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Abdellatif Agouzal
Laurence Lamoulie
Jean-Marie Thomas

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3D DOMAIN DECOMPOSITION METHOD COUPLING CONFORMING AND NONCONFORMING FINITE ELEMENTS

Abdellatif Agouzal[1], Laurence Lamoulie[2] and Jean-Marie Thomas[3]

Abstract. This paper deals with the solution of problems involving partial differential equations in $\mathbb{R}^3$. For three dimensional case, methods are useful if they require neither domain boundary regularity nor regularity for the exact solution of the problem. A new domain decomposition method is therefore presented which uses low degree finite elements. The numerical approximation of the solution is easy, and optimal error bounds are obtained according to suitable norms.

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1. Introduction

In three dimensions, the domain boundary regularity is obviously root of difficulties. For example some bounded polyhedral domain have a continuous but not Lipschitzian boundary (*cf.* Fig. 1).

In this example, near the contact points between the two bars, nor Lipschitzian function exists that can fit the domain boundary.

Numerical solution is harder because of the lack of regularity. Domain decomposition sometimes allows to overcome this problem: a clever decomposition may lead to regular bounded subdomains, from an initial regular domain. For Figure 1, a good choice is obviously the following one (*cf.* Fig. 2).

Matching unknowns on the interface is one of the difficulties one is faced with in domain decomposition methods. Mesh parameters must not imposed on both sides of the interface by the chosen matching conditions. Moreover numerical solution must be easy on sequential or parallel computers.

In the context of non matching grids, to our knowledge, three approaches are considered in literature: mortar element methods, in a primal formulation [6] or in a dual formulation [2], hybrid methods [3,13], and primal-dual coupling methods [4,10].

Allowing to achieve this goal, a domain decomposition method using low degree finite elements is presented here. For the sake of shortness and simplicity, the method is exposed for the following model problem.

Assuming that f is a given function in $L^2(\Omega)$, find u solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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[1] Université Claude Bernard - Lyon I and UMR 5585, 43 bd. du 11 Novembre 1918, 69622 Villeurbanne, France.
e-mail: agouzal@iris.univ-lyon1.fr

[2] Ecole d'Ingénieurs en Génie des Systèmes Industriels, 17041 La Rochelle, France.

[3] Laboratoire de Mathématiques Appliquées pour l'Industrie, UPRES A 5033, Université de Pau, 64000 Pau, France.

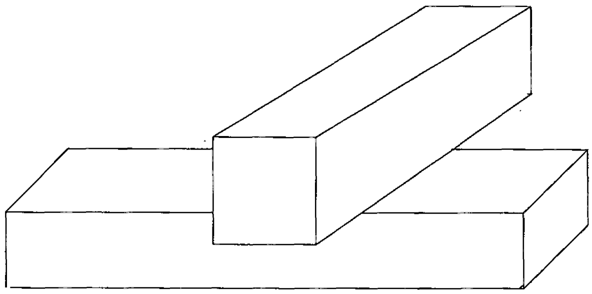

FIGURE 1. A non Lipschitzian domain

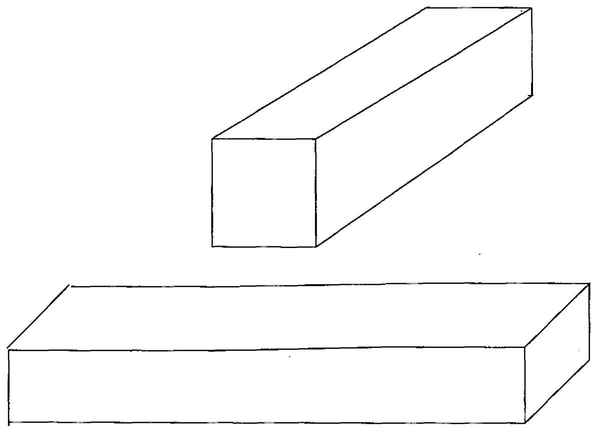

FIGURE 2. Domain decomposition with regular bounded subdomains.

Interpolation results are firstly proved, which allows to analyse a domain decomposition method coupling conforming and nonconforming finite elements. A new formulation is then established that couples primal and dual variables. The idea was first introduce in [4,10], and used for example in [5] for the coupling of Stokes equations. Using low degree spaces for numerical approximation, optimal error bounds are obtained without any condition on meshes and restricting regularity requirements on the solution.

