ESAIM: MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

TAHAR-ZAMÈNE BOULMEZAOUD
YVON MADAY
TAHAR AMARI

On the linear force-free fields in bounded and unbounded three-dimensional domains

ESAIM: Modélisation mathématique et analyse numérique, tome 33, n° 2 (1999), p. 359-393
<http://www.numdam.org/item?id=M2AN_1999__33_2_359_0>

Mathematical Modelling and Numerical Analysis M2AN, Vol. 33 N° 2, 1999, p. 359–393
Modélisation Mathématique et Analyse Numérique

ON THE LINEAR FORCE-FREE FIELDS IN BOUNDED AND UNBOUNDED THREE-DIMENSIONAL DOMAINS

TAHAR-ZAMÈNE BOULMEZAOUD[1], YVON MADAY [2] AND TAHAR AMARI[3]

Abstract. Linear Force-free (or Beltrami) fields are three-components divergence-free fields solutions of the equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$, where α is a real number. Such fields appear in many branches of physics like astrophysics, fluid mechanics, electromagnetics and plasma physics. In this paper, we deal with some related boundary value problems in multiply-connected bounded domains, in half-cylindrical domains and in exterior domains.

Résumé. Les champs de Beltrami (ou sans-force) linéaires sont des champs tri-dimensionnels à divergence nulle et vérifiant l'équation $\mathbf{rot}\mathbf{B} = \alpha\mathbf{B}$ où α est une constante réelle connue ou inconnue. Ces champs apparaissent dans plusieurs domaines de la physique tels que la mécanique des fluides, la physique des plasmas, l'astrophysique et l'électromagnétisme. Dans ce papier, nous présentons quelques nouveaux résultats concernant des problèmes aux limites associés dans des domaines tri-dimensionnels bornés (simplement ou multiplement connexes) et non-bornés (cylindre semi-infini et extérieur d'un domaine). Ces résultats concernent essentiellement l'existence, l'unicité, la régularité et les propriétés d'énergie des solutions.

AMS Subject Classification. 35F05, 35F15, 35M10, 35Q35, 35Q72, 85A30, 76C05.

Received: November 7, 1997. Revised: May 28, 1998.

INTRODUCTION

A three-components field function $\mathbf{B}$ is called Beltrami (or force-free) if $\mathbf{B}$ is solution of the system

$$\mathbf{curl}\,\mathbf{B} \times \mathbf{B} = \mathbf{0}, \tag{1}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{2}$$

Such fields play a prominent role in solar physics (see, *e.g.*, [5,42]), in plasma physics (see [27,45]), in fluid mechanics -they are solutions to Euler's equation (see [7,15,19,38,49]), in superconducting materials (see [24])

Keywords and phrases. Beltrami flows, Force-free fields, curl operator, hydromagnetics, stars: corona, magnetic fields.

[1] Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France. e-mail: `boulmeza@ann.jussieu.fr`

[2] Laboratoire ASCI-CNRS, Bâtiment 506, Université Paris Sud, 91405 Orsay, Cedex, France.

[3] (CNRS) DSM/DAPNIA Service d'Astrophysique, C-E Saclay, 91191 Gif sur Yvette Cedex, France.

and in electromagnetic waves theory. The basic equation (1) is often replaced by:

$$\mathbf{curl}\,\mathbf{B} = \alpha(\mathbf{x})\mathbf{B}, \tag{3}$$

where $\mathbf{B}$ as well as the scalar function $\alpha(\mathbf{x})$ are unknown. Two situations are commonly distinguished: α constant everywhere and α a variable function. In the first case, equation (3) reduces to a linear equation called also the Trkalian equation. The case $\alpha = 0$ corresponds to the well known potential field theory.

Recently, linear Beltrami fields have been investigated by many authors; see, *e.g.*, [9, 10, 14, 15, 30–33, 35, 36, 38, 39, 41]. Note that they are also subject of a very intensive research in astrophysics and especially in solar physics (see [42], reviews by [5] or [43] and references therein).

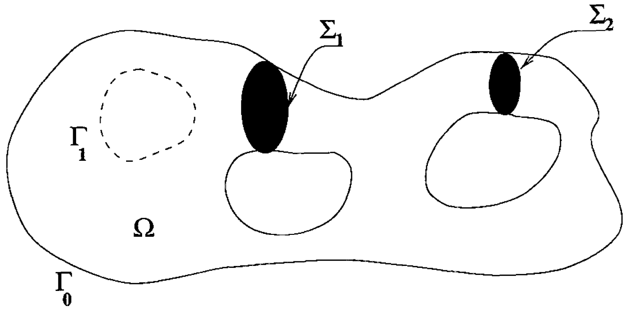
In this paper, we shall present some new results about existence and uniqueness of 3-D Beltrami fields in a bounded domain, in half-cylinder and in an exterior domain. The sequel of this paper is divided into three separated and independent parts:

- SECTION 1. We present a new and a general theorem concerning the existence, uniqueness and regularity of linear force-free fields subject to appropriate boundary conditions in a bounded region. The proof of this theorem is mainly based on Fredholm alternative and spectral theory. A similar problem with a given helicity-like data instead of α, which is unknown, is also studied. Note that in this last situation, our approach is different from the minimization one presented by [33].
- SECTION 2. We deal with linear force-free fields in a semi-infinite cylindrical domain $\tilde{\Omega}\times]0, +\infty[$. A diagonalization method is used for deriving an explicit formula of the general solution.
- SECTION 3. We discuss the existence or not of linear force-free fields in exterior domains using a new approach based on weighted Sobolev spaces.

1. Linear Beltrami fields in a bounded domain

1.1. Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^3$ with boundary Γ. We make the following assumptions on Ω: Ω is bounded, connected but eventually multiply-connected and its boundary Γ is of class $\mathcal{C}^2$. Let Γ_0 be the exterior boundary of Ω and $\Gamma_1, ..., \Gamma_p$ the other components of Γ.



We assume that there exists m manifolds of dimension 2, $\Sigma_1, ..., \Sigma_m$, such that $\Omega_0 = \Omega\backslash\cup_{i=1}^m\Sigma_i$ is smooth and simply-connected and $\Sigma_i \cap \Sigma_j = \emptyset$ if $i \neq j$ (m describes the connectedness of Ω, and $\Sigma_i, ..., \Sigma_m$ are regular cuts linking $(\Gamma_i)_{1\leq i\leq p}$). We set $m = 0$ when Ω is simply-connected.

In the sequel we shall denote by $(.,.)$ both the scalar product in $L^2(\Omega)$ and in $L^2(\Omega)^3$. The duality product between $H^{-\frac{1}{2}}(\Gamma_i)$ and $H^{\frac{1}{2}}(\Gamma_i)$ will be denoted by $\langle .,. \rangle_{\Gamma_i}$.

Define the following spaces:

$$\begin{aligned} V &= \{\mathbf{v} \in L^2(\Omega)^3, \operatorname{div}\mathbf{v} \in L^2(\Omega), \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3, \mathbf{v}.\mathbf{n} = 0 \text{ on } \Gamma\},\\ U &= \{\mathbf{v} \in L^2(\Omega)^3, \operatorname{div}\mathbf{v} \in L^2(\Omega), \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\},\end{aligned}$$

equipped with the norm:

$$\|\mathbf{v}\| = (\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2)^{\frac{1}{2}}. \tag{4}$$

We need the following result due to [23] (see also [22,26]):

Lemma 1. *The spaces U and V are Hilbert spaces. Moreover, one has the following identities topologically and algebraically:*

$$\begin{aligned} V &= \{\mathbf{v} \in H^1(\Omega)^3, \mathbf{v}.\mathbf{n} = 0 \text{ on } \Gamma\},\\ U &= \{\mathbf{v} \in H^1(\Omega)^3, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.\end{aligned}$$

Now, define the spaces:

$$\begin{aligned} H &= \{\mathbf{v} \in V,\ \operatorname{div}\mathbf{v} = 0,\ \mathbf{curl}\,\mathbf{v} = \mathbf{0}\},\\ G &= \{\mathbf{v} \in U,\ \operatorname{div}\mathbf{v} = 0,\ \mathbf{curl}\,\mathbf{v} = \mathbf{0}\}.\end{aligned}$$

One has also the following result (see [18]):

Lemma 2. *The space H (resp. G) has a finite dimension m (resp. p) and there exists a base $(\mathbf{q}_i)_{i=1,\dots,m}$ (resp. $(\mathbf{f}_i)_{i=1,\dots,p}$) such that:*

$$\begin{aligned} &\int_{\Sigma_j} \mathbf{q}_i.\mathbf{n}d\sigma = \delta_{i,j} \quad i,\ j = 1, \dots, m,\\ &\int_{\Gamma_j} \mathbf{f}_i.\mathbf{n}d\sigma = \delta_{i,j}, \quad i, j = 1, \dots, p.\end{aligned} \tag{5}$$

Thus, we denote by $\mathbf{P}_H$ (resp. $\mathbf{P}_G$) the orthogonal projection from V on H (resp. from U on G) with respect to inner product associated with the norm (4). The two next lemmas will be useful throughout this section (see [18]):

Lemma 3. *For any vector field $\mathbf{v}$ in $L^2(\Omega)^3$ verifying*

$$\operatorname{div}\mathbf{v} = 0,$$

we have

- *If $\mathbf{v} \in V$ then*

$$\mathbf{P}_H\mathbf{v} = \sum_{i=1}^{m}(\int_{\Sigma_i}\mathbf{v}.\mathbf{n}d\sigma)\mathbf{q}_i.$$

- *If $\mathbf{v} \in U$ then*

$$\mathbf{P}_G\mathbf{v} = \sum_{i=1}^{p}(\int_{\Gamma_i}\mathbf{v}.\mathbf{n}d\sigma)\mathbf{f}_i.$$

Lemma 4. *The mapping* $\mathbf{v} \longrightarrow \|\mathbf{v}\|_V = (\|\text{div}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{P}_H\mathbf{v}\|_{0,\Omega}^2)^{\frac{1}{2}}$ *(resp. the mapping* $\mathbf{v} \longrightarrow \|\mathbf{v}\|_U = (\|\text{div}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{P}_G\mathbf{v}\|_{0,\Omega}^2)^{\frac{1}{2}}$ *) is a norm on* V *(resp. on* U*) equivalent to the norm (4).*

In the remaining of this paper, we shall denote by $\alpha_0(\Omega)$ and $\alpha_1(\Omega)$ the constants defined by:

$$\alpha_0(\Omega) = \inf_{\mathbf{v}\in V, \mathbf{v}\neq \mathbf{0}} \frac{\|\mathbf{v}\|_V}{\|\mathbf{v}\|_{0,\Omega}}, \tag{6}$$

$$\alpha_1(\Omega) = \inf_{\mathbf{v}\in U, \mathbf{v}\neq \mathbf{0}} \frac{\|\mathbf{v}\|_U}{\|\mathbf{v}\|_{0,\Omega}}. \tag{7}$$

$\alpha_0(\Omega)$ and $\alpha_1(\Omega)$ are positive and *not equal to zero* because of Lemma 4. An estimate of these constants will be given in Lemma 12.

1.2. Statement of the problem when α is known

In this section, we assume α to be a given real number *not equal to zero* (the case $\alpha = 0$ corresponds to the classical potential theory). We propose to study the following boundary value problem:

$$\left\{\begin{array}{rcll} \mathbf{curl}\,\mathbf{B} &=& \alpha\mathbf{B} & \text{in } \Omega, \\ \text{div}\,\mathbf{B} &=& 0 & \text{in } \Omega, \\ \mathbf{B}.\mathbf{n} &=& \text{g} & \text{on } \Gamma, \\ \int_\Gamma (\mathbf{B}\times\mathbf{n}).\mathbf{q}_i d\sigma &=& \alpha a_i, & i = 1,...,m, \end{array}\right. \tag{8}$$

where $(a_1, a_2, ..., a_m) \in \mathbb{R}^m$ and $\text{g} \in H^{\frac{1}{2}}(\Gamma)$ are given. Note that g must verify the compatibility condition

$$\int_{\Gamma_i} \text{g} d\sigma = 0, \quad \text{for } 0 \leq i \leq p. \tag{9}$$

In fact, let $\mathbf{B} \in H^1(\Omega)^3$ solution of (8). For $0 \leq i \leq p$, let κ_i be a function of $\mathcal{D}(\mathbb{R}^3)$ satisfying $\kappa_i(\mathbf{r}) = \delta_{i,j}$ in a neighborhood of Γ_j. One has

$$\mathbf{curl}\,(\kappa_i\mathbf{B}) = \alpha\kappa_i\mathbf{B} + \nabla\kappa_i \times \mathbf{B}.$$

Thus $\alpha \int_{\Gamma_i} \mathbf{B}.\mathbf{n} d\sigma = \int_\Gamma \mathbf{curl}\,(\kappa_i\mathbf{B}).\mathbf{n} = \int_\Omega \text{div}\,\mathbf{curl}\,(\kappa_i\mathbf{B}) d\Omega = 0.$

Note that the boundary condition

$$\alpha a_i = \int_\Gamma (\mathbf{B}\times\mathbf{n}).\mathbf{q}_i d\sigma = \int_\Omega \mathbf{curl}\,\mathbf{B}.\mathbf{q}_i d\Omega = \alpha \int_\Omega \mathbf{B}.\mathbf{q}_i d\Omega,$$

means that the orthogonal projection of $\mathbf{B}$ on H is given.

1.3. A general existence and uniqueness result

In this section, we deal with the problems of existence, uniqueness and regularity of solutions to (8). The approach we propose in a first time is based on the use of Fredholm alternative. It is divided into several steps.

