A2)**, (6.16, 6.4), Sobolev's embedding Theorem, Hölder's inequality, (6.6), and Young's inequality, we conclude

$$
\begin{aligned}
A_{10} = &\int_\Omega \left(\sigma'(\chi)-\sigma'(\widehat{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x + \int_\Omega \left(\sigma_d'(\widehat{\chi}^Z,\widehat{\chi}^Z)-\sigma_d'(\overline{\chi}^Z,\underline{\chi}^Z)\right)\left(\chi-\widehat{\chi}^Z\right)\,\mathrm{d}x\\
\leq &C_{21}\left\|\chi-\widehat{\chi}^Z\right\|_2^2 + C_{22}\left|Z\right|^2 \quad \text{a.e. in } (0,T). \qquad (6.32)
\end{aligned}
$$

In the light of (6.30), **(A6)**, the generalized Hölder's inequality, **(A2)**, (6.16, 6.3, 6.4), Sobolev's embedding Theorem, (6.6), and Young's inequality, we see that

$$\begin{aligned} A_{11} =& - \int_\Omega \Big(\lambda'(\chi)\left(u - \overline{u}^Z\right) + \left(\lambda'(\chi) - \lambda'(\widehat{\chi}^Z)\right)\overline{u}^Z \Big) \left(\chi - \widehat{\chi}^Z\right) \mathrm{d}x \\ & - \int_\Omega \Big(\lambda_d'(\widehat{\chi}^Z, \widehat{\chi}^Z) - \lambda_d'(\overline{\chi}^Z, \underline{\chi}^Z) \Big) \overline{u}^Z \left(\chi - \widehat{\chi}^Z\right) \mathrm{d}x \\ \le & C_{23} \left\| \left(u - \overline{u}^Z\right)\left(\chi - \widehat{\chi}^Z\right)\right\|_1 + C_{24} \left\|\chi - \widehat{\chi}^Z\right\|_2^2 + C_{25} |Z|^2 . \end{aligned} \tag{6.33}$$

Combining (6.30–6.33), integrating in time, using **(A3)**, (2.1h, 6.2h, 6.11, 6.6), and adding the resulting estimate to (6.28), we get

$$\begin{aligned} & \frac{C_{16}}{2} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 + C_{26} \int_0^s \left(\left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2 + \left\|\theta - \overline{\theta}^Z\right\|_{L^1(\Omega)}^2 \right) \mathrm{d}t \\ & + \frac{c_0}{2} \int_0^s \left\| \frac{u - \overline{u}^Z}{\sqrt{u \overline{u}^Z}} \right\|_2^2 \mathrm{d}t + \frac{1}{2} \left\| \frac{1}{\sqrt{\gamma(s)}} \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2 \\ & + \frac{c_\eta}{2} \left\|\chi(s) - \widehat{\chi}^Z(s)\right\|_2^2 + \frac{\varepsilon}{2} \int_0^s \left\|\nabla\left(\chi - \widehat{\chi}^Z\right)\right\|_2^2 \mathrm{d}t + \frac{\varepsilon}{2} \int_0^s \left\|\chi - \widehat{\chi}^Z\right\|_{H^1(\Omega)}^2 \mathrm{d}t \\ \le & A_{12} + C_{18} |Z| \left\|u - \overline{u}^Z\right\|_{L^2(0,T;L^2(\Omega))} + C_{27} |Z|^2 \\ & + \frac{1}{2} \int_0^s \left\| \int_0^t \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{H^1(\Omega)}^2 \mathrm{d}t + \left(\frac{\varepsilon}{2} + C_{21} + C_{24}\right) \int_0^s \left\|\chi - \widehat{\chi}^Z\right\|_2^2 \mathrm{d}t \\ & + C_{20} \int_0^s \left\| \frac{1}{\sqrt{\gamma(t)}} \int_0^t \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \end{aligned} \tag{6.34}$$

with

$$A_{12} := (C_{17} + C_{23}) \int_0^s \left\| \left(\chi - \widehat{\chi}^Z\right)\left(u - \overline{u}^Z\right)\right\|_1 \mathrm{d}t .$$

Using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we obtain

$$A_{12} \le \frac{C_{26}}{2} \int_0^s \left\|u - \overline{u}^Z\right\|_{\frac{3}{2}}^2 \mathrm{d}t + \frac{\varepsilon}{4} \int_0^s \left\|\chi - \widehat{\chi}^Z\right\|_{H^1(\Omega)}^2 \mathrm{d}t + C_{28} \int_0^s \left\|\chi - \widehat{\chi}^Z\right\|_2^2 \mathrm{d}t .$$

Hence, (6.34), Gronwall's lemma, and **(A3)** yield that (6.14) is satisfied. Combining this with (6.16, 6.3), we deduce that (6.15) is satisfied. □