2. A THREE DIMENSIONAL INTERPOLATION OPERATOR

Let T be a tetrahedron, let us denote its faces by F_i, $i = 1, ..., 4$. It is well known and immediate to prove that for each $v \in H^1(T)$, there exists a unique affine function $\Pi_h v \in P_1(T)$ such that:

$$\forall i = 1, ..., 4 \qquad \int_{F_i} \Pi_T v \mathrm{d}\sigma = \int_{F_i} v \mathrm{d}\sigma.$$

So Π_T define an interpolation operator; associated interpolation error bounds are now to be determined. The following result will be helpful:

Lemma 2.1. *As operator from $H^1(T)$ onto $P_1(T)$, Π_T is continuous and for all $u \in H^1(T)$ the following inequality holds:*

$$|\Pi_T u|_{1,T} \leq |u|_{1,T}.$$

Proof. Assuming that $u \in H^1(T)$, by Green formula we have for all regular test function v:

$$(u - \Pi_T u, v)_{1,T} = -\int_T (u - \Pi_T u)\Delta v \mathrm{d}x + \sum_{i=1}^{4} \int_{F_i} \left\{ \frac{\partial v}{\partial n_i}(u - \Pi_T u)\mathrm{d}\sigma \right\}.$$

Then for all affine function v, sine $\Delta v = 0$ and $\int_{F_i}(u - \Pi_T u)\mathrm{d}\sigma = 0, i = 1, ..., 4$, this implies that

$$(u - \Pi_T u, v)_{1,T} = 0.$$

So $\Pi_T u$ is he orthogonal projection of u onto $P_1(T)$ for the scalar product of $H^1(T)$ and so

$$|\Pi_T u|_{1,T} \leq |u|_{1,T}.$$

Using Lemma 2.1, standard arguments of finite elements methods theory and functional interpolation of Hilbert spaces, the following interpolation error bounds holds (see also [1] for similar result in two-dimensional case).

Theorem 2.1. *There exists a constant C such that, for all $u \in H^{1+\sigma}(T)$, $0 \leq \sigma \leq 1$, we have*

$$|u - \Pi_T u|_{1,T} \leq Ch^\sigma |u|_{1+\sigma,T}.$$

3. Conforming-nonconforming hybrid formulation

The following model problem is studied:

$\mathcal{P}$: Assuming that f is a given function in $L^2(\Omega)$, find u solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that Ω is an open connected subset of $\mathbb{R}^3$, which boundary $\partial\Omega$ is regular enough but may be not Lipschitzian. Ω is divided into two non overlapping subdomains Ω_1 and Ω_2 supposed connected for simplicity. The interface between Ω_1 and Ω_2 is called Σ, and $\Gamma_i = \partial\Omega_i/\Sigma$, i=1,2. For sake of simplicity, it is supposed that $\Gamma_i \neq \emptyset$, $i = 1, 2$.

Let u_i, $i = 1, 2$ denote the restriction of u to the subdomain called Ω_i, $i = 1, 2$.

3.1. Continuous formulation

Few notations are needed:

$$\begin{aligned} V_i = H^1_{0,\Gamma_i}(\Omega_i) &= \{v \in H^1(\Omega_i) \text{ such that } v_{|\Gamma_i} = 0\}, \quad i = 1, 2, \\ H^{1/2}_{0,0}(\Sigma) &= \{v_{|\Sigma}, v \in V_1\} = \{v_{|\Sigma}, v \in V_2\} \\ \Lambda &= H^{-1/2}(\Sigma) \quad \text{dual space of } \; H^{1/2}_{0,0}(\Sigma). \end{aligned}$$

A continuous formulation of $\mathcal{P}$ is established dualizing the traces matching equation on the interface Σ.

Continuous hybrid formulation:

$\mathcal{P}_H$: Find $(u_1, u_2, \lambda) \in V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \forall (v_1, v_2) \in V_1 \times V_2, \sum_{i=1}^{2} \int_{\Omega_i} \mathbf{grad}\, u_i . \mathbf{grad}\, v_i \mathrm{d}x - \langle \lambda, (v_1 - v_2)_{|\Sigma} \rangle = \sum_{i=1}^{2} \int_{\Omega_i} f v_i \mathrm{d}x, \\ \forall \mu \in \Lambda, \langle \mu, (u_1 - u_2)_{|\Sigma} \rangle = 0. \end{cases}$$

where $\langle ., . \rangle$ denotes the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}_{0,0}(\Sigma)$.

As it was shown by [12], the following result holds:

Theorem 3.1. *The problem* $(\mathcal{P}_H)$ *has a unique solution satisfying the following equalities:*

$$\begin{cases} u_i = u_{|\Omega_i}, & i = 1,2 \\ \lambda = \dfrac{\partial u}{\partial n_1} & in\, H^{-1/2}(\Sigma). \end{cases}$$

Remark 1. The solution would be also unique if a Dirichlet condition were imposed only on a subset of non zero measure of Γ_1 or Γ_2. The local problems are well-posed if $\text{meas}(\Gamma_i) \neq 0$, $i = 1, 2$.