1.3.1. *An equivalent problem*

For any $\text{g} \in H^{\frac{1}{2}}(\Gamma)$ verifying (9) and $(a_1, ..., a_m) \in \mathbb{R}^m$, define a potential field $\mathbf{B}_0 \in H^1(\Omega)^3$ by $\mathbf{B}_0 = \nabla\varphi_0 - \sum_{i=1}^m a_i\mathbf{q}_i$, where $\varphi_0 \in H^2(\Omega)/\mathbb{R}$ is solution of the Neumann problem:

$$\Delta\varphi_0 = 0, \qquad \frac{\partial\varphi_0}{\partial n} = \text{g} \quad \text{on } \Gamma. \tag{10}$$

Observe that $\mathbf{B}_0$ verifies

$$\mathbf{curl}\,\mathbf{B}_0 = 0,\ \ \operatorname{div}\mathbf{B}_0 = 0,\ \ \mathbf{B}_0.\mathbf{n} = \mathrm{g} \text{ on } \Gamma, \text{ and } \int_\Omega \mathbf{B}_0.\mathbf{q}_i d\Omega = -a_i \quad \text{for } i = 1, ..., m.$$

In the sequel, we define the energy $E_0 \geq 0$ by

$$E_0 = \|\mathbf{B}_0\|_{0,\Omega}^2 = \|\nabla\varphi_0\|_{0,\Omega}^2 + \sum_{i=1}^m a_i^2. \tag{11}$$

Now, $\mathbf{B} \in H^1(\Omega)^3$ is solution of (8) if and only if $\mathbf{b} = \mathbf{B} - \mathbf{B}_0 \in H^1(\Omega)^3$ is solution of

$$\left\{\begin{array}{rcll} \mathbf{curl}\,\mathbf{b} &=& \alpha\mathbf{b} + \alpha\mathbf{B}_0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{b} &=& 0 & \text{in } \Omega, \\ \mathbf{b}.\mathbf{n} &=& 0 & \text{on } \Gamma, \\ \mathbf{P}_H\mathbf{b} &=& \mathbf{0}. & \end{array}\right. \tag{12}$$

The equivalence between this system and the original problem (8) is obvious. In fact, let $\mathbf{B} \in H^1(\Omega)^3$ be solution of (8). Then, it is clear that $\mathbf{b} = \mathbf{B} - \mathbf{B}_0$ belongs to V and verifies the first three equations of (12). In addition,

$$\alpha a_i = \int_\Gamma (\mathbf{B} \times \mathbf{n}).\mathbf{q}_i d\sigma = -\int_\Omega \mathbf{curl}\,\mathbf{B}.\mathbf{q}_i d\Omega = -\int_\Omega \alpha(\mathbf{b} + \mathbf{B}_0).\mathbf{q}_i d\Omega = -\alpha(\mathbf{P}_H\mathbf{b}, \mathbf{q}_i) + \alpha a_i.$$

Hence $\mathbf{P}_H\mathbf{b} = \mathbf{0}$. Conversely, if $\mathbf{b}$ is solution of (12), then the same calculus asserts that $\mathbf{B}$ is solution of (8).

Now, in order to give a new formulation of (12), let us introduce the space

$$X = \{\mathbf{v} \in L^2(\Omega)^3,\ \ \operatorname{div}\mathbf{v} = 0 \text{ and } \langle \mathbf{v}.\mathbf{n}, 1\rangle_{\Gamma_i} = 0,\ 1 \leq i \leq p\}. \tag{13}$$

X is a closed subspace of $H(\operatorname{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^3,\ \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$. Hence, it is a Hilbert space equipped with the norm of $L^2(\Omega)^3$.

As a first step of our investigation, we consider the *curl-div* system:
Given $\mathbf{j} \in X$*, Find* $\mathbf{u} \in V$ *such that:*

$$\mathbf{curl}\,\mathbf{u} = \mathbf{j},\ \ \operatorname{div}\mathbf{u} = 0,\ \ \mathbf{P}_H\mathbf{u} = 0. \tag{14}$$

Lemma 5. $\mathbf{u} \in V$ *is solution of (14) if and only if* $\mathbf{u}$ *is solution of the variational problem:*

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\mathbf{P}_H\mathbf{u}, \mathbf{P}_H\mathbf{v}) = (\mathbf{j}, \mathbf{curl}\,\mathbf{v}), \quad \forall \mathbf{v} \in V. \tag{15}$$

In addition, this problem admits one and only one solution $\mathbf{u} \in V$*, and there exists a constant* $C(\Omega)$ *such that:*

$$\|\mathbf{u}\|_{H^1(\Omega)^3} \leq C(\Omega)\|\mathbf{j}\|_{0,\Omega}. \tag{16}$$

Proof. First, it is quite obvious that if $\mathbf{u}$ is solution of (14), then it is also solution of the variational problem (15).
The converse is slightly more complicated; let $\mathbf{u}$ be solution of (15). On one hand, taking $\mathbf{v} = \mathbf{P}_H\mathbf{u}$ in (15), one obtains $\mathbf{P}_H\mathbf{u} = \mathbf{0}$. On the other hand, let Φ be solution of the Neumann problem:

$$\Delta\Phi = \operatorname{div}\mathbf{u} \text{ in } \Omega,\ \ \frac{\partial\Phi}{\partial n} = 0 \quad \text{on } \Gamma.$$

This problem admits a unique solution in $H^1(\Omega)/\mathbb{R}$ since $(\operatorname{div}\mathbf{u}, 1) = 0$ by Green's formula. Now, taking $\mathbf{v} = \nabla\Phi$ in (15) one yields $\operatorname{div}\mathbf{u} = 0$ (almost everywhere).

It remains to prove that $\mathbf{curl}\,\mathbf{u} = \mathbf{j}$. We set $\mathbf{w} = \mathbf{curl}\,\mathbf{u} - \mathbf{j}$. Then $\operatorname{div}\mathbf{w} = 0$ and (15) implies:

$$\mathbf{curl}\,\mathbf{w} = \mathbf{0}, \quad \langle \mathbf{w}\times\mathbf{n}, \mathbf{v}\rangle_\Gamma = \mathbf{0}, \quad \forall \mathbf{v} \in V.$$

Thus $\mathbf{w}$ belongs to G. In addition, $\langle \mathbf{w}.\mathbf{n}, 1\rangle_{\Gamma_i} = \langle \mathbf{curl}\,\mathbf{u}.\mathbf{n}, 1\rangle_{\Gamma_i} - \langle \mathbf{j}, 1\rangle_{\Gamma_i} = 0$. Hence $\mathbf{P}_G\mathbf{w} = 0$ and therefore $\mathbf{w} = \mathbf{0}$.

The existence and uniqueness of solution are a direct consequence of Lax-Milgram's theorem and the Lemma 4. □

This existence lemma allows us to introduce a bounded linear operator

$$K: \ \mathbf{j} \in X \mapsto \mathbf{u} \in X \text{ solution of (14).}$$

Moreover, one can observe that K is a product of a linear continuous operator $\mathbf{j} \in X \mapsto \mathbf{u} \in V$ solution of (14) and the imbedding $V \hookrightarrow X$ which is obviously compact since the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Thus, we have the lemma

Lemma 6. *K is a compact operator.*

Now, we can rewrite the system (12) into the form

$$\textit{Find } \mathbf{b} \in X \textit{ such that: } \quad \mathbf{b} - \alpha K\mathbf{b} = \alpha K\mathbf{B}_0. \tag{17}$$

In order to use the Fredholm alternative, let us introduce the Adjoint problem.

1.3.2. *The Adjoint problem*

The following lemma will be useful here (see [18]):

Lemma 7. *Let $\mathbf{u}$ be a given field of $L^2(\Omega)^3$ such that*

$$\operatorname{div}\mathbf{u} = 0 \textit{ and } \int_{\Sigma_i} \mathbf{u}.\mathbf{n}d\sigma = 0, i = 1, ..., m.$$

Then, given $d = (d_1, ..., d_p) \in \mathbb{R}^p$, there exists a unique vector potential $\Phi \in H^1(\Omega)^3$ that satisfies:

$$\mathbf{curl}\,\Phi = \mathbf{u},\ \operatorname{div}\Phi = 0,\ \ \Phi\times\mathbf{n} = 0 \textit{ on } \Gamma \textit{ and } \int_{\Gamma_i} \Phi d\sigma = d_i, \ \ i = 1, ..., p. \tag{18}$$

In particular, this lemma implies that any vector field $\mathbf{j}$ in X admits a unique Weyl-Helmoltz decomposition into the form (see [16, 20, 23]):

$$\mathbf{j} = \nabla s + \sum_{i=1}^{m} c_i\mathbf{q}_i + \mathbf{curl}\,\Phi, \tag{19}$$

where $s \in H^1(\Omega)$ is solution of the Neumann problem $\Delta s = 0$, $\dfrac{\partial s}{\partial n} = \mathbf{j}.\mathbf{n}$ on Γ. The numbers $c_1, ..., c_m$ are given by

$$c_i = \int_{\Sigma_i} (\mathbf{j} - \nabla s).\mathbf{n}d\sigma.$$

The vector field Φ belongs to $H^1(\Omega)^3$ and verifies

$$\operatorname{div}\Phi = 0, \ \ \Phi\times\mathbf{n} = \mathbf{0}, \ \int_{\Gamma_i} \Phi.\mathbf{n}d\sigma = 0, \ 1 \le i \le p.$$

Φ exists by virtue of Lemma 7. Furthermore, it is characterized by the variational problem:

Lemma 8. *For any* $\mathbf{j} \in X$, *the vector* Φ *in the decomposition (19) is the unique solution in* U *of the variational problem:*

$$(\mathbf{curl}\,\Psi, \mathbf{curl}\,\mathbf{v}) + (\operatorname{div}\Psi, \operatorname{div}\mathbf{v}) + (\mathbf{P}_G\Psi, \mathbf{P}_G\mathbf{v}) = (\mathbf{j}, \mathbf{curl}\,\mathbf{v}), \quad \forall \mathbf{v} \in U. \tag{20}$$

The proof of this lemma is similar to that of Lemma 5.

Now, consider the operator

$$K^* : \mathbf{j} \in X \mapsto \Phi \in X \text{ solution of (20)}\ .$$

Lemma 9. K^* *is the adjoint operator of* K.

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be two elements of X. $\mathbf{v}$ can be decomposed into the form

$$\mathbf{v} = \nabla s + \sum_{i=1}^{m} c_i \mathbf{q}_i + \mathbf{curl}\,(K^*\mathbf{v}).$$

Hence

$$\int_\Omega K\mathbf{u}.\mathbf{v} d\Omega = \int_\Omega K\mathbf{u}[\nabla s + \sum_{i=1}^{m} c_i \mathbf{q}_i + \mathbf{curl}\,(K^*\mathbf{v})] d\Omega.$$

But, one has

$$\int_\Omega K\mathbf{u}.\nabla s = -\int_\Omega \operatorname{div}(K\mathbf{u}) s + \int_\Gamma s(K\mathbf{u}).\mathbf{n} = 0,$$

$$\int_\Omega K\mathbf{u}.\mathbf{q}_i d\Omega = (\mathbf{P}_H K\mathbf{u}, \mathbf{q}_i) = 0, \quad \text{for } i = 1, ..., p.$$

Thus

$$\int_\Omega K\mathbf{u}.\mathbf{v} d\Omega = \int_\Omega K\mathbf{u}.\mathbf{curl}\,(K^*\mathbf{v}) d\sigma = \int_\Omega \mathbf{curl}\,(K\mathbf{u}).K^*\mathbf{v} d\sigma = \int_\Omega \mathbf{u}.K^*\mathbf{v} d\sigma.$$

Therefore K^* is the adjoint operator of K. □

The homogeneous adjoint equation can be written into the form

$$\textit{Find } \varphi \textit{ in } X \textit{ such that:} \quad (I - \alpha K^*)\varphi = \mathbf{0}. \tag{21}$$

In other words, *Find* $\varphi \in X$, $s \in H^1(\Omega)/\mathbb{R}$, $(\gamma_1, ..., \gamma_m) \in \mathbb{R}^m$, *such that*

$$\nabla s + \sum_{i=1}^{m} \gamma_i \mathbf{q}_i + \frac{1}{\alpha}\mathbf{curl}\,\varphi = \varphi, \quad \varphi \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \tag{22}$$

Remark 1. One can also prove that φ is solution of the homogeneous adjoint problem if and only if $\varphi \in X$ and

$$\mathbf{curl}\,\mathbf{curl}\,\varphi - \alpha \mathbf{curl}\,\varphi = \mathbf{0}, \quad \varphi \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma. \tag{23}$$

In fact, (22) means that $K^*(\mathbf{curl}\,\varphi - \alpha\varphi) = 0$, say, by lemma 8, $\operatorname{div}\varphi = 0$ and

$$(\mathbf{curl}\,\varphi - \alpha\varphi, \mathbf{curl}\,\mathbf{v}) = 0, \quad \forall \mathbf{v} \in U$$

which is equivalent to (23).

1.3.3. *Fredholm alternative*

Since K is compact, its adjoint operator K^* is also compact. In addition, according to classical Riesz-Fredholm theory, the direct and the adjoint homogeneous problems admit two finite dimensional spaces of solutions with the same dimension $\mathbf{n}$. The following lemma display the relationship between their solutions:

Lemma 10. *Let $\varphi \in X$ be solution of the homogeneous adjoint problem*

$$\varphi - \alpha K^* \varphi = 0. \tag{24}$$

Then $\mathbf{curl}\,\varphi$ is solution to the direct homogeneous problem

$$\boldsymbol{\xi} - \alpha K \boldsymbol{\xi} = 0. \tag{25}$$

Conversely, if $\boldsymbol{\xi}$ is solution of (25), then there exists a unique φ in X, solution of (24) and such that $\mathbf{curl}\,\varphi = \boldsymbol{\xi}$.