6.3. **Proof of Theorem 2.2**

Proof. Thanks to the estimates (6.3, 6.4), Sobolev's embedding Theorem, and compactness (see, *e.g.* Props. 23.7, 23.19, Prob. 23.12 in [31]), we get $(\theta, u, \chi, \xi, \lambda^*)$ fulfilling (2.1b, c), (2.8–2.10), and

$$\theta \in H^1(0,T;L^2(\Omega)), \quad u \in L^\infty(0,T;H^1(\Omega)), \quad \lambda^* \in W^{1,\infty}(0,T;L^2(\Omega)),$$

such that we have, for some subsequence with $|Z| \to 0$, the convergences (2.11–2.19), and

$$\widehat{\lambda}^Z \longrightarrow \lambda^* \quad \text{weakly–star in} \quad W^{1,\infty}(0,T;L^2(\Omega)). \tag{6.35}$$

We obtain the convergences (2.11–2.19) for the whole sequence, if we can show that (θ, u, χ, ξ) is the unique solution to the Penrose–Fife system **(PF)**. Hence, we need only to prove this, to finish the proof of Theorem 2.2.

Thanks to the convergences for $\widehat{\chi}^Z$ in (2.17, 6.4), the Aubin compactness lemma (see, *e.g.* p. 58 in [24]), and (6.6), we also get

$$\widehat{\chi}^Z \longrightarrow \chi, \quad \overline{\chi}^Z \longrightarrow \chi, \quad \underline{\chi}^Z \longrightarrow \chi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{6.36}$$

Hence, after possibly extracting a further subsequence, we have

$$\overline{\chi}^Z \longrightarrow \chi, \quad \underline{\chi}^Z \longrightarrow \chi \qquad \text{a.e. in } \Omega_T.$$

This yields, thanks to **(A2, A6)**, (6.4), and the Lebesgue dominated convergence theorem, that

$$\lambda(\overline{\chi}^Z) \longrightarrow \lambda(\chi), \quad \lambda_d'(\overline{\chi}^Z, \underline{\chi}^Z) \longrightarrow \lambda'(\chi), \quad \sigma_d'(\overline{\chi}^Z, \underline{\chi}^Z) \longrightarrow \sigma'(\chi) \quad \text{strongly in} \quad L^2(\Omega_T). \tag{6.37}$$

Thus, (6.35, 6.6, 6.7) yield that $\lambda^* = \lambda(\chi)$ a.e. on Ω_T. Hence, using (2.11–2.19, 6.35–6.37, 6.3–6.10), we can pass to the limit in (6.2a–6.2h) and obtain that (θ, u, χ, ξ) is a solution to the Penrose–Fife system **(PF)**. Details can be found in Section 8 of [16]. It remains to show that this solution is unique.

Let $(\theta^*, u^*, \chi^*, \xi^*)$ be any solution to the Penrose–Fife system **(PF)**. Since we can apply Lemma 6.3 for this solution, using (6.15) and the convergences (2.11–2.18) yields that

$$\theta^* = \theta, \quad u^* = u, \quad \chi^* = \chi \quad \text{a.e. in } \Omega_T.$$

Comparing the terms in (2.1f), we see that the two solutions coincide. □

6.4. **Proof of Theorem 2.3**

Proof. Thanks to (2.1d, 6.2d), Hölder's inequality, (2.8, 2.9, 6.3), we have

$$\left\|u - \overline{u}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\theta - \overline{\theta}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} \leq C_{29} \left\|\frac{u - \overline{u}^Z}{\sqrt{u\overline{u}^Z}}\right\|^2_{L^2(0,T;L^2(\Omega))}$$

Moreover, we have $\chi - \widehat{\chi}^Z \in C([0,T];L^2(\Omega))$, because of (6.2b, 2.1b). Hence, we obtain from (6.14) and Young's inequality that

$$\max_{0\leq s\leq T} \left\| \int_0^s \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|^2_{H^1(\Omega)} + \max_{0\leq s\leq T} \left\| \int_0^s \gamma(\tau) \left(u(\tau) - \overline{u}^Z(\tau)\right) \mathrm{d}\tau \right\|^2_{\Gamma} + \frac{1}{2C_{29}} \left\|u - \overline{u}^Z\right\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ \frac{1}{C_{29}} \left\|\theta - \overline{\theta}^Z\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\chi - \widehat{\chi}^Z\right\|^2_{L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))} \leq C_{30} \left|Z\right|^2. \tag{6.38}$$

Therefore, by comparing the terms in (6.17), and using (6.10, 6.3), we get

$$\left\| c_0 \left(\theta - \widehat{\theta}^Z\right) + \lambda(\chi) - \widehat{\lambda}^Z \right\|^2_{L^\infty(0,T;H^1(\Omega)^*)} \leq C_{31} \left|Z\right|^2. \tag{6.39}$$

Now, **(A2)**, (6.16, 6.4, 6.6, 6.7), $L^2(\Omega) \subset H^1(\Omega)^*$, (6.38, 6.2a, 2.1a) yield that

$$\left\| \theta - \widehat{\theta}^Z \right\|^2_{C([0,T];H^1(\Omega)^*)} \leq C_{32} \left|Z\right|^2. \tag{6.40}$$

Combining this with (6.38, 6.5), we see that (2.20) is satisfied. □

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A. Appendix

For convenience, we list some inequalities and equalities used throughout this paper.