Remark 2. The variational problem $(\mathcal{P}_H)$ is equivalent to the saddle point problem for the Lagrangian

$$\mathcal{L}(v_1, v_2, \mu) = \sum_{i=1}^{2} |v_i\|_{1,\Omega_i}^2 \mathrm{d}x - 2\sum_{i=1}^{2} \int_{\Omega_i} f v_i \mathrm{d}x - 2\langle \mu, (v_1 - v_2)_{|\Sigma}\rangle.$$

3.2. Discrete formulation

Assume that Ω, Ω_1 and Ω_2 are now polyhedral open sets, without any crack. For $i = 1, 2$, let $\mathcal{T}_i$ denote a regular triangulation according to Ciarlet [8] of Ω_i, which mesh size is called h_i. Neither compatibility condition between triangulations, nor between mesh sizes h_1 and h_2 are imposed on both sides of Σ.

The discretization will need the following spaces and subsets:

$$V_{1,h} = \{v_h \in V_1; \forall T \in \mathcal{T}_{1,h},\ v_{|T} \in P_1(T)\} \cdot$$

Let S_h denote the set of tetrahedral faces included in Ω_2 *i.e.*:

$$S_h = \{F = \partial T_1 \cap \partial T_2, (T_1, T_2) \in (\mathcal{T}_{2,h})^2 \ \text{ and } \ \text{meas}(\partial T_1 \cap \partial T_2) \neq 0\}$$

and S_h^Γ (S_h^Σ) denote the set of tetrahedral faces included in Γ_2 (in Σ) *i.e.*:

$$S_h^\Gamma = \{F = \partial T \cap \Gamma_2, T \in \mathcal{T}_{2,h} \ \text{ and } \ \text{meas}(\partial T \cap \Gamma_2) \neq 0\};$$
$$S_h^\Sigma = \{F = \partial T \cap \Sigma, T \in \mathcal{T}_{2,h} \ \text{ and } \ \text{meas}(\partial T \cap \Sigma) \neq 0\} \cdot$$

The spaces $V_{2,h}$ and Λ_h are defined by:

$$V_{2,h} = \{v_h \in L^2(\Omega_2); \forall T \in \mathcal{T}_{2,h}, v_{h|T} \in P_1(T);$$
$$\forall F = \partial T_1 \cap \partial T_2 \in S_h, \int_F v_{h|T_1} \mathrm{d}\sigma = \int_F v_{h|T_2} \mathrm{d}\sigma, \forall F \in S_h^\Gamma, \int_F v_h \mathrm{d}\sigma = 0\}$$

and

$$\Lambda_h = \{\mu_h \in L^2(\Sigma); \forall F \in S_h^\Sigma, \mu_{h|F} \in P_0(F)\} \cdot$$

As $V_{2,h}$ is of nonconforming space type, it must equipped with the following broken norm

$$\forall v_{2,h} \in V_{2,h}, \qquad |v_{2,h}|_{1,h} = \left(\sum_{T \in \mathcal{T}_{2,h}} |v_{2,h}|_{1,T}^2 \right)^{\frac{1}{2}} .$$

The following discrete formulation is obtained from the $(\mathcal{P}_H)$ formulation:

Discrete Hybrid conforming-nonconforming formulation:

$\mathcal{P}_{HD}$: Find $(u_{1,h}, u_{2,h}, \lambda_h) \in V_{1,h} \times V_{2,h} \times \Lambda_h$ such that

$$\begin{cases} \forall (v_{1,h}, v_{2,h}) \in V_{1,h} \times V_{2,h}, \\ \int_{\Omega_1} \mathbf{grad}\, u_{1,h}.\, \mathbf{grad}\, v_{1,h} \mathrm{d}x + \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_{2,h} \mathrm{d}x - \int_\Sigma \lambda_h (v_{1,h} - v_{2,h}) \mathrm{d}\sigma = \\ \qquad \int_{\Omega_1} f v_{1,h} \mathrm{d}x + \int_{\Omega_2} \overline{f} v_{2,h} \mathrm{d}x, \\ \forall \mu_h \in \Lambda_h, \qquad \int_\Sigma \mu_h (u_{1,h} - u_{2,h}) \mathrm{d}\sigma = 0 \end{cases}$$

where $\overline{f}$ denotes the constant by triangle function, that is $L^2(\Omega_2)$-projection of f on the piecewise constant finite element space. Modifying f in the right-hand side of the equation on Ω_2 is equivalent to evaluate $\int_{\Omega_2} f v_{2,h}$ using numerical integration formula.