Proof. Let φ be a solution of (24). Applying the curl operator to (22), one gets

$$\mathbf{curl}\,(\mathbf{curl}\,\varphi) = \alpha \mathbf{curl}\,\varphi.$$

In addition, $\mathbf{curl}\,\varphi$ is divergence-free and verifies

$$\mathbf{curl}\,\varphi.\mathbf{n} = 0 \text{ on } \Gamma \text{ and } \int_{\Sigma_i} \mathbf{curl}\,\varphi.\mathbf{n} d\sigma = 0.$$

Hence $\mathbf{curl}\,\varphi$ is solution of the direct homogeneous equation.

Conversely, let $\boldsymbol{\xi}$ be solution of (25). We set $\varphi = K^* \boldsymbol{\xi} \in X$. Since $\operatorname{div} \boldsymbol{\xi} = 0$ and $\mathbf{P}_H \boldsymbol{\xi} = 0$, then $\boldsymbol{\xi} = \mathbf{curl}\,\varphi$ and φ verifies (23). □

Now, observe that the homogeneous problem admits a non trivial solution if and only if $1/\alpha$ belongs to $\sigma(K)$, the spectrum of K. Since K is compact, $\sigma(K)$ contains 0 and $\sigma(K)/\{0\}$ is an empty, a finite or a countable set of eigenvalues contained in $[-\|K\|, \|K\|]$. To settle this question, we use the following result due to [48]:

Lemma 11. *The operator S, defined in the Hilbert Space*

$$X_1 = \{\mathbf{v} \in X, \ \ \mathbf{v}.\mathbf{n} = 0 \ on \ \Gamma, \mathbf{P}_H \mathbf{v} = \mathbf{0}\},$$

by $S\mathbf{u} = \mathbf{curl}\,\mathbf{u}$, for $\mathbf{u} \in D(S) = \{\mathbf{u} \in X_1, \ \ \mathbf{curl}\,\mathbf{u} \in X_1\}$, is self-adjoint and its spectrum $\sigma(S)$ consists of countable sequence of eigenvalues.

Since every eigenfunction of K belongs to $D(S)$ and is an eigenfunction of S, with an inverse eigenvalue, and conversely, we conclude that

$$\sigma(K) = \{0\} \cup \{\frac{1}{\mu}, \ \mu \in \sigma(S)\} \in [-\|K\|, \|K\|]. \tag{26}$$

Now, applying the Fredholm alternative to the inhomogeneous problem (17) yields:

- If $1/\alpha \notin \sigma(K)$, then the inhomogeneous problem admits one and only one solution (the direct and the adjoint homogeneous problems don't admit any non-trivial solution).

- $1/\alpha \in \sigma(K)$, then the adjoint homogeneous problem (22) admits a finite dimensional space of non-trivial solutions. The inhomogeneous problem (17) is solvable if and only if the right hand side verifies

$$\int_\Omega K\mathbf{B}_0.\Psi = 0, \tag{27}$$

for any Ψ solution of (22). If this solvability condition is fulfilled, then (17) has a general solution of the form

$$\mathbf{b} = \tilde{\mathbf{b}} + \mathbf{curl}\,\Psi,$$

where $\tilde{\mathbf{b}}$ is a particular solution and Ψ is a solution of the homogeneous adjoint problem (22) (since by Lemma 10, $\mathbf{curl}\,\Psi$ is solution of homogeneous problem (25)).

Now, let us rewrite the solvability condition (27) differently. Let $(\Psi, s, \gamma = (\gamma_1, ..., \gamma_m))$ be solution of (22). Then

$$\begin{aligned}(K\mathbf{B}_0, \Psi) &= (\mathbf{B}_0, K^*\Psi) = \frac{1}{\alpha}(\mathbf{B}_0, \Psi),\\ &= \frac{1}{\alpha}(\mathbf{B}_0, \mathbf{curl}\,\frac{\Psi}{\alpha} + \nabla s + \sum_{i=1}^m \gamma_i \mathbf{q}_i),\\ &= \frac{1}{\alpha}(\int_\Gamma s\mathbf{B}_0.\mathbf{n}d\sigma + \sum_{i=1}^m \gamma_i \int_\Omega \mathbf{B}_0.\mathbf{q}_i d\Omega),\\ &= \frac{1}{\alpha}(\int_\Gamma s\mathrm{g}d\sigma - \sum_{i=1}^m \gamma_i a_i).\end{aligned}$$

Hence, we can rewrite the solvability condition (27) into the form

$$\int_\Gamma s\mathrm{g}d\sigma - \sum_{i=1}^m \gamma_i a_i = 0.$$

Finally, we summarize our investigation in the following general existence and uniqueness result:

Theorem 1. *There exists a countable sequence of real values $\{\alpha_i, i \in \mathbb{N}\}$, verifying*

- $\forall i \in \mathbb{N}, \quad |\alpha_i| \geq \alpha_0,$
- *the sequence $(\alpha_i^{-1})_{i\in\mathbb{N}}$ converges to zero,*

and such that:

(i) *If $\alpha \notin \{\alpha_i, i \in \mathbb{N}\}$ (in particular if $|\alpha| < \alpha_0$), then the problem (8) admits one and only one solution $\mathbf{B} \in H^1(\Omega)^3$ for any $\mathrm{g} \in H^{\frac{1}{2}}(\Gamma)$ verifying (9) and $(a_1, ..., a_m)$ in $\mathbb{R}^m$.*

(ii) *If $\alpha = \alpha_i$ for some $i \in \mathbb{N}$, then the adjoint homogeneous problem (22) admits a finite dimensional space of solutions, and the problem (8) is solvable if and only if the data g and $(a_1, ..., a_m)$ verify the condition:*

$$\int_\Gamma s\mathrm{g} - \sum_{i=1}^m \gamma_i a_i = 0, \tag{28}$$

for any $(\varphi, s, \gamma = (\gamma_1, ..., \gamma_m))$ solution of (22). If this solvability condition is fulfilled, then (8) has a general solution of the form

$$\mathbf{B} = \tilde{\mathbf{B}} + \mathbf{curl}\,\Psi,$$

where $\tilde{\mathbf{B}}$ is a particular solution and Ψ is a solution of the homogeneous adjoint problem (22).

Remark 2. In Theorem 1, the estimate $|\alpha_i| \geq \alpha_0$ is in fact deduced from the next lemma which gives some inequalities relating $\|K\|$, α_0, α_1. λ_1 and $\tilde{\lambda}_2$ are the first and the second eigenvalues of the Laplace operator associated with the homogeneous Dirichlet and Neumann boundary conditions respectively (see Appendix):

Lemma 12. *We have:*

(i) *The following inequalities hold*

$$\alpha_0(\Omega) \leq \inf(\tilde{\lambda}_2^{\frac{1}{2}}, \|K\|^{-1}), \tag{29}$$

$$\alpha_1(\Omega) \leq \inf(\lambda_1^{\frac{1}{2}}, \|K\|^{-1}). \tag{30}$$

(ii) *If Γ is connected then*

$$\alpha_0 \leq \alpha_1 = \frac{1}{\|K\|}. \tag{31}$$

(iii) *If, in addition Ω is simply-connected then*

$$\alpha_0 = \inf(\tilde{\lambda}_2^{\frac{1}{2}}, \|K\|^{-1}). \tag{32}$$

(iv) *If Ω is star-shaped, then*

$$\alpha_1 d(\Omega) \geq 1, \tag{33}$$

where $d(\Omega) = \sup_{\mathbf{x},\mathbf{y}\in\Omega} |\mathbf{x} - \mathbf{y}|$ is the diameter of Ω.

1.4. Regularity of solutions

Corollary 1. *Assume Γ to be of class $C^{m+1,1}$ and $g \in H^{m+\frac{1}{2}}(\Gamma)$ for some $m \geq 0$ Let $\mathbf{B}$ solution of (8). Then $\mathbf{B} \in H^{m+1}(\Omega)$. Moreover, $\mathbf{B} - \mathbf{B}_0 \in H^{m+2}(\Omega)$.*

This corollary stems from the following lemma (see for instance [22]) combined with the basic equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$ (by induction):

Lemma 13. *Assume Γ to be of class $C^{m+1,1}$. Then*

$$H^{m+1}(\Omega)^3 = \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div}\mathbf{v} \in H^m(\Omega), \mathbf{curl}\,\mathbf{v} \in H^m(\Omega), \mathbf{v}.\mathbf{n} \in H^{m+\frac{1}{2}}(\Gamma)\}.$$

1.5. A variational formulation and energy estimate

When $|\alpha| < \alpha_0$, the problem (12) admits a variational formulation which brings all the equations of this problem together.

Proposition 1. *If $|\alpha| < \alpha_0(\Omega)$, then $\mathbf{B}$ is solution of (8) if and only if $\mathbf{b} = \mathbf{B} - \mathbf{B}_0$ is solution of the variational problem*

$$(\mathbf{curl}\,\mathbf{b} - \alpha\mathbf{b}, \mathbf{curl}\,\mathbf{v}) + (\operatorname{div}\mathbf{b}, \operatorname{div}\mathbf{v}) + (\mathbf{P}_H\mathbf{b}, \mathbf{P}_H\mathbf{v}) = (\alpha\mathbf{B}_0, \mathbf{curl}\,\mathbf{v}). \tag{34}$$

Moreover, this problem admits one and only one solution $\mathbf{B} \in H^1(\Omega)^3$, and we have the energy estimate

$$E_0 \leq \|\mathbf{B}\|_{0,\Omega}^2 \leq \frac{1}{1-r^2}E_0 \tag{35}$$

where E_0 is defined by (11) and $r = \dfrac{\alpha}{\alpha_0}$.

Proof. First, it is obvious that (12) implies (34). Conversely, let $\mathbf{b}$ a solution of (34). Then, one can prove exactly as in the proof of Lemma 5 that $\mathbf{P}_H\mathbf{b} = \mathbf{0}$ and $\operatorname{div}\mathbf{b} = 0$. It remains to prove that $\mathbf{curl}\,\mathbf{b} - \alpha\mathbf{b} = \alpha\mathbf{B}_0$. We use the following lemma (see, *e.g.*, [25], p. 47):

Lemma 14. *Let* $\mathbf{w} \in L^2(\Omega)^3$ *such that*

$$\operatorname{div}\mathbf{w} = 0 \text{ and } \langle \mathbf{w}.\mathbf{n}, 1\rangle_{\Gamma_i} = 0,\ 1 \le i \le p.$$

Then, there exists $\mathbf{A} \in H^1(\Omega)^3$ *such that*

$$\mathbf{curl}\,\mathbf{A} = \mathbf{w},\ \operatorname{div}\mathbf{A} = 0,\ \mathbf{A}.\mathbf{n} = 0.$$

The identity $\mathbf{curl}\,\mathbf{b} - \alpha\mathbf{b} - \alpha\mathbf{B}_0 = 0$ is deduced by taking $\mathbf{v} = \mathbf{b} - \alpha\mathbf{A} \in V$ in (34), where $\mathbf{A}$ is in $H^1(\Omega)^3$ and satisfies $\mathbf{curl}\,\mathbf{A} = \mathbf{b} - \mathbf{B}_0$, $\operatorname{div}\mathbf{A} = 0, \mathbf{A}.\mathbf{n} = 0$.
The existence and uniqueness of solution of the variational problem are a direct consequence of the Lax-Milgram's theorem since the left hand side in (34) is a bilinear form $a(.,.)$ which verifies $|a(\mathbf{v},\mathbf{v})| \ge (1 - \frac{|\alpha|}{\alpha_0})\|\mathbf{v}\|_V^2$, $\forall \mathbf{v} \in V$. Note that the energy estimate (35) is slightly more accurate than the one obtained directly from the variational formulation and the ellipticity of a. In fact, to prove it, one remarks that

$$(\mathbf{curl}\,\mathbf{b}, \mathbf{b}) = \alpha\|\mathbf{b}\|_{0,\Omega}^2,$$

since $(\mathbf{B}_0, \mathbf{b}) = 0$. Thus

$$\alpha^2\|\mathbf{B}_0^2\|_{0,\Omega}^2 = \|\mathbf{curl}\,\mathbf{b} - \alpha\mathbf{b}\|_{0,\Omega}^2 = \|\mathbf{curl}\,\mathbf{b}\|_{0,\Omega} - \alpha^2\|\mathbf{b}\|_{0,\Omega}^2 \ge (\alpha_0^2 - \alpha^2)\|\mathbf{b}\|_{0,\Omega}^2.$$

Hence $\|\mathbf{b}\|_{0,\Omega}^2 \le \dfrac{\alpha^2}{\alpha_0^2 - \alpha^2} E_0$. (35) is then deduced by observing that

$$\|\mathbf{B}\|_{0,\Omega}^2 = \|\mathbf{b}\|_{0,\Omega}^2 + \|\mathbf{B}_0\|_{0,\Omega}^2.$$

□

Remark 3. If α is a not a constant function, then the variational problem (34) is not in general equivalent to the problem (12). In fact, following exactly the same steps of the proof above (see [12]), one can show that if $\mathbf{b}$ is solution of (12) with $\alpha(\mathbf{x})$ a variable function, then $\mathbf{b}$ is divergence-free and there exists a function $p \in H_0^1(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{b} = \alpha(\mathbf{x})(\mathbf{B} + \mathbf{B}_0) + \nabla p.$$

Thus, applying the divergence operator to this equation yields

$$\Delta p = \operatorname{div}(\alpha\mathbf{B}) = \nabla\alpha.\mathbf{B}.$$

Hence, if α is constant p is necessary equal to zero.

1.6. A vector potential formulation

Another variational formulation based on the use of vector potential is possible. The vector potential can be introduced using Lemma 7.