Lemma AP.1 (Young's inequality). *For $a \geq 0$, $b \geq 0$, $\sigma > 0$, $p > 1$, $q := p/(p-1)$, it holds*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \quad ab \leq \frac{1}{p}\sigma^{-(p-1)}a^p + \frac{1}{q}\sigma b^q,$$
$$a^{ps}b^{p(1-s)} \leq s\left(\frac{\sigma}{1-s}\right)^{\frac{s-1}{s}} a^p + \sigma b^p, \quad \forall\, 0 < s < 1.$$

Lemma AP.2 (Generalized Hölder's inequality). *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, p, p_1, p_2, $p_3 \in [1,\infty]$, $f_1 \in L^{p_1}(\Omega)$, $f_2 \in L^{p_2}(\Omega)$, and $f_3 \in L^{p_3}(\Omega)$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p},$$

we have $f_1 \cdot f_2 \cdot f_3 \in L^p(\Omega)$ and

$$\left\| f_1 \cdot f_2 \cdot f_3 \right\|_{L^p(\Omega)} \leq \left\| f_1 \right\|_{L^{p_1}(\Omega)} \left\| f_2 \right\|_{L^{p_2}(\Omega)} \left\| f_3 \right\|_{L^{p_3}(\Omega)}.$$

Thanks to Sobolev's embedding Theorem, we have:

Lemma AP.3. *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ and Lipschitz boundary, there is a positive constant C such that*

$$\left\| |v|^p \right\|_{L^{\frac{6}{p}}(\Omega)} = \left\| v \right\|^p_{L^6(\Omega)} \leq C^p \left\| v \right\|^p_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),\ p \in (0,6]. \tag{AP.1}$$

The following classical elliptic estimate can be found in Remark 9.3d of [1].

Lemma AP.4. *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$ and $\partial\Omega$ smooth there is a positive constant C such that*

$$\|v\|^2_{H^2(\Omega)} \le C \left(\|\Delta v\|^2_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial n} \right\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|v\|^2_{L^2(\Omega)} \right), \quad \forall\, v \in H^2(\Omega).$$

In particular, for all $v \in H^2(\Omega)$ with $\partial v/\partial n = 0$ a.e. on Γ,

$$\|v\|^2_{H^2(\Omega)} \le C \left(\|\Delta v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right).$$

The following version of the Gagliardo–Nirenberg inequality is a special case of those considered in Theorem 1.1.4ii of [32].

Lemma AP.5. *Let $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ be a bounded, open domain with $\partial\Omega$ smooth. Let $2 \le p \le 6$ be given. and $a := 3/2 - 3/p$, Then there is a positive constant C such that*

$$\|u\|_{L^p(\Omega)} \le C\, \|u\|^a_{H^1(\Omega)}\, \|u\|^{1-a}_{L^2(\Omega)}, \quad \|u\|_{L^p(\Omega)} \le C\, \|u\|^{\frac{a}{2}}_{H^2(\Omega)}\, \|u\|^{1-\frac{a}{2}}_{L^2(\Omega)}.$$

If $\Omega \subset \mathbb{R}^2$, then the first estimate is also satisfied for $a = 1 - 2/p$.

Elementary calculations lead to:

Lemma AP.6. *For $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_n \in \mathbb{R}$, we have*

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) - \sum_{j=1}^{n-1} b_{j+1} \sum_{i=1}^{j} a_i, \tag{AP.2}$$

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} a_j = \frac{1}{2}\left(\sum_{i=1}^{n} a_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n} a_i^2, \tag{AP.3}$$

$$\sum_{i=1}^{n} a_i (b_i - b_{i-1}) = a_n b_n - a_1 b_0 - \sum_{i=1}^{n-1} (a_{i+1} - a_i)\, b_i. \tag{AP.4}$$

Lemma AP.7. *Let H be a Hilbert space with scalar-product $\langle \cdot,\cdot \rangle_H$ and norm $\|\cdot\|_H$. Then we have*

$$\langle a, a-b \rangle_H = \frac{1}{2}\|a\|^2_H - \frac{1}{2}\|b\|^2_H + \frac{1}{2}\|a-b\|^2_H, \quad \forall\, a,b \in H. \tag{AP.5}$$

The next lemma follows from elementary analysis.

Lemma AP.8. *Let $a, b > 0$ be given. Then there exists a constant $C > 0$, such that*

$$\frac{a}{2}s + b\,|\ln s| \le as - b\ln s + C, \quad \forall\, s > 0.$$

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