The following result will be helpful:

Lemma 3.1. *There exists a constant $C > 0$, independent of mesh parameters such that, if $\overline{b}(.,.)$ is the bilinear form defined on $V_2 \times \Lambda_h$ by*

$$\overline{b}(v_2, \mu_h) = \int_\Sigma \mu_h v_2 \mathrm{d}\sigma,$$

then

$$\sup_{v_2 \in V_2} \frac{\overline{b}(v_2, \mu_h)}{\|v_2\|_{1,\Omega_2}} \geq C \|\mu_h\|_{-1/2,\Sigma}, \quad \textit{for all} \quad \mu_h \in \Lambda_h.$$

Proof. Introducing v_2 solution to:

$$\begin{cases} -\Delta v_2 = 0 & \text{in } \Omega_2 \\ \partial v_2/\partial n = \mu_h & \text{on } \Sigma \\ v_2 = 0 & \text{on } \Gamma_2 \end{cases}$$

the Green formula leads to

$$\overline{b}(v_2, \mu_h) = |v_2|^2_{1,\Omega_2}.$$

Moreover trace theorem provides

$$\|\mu_h\|_{-1/2,\Sigma} \leq \|v_2\|_{1,\Omega_2}.$$

Equivalence of norm and semi-norm in $H^1_{0,\Gamma_2}(\Omega_2)$ allows to conclude.

The following theorem can then be proved.

Theorem 3.2. *Problem $(\mathcal{P}_{HD})$ has a unique solution.*

Proof. In the formulation of (P_{HD}) appears the bilinear form $a(.,.)$:

$$a((u_{1,h}, u_{2,h}), (v_{1,h}, v_{2,h})) = \int_{\Omega_1} \mathbf{grad}\, u_{1,h}.\, \mathbf{grad}\, v_{1,h} \mathrm{d}x + \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_{2,h} \mathrm{d}x.$$

Therefore for each $(v_{1,h}, v_{2,h})$:

$$a((v_{1,h}, v_{2,h}), (v_{1,h}, v_{2,h})) = \int_{\Omega_1} \|\, \mathbf{grad}\, v_{1,h}\|^2 + \sum_{T\in\mathcal{T}_{2,h}} \int_T \|\, \mathbf{grad}\, v_{2,h}\|^2 \mathrm{d}x$$

(P_{HD}) solution $(u_{1,h}, u_{2,h})$ belongs to the so-called V_h product space defined by

$$V_h = V_{1,h} \times V_{2,h}.$$

The bilinear form $a(.,.)$ is continuous and elliptic with respect to V_h norm.

If $a((u_{1,h}, u_{2,h}), (u_{1,h}, u_{2,h})) = 0$, then $|u_{1,h}|_{1,\Omega_1} = 0$ and, for all $T \in \mathcal{T}_{2,h}$, $|u_{2,h}|_{1,T} = 0$, so $u_{1,h} = 0$ and $u_{2,h} = 0$.

We shall now prove that the bilinear form $b(.,.)$ defined by

$$b((v_{1,h}, v_{2,h}), \mu_h) = \int_\Sigma \mu_h (v_{1,h} - v_{2,h}) \mathrm{d}\sigma$$

must satisfy the following inf-sup condition: there exist a positive constant C, independent of h, such that:

$$\inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{(v_{1,h}, v_{2,h}) \in V_h} \frac{b((v_{1,h}, v_{2,h}), \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|(v_{1,h}, v_{2,h})\|_{V_h}} \geq C.$$

One can notice that this condition is satisfied as soon as we have

$$\inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{(0, v_{2,h}) \in V_h} \frac{b((0, v_{2,h}), \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|v_{2,h}\|_{V_{2,h}}} \geq C$$

that is to say

$$\inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{v_{2,h} \in V_{2,h}} \frac{\bar{b}(v_{2,h}, \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|v_{2,h}\|_{V_{2,h}}} \geq C.$$

Fortin's lemma is used to demonstrate this result [7]. Indeed

$$\inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{v_{2,h} \in V_{2,h}} \frac{\bar{b}(v_{2,h}, \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|v_{2,h}\|_{V_{2,h}}} \geq \inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{v_2 \in V_2} \frac{\bar{b}(\Pi_h v_2, \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|\Pi_h v_2\|_{V_{2,h}}}$$

so

$$\inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{v_{2,h} \in V_{2,h}} \frac{\bar{b}(v_{2,h}, \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} \|v_{2,h}\|_{V_{2,h}}} \geq \inf_{\mu_h \in \Lambda_h, \mu_h \neq 0} \sup_{v_2 \in V_2} \frac{\bar{b}(v_2, \mu_h)}{\|\mu_h\|_{-1/2,\Sigma} |v_2|_{1,\Omega_2}}$$

because the interpolation operator is continuous. Lemma 2.1 is then providing the result.

Existence and uniqueness of the solution are established.