Thus, given $d = (d_1, ..., d_p) \in \mathbb{R}^p$, one can introduce $\Phi \in U$ such that $\mathbf{curl}\,\Phi = \mathbf{b}$, where $\mathbf{b}$ is solution of (12). The next lemma characterizes Φ as the unique solution of a variational problem:

Proposition 2. *If* $|\alpha| < \alpha_1$*, then* $\mathbf{b}$ *is solution of (12) if and only if* Φ *is solution of the variational problem:* $\forall \mathbf{v} \in U$

$$(\mathbf{curl}\,\Phi, \mathbf{curl}\,\mathbf{v} - \alpha\mathbf{v}) \quad + \quad (\mathrm{div}\,\Phi, \mathrm{div}\,\mathbf{v}) + (\mathbf{P}_G\Phi, \mathbf{P}_G\mathbf{v}) = \alpha(\mathbf{B}_0, \mathbf{v}) + \sum_{i=1}^{p} d_i(f_i, \mathbf{P}_G\mathbf{v}).$$

Proof. First, if $\mathbf{b}$ is solution of (12) then it is obvious that Φ is solution of (36). Conversely, the Lemma 4 ensures us that the bilinear form $a_1(.,.)$ defined by:

$$a_1(\mathbf{u}, \mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v} - \alpha\mathbf{v}) + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) + (\mathbf{P}_G\mathbf{u}, \mathbf{P}_H\mathbf{v}),$$

is continuous and U-elliptic since $|\alpha| < \alpha_1$. Hence, by virtue of Lax-Milgram's theorem, (36) admits a unique solution which is necessarily the vector potential of the unique solution of (12). □

1.7. **Example.** *Ω is a sphere*

Here we consider the particular case where Ω is the unit sphere. In this simple geometry, following [14], we shall see that it is possible to give an explicit expression of the solution of the interior boundary value problem.

First, set $\mathbf{x} = (x, y, z)$ and introduce the toroidal-poloidal decomposition (see [34])

$$\mathbf{B} = \mathbf{curl}\,(T\mathbf{x}) + \mathbf{curl}\,\mathbf{curl}\,(P\mathbf{x}), \tag{36}$$

where T and P are two unknown functions. One can verify easily that if T and P are solutions of

$$\begin{cases} T = \alpha P, \\ \Delta P + \alpha^2 P = 0 \quad \text{in } \Omega, \\ \Delta_S P = -\mathrm{g} \quad \text{on } S, \end{cases} \tag{37}$$

then $\mathbf{B}$ is linear force-free. S is the surface of the unit sphere Ω and Δ_S is the Laplace-Beltrami operator on S defined by:

$$\Delta_S u = \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right). \tag{38}$$

In the sequel, we shall denote $(.,.)_S$ the scalar product $L^2(S)$ and by Y_l^m, $l \geq 0$ and $-l \leq m \leq l$, the spherical harmonics on S. They constitute an orthonormal basis of $L^2(S)$ and an orthogonal basis of $H^1(S)$. Recall that

$$Y_l^m(\theta, \varphi) = (-1)^m \left[\frac{(l+\frac{1}{2})(l-m)!}{2\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi} P_l^m(\cos\theta)$$

where P_l^m is the Legendre function of order m and degree l. For any $l \geq 0$, $-l \leq m \leq l$, Y_l^m verifies

$$\Delta_S Y_l^m + l(l+1)Y_l^m = 0. \tag{39}$$

In addition, for any distribution $u \in \mathcal{D}(S)$, we set $u_l^m = \langle u, Y_l^m\rangle_S$. $u \in H^s(\Omega)$ where s is a real number if and only if:

$$\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}(l+1)^{2s}|u_l^m|^2 < +\infty.$$

In this case, $u_l^m = \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} u_l^m Y_l^m$. Now, we decompose $\mathrm{g} \in H^{-\frac{1}{2}}(S)$ on this basis:

$$\mathrm{g}(\theta,\varphi) = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \mathrm{g}_l^m Y_l^m,$$

with $\mathrm{g}_l^m = \langle \mathrm{g}, Y_l^m\rangle_S$. Note that $\overline{\mathrm{g}_l^m} = (-1)^m \mathrm{g}_l^{-m}$, for any l and m, and $\mathrm{g}_0^0 = \langle \mathrm{g}, Y_0^0\rangle_S = 0$ since $\langle \mathrm{g}, 1\rangle_S = 0$.

We thus use a decomposition on spherical harmonics and deduce easily that $P_{|S} \in H^{\frac{3}{2}}(S)$ can be written in the form

$$P_{|S}(\theta,\varphi) = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \frac{\mathrm{g}_l^m}{l(l+1)} Y_l^m + C, \tag{40}$$

where C is a constant. After what P is obtained in the interior of the sphere by solving the Helmoltz's equation with a Dirichlet boundary condition (see, *e.g.*, [40])

$$P(r,\theta,\varphi) = \sum_{l=1}^{+\infty} \frac{1}{l(l+1)} \sum_{m=-l}^{l} k_l^m(\alpha r) Y_l^m + k_0(\alpha r). \tag{41}$$

The functions k_l, for $l \geq 0$, are defined by:

$$k_l^m(\alpha r) = \begin{cases} \dfrac{j_l(\alpha r)}{j_l(\alpha)} \mathrm{g}_l^m & \text{if } j_l(\alpha) \neq 0, \\ C_l^m j_l(\alpha r) & \text{if } j_l(\alpha) = 0 \text{ and } \mathrm{g}_l^m = 0 \ \forall m \in \{-l, ..., +l\}, \end{cases} \tag{42}$$

where $(j_l)_{l\geq 0}$ are the usual spherical Bessel functions defined by:

$$j_l(r) = (-r)^l \left(\frac{1}{r}\frac{d}{dr}\right)^l \left(\frac{\sin r}{r}\right).$$

Note that, in accordance with Theorem 1, if $j_l(\alpha) = 0$, then the condition $\mathrm{g}_l^m = 0 \ \forall m \in \{-l, ..., +l\}$ is necessary and the constants C_l^m are in this case arbitrary.

Finally, $\mathbf{B}$ is given by

$$\mathbf{B}(r,\theta,\varphi) = \sum_{l=1}^{l=+\infty}\sum_{m=-l}^{l} \frac{1}{l(l+1)} \mathbf{b}_l^m,$$

where $\mathbf{b}_l^m$ is defined by

$$\begin{aligned} \mathbf{b}_l^m(r,\theta,\varphi) \quad = \quad & \frac{l(l+1)}{r} k_l^m(\alpha r) Y_l^m \mathbf{e}_r \\ & + \left[\frac{\alpha}{\sin\theta} k_l^m(\alpha r) \frac{\partial Y_l^m}{\partial\varphi} + \frac{1}{r}\frac{d}{dr}\left(r k_l^m(\alpha r)\right) \frac{\partial Y_l^m}{\partial\theta}\right] \mathbf{e}_\theta \\ & - \left[\alpha k_l^m(\alpha r) \frac{\partial Y_l^m}{\partial\theta} - \frac{1}{r\sin\theta}\frac{d}{dr}\left(r k_l^m(\alpha r)\right) \frac{\partial Y_l^m}{\partial\varphi}\right] \mathbf{e}_\varphi. \end{aligned} \tag{43}$$

□

1.8. **Two problems with α unknown.**

In the sections before we dealt with the existence of a linear force-free field submitted to appropriate boundary conditions when the constant α is known. However, physical situations in which α is unknown are quite possible. A common example is that of a closed system for confining a plasma by a magnetic field. In such a system, if the plasma is perfectly conducting, the magnetic field is subject to a topological constraint. To express this constraint, consider a vector potential $\mathbf{A}$ corresponding to $\mathbf{B}$ and set

$$H(\mathbf{B}) = (\mathbf{A}, \mathbf{B}). \tag{44}$$

This quantity, called *the helicity*, is physically meaningful and jauge invariant when $\mathbf{B}$ verifies the conditions:

$$\mathbf{B}.\mathbf{n} = 0 \text{ on } \Gamma, \tag{45}$$

and (if the domain is multiply-connected)

$$\int_{\Sigma_i} \mathbf{B}.\mathbf{n} d\sigma = 0, \quad \text{for } i = 1, ..., m. \tag{46}$$

On one hand, the helicity describes the linkage of lines of force of $\mathbf{B}$ with one another (see [8,37]). On the other hand, it is an invariant of any perfect MHD motion of the plasma (see [50]; see also [46,47]).
When one of the two conditions (45) and (46) is not satisfied, the helicity, as defined by (44), loses its jauge invariance. In that case, Berger and Field [8] (see also [29]) introduced the notion of *relative helicity* defined as follows:

$$H_r(\mathbf{B}) = (\mathbf{A}, \mathbf{B}) - (\mathbf{A}_0, \mathbf{curl}\, \mathbf{A}_0), \tag{47}$$

where $\mathbf{A}$ is vector potential of $\mathbf{B}$ and $\mathbf{A}_0$ is solution of the system:

$$\mathbf{curl}\,\mathbf{curl}\, \mathbf{A}_0 = \mathbf{0}, \ \operatorname{div} \mathbf{A}_0 = 0, \ \mathbf{A}_0 \times \mathbf{n} = \mathbf{A} \times \mathbf{n} \text{ on } \Gamma. \tag{48}$$

It is worth noting that the quantity $H_r(\mathbf{B})$ is jauge invariant and generalizes the concept of helicity since $H_r(\mathbf{B}) = H(\mathbf{B})$ when $\mathbf{B}$ verifies (45) and (46).
Another possible data which can be given instead of α is the integral

$$m_0 = \int_\Omega \mathbf{B}.\mathbf{curl}\, \mathbf{B} d\Omega. \tag{49}$$

This quantity lacks a clear physical meaning. However, when $\mathbf{B}$ is force-free, m_0 looks like the helicity since $\alpha^{-1}\mathbf{B}$ is a vector potential of $\mathbf{B}$.
Our aim here is to treat the two following boundary value problems in which α is unknown and g is given in $H^{\frac{1}{2}}(\Gamma)$ and verifies (9):

Problem A. *Find $\alpha \in \mathbb{R}$ and $\mathbf{B} \in H^1(\Omega)^3$ such that:*

$$\begin{cases} \mathbf{curl}\, \mathbf{B} & = & \alpha \mathbf{B} & \text{in } \Omega, \\ \operatorname{div} \mathbf{B} & = & 0 & \text{in } \Omega, \\ \mathbf{B}.\mathbf{n} & = & 0 & \text{on } \Gamma, \\ (\mathbf{B}.\mathbf{n}, 1)_{\Sigma_i} & = & -a_i, & i = 1, ..., m, \\ H_r(\mathbf{B}) & = & H_0 & \text{(prescribed helicity)}, \end{cases} \tag{50}$$

where $H_0, a_1, a_2, ..., a_m$ are given real numbers.

Problem B. *Find* $\alpha \in \mathbb{R}$ *and* $\mathbf{B} \in H^1(\Omega)^3$ *such that:*

$$\left\{\begin{array}{rcll} \mathbf{curl}\,\mathbf{B} & = & \alpha\mathbf{B} & \text{in } \Omega, \\ \operatorname{div}\mathbf{B} & = & 0 & \text{in } \Omega, \\ \mathbf{B}.\mathbf{n} & = & \mathrm{g} & \text{on } \Gamma, \\ (\mathbf{B}\times\mathbf{n}, \mathbf{q}_i)_\Gamma & = & \alpha a_i, & i = 1, ..., m, \\ (\mathbf{curl}\,\mathbf{B}, \mathbf{B}) & = & m_0, & \end{array}\right. \tag{51}$$

where m_0 is a given real number, $(a_1, a_2, ..., a_m) \in \mathbb{R}^m$.

In both the problems, we still denote by $\mathbf{B}_0$ the potential field corresponding to the data and E_0 its energy defined as in Section 1.3. We have the following results:

Theorem 2. *Assume that* $H_0 \neq 0$ *and* $E_0 \neq 0$. *Then, the problem (50) admits at least one solution* $(\alpha, \mathbf{B}) \in \mathbb{R} \times H^1(\Omega)^3$. *This solution satisfies the estimates*

$$|\alpha| \le r\alpha_0 \tag{52}$$

$$E_0^2 \le \|\mathbf{B}\|_{0,\Omega}^2 \le \mu E_0, \tag{53}$$

where $\mu = \sqrt{1+\delta}$, $r = \dfrac{\mu - 1}{\mu}$ *with* $\delta = \dfrac{\alpha_0|H_0|}{E_0^2}$. *In addition, if* Ω *is* $C^{m+1,1}$ *and* $\mathrm{g} \in H^{m+\frac{1}{2}}(\Gamma)$, *then* $\mathbf{B} \in H^{m+1}(\Omega)$.