Theorem 3.3. *If the solution to P_H is such that $u_i = u_{|\Omega_i} \in H^{\sigma_i}(\Omega_i)$ with $1 < \sigma_i \leq 2$, $i = 1, 2$, then we have the following error bound for discrete problem P_{HD}:*

$$|u_1 - u_{1,h}|_{1,\Omega_1} + \left(\sum_{T \in \mathcal{T}_{2,h}} |u_2 - u_{2,h}|_{1,T}^2 \right)^{1/2} \leq C(h_1^{\sigma_1 - 1} \|u_1\|_{\sigma_1,\Omega_1} + h_2^{\sigma_2 - 1} \|u_2\|_{\sigma_2,\Omega_2}).$$

Proof. A new formulation is necessary to make the proof easier: the weak matching condition is relaxed using a primal hybrid formulation [7, 12]. Arguments introduced by [12] allow to conclude the proof. In order to get rid of regularity conditions on exact solution, introducing interpolation operators is useful, namely:

- for nonconforming variable, Π_h operator, for which error interpolation bounds have been given in Theorem 1,
- for conforming variable, to avoid supposing that $u \in H^{3/2+\epsilon}(\Omega_1)$, one can use a Clément interpolation operator [9], which is defined from V_1 onto $V_{1,h}$.

4. Equivalence with low degree semi-primal semi-dual formulation for 3D

Our aim is to establish a semi-primal semi-dual formulation [4,10], that is a formulation using for example the primal unknown u on Ω_1 and the dual unknown $\mathbf{p}$ on Ω_2.
The discrete formation comes from the conforming-nonconforming hybrid formulation (P_{HD}), previously established. Indeed, if one chooses:

- $u_{1,h}$ used in the conforming-nonconforming hybrid formulation
- $\mathbf{p}_{2,h}$ defined on Ω_2 by:

$$\mathbf{p}_{2,h} = \mathbf{grad}\, u_{2,h} - \frac{\overline{f}}{3}(\mathbf{x} - \mathbf{x}_T), \qquad \text{for all } T \in \mathcal{T}_{2,h} \tag{4.1}$$

where $u_{2,h}$ is the nonconforming unknown involed in the conforming-nonconforming hybrid formulation; $\mathbf{x}_T$ the triangle barycenter and $\overline{f}$ the mean value of f on T. Then $(u_{1,h},\, \mathbf{p}_{2,h})$ is solution of semi-primal semi-dual formulation associated to problem P. The above construction of $\mathbf{p}_{2,h}$ from $u_{2,h}$ is similar in 3D to the one given in 2D by [11].

Let $W_{2,h}$ defined by

$$W_{2,h} = \{\, \mathbf{p}_h \in H(\mathrm{div}, \Omega_2),\ \forall T \in \mathcal{T}_{2,h},\ \mathbf{p}_{h|T} \in RT_1(T)\} \tag{4.2}$$

where $RT_1(T) = (P_0(T))^3 + \mathbf{x}\, P_0(T)$.

Some preliminary results are to be shown, what is done in the following lemmas.

Lemma 4.1. *The vectorial function $\mathbf{p}_{2,h}$ defined by (4.1) belongs to $H(\mathrm{div}, \Omega_2)$.*

Proof. On each $T \in \mathcal{T}_{2,h},\ \mathbf{p}_{2,h} \in (L^2(T))^2$ and $-\mathrm{div}\,\mathbf{p}_{2,h} = \overline{f}$. One just needs then to show that

$$\forall T_1, T_2 \in \mathcal{T}_{2,h}, \text{ with } \mathrm{meas}(\partial T_1 \cap \partial T_2) \neq 0,\ \mathbf{p}_{2,h}.n_1 + \mathbf{p}_{2,h}.n_2 = 0 \text{ on } F = \partial T_1 \cap \partial T_2.$$

Let $[\,\mathbf{p}_{2,h}.n]$ denote the normal jump through any face F defined by

$$[\,\mathbf{p}_{2,h}.n] = \mathbf{p}_{2,h}.n_1 + \mathbf{p}_{2,h}.n_2.$$

To obtain $[\,\mathbf{p}_{2,h}.n] = 0$, one can show that $\int_F [\,\mathbf{p}_{2,h}.n] v_h \mathrm{d}\sigma = 0$, for the basis function $v_h \in V_{2,h}$ associated to F defined by

$$\int_F v_h \mathrm{d}\sigma = 1$$

and

$$\text{for all face } F' \neq F,\ \int_{F'} v_h \mathrm{d}\sigma = 0.$$

With the definition of $[\,\mathbf{p}_{2,h}.n]$ and Green formulae, we have

$$\int_F [\,\mathbf{p}_{2,h}.n] v_h \mathrm{d}\sigma = \int_{T_1} \mathbf{p}_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x + \int_{T_2} \mathbf{p}_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x + \int_{T_1} \mathrm{div}\, \mathbf{p}_{2,h} v_h \mathrm{d}x + \int_{T_2} \mathrm{div}\, \mathbf{p}_{2,h} v_h \mathrm{d}x$$

and from $\mathbf{p}_{2,h}$ definition, we obtain

$$\begin{aligned}\int_F [\,\mathbf{p}_{2,h}.n] v_h \mathrm{d}\sigma &= \int_{T_1} \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x + \int_{T_2} \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x \\ &\quad - \frac{\overline{f}}{3} \int_{T_1} (\mathbf{x} - \mathbf{x}_{T_1})\, \mathbf{grad}\, v_h \mathrm{d}x - \frac{\overline{f}}{3} \int_{T_2} (\mathbf{x} - \mathbf{x}_{T_1})\, \mathbf{grad}\, v_h \mathrm{d}x - \int_{T_1} \overline{f} v_h \mathrm{d}x - \int_{T_2} \overline{f} v_h \mathrm{d}x.\end{aligned}$$

Using the facts that $u_{2,h}$, being the solution with right hand side $\overline{f}$, verifies:

$$\int_{T_1} \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x + \int_{T_2} \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_h \mathrm{d}x = \int_{T_1} \overline{f} v_h \mathrm{d}x + \int_{T_2} \overline{f} v_h \mathrm{d}x$$

and that $\forall T \in \mathcal{T}_{2,h}$, $\int_T (\mathbf{x} - \mathbf{x}_T) \mathrm{d}x = 0$, we get the result

$$\int_F [\mathbf{p}_{2,h}.n] v_h \mathrm{d}\sigma = 0.$$

Lemma 4.2. *The vectorial function* $\mathbf{p}_{2,h}$ *belongs to* $W_{2,h}$*, where the space* $W_{2,h}$ *is defined in (3.2).*

Lemma 4.3. *The function* $\mathbf{p}_{2,h}$ *satisfies:*

$$\forall\, \mathbf{q}_{2,h} \in W_{2,h}, \text{ with } \operatorname{div} \mathbf{q}_{2,h} = 0, \qquad \int_{\Omega_2} \mathbf{p}_{2,h}\, \mathbf{q}_{2,h} \mathrm{d}x = \int_\Sigma u_{1,h}\, \mathbf{q}_{2,h}.n \mathrm{d}\sigma.$$

Proof. Using Lemma 4.4 and 4.5, we have $\mathbf{p}_{2,h} \in W_{2,h}$

$$\int_{\Omega_2} \mathbf{p}_{2,h}\, \mathbf{q}_{2,h} \mathrm{d}x = \sum_{T \in \mathcal{T}_{2,h}} \int_T (\mathbf{grad}\, u_{2,h} - \frac{\overline{f}}{3}(\mathbf{x} - \mathbf{x}_T))\, \mathbf{q}_{2,h} \mathrm{d}x$$

$$= \sum_{T \in \mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{q}_{2,h} \mathrm{d}x - \sum_{T \in \mathcal{T}_{2,h}} \int_T \frac{\overline{f}}{3} (\mathbf{x} - \mathbf{x}_T)\, \mathbf{q}_{2,h} \mathrm{d}x.$$

Because $\mathbf{q}_{2,h} \in W_{2,h}$, $\mathbf{q}_{2,h_T}$ can be written as $\mathbf{q}_{2,h} = \underline{\alpha} + \beta x$ with $\underline{\alpha} \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$; $\operatorname{div} \mathbf{q}_{2,h} = 0$ implies that $\beta = 0$ and $\mathbf{q}_{2,h} = \underline{\alpha}$.

It follows that

$$\int_T (\mathbf{x} - \mathbf{x}_T)\, \mathbf{q}_{2,h} \mathrm{d}x = \mathbf{q}_{2,h} \int_T (\mathbf{x} - \mathbf{x}_T) \mathrm{d}x = 0$$

so as $\operatorname{div} \mathbf{q}_{2,h} = 0$ and $q_{2,h|T} = \underline{\alpha}$, forall $T \in \mathcal{T}_{2,h}$,

$$\sum_{T \in \mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{q}_{2,h} \mathrm{d}x = \sum_{T \in \mathcal{T}_{2,h}} \int_{\partial T} u_{2,h}\, \mathbf{q}_{2,h}.n_T \mathrm{d}\sigma = \sum_{T \in \mathcal{T}_{2,h}} \sum_{e \text{ edge of } \partial T} \overline{u_{2,h}}_{|e} \int_e \mathbf{q}_{2,h}.n_T \mathrm{d}\sigma$$

where $\overline{u_{2,h}}_{|e} = \int_e u_{2,h} \mathrm{d}\sigma$. Since $\mathbf{q}_{2,h} \in H(\operatorname{div}, \Omega_2)$, we have

$$\mathbf{q}_{2,h}.n_{T_1}{}_{|T_1} + \mathbf{q}_{2,h}.n_{T_2}{}_{|T_2} = 0 \quad \text{ on } e = \partial T_1 \cap \partial T_2,$$

and $u_{2,h} \in V_{2,h}$ implies

$$\overline{(u_{2,h|T_1})_{|e}} = \overline{(u_{2,h|T_2})_{|e}}.$$

On the faces e included in $\partial\Omega$, the contribution equals zero due to Drichlet condition. The non zero terms corresponding to faces included in Σ, we have finally

$$\int_{\Omega_2} \mathbf{p}_{2,h}.\, \mathbf{q}_{2,h} \mathrm{d}x = \int_\Sigma u_{2,h}\, \mathbf{q}_{2,h}.n_\Sigma \mathrm{d}\sigma.$$