Proof. Let us first introduce the vector potential $\mathbf{a}_0 \in H^1(\Omega)^3$ defined as the unique solution of:

$$\mathbf{curl}\,\mathbf{a}_0 = \mathbf{B}_0, \ \ \operatorname{div}\mathbf{a}_0 = 0, \ \ \mathbf{a}_0.\mathbf{n} = 0 \text{ on } \Gamma, \mathbf{P}_H\mathbf{a}_0 = \mathbf{0}. \tag{54}$$

Note that $\mathbf{a}_0$ is also solution of the variational problem:

$$(\mathbf{curl}\,\mathbf{a}_0, \mathbf{curl}\,v) + (\operatorname{div}\mathbf{a}_0, \operatorname{div}\mathbf{v}) + (\mathbf{P}_H\mathbf{a}_0, \mathbf{P}_H\mathbf{v}) = (\mathbf{B}_0, \mathbf{curl}\,\mathbf{v}), \ \ \forall \mathbf{v} \in V,$$

and thus verifies the estimate:

$$\|\mathbf{a}_0\|_{0,\Omega}^2 \le \frac{1}{\alpha_0^2}\|\mathbf{B}_0\|_{0,\Omega}^2 = \frac{E_0^2}{\alpha_0^2}\,. \tag{55}$$

Now, we set $\mathbf{b} = \mathbf{B} - \mathbf{B}_0$ and we define $\mathbf{a} \in H^1(\Omega)^3$ by:

$$\mathbf{curl}\,\mathbf{a} = \mathbf{b}, \ \ \operatorname{div}\mathbf{a} = 0, \ \ \mathbf{a}\times\mathbf{n} = 0 \text{ on } \Gamma, \mathbf{P}_G\mathbf{a} = \mathbf{0}. \tag{56}$$

With $\mathbf{A} = \mathbf{a} + \mathbf{a}_0$ and $\mathbf{A}_0 = \mathbf{a}_0$, one proves easily that the relative helicity $H_r(\mathbf{B})$ can be rewritten as follows:

$$H_r(\mathbf{B}) = \int_\Omega \mathbf{a}.\mathbf{b}d\Omega + 2\int_\Omega a.\mathbf{B}_0 d\Omega. \tag{57}$$

Furthermore, if $\mathbf{B} = \mathbf{b} + \mathbf{B}_0$ verifies $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$, then

$$\int_\Omega |\mathbf{b}|^2 d\Omega = \int_\Omega \mathbf{curl}\,\mathbf{b}.\mathbf{a}d\Omega = \alpha\int_\Omega \mathbf{a}.(\mathbf{b} + \mathbf{B}_0)d\Omega.$$

Thus, since $(\mathbf{a}, \mathbf{B}_0) = (\mathbf{a}, \mathbf{curl}\, \mathbf{a}_0) = (\mathbf{curl}\, \mathbf{a}, \mathbf{a}_0) = (\mathbf{b}, \mathbf{a}_0)$, we get

$$\int_\Omega |\mathbf{b}|^2 d\Omega = \alpha H_r(\mathbf{B}) - \alpha \int_\Omega \mathbf{a}_0.\mathbf{b} d\Omega. \tag{58}$$

We consider now the following subspace of $H(\mathrm{div}\,;\,\Omega)$:

$$Y = \left\{ \mathbf{v} \in L^2(\Omega)^3;\ \mathrm{div}\, \mathbf{v} = 0,\ \mathbf{v}.\mathbf{n} = 0 \text{ on } \Gamma;\ \int_\Omega \mathbf{v}.\mathbf{q}_i = 0, i = 1, ..., m \right\},$$

the ball

$$\mathcal{B} = \{\mathbf{v} \in Y;\ \|\mathbf{v}\|_{0,\Omega} \le (\mu - 1)E_0\},$$

and the mapping $L : \mathbf{j} \in \mathcal{B} \longrightarrow \mathbf{u} \in \mathcal{B}$ where $\mathbf{u}$ is the unique solution of:

$$\mathbf{curl}\, \mathbf{u} = \alpha(\mathbf{j} + \mathbf{B}_0),\ \mathrm{div}\, \mathbf{u} = 0,\ \mathbf{u}.\mathbf{n} = 0,\ \mathbf{P}_H \mathbf{u} = \mathbf{0},$$

with

$$\alpha = \frac{\|\mathbf{j}\|^2_{0,\Omega}}{H_0 - \displaystyle\int_\Omega \mathbf{a}_0.\mathbf{j} d\Omega}. \tag{59}$$

This mapping is well defined since

$$\begin{aligned}\left| H_0 - \int_\Omega \mathbf{a}_0.\mathbf{j} d\Omega \right| &\ge\ \big||H_0| - \|\mathbf{a}_0\|_{0,\Omega}.\|\mathbf{j}\|_{0,\Omega}\big|, \\ &\ge\ \left||H_0| - \frac{E_0}{\alpha_0}(\mu - 1)E_0\right|, \\ &\ge\ \frac{E_0^2}{\alpha_0}(\mu^2 - \mu) \ge 0.\end{aligned}$$

Thus, α is well defined, $|\alpha| \le r\alpha_0$ and

$$\alpha_0 \|\mathbf{u}\|_{0,\Omega} \le \|\mathbf{curl}\, \mathbf{u}\|_{0,\Omega} \le |\alpha|(\|\mathbf{j}\|_{0,\Omega} + \|\mathbf{B}_0\|_{0,\Omega}) \le \alpha_0(\mu - 1)E_0.$$

Furthermore, T is clearly continuous and compact. Indeed, let $\mathbf{j}_n$ be a sequence in $\mathcal{B}$. The corresponding sequences $\alpha^{(n)}$ and $(\mathbf{u}^{(n)})$ are also bounded in $\mathbb{R}$ and in $H^1(\Omega)^3$. By compacity of the inclusion $H^1(\Omega)^3 \hookrightarrow L^2(\Omega)^3$, a subsequence, still denoted $(\mathbf{u}^{(n)})$, converging in $L^2(\Omega)^3$, can be extracted. The proof is achieved by applying Schauder's fixed point theorem.
The proof of regularity remains the same as in Corollary 1. □

Theorem 3. *If the data* g, $\mathbf{a} = (a_1, ..., a_m)$ *and* m_0 *verify the condition*

$$\tau = \frac{|m_0|}{\alpha_0 E_0} < 1, \tag{60}$$

then the problem (51) admits one and only one solution $(\alpha, \mathbf{B}) \in \mathbb{R} \times H^1(\Omega)^3$. *Moreover,* α *and* $\mathbf{B}$ *verify:*

$$\frac{2\tau}{1 + \sqrt{1 + 4\tau^2}} \le \frac{|\alpha|}{\alpha_0} \le \tau,\quad E_0 \le \|\mathbf{B}\|^2_{0,\Omega} \le (1 + \sqrt{1 + 4\tau^2})\frac{E_0}{2}\cdot \tag{61}$$

In addition, if Ω *is* $\mathcal{C}^{m+1,1}$ *and* $\mathrm{g} \in H^{m+\frac{1}{2}}(\Gamma)$, *then* $\mathbf{B} \in H^{m+1}(\Omega)$.

Proof. We consider the following sequence:

- $\mathbf{B}^{(0)} = \mathbf{B}_0$
- For $n \geq 0$, $\alpha^{(n)} = \dfrac{m_0}{\|\mathbf{B}^{(n)}\|^2_{0,\Omega}}$.
- For $n \geq 1$, $\mathbf{B}^{(n+1)} = \mathbf{B}_0 + \mathbf{b}^{(n+1)}$, where $\mathbf{b}^{(n+1)}$ is solution of the curl-div system:

$$\begin{cases} \mathbf{curl}\,\mathbf{b}^{(n+1)} = \alpha^{(n)}\mathbf{B}^{(n)}, \\ \operatorname{div}\mathbf{b}^{(n+1)} = 0, \\ \mathbf{b}^{(n+1)}.\mathbf{n} = 0 \quad \text{on } \Gamma. \\ \mathbf{P}_H\mathbf{b}^{(n+1)} = \mathbf{0}. \end{cases} \tag{62}$$

This sequence $(\alpha^{(n)}, \mathbf{B}^{(n)})$ is well defined, since

$$|\alpha^{(n)}| = \frac{|m_0|}{\|\mathbf{B}^{(n)}\|^2_{0,\Omega}} \leq \frac{|m_0|}{\tilde{E}_0^2} < \alpha_0(\Omega).$$

For $n \geq 0$, define $\mathbf{v}^{(n+1)} = \mathbf{B}^{(n+1)} - \mathbf{B}^{(n)} \in V$. Hence

$$\begin{aligned} \mathbf{curl}\,\mathbf{v}^{(n+1)} &= \alpha^{(n)}\mathbf{B}^{(n)} - \alpha^{(n-1)}\mathbf{B}^{(n-1)}, \\ \|\mathbf{curl}\,\mathbf{v}^{(n+1)}\|^2_{0,\Omega} &= {\alpha^{(n)}}^2\|\mathbf{B}^{(n)}\|^2_{0,\Omega} + {\alpha^{(n-1)}}^2\|\mathbf{B}^{(n-1)}\|^2_{0,\Omega} - 2\alpha^{(n)}\alpha^{(n-1)}(\mathbf{B}^{(n)}, \mathbf{B}^{(n-1)}), \\ &= m_0^2\frac{\|\mathbf{B}^{(n)}\|^2_{0,\Omega} + \|\mathbf{B}^{(n-1)}\|^2_{0,\Omega} - 2(\mathbf{B}^{(n)}, \mathbf{B}^{(n-1)})}{\|\mathbf{B}^{(n)}\|^2_{0,\Omega}\|\mathbf{B}^{(n-1)}\|^2_{0,\Omega}}, \\ &= \frac{m_0^2\|\mathbf{v}^{(n)}\|^2_{0,\Omega}}{\|\mathbf{B}^{(n)}\|^2_{0,\Omega}\|\mathbf{B}^{(n-1)}\|^2_{0,\Omega}}. \end{aligned}$$

Thus by (6):

$$\|\mathbf{v}^{(n+1)}\|_V \leq \frac{|m_0|}{\alpha_0\tilde{E}_0^2}\|\mathbf{v}^{(n)}\|_V, \tag{63}$$

By using the condition (60), it follows that $(\mathbf{B}^{(n)})_{n\in N}$ is a Cauchy sequence in $H^1(\Omega)^3$ and converges to $\mathbf{B}$, solution of (51). Let $E = \|\mathbf{B}\|_{0,\Omega}$. Then, E verifies

$$E^2 \leq \frac{1}{1-r^2}\tilde{E}_0^2$$

where $r = \dfrac{|\alpha|}{\alpha_0} = \dfrac{|m_0|}{E^2\alpha_0}$. Hence, $x^2 - x - \tau^2 \leq 0$, where $x = \dfrac{E^2}{\tilde{E}_0^2}$ and $\tau = \dfrac{|m_0|}{\alpha_0\tilde{E}_0^2}$. Thus x is between the two roots of the polynomial $X^2 - X - \tau^2$ and this implies (61).

It remains to prove uniqueness. Let $(\alpha_1, \mathbf{B}_1)$, $(\alpha_2, \mathbf{B}_2)$ be two solutions to (51). Then doing exactly the same calculus as above, where $\mathbf{B}^{(n)}$ and $\mathbf{B}^{(n+1)}$ are replaced by $\mathbf{B}_1$ and $\mathbf{B}_2$, one obtains the following inequality analogous to (63).

$$\|\mathbf{B}_2 - \mathbf{B}_1\|_V \leq \frac{|m_0|}{\alpha_0\tilde{E}_0^2}\|\mathbf{B}_2 - \mathbf{B}_1\|_V. \tag{64}$$

Since $\dfrac{|m_0|}{\alpha_0 E_0^2} < 1$, then $\mathbf{B}_2 - \mathbf{B}_1 = \mathbf{0}$. □

Remark 4. *Toroidal geometry.* The case in which Ω is a torus plays a prominent role in plasma plasma confining experiments. The toroidal pinch experiments are the best illustrations of such a situation (see [46,47] and references therein). It is worth noting that the torus is a multiply-connected domain with $m = 1$; only one cut Σ_1 suffices to make it simply-connected. The flux throughout Σ_1 is nothing but the classical well known "toroidal flux". Theorems 1 and 2 and Proposition 1 give a complete palette of results about existence, uniqueness, regularity and à priori estimates which can be very useful in plasma confinement.

Remark 5. *Eigenfields of Maxwell operator and Beltrami fields.* The purpose of this remark is to show a simple manner of deriving linear force-free fields from the Maxwell spectrum; let $\mathcal{A}$ the Maxwell operator defined by:

$$\begin{aligned}\mathcal{D}(\mathcal{A}) \quad = \quad & \{(\mathbf{E},\mathbf{H}) \in L^2(\Omega)^6,\ \mathbf{curl}\,\mathbf{E} \text{ and } \mathbf{curl}\,\mathbf{H} \in L^2(\Omega)^3, \\ & \operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0,\ \mathbf{E}\times\mathbf{n} = 0,\ \mathbf{H}.\mathbf{n} = 0 \text{ on } \Gamma\}.\end{aligned}$$

$$\mathcal{A} = \begin{pmatrix} 0 & -\mathbf{curl} \\ \mathbf{curl} & 0 \end{pmatrix}. \tag{65}$$

It is well-known that the operator $i\mathcal{A}$ admits a discrete set of real eigenvalues with finite multiplicity. Let w be an eigenvalue of $i\mathcal{A}$ and let $(\mathbf{E},\mathbf{H}) \in L^2(\Omega)^6$ be the corresponding eigenvector. One has:

$$\mathbf{curl}\,\mathbf{H} = i\omega\mathbf{E},\ \ \mathbf{curl}\,\mathbf{E} = -i\omega\mathbf{H},\ \ \operatorname{div}\mathbf{H} = \operatorname{div}\mathbf{E} = 0,\ \ \mathbf{H}.\mathbf{n} = 0 \ \text{ and }\ \mathbf{E}\times\mathbf{n} = \mathbf{0}.$$

Then one can observe that for any complex number μ, $\mathbf{B}_1 = Re[\mu(\mathbf{E} - i\mathbf{H})]$ and $\mathbf{B}_2 = Re[\mu(\mathbf{E} + i\mathbf{H})]$ verify:

$$\mathbf{curl}\,\mathbf{B}_1 = \omega\mathbf{B}_1,\ \mathbf{curl}\,\mathbf{B}_2 = -\omega\mathbf{B}_2. \tag{66}$$

Hence, $\mathbf{B}_1$ and $\mathbf{B}_2$ are linear Beltrami fields in Ω.

2. Linear Beltrami fields in a half-cylinder

In modelling natural phenomena physicists are often led to deal with unbounded regions of space. In the context of force-free field, the studies of solar atmosphere supply several nice examples of this situation. For example, in reconstructing the coronal magnetic field, the region above a small part of the solar photosphere is often likened to a half-space [2,5] since the curvature of the sun's surface can be neglected. Furthermore, some recent models for coronal heating are developed in a semi-infinite cylindrical part of space (see [3] for a study of existence and stability of axisymmetrical linear force-free fields). Our aim here is to solve explicitly the problem of existence of linear force-free fields in a semi-infinite cylinder with a general section (and without any assumption on the axisymmetry of the data or the solution).

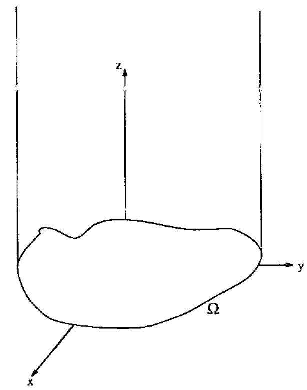
Let $\tilde{\Omega}$ be a bounded connected domain of $\mathbb{R}^2$ with boundary Γ of class $\mathcal{C}^{1,1}$ (or assume $\tilde{\Omega}$ a convex polygon). Let $\mathcal{D}$ be the half-cylinder of $\mathbb{R}^3$ defined by $\mathcal{D} = \tilde{\Omega}\times]0,+\infty[$. For any field $\mathbf{v} = (v_i)_{i=1,2,3}$, we set $\mathbf{v_h} = (v_1, v_2)$ the horizontal part of $\mathbf{v}$. We denote also by ∇_h, div_h, $\mathbf{curl}_h$ and Δ_h the horizontal gradient, divergence, curl and Laplace operators with respect to coordinates (x, y).

Define as usual the Sobolev space $H_0^1(\tilde{\Omega})$ by:

$$H_0^1(\tilde{\Omega}) = \{u \in H^1(\tilde{\Omega});\ u = 0 \text{ on } \Gamma\},$$

which is a Hilbert space for the norm

$$|v|_{1,\tilde{\Omega}} = \|\nabla_h v\|_{0,\tilde{\Omega}},$$



thanks to Poincaré's inequality on $\tilde{\Omega}$. Introduce also the subspaces of $L^2(\tilde{\Omega})^2$:

$$\begin{aligned} H(\mathbf{curl}_{\mathrm{h}}, \tilde{\Omega}) &= \{\mathbf{u} \in L^2(\tilde{\Omega})^2; \mathbf{curl}_{\mathrm{h}}\mathbf{u} \in L^2(\tilde{\Omega})\}, \\ H(\mathrm{div}_{\mathrm{h}}, \tilde{\Omega}) &= \{\mathbf{u} \in L^2(\tilde{\Omega})^2; \mathrm{div}_{\mathrm{h}}\, \mathbf{u} \in L^2(\tilde{\Omega})\}, \end{aligned} \tag{67}$$

and set

$$H = H(\mathrm{div}_{\mathrm{h}}, \tilde{\Omega}) \cap H(\mathbf{curl}_{\mathrm{h}}, \tilde{\Omega}),$$

H is a Hilbert space for the norm

$$\|\mathbf{u}\|_H = (\|\mathbf{u}\|_{0,\tilde{\Omega}} + \|\mathrm{div}_{\mathrm{h}}\, \mathbf{u}\|_{0,\tilde{\Omega}} + \|\mathbf{curl}_{\mathrm{h}}\mathbf{u}\|_{0,\tilde{\Omega}})^{\frac{1}{2}}.$$

Finally, for any Banach space X, define the functional space

$$L^2(0, +\infty, X) = \left\{ \mathbf{v} :]0, +\infty[\longrightarrow X;\ \mathbf{v} \text{ is measurable and } \int_0^{+\infty} \|\mathbf{v}(t)\|_X^2 dt < +\infty \right\},$$

equipped with the norm:

$$\|\mathbf{v}\|_{L^2(0,+\infty,X)} = (\int_0^{+\infty} \|\mathbf{v}(t)\|_X^2 dt)^{\frac{1}{2}}.$$

2.1. Statement of the problem

Given a real number α, we want to find a bounded three-dimensional field $\mathbf{B}$ satisfying $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$ in the half-cylinder $\mathcal{D}$ with a given vertical component B_z on $\tilde{\Omega}$ at $z = 0$.

First, we introduce the following closed subspace of $H \times H_0^1(\tilde{\Omega})$:

$$V_\alpha = \{\mathbf{v} \in H \times H_0^1(\tilde{\Omega});\ \mathbf{curl}_{\mathrm{h}}\mathbf{v}_{\mathbf{h}} = \alpha v_z\}.$$

It is worth noting that the equation $\mathbf{curl}_{\mathrm{h}}\mathbf{v}_{\mathbf{h}} = \alpha v_z$ is nothing but the z-component of the Beltrami equation. Now, given a scalar function $\mathrm{g} = \mathrm{g}(x, y) \in H_0^1(\tilde{\Omega})$, we seek $\mathbf{B}$ satisfying:

$$\begin{cases} \mathbf{B} \in L^2(0, +\infty, V_\alpha), pt_z\mathbf{B} \in L^2(0, +\infty, L^2(\tilde{\Omega})^3), \\ \partial_z B_x = \partial_x B_z + \alpha B_y \quad \text{in } \mathcal{D}, \\ \partial_z B_y = \partial_y B_z - \alpha B_x \quad \text{in } \mathcal{D}, \\ \partial_z B_z = -\partial_x B_x - \partial_y B_y \quad \text{in } \mathcal{D}, \\ B_z = \mathrm{g} \quad \text{at } z = 0. \end{cases} \tag{68}$$

This system is a non standard evolution problem, in which the vertical coordinate z plays the same role as time, and the boundary condition on $\tilde{\Omega}$ corresponds to initial conditions. Note that the "initial condition" at $z = 0$ is only on B_z. Note also that the fourth equation and the description of behavior at $z = +\infty$ are included in the definition of the space in which the solution is required to be.

Our purpose in the next section is to prove that the boundary-value problem (68) has one and only one solution.

2.2. Existence and uniqueness of solution

In this section, we shall extensively use the spectrum of $-\Delta_h$ in Ω. It is well known that this operator admits an infinite countable set of eigenfunctions $\{\omega_j\}_{j=1,2,\ldots,+\infty} \in H_0^1(\tilde{\Omega})$ with a sequence $\{\lambda_j\}_{j=1,2,\ldots,+\infty}$ of strictly positive eigenvalues conventionally ordered such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots < +\infty.$$

Moreover the family $\{\omega_j\}_{j=1,2,\ldots,+\infty}$ can be chosen to form an orthonormal basis of $L^2(\tilde{\Omega})$ and an orthogonal basis of $H_0^1(\tilde{\Omega})$. So, one has for any $i, k \geq 1$:

$$(\omega_i, \omega_k) = \delta_{i,k}, \tag{69}$$

$$(\nabla_h \omega_i, \nabla_h \omega_k) = \lambda_k \delta_{i,k}. \tag{70}$$

Theorem 4. *If $\alpha^2 < \lambda_1$, then for any $\mathrm{g} \in H_0^1(\tilde{\Omega})$, the problem (68) admits a unique solution. Moreover, this solution can be written in the form*

$$\mathbf{B} = \sum_{k=1}^{+\infty} (\mathrm{g}, \omega_k) e^{-\beta_k z} \mathbf{b}_k, \tag{71}$$

where β_k and $\mathbf{b}_k$ are defined by:

$$\beta_k = \sqrt{\lambda_k - \alpha^2}, \tag{72}$$

$$\mathbf{b}_k = \begin{pmatrix} \frac{1}{\lambda_k}(\beta_k \frac{\partial \omega_k}{\partial x} + \alpha \frac{\partial \omega_k}{\partial y}) \\ \frac{1}{\lambda_k}(\beta_k \frac{\partial \omega_k}{\partial y} - \alpha \frac{\partial \omega_k}{\partial x}) \\ w_k \end{pmatrix}. \tag{73}$$

Proof. **Uniqueness**

Let $\mathbf{B}_1$, $\mathbf{B}_2 \in L^2(0, +\infty, V_\alpha)$ be two solutions of problem (68). Their difference $\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2$ satisfies the same problem with an homogeneous boundary condition $B_z(z = 0) = 0$. Multiplying the two first equations of (68) by $\nabla_h B_z$ and integrating by parts over $\tilde{\Omega}$ yield the identity

$$\frac{1}{2} \frac{d^2}{dz^2} \|B_z\|_{0,\tilde{\Omega}}^2 = \|\frac{\partial B_z}{\partial z}\|_{0,\tilde{\Omega}}^2 + \|\nabla_h B_z\|_{0,\tilde{\Omega}}^2 - \alpha^2 \|B_z\|_{0,\tilde{\Omega}}^2.$$

Then, using the Poincaré's inequality gives

$$\frac{1}{2} \frac{d^2}{dz^2} \|B_z\|_{0,\tilde{\Omega}}^2 \geq (\lambda_1 - \alpha^2) \|B_z\|_{0,\tilde{\Omega}}^2 \geq 0.$$

On the other hand, we have

$$\|B_z(z=0)\|_{0,\tilde{\Omega}} = 0 \quad \text{and} \quad \int_0^{+\infty} \|B_z\|_{0,\tilde{\Omega}}^2 dz < +\infty.$$

Thus, $\|B_z\|_{0,\tilde{\Omega}} = 0$ on $]0, +\infty[$.

If we multiply now the system (68) by $(B_x, B_y, -B_z)$, then we get

$$\frac{d}{dz}(\|B_x\|_{0,\tilde{\Omega}}^2 + \|B_y\|_{0,\tilde{\Omega}}^2) = \frac{d}{dz}\|B_z\|_{0,\tilde{\Omega}}^2 = 0.$$

Therefore, $\|B_x\|_{0,\tilde{\Omega}}^2 = \|B_y\|_{0,\tilde{\Omega}}^2 = 0$ on $]0, +\infty[$ since $\mathbf{B_h} \in L^2(0, +\infty, L^2(\tilde{\Omega})^2)$. Hence, $\mathbf{B} = 0$.

Existence

We proceed in several steps:

Step 1. Construction of the Galerkin Basis

We consider the following special eigenvalue problem: find the eigenvalues γ_j and the eigenfunctions $(u_j, v_j, w_j) \in V_\alpha$ such that:

$$\begin{cases} \dfrac{\partial w_j}{\partial x} + \alpha v_j &= \gamma_j u_j, \\ \dfrac{\partial w_j}{\partial y} - \alpha u_j &= \gamma_j v_j, \\ \dfrac{\partial u_j}{\partial x} + \dfrac{\partial v_j}{\partial y} &= -\gamma_j w_j. \end{cases} \tag{74}$$

The two first equations can be arranged in the form

$$\begin{aligned} (\alpha^2 + \gamma_j^2) u_j &= \left(\gamma_j \frac{\partial \omega_j}{\partial x} + \alpha \frac{\partial \omega_j}{\partial y}\right) \\ (\alpha^2 + \gamma_j^2) v_j &= \left(\gamma_j \frac{\partial \omega_j}{\partial y} - \alpha \frac{\partial \omega_j}{\partial x}\right). \end{aligned}$$

Then, using the identity

$$\alpha w_j = \frac{\partial v_j}{\partial x} - \frac{\partial u_j}{\partial y}.$$

one obtains the equation

$$-\Delta_h w_j = (\gamma_j^2 + \alpha^2) w_j,$$

which means that w_j is an eigenfunction of $-\Delta_h$ and $\gamma_j^2 + \alpha^2$ is the corresponding eigenvalue. Therefore, for every eigenfuction ω_j, there are two possible values of γ_j; $\gamma_{j,\epsilon} = \epsilon \beta_j$ where $\epsilon \in \{-1, 1\}$ and $\beta_j = \sqrt{\lambda_j - \alpha^2}$. Then, we have:

$$\begin{aligned} u_{j,\epsilon} &= \frac{1}{\lambda_j}\left(\gamma_{j,\epsilon} \frac{\partial \omega_j}{\partial x} + \alpha \frac{\partial \omega_j}{\partial y}\right) \\ v_{j,\epsilon} &= \frac{1}{\lambda_j}\left(\gamma_{j,\epsilon} \frac{\partial \omega_j}{\partial y} - \alpha \frac{\partial \omega_j}{\partial x}\right). \end{aligned}$$

Remark 6. Taking into account (69) and (70), one can deduce the relations:

$$(u_{i,\epsilon}, u_{k,\epsilon}) + (v_{i,\epsilon}, v_{k,\epsilon}) = \delta_{ik}, \tag{75}$$

$$(u_{i,\epsilon}, v_{k,\epsilon}) - (v_{i,\epsilon}, u_{k,\epsilon}) = 0, \tag{76}$$

for any integers i and k.

Step 2. Approximated solutions and convergence
The function $\mathrm{g} \in H_0^1(\tilde{\Omega})$ can be expanded on the basis ω_i:

$$\mathrm{g} = \sum_{i=1}^{+\infty} (\mathrm{g}, \omega_i)\omega_i \quad \text{with} \quad \sum_{i=1}^{+\infty} \lambda_i (\mathrm{g}, \omega_i)^2 < +\infty. \tag{77}$$

We look for a solution $\mathbf{B}$ of (68) into the form

$$\mathbf{B} = \sum_{i=1}^{+\infty} c_{i,1}(z)\mathbf{b}_{i,1} + \sum_{i=1}^{+\infty} c_{i,-1}(z)\mathbf{b}_{i,-1},$$

where the vector fields $\mathbf{b}_{i,\epsilon}$ are defined by $\mathbf{b}_{i,\epsilon} = (u_{i,\epsilon}, v_{i,\epsilon}, \omega_i)$.