As $\mathbf{q}_{2,h}.n_T \in \Lambda_h$ we get, from second equations of problem (P_{HD}):

$$\int_\Sigma u_{2,h}\, \mathbf{q}_{2,h}.n_\Sigma \mathrm{d}\sigma = \int_\Sigma u_{1,h}\, \mathbf{q}_{2,h}.n_\Sigma \mathrm{d}\sigma$$

and then the result

$$\int_{\Omega_2} \mathbf{p}_{2,h}\, \mathbf{q}_{2,h} \mathrm{d}x = \int_\Sigma u_{1,h}\, \mathbf{q}_{2,h}.n_\Sigma \mathrm{d}\sigma.$$

Lemma 4.4. *The function* $\mathbf{p}_{2,h}$ *satisfies:*

$$\sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_{2,h} \mathrm{d}x = \int_\Sigma \mathbf{p}_{2,h}.n_\Sigma v_{2,h} \mathrm{d}\sigma + \int_{\Omega_2} \overline{f} v_{2,h} \mathrm{d}x, \ \forall v_{2,h} \in V_{2,h}.$$

Proof. For every $v_{2,h} \in V_{2,h}$, since $\mathbf{grad}\, v_{2,h} \in (P_0(T))^3$ and $\int_T(\mathbf{x} - \mathbf{x}_T)\mathrm{d}x = 0$, we have

$$\begin{aligned} \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{grad}\, u_{2,h}\, \mathbf{grad}\, v_{2,h} &= \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{p}_{2,h}\, \mathbf{grad}\, v_{2,h} + \sum_{T\in\mathcal{T}_{2,h}} \frac{\overline{f}}{3} \int_T (\mathbf{x} - \mathbf{x}_T)\mathrm{d}x \\ &= \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathbf{p}_{2,h}\, \mathbf{grad}\, v_{2,h} - \sum_{T\in\mathcal{T}_{2,h}} \int_T \mathrm{div}\, \mathbf{p}_{2,h} v_{2,h} \mathrm{d}x + \int_\Sigma \mathbf{p}_{2,h}.n_\Sigma v_{2,h} \mathrm{d}\sigma \\ &= \int_\Sigma \mathbf{p}_{2,h}.n_\Sigma v_{2,h} \mathrm{d}\sigma + \int_{\Omega_2} \overline{f} v_{2,h} \mathrm{d}x. \end{aligned}$$

Lemma 4.5. *The function* $\mathbf{p}_{2,h}$ *satisfies:*

$$\mathbf{p}_{2,h}.n_\Sigma = -\lambda_h \text{on } \Sigma.$$

Proof. Comparing the equation established in Lemma 4.7 with the discrete hybrid conforming-nonconforming formulation, and remarking that λ_h is unique, and $\mathbf{p}_{2,h}.n_\Sigma \in \Lambda_h$, the result is deduced.

Theorem 4.1. *The functions* $(u_{1,h}, \mathbf{p}_{2,h})$ *are solutions of the following semi-primal semi-dual formulation* (P_s)*:*

$$\begin{cases} \textit{Find } (u_{1,h}, \mathbf{p}_{2,h}) \in V_{1,h} \times W_{2,h} \textit{ such that} \\ \forall v_{1,h} \in V_{1,h},\ \int_{\Omega_1} \mathbf{grad}\, u_{1,h}\, \mathbf{grad}\, v_{1,h} \mathrm{d}x + \int_\Sigma \mathbf{p}_{2,h}.n_\Sigma v_{1,h} \mathrm{d}\sigma = \int_{\Omega_1} f v_{1,h} \mathrm{d}x, \\ -\mathrm{div}\, \mathbf{p}_{2,h} = \overline{f}, \quad \textit{on } \Omega_2, \\ \forall\, \mathbf{q}_{2,h} \in W_{2,h}, \textit{with } \mathrm{div}\, \mathbf{q}_{2,h} = 0, \quad \int_{\Omega_2} \mathbf{p}_{2,h}.\, \mathbf{q}_{2,h} \mathrm{d}x - \int_\Sigma u_{1,h}\, \mathbf{q}_{2,h}.n_\Sigma \mathrm{d}\sigma = 0. \end{cases}$$

Theorem 4.2. *If the solution u to (P_H) is such that $u_i = u_{|\Omega_i} \in H^{\sigma_i}(\Omega_i)$ with $1 < \sigma_i \le 2$, i=1,2, then we have the following error bound:*

$$|u - u_{1,h}|_{1,\Omega_1} + \|\,\mathbf{p} - \mathbf{p}_{2,h}\|_{0,\Omega_2} \le C\{h_1^{\sigma_1-1}|u|_{\sigma_1,\Omega_1} + h_2^{\sigma_2-1}|u|_{\sigma_2,\Omega_2} + h_2|f|_{0,\Omega_2}\},$$

where $\mathbf{p} = \mathbf{grad}\, u$.