Formally, if we substitute in (68), we deduce that the coefficients $c_{i,\epsilon}$ are solutions to

$$\begin{cases} \dfrac{dc_{i,\epsilon}}{dz} + \epsilon\beta_i c_{i,\epsilon} = 0, \quad \epsilon = 1, -1, \\ c_{i,1}(0) + c_{i,-1}(0) = (\mathrm{g}, \omega_i). \end{cases}$$

Therefore,

$$c_{i,1}(z) = (\mathrm{g}, \omega_i)e^{-\beta_i z}, \quad c_{i,-1}(z) = 0,$$

since we search a bounded solution at infinity. Hence,

$$\mathbf{B} = \sum_{i=1}^{+\infty} (\mathrm{g}, \omega_i)e^{-\beta_i z}\mathbf{b}_i, \tag{78}$$

where we write $\mathbf{b}_i$ instead of $\mathbf{b}_{i,1}$ (for clearness). It remains to prove that the field $\mathbf{B}$, given by the sum (78), is well defined and is the solution of the problem (68):

(i) Setting

$$\mathbf{B}_m = \sum_{i=1}^{\mathbf{m}} (\mathrm{g}, \omega_i)e^{-\beta_i z}\mathbf{b}_i, \tag{79}$$

one gets for any $z > 0$

$$\|\mathbf{B}_{\mathbf{h},m+p} - \mathbf{B}_{\mathbf{h},m}\|_{0,\tilde{\Omega}}^2 = \sum_{i=m+1}^{m+p} (\mathrm{g}, \omega_i)^2 e^{-2\beta_i z}, \tag{80}$$

$$\|\mathrm{div_h}\, \mathbf{B}_{\mathbf{h},m+p} - \mathrm{div_h}\, \mathbf{B}_{\mathbf{h},m}\|_{0,\tilde{\Omega}}^2 = \sum_{i=m+1}^{m+p} \beta_i^2 (\mathrm{g}, \omega_i)^2 e^{-2\beta_i z}, \tag{81}$$

$$\|\mathbf{curl}_{\mathrm{h}} \mathbf{B}_{\mathbf{h},m+p} - \mathbf{curl}_{\mathrm{h}} \mathbf{B}_{\mathbf{h},m}\|_{0,\tilde{\Omega}}^2 = \alpha^2 \sum_{i=m+1}^{m+p} (\mathrm{g}, \omega_i)^2 e^{-2\beta_i z}, \tag{82}$$

$$|B_{z,m+p} - B_{z,m}|_{1,\tilde{\Omega}}^2 = \sum_{i=m+1}^{m+p} \lambda_i (\mathrm{g}, \omega_i)^2 e^{-2\beta_i z}. \tag{83}$$

Therefore for any $z > 0$, $(\mathbf{B}_m)_{m \geq 1}$ is a Cauchy sequence in V_α. Hence, the series (78) converges for every $z > 0$ and $\mathbf{B}$ is well defined. Furthermore, it is clear that $\mathbf{B} \in L^2(0, +\infty, V_\alpha)$.

(ii) Now, we show that $B_z(.,z=0) = \mathrm{g}$ in $H_0^1(\tilde{\Omega})$.
First, given $T > 0$,

$$\begin{aligned} \|B_{z,m+p} - B_{z,m}\|_{\mathcal{C}^0([0,T],H_0^1(\tilde{\Omega}))} &= \sup_{[0,T]} \Big(\sum_{k=m+1}^{m+p} \lambda_k (\mathrm{g},\omega_k)^2 e^{-2\beta_k z} \Big) \\ &\leq \sum_{k=m+1}^{m+p} \lambda_k (\mathrm{g},\omega_k)^2. \end{aligned}$$

Hence, $B_z \in \mathcal{C}^0([0,T], H_0^1(\tilde{\Omega}))$ and $B_z(.,z=0)$ is defined. Moreover

$$\sum_{k=1}^{+\infty} \lambda_k \|(g,\omega_k)e^{-\gamma_k z} - (g,\omega_k)\|^2 \leq (1-e^{-\gamma_m z})^2 \sum_{k=1}^{m} \lambda_k (g,\omega_k)^2 + \sum_{k=m+1}^{+\infty} \lambda_k (g,\omega_k)^2,$$

for any integer m. Using (77) , one can deduce that

$$\lim_{z \longrightarrow 0} (B_z(z) - \mathrm{g}) = 0, \quad \text{in } H_0^1(\tilde{\Omega}).$$

Thus, $B_z(z=0) = \mathrm{g}$ in $H_0^1(\tilde{\Omega})$.

(iii) Finally, $\dfrac{\partial \mathbf{B}}{\partial z} \in L^2(0,+\infty, L^2(\tilde{\Omega})^3)$ since

$$\int_0^{+\infty} \|\frac{\partial \mathbf{B}_m}{\partial z}\|_{0,\tilde{\Omega}}^2 = 2\sum_{i=1}^{m} (\lambda_i - \alpha^2)(\mathrm{g},\omega_i)^2 e^{-2\gamma_i z},$$

and $\mathbf{B}$ is a classical solution for (68).

□

Remark 7. *Energy estimate.*
The energy of the solution

$$E = \int_\Omega |\mathbf{B}|^2 d\Omega,$$

can be easily computed if one uses the orthogonality relation:

$$(\mathbf{b}_k, \mathbf{b}_j)_{L^2(\tilde{\Omega})} = 2\delta_{k,j}, \quad \text{for any } j,k \geq 1. \tag{84}$$

In fact,

$$\begin{aligned} (\mathbf{b}_k, \mathbf{b}_j)_{L^2(\tilde{\Omega})} &= \left(\frac{\beta_k}{\lambda_k} \nabla\omega_k + \frac{\alpha}{\lambda_k} \nabla\omega_k \times \mathbf{e}_z,\ \frac{\beta_j}{\lambda_j} \nabla\omega_j + \frac{\alpha}{\lambda_j} \nabla\omega_j \times \mathbf{e}_z \right) \\ &= \frac{\beta_k\beta_j + \alpha^2}{\lambda_k\lambda_j} (\nabla\omega_k, \nabla\omega_j) + \frac{\alpha(\beta_k - \beta_j)}{\lambda_k\lambda_j} (\nabla\omega_k, \nabla\omega_j \times \mathbf{e}_z) + (\omega_k,\omega_j), \\ &= 2\delta_{k,j} + \frac{\alpha(\beta_k - \beta_j)}{\lambda_k\lambda_j} (\nabla\omega_k, \mathbf{curl}_h \omega_j), \\ &= 2\delta_{k,j}. \end{aligned}$$

Thus,

$$E = \int_\Omega |\mathbf{B}|^2 d\Omega = \sum_{k=1}^{+\infty} \frac{(\mathrm{g},\omega_k)^2}{\beta_k}. \tag{85}$$

Remark 8. Let us clarify some points about the necessity of the condition $\alpha^2 < \lambda_1$, involved in theorem 4 for existence of solution; let $\mathbf{B}$ be a solution of the problem (68). Multiplying by B_z the equation $\mathbf{curl}_h \mathbf{B_h} = \alpha B_z$, taken at $z = 0$, and integrating over $\tilde{\Omega}$, one gets

$$\alpha \|g\|_{0,\tilde{\Omega}}^2 = \left(B_x, \frac{\partial g}{\partial y}\right) - \left(B_y, \frac{\partial g}{\partial x}\right).$$

Thus

$$|\alpha| \leq \frac{|g|_{1,\tilde{\Omega}} \|\mathbf{B_h}(z=0)\|_{0,\tilde{\Omega}}}{\|g\|_{0,\tilde{\Omega}}^2}. \tag{86}$$

Combining the two first equations of (68) leads to

$$\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) B_y - \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) B_x = 0.$$

Using the identity $\dfrac{\partial B_z}{\partial z} = -\mathrm{div}_h \, \mathbf{B_h}$ and integrating over $\tilde{\Omega}$ gives

$$\|\mathbf{B_h}(z=0)\|_{0,\tilde{\Omega}} = \|B_z(z=0)\|_{0,\tilde{\Omega}} = \|g\|_{0,\tilde{\Omega}},$$

and finally (86) becomes

$$|\alpha| \leq \frac{|g|_{1,\tilde{\Omega}}}{\|g\|_{0,\tilde{\Omega}}}.$$

A necessary condition for the existence of solutions to (68) for any $g \in H_0^1(\tilde{\Omega})$ is that $\alpha^2 \leq \lambda_1$.

Note that for a particular g given in $H_0^1(\tilde{\Omega})$ the boundary value problem (68) still has a solution if $\alpha^2 < \lambda_m$, where m is the smallest integer such that $(g, \omega_m) \neq 0$.

Remark 9. The Dirichlet condition $B_z = 0$ included implicitly in the definition of V_α can be replaced by the Neumann condition $\dfrac{\partial B_z}{\partial n} = 0$ on $\Gamma \times \{z > 0\}$ (the derivative $\dfrac{\partial}{\partial n} = 0$ is of course with respect to the normal of Γ). In this case, one must replace $(\omega_j)_{j \geq 1}$ and $(\lambda_j)_{j \geq 1}$ by the eigenfunctions and the eigenvalues of $-\Delta_h$ with Neumann boundary condition.

2.3. Approximation and error estimate

Let α a given real number such that $|\alpha| < \sqrt{\lambda_1}$ and $g \in H_0^1(\tilde{\Omega})$. Let $\mathbf{B}$ be the unique solution of (68) and consider its approximation $\mathbf{B}_m$ defined by:

$$\mathbf{B}_m = \sum_{i=1}^{m} (g, \omega_i) e^{-\beta_i z} \mathbf{b}_i. \tag{87}$$

Denote by g_m the approximation of g defined by

$$g_m = \sum_{i=1}^{m} (g, \omega_i) \omega_i. \tag{88}$$

We have the

Theorem 5. *There exists two constants a_0 and $C(\tilde{\Omega})$ such that for any integer m, the following inequalities yield*

$$\|\mathbf{B}-\mathbf{B}_m\|_{0,\tilde{\Omega}} \leq e^{-a_0 m z}\|\mathrm{g}-\mathrm{g}_m\|_{0,\tilde{\Omega}}, \tag{89}$$

$$\|\mathbf{B}-\mathbf{B}_m\|_H \leq C(\tilde{\Omega})e^{-a_0 m z}|\mathrm{g}-\mathrm{g}_m|_{1,\tilde{\Omega}}. \tag{90}$$

Proof. Let m be an integer. From (80)-(83) it stems

$$\|\mathbf{B}-\mathbf{B}_m\|_{0,\tilde{\Omega}} \leq e^{-\beta_{m+1} z}\|\mathrm{g}-\mathrm{g}_m\|_{0,\tilde{\Omega}},$$
$$\|\mathbf{B}-\mathbf{B}_m\|_H \leq C(\tilde{\Omega})e^{-\beta_{m+1} z}|\mathrm{g}-\mathrm{g}_m|_{1,\tilde{\Omega}}.$$

Then, using the 2-D Weyl formula

$$\lambda_n \sim \frac{4\pi}{meas(\tilde{\Omega})}n, \text{ when } n \longrightarrow +\infty$$

we get (89) and (90). □

Remark 10. *Bi-periodic Beltrami fields*

A treatment of Beltrami fields bi-periodic with periods L_i ($i = 1$ or 2) in each horizontal direction $\mathbf{e}_x$ and $\mathbf{e}_y$ can also be done (for periodicity in the three directions the reader can see [15]). Bi-periodic Beltrami fields may be useful for modelling some physical problems such as Prominences on Sun's surface (see [17]). Let us use the Fourier expansion

$$\mathbf{B}(\mathbf{x}) = \mathbf{b}_0(z) + \sum_{\mathbf{k}\neq\mathbf{0}} \mathbf{b}_\mathbf{k}(z)e^{i\mathbf{k}.\mathbf{x}}, \tag{91}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{k} = \left(\frac{2\pi n}{L_1}, \frac{2\pi m}{L_2}, 0\right)$, $n, m \in \mathbb{N}$. Given α a real number not equal to zero, we look for $\mathbf{B}$ such that $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$.

First, for any real vector $\mathbf{k} = (k_1, k_2, 0)$ such that $k = |\mathbf{k}| \neq 0$, we introduce the vectors:

$$\mathbf{e}_1(\mathbf{k}) = i\frac{\mathbf{k}}{k}, \;\; \mathbf{e}_2(\mathbf{k}) = i\frac{\mathbf{k}}{k} \times \mathbf{e}_z, \;\; \mathbf{e}_3(\mathbf{k}) = \mathbf{e}_z.$$

The following properties are easily verified:

$$\mathbf{e}_i(\mathbf{k})\bar{\mathbf{e}}_j(\mathbf{k}) = \delta_{i,j} \;\text{ for } i, j \in \{1, 2, 3\}, \tag{92}$$

$$i\mathbf{k} \times \mathbf{e}_1(\mathbf{k}) = 0, \; i\mathbf{k} \times \mathbf{e}_2(\mathbf{k}) = k\mathbf{e}_3(\mathbf{k}), \; i\mathbf{k} \times \mathbf{e}_3(\mathbf{k}) = ke_2(\mathbf{k}). \tag{93}$$

Thus, $(\mathbf{e}_1(k), \mathbf{e}_2(k), \mathbf{e}_3(k))$ is a basis of $\mathbb{R}^3$ and for $k \neq 0$ one can decompose $\mathbf{b}_\mathbf{k}(z)$ on this basis:

$$\mathbf{b}_\mathbf{k}(z) = \gamma_\mathbf{k}(z)\mathbf{e}_1(\mathbf{k}) + \mu_\mathbf{k}(z)\mathbf{e}_2(\mathbf{k}) + \lambda_\mathbf{k}(z)\mathbf{e}_3(\mathbf{k}). \tag{94}$$

The introduction of this decomposition is useful here since the equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$ becomes:

$$k\mu_\mathbf{k}(z) = \lambda_\mathbf{k}(z), \; k\gamma_\mathbf{k}(z) = \lambda'_\mathbf{k}(z),$$
$$\lambda''_\mathbf{k}(z) + (\alpha^2 - k^2)\lambda_\mathbf{k}(z) = 0, \;\; \mathbf{curl}\,\mathbf{b}_0(z) = \alpha\mathbf{b}_0(z).$$

One gets after solving this system:

$$\mathbf{b_0} = b_{01}\mathbf{e}^{i\alpha z}\begin{pmatrix} 1 \\ i \end{pmatrix} + b_{02}\mathbf{e}^{-i\alpha z}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{95}$$

$$\lambda_{\mathbf{k}}(z) = \begin{cases} \lambda_{\mathbf{k},0}e^{i\beta_{\mathbf{k}}z} + \lambda_{\mathbf{k},1}e^{-i\beta_{\mathbf{k}}z}, & \text{if } \alpha^2 - k^2 \geq 0, \\ \lambda_{\mathbf{k},0}e^{\beta_{\mathbf{k}}z} + \lambda_{\mathbf{k},1}e^{-\beta_{\mathbf{k}}z}, & \text{if } \alpha^2 - k^2 \leq 0, \end{cases} \tag{96}$$

where $\beta_k = \sqrt{|\alpha^2 - k^2|}$, b_{01}, b_{02}, $\lambda_{\mathbf{k},0}$ and $\lambda_{\mathbf{k},1}$ are complex constants chosen such that $\mathbf{B}$ is a real vector. They can be fixed using for example a boundary condition on B_z at $z = 0$ and a behaviour condition at infinity ($\lim_{z \longrightarrow +\infty} |\mathbf{B}| = 0$ for example).