Proof. From the $\mathbf{p}_{2,h}$ definition one gets:

$$\|\,\mathbf{p} - \mathbf{p}_{2,h}\|_{0,T} \le \sum_{T\in\mathcal{T}_{2,h}} \int_T |u - u_{2,h}|_{1,T} + \sum_{T\in\mathcal{T}_{2,h}} \left\| \frac{\overline{f}}{3}(x - x_T) \right\|_{0,T}, \text{ where } \mathbf{p} = \mathbf{grad}\, u,$$

and global error splits into three terms:

$$|u - u_{1,h}|_{1,\Omega_1} + \| \mathbf{p} - \mathbf{p}_{2,h} \|_{0,\Omega_2} \leq |u - u_{1,h} - |_{1,\Omega_1} + \sum_{T \in \mathcal{T}_{2,h}} \int_T |u - u_{2,h}|_{1,T} + \sum_{T \in \mathcal{T}_{2,h}} \| \frac{\overline{f}}{3}(\mathbf{x} - \mathbf{x}_T) \|_{0,T}.$$

The final result follows from the error bounds established in Theorem 4 and easy calculations.

5. Conclusion

Theorem 4.9. shows that $(u_{1,h}, \mathbf{p}_{2,h})$ is solution of a semi-primal semi-dual formulation, therefore both of the unknowns are the approximation of the exact solution restriction to each subdomain. The continuity of $\mathbf{p}_{2,h}$ across tetrahedrons interfaces is obtained by local correction of $\mathbf{grad}\, u_{2,h}$. This property is interesting, because numerically a nonconforming allows to obtain a more powerful approximation, verifying flux compatibility.

Theorem 4.10 gives approximation error bounds, which are optimal without any compatibility condition on the meshes. Moreover, one can note that the last term in the bound is dominated by the other two. In order to provide a very flexible method, regularity required on the exact solution is weak, and may not be the same on the two subdomains.

References

[1] A. Agouzal, *A Posteriori* Error Estimator for Nonconforming Finite Element Methods. *Appl. Math. Lett.* **7** (1994) 61–66.
[2] A. Agouzal, Méthode de décomposition de domaines en formulation mixte. *Jap. Math.* **43** (1994) 31–35.
[3] A. Agouzal and J.-M.Thomas, Une méthode d'éléments finis hybrides en décomposition de domaines. *RAIRO Modél. Math. Anal. Numér.* **29** (1995) 749–764.
[4] A. Agouzal and L. Lamoulie, Un algorithme de résolution pour une méthode de décomposition de domaines par éléments finis. *C.R. Acad. Sci. Paris Sér. I* **318** (1994) 117–176.
[5] A. Alonso and A. Valli, A new approach to the coupling of viscous and inviscid Stokes equations. *East-West J. Numer. Math.* **3** (1995) 29–42.
[6] C. Bernardi, Y. Maday and A. Patera, A new nonconforming approach to domain decomposition: The mortar element method, in *Nonlinear Partial Differential Equations and their Applications*, H. Brezis and J.L. Lions Eds., Pitman (1989).
[7] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer Verlag, New York (1991).
[8] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems, in *Handbook of Numerical Analysis, Vol II, Finite Element Methods,* P.G. Ciarlet and J.L. Lions Eds., North-Holland, Amsterdam (1991) 17–352.
[9] P. Clement, Approximation by finite elements functions using local regularization. *RAIRO Anal. Numér.* **9** (1975) 77–84.
[10] L. Lamoulie and J.-M.Thomas, Couplage de méthodes primales et duales d'éléments finis pour les problèmes elliptiques du second ordre. *C.R. Acad. Sci. Paris Sér. I* **318** (1994) 269–274.
[11] D. Marini, An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method. *SIAM J. Numer. Anal.* **22** (1985) 493–496.
[12] J.E. Roberts and J.-M.Thomas, Mixed and Hybrid Methods, in *Handbook of Numerical Analysis, Vol II, Finite Element Methods*, P.G. Ciarlet and J.L. Lions Eds., North-Holland, Amsterdam (1991) 523–639.
[13] J.-M. Thomas, Finite Element Matching Methods, in *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations,* D. Keys, T. Chan, G. Meurant, J. Scroggs and R. Voigt Eds., SIAM, Philadelphia (1992) 99–105.