3. Linear Beltrami fields in exterior domains

In this last part, our investigation concerns the existence of Beltrami fields in exterior domains. The behavior at infinity is described by setting the problem in a weighted Sobolev space. We show that the Beltrami equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$, with a given normal component, admits an infinity of solutions in this space, and by the way we show also that nor one of those solutions is finite energy, as proved by [44] otherwise. Finally, we prove that the addition of a well chosen boundary condition allows one to get well posedness of the problem.

Let Ω^e be the exterior of the *unit sphere* of $\mathbb{R}^3$, $r = |\mathbf{x}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ the distance to its center and $S = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| = 1\}$ its surface. For any real number k, define the space $W_k(\Omega^e)$ as:

$$W_k(\Omega^e) = \left\{ u \in \mathcal{D}'(\Omega^e);\ \frac{u}{r^k} \in L^2(\Omega^e),\ \frac{\nabla u}{r^k} \in L^2(\Omega^e)^3 \right\}$$

which is a Hilbert space for the norm

$$\|\mathbf{u}\|_{W_k(\Omega^e)} = \{\|\frac{u}{r^k}\|^2_{L^2(\Omega^e)} + \|\frac{\nabla u}{r^k}\|^2_{L^2(\Omega^e)}\}^{\frac{1}{2}}.$$

Observe that in the neighborhood of $S = \partial\Omega^e$, the functions of $W_k(\Omega^e)$ have the same regularity as those of the classical Sobolev space H^1. Therefore, the trace on $\partial\Omega^e$ can be defined like in H^1 and the usual trace theorem holds. As in a bounded domain, let α be a real number *not equal to zero* and consider the problem: *find* $\mathbf{B} \in W_k(\Omega^e)$ *(k will be specified) such that*

$$\begin{cases} \mathbf{curl}\,\mathbf{B} &= \alpha\mathbf{B} & \text{in } \Omega^e, \\ \mathrm{div}\,\mathbf{B} &= 0 & \text{in } \Omega^e, \\ \mathbf{B}.\mathbf{e}_r &= \mathrm{g} & \text{at } |\mathbf{x}| = 1, \end{cases} \tag{97}$$

where g is a given function. If we denote by B_r, B_φ and B_θ the components of $\mathbf{B}$ in spherical coordinates, then rB_r verifies the Helmoltz equation

$$\Delta(rB_r) + \alpha^2 rB_r = 0. \tag{98}$$

In fact, this equation can be obtained by applying the operator $\mathbf{x}.\mathbf{curl}$ to the equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$:

$$\mathbf{x}.\mathbf{curl}\,\mathbf{curl}\,\mathbf{B} = \mathbf{x}.\mathbf{curl}\,(\alpha\mathbf{B}) = \alpha^2\mathbf{x}.\mathbf{B}.$$

Then, we get (98) using $\operatorname{div}\mathbf{B} = 0$ and the identity:

$$\mathbf{x}.\mathbf{curl}\,\mathbf{curl}\,\mathbf{B} = -\mathbf{x}.\Delta\mathbf{B} + \mathbf{x}.\nabla(\operatorname{div}\mathbf{B}) = -\Delta(\mathbf{x}.\mathbf{B}) - 2\operatorname{div}\mathbf{B} + \mathbf{x}.\nabla(\operatorname{div}\mathbf{B}).$$

Now, a decomposition of rB_r on the spherical harmonics leads to:

$$B_r = \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=l} \frac{h_l^m(r)}{r} Y_l^m(\theta,\varphi),$$

where $h_l^m(r)$ are solutions of the spherical Bessel equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dh}{dr}\right) + (1 - \frac{l(l+1)}{r^2})h(r) = 0. \tag{99}$$

The function $h_l^m(r)$ can be written into the form

$$h_l^m(r) = \lambda_l^m h_l(\alpha r) + \mu_l^m \overline{h}_l(\alpha r), \tag{100}$$

where h_l is the spherical Hankel function given by:

$$h_l(r) = (-r)^l\left(\frac{1}{r}\frac{d}{dr}\right)^l\left(\frac{e^{\imath r}}{r}\right), \tag{101}$$

while $(\lambda_l^m)_{l\geq 1,|m|\leq l}$, $(\mu_l^m)_{l\geq 1,|m|\leq l}$ are two families of complex numbers such that:

(i) $\overline{\lambda}_l^m = (-1)^m \mu_l^{-m}$, for any $l \geq 0$ and $m \in \{-l,...,l\}$ (to guarantee real value radial component B_r),
(ii) $\lambda_l^m h_l(\alpha) + \mu_l^m \overline{h}_l(\alpha) = \mathrm{g}_l^m$ for any $l \geq 0$ and $m \in \{0,...,l\}$ (boundary condition at $|\mathbf{x}| = 1$),

where $\mathrm{g}_l^m = \langle \mathrm{g}, Y_l^m\rangle$. Observe that if $\mathrm{g}_l^m \neq 0$ and $h_l(\alpha) = 0$ then the equation $\lambda_l^m h_l^{(1)}(\alpha) + \mu_l^m h_l^{(2)}(\alpha) = \mathrm{g}_l^m$ does not admit any solution (λ_l^m, μ_l^m). Thus, we shall assume that:

$$\mathrm{g}_l^m = 0,\ \forall m \in \{-l,...,l\} \text{ if } h_l(\alpha) = 0. \tag{102}$$

Under this assumption, there exists an infinity of solutions $(\lambda_l^m, \mu_l^m)_{l\geq 1,|m|\leq l}$ verifying (i) and (ii). Let us fix one among them. Then, B_φ and B_θ are solutions of the following differential system;

$$\frac{\partial}{\partial r}\begin{bmatrix} rB_\varphi \\ rB_\theta \end{bmatrix} = \begin{bmatrix} 0 & -\alpha, \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} rB_\varphi \\ rB_\theta \end{bmatrix} + \begin{bmatrix} \dfrac{1}{\sin\theta}\dfrac{\partial B_r}{\partial\varphi} \\ \dfrac{\partial B_r}{\partial\theta} \end{bmatrix}. \tag{103}$$

This system is obtained by projection of the basic equation $\mathbf{curl}\,\mathbf{B} = \alpha\mathbf{B}$ on $\mathbf{e}_\varphi$ and $\mathbf{e}_\theta$. Its homogeneous solution is given by

$$\begin{aligned} b_{h,\theta} &= \psi_0(\theta,\varphi)\frac{\cos(\alpha r)}{r} + \psi_1(\theta,\varphi)\frac{\sin(\alpha r)}{r}, \\ b_{h,\varphi} &= \psi_0(\theta,\varphi)\frac{\sin(\alpha r)}{r} - \psi_1(\theta,\varphi)\frac{\cos(\alpha r)}{r}. \end{aligned}$$

A particular solution is

$$b_\theta = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{1}{l(l+1)}\left[\frac{\alpha}{\sin\theta}h_l^m\frac{\partial Y_l^m}{\partial\varphi}+\frac{1}{r}\frac{\partial}{\partial r}(rh_l^m)\frac{\partial Y_l^m}{\partial\theta}\right], \tag{104}$$

$$b_\varphi = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{1}{l(l+1)}\left[-\alpha h_l^m\frac{\partial Y_l^m}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rh_l^m)\frac{\partial Y_l^m}{\partial\varphi}\right]. \tag{105}$$

Hence, one deduces that (B_θ, B_φ) are on the form

$$B_\theta = b_\theta + b_{h,\theta},$$
$$B_\varphi = b_\varphi + b_{h,\varphi}.$$

Observe now that the system (98)+(103) is not equivalent to the equation $\mathbf{curl}\,\mathbf{B}-\alpha\mathbf{B}=\mathbf{0}$. In fact, the projection of this one on $\mathbf{r}$ must be verified:

$$0=\frac{1}{\sin\theta}\left[\frac{\partial(\sin\theta\mathbf{B}_\varphi)}{\partial\theta}-\frac{\partial\mathbf{B}_\theta}{\partial\varphi}\right]-\alpha rB_r=\frac{1}{\sin\theta}\left[\frac{\partial(\sin\theta\mathbf{b}_{h,\varphi})}{\partial\theta}-\frac{\partial\mathbf{b}_{h,\theta}}{\partial\varphi}\right],$$

since $\dfrac{\partial(\sin\theta\mathbf{b}_\varphi)}{\partial\theta}-\dfrac{\partial\mathbf{b}_\theta}{\partial\varphi}=\alpha r\sin\theta B_r$. Thus, ψ_0 and ψ_1 must verify the system:

$$\frac{\partial}{\partial\theta}(\sin\theta\psi_1)-\frac{\partial\psi_2}{\partial\varphi} = 0,$$
$$\frac{\partial}{\partial\theta}(\sin\theta\psi_2)+\frac{\partial\psi_1}{\partial\varphi} = 0.$$

Hence $\psi_1=\dfrac{1}{\sin\theta}\dfrac{\partial\xi}{\partial\varphi}$ and $\psi_2=\dfrac{\partial\xi}{\partial\theta}$, where ξ is solution of the equation

$$\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\xi}{\partial\theta})+\frac{1}{\sin\theta}\frac{\partial^2\xi}{\partial\varphi^2}=0.$$

Therefore, $\Delta_S\xi=0$, where Δ_S is the surfacic Laplace-Beltrami operator defined by (38). So ξ is a constant function and $\psi_1=\psi_2=0$. It follows that $B_\theta=\mathbf{b}_\theta$ and $B_\varphi=\mathbf{b}_\varphi$ where $\mathbf{b}_\theta$ and $\mathbf{b}_\varphi$ are given by (104) and (105). Let us check if $\mathbf{B}=(B_r,B_\theta,B_\varphi)\in L^2(\Omega^e)^3$. First, remark that $\mathbf{B}$ can be written into the form:

$$\mathbf{B}=\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{1}{l(l+1)}\mathbf{b}_l^m, \tag{106}$$

with

$$\mathbf{b}_l^m=\alpha h_l^m(r)\nabla_S Y_l^m\times\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial r}(rh_l^m(r))\nabla_S Y_l^m+l(l+1)\frac{h_l^m(r)}{r}Y_l^m(\theta,\varphi)\mathbf{e}_r, \tag{107}$$

where $\nabla_S u=\dfrac{1}{\sin\theta}\dfrac{\partial u}{\partial\varphi}\mathbf{e}_\varphi+\dfrac{\partial u}{\partial\theta}\mathbf{e}_\theta$ denotes the surfacic gradient of u. Since

$$(Y_{l_1}^{m_1},Y_{l_2}^{m_2})_S = \delta_{l_1,l_2}\delta_{m_1,m_2},$$
$$(\nabla_S Y_{l_1}^{m_1},\nabla_S Y_{l_2}^{m_2})_S = l_1(l_1+1)\delta_{l_1,l_2}\delta_{m_1,m_2},$$
$$(\nabla_S Y_{l_1}^{m_1}\times\mathbf{e}_r,\nabla_S Y_{l_2}^{m_2})_S = (\mathbf{curl}_S Y_{l_1}^{m_1},\nabla_S Y_{l_2}^{m_2})_S=0,$$

one has

$$(\mathbf{b}_{l_1}^{m_1}, \mathbf{b}_{l_2}^{m_2})_S = 0 \text{ if } l_1 \neq l_2 \text{ or } m_1 \neq m_2, \tag{108}$$

$$(\mathbf{b}_l^m, \mathbf{b}_l^m)_S = l(l+1)\left[\alpha^2|h_l^m(r)|^2 + |\frac{1}{r}\frac{d}{dr}(rh_l^m(r))|^2\right] + l^2(l+1)^2\frac{|h_l^m(r)|^2}{r^2}. \tag{109}$$

At infinity the function h_l looks like:

$$h_l(\alpha r) \sim (-i)^l \frac{e^{i\alpha r}}{\alpha r}. \tag{110}$$

It follows

$$(\mathbf{b}_l^m.\mathbf{e}_r, \mathbf{b}_l^m.\mathbf{e}_r) \sim \frac{C_l}{r^4}, \tag{111}$$

$$(\mathbf{b}_l^m, \mathbf{b}_l^m)_S \sim 2l(l+1)\frac{|\lambda_l^m|^2 + |\mu_l^m|^2}{r^2}. \tag{112}$$

Thus, $\mathbf{b}_l^m$ does not belong to $L^2(\Omega^e)^3$ if $h_l^m(r) \not\equiv 0$. Hence

There is no finite-energy non-trivial linear Force-free fields in an exterior domain with $\alpha \neq 0$.

However, $\mathbf{b}_l^m$ belongs to any $W_k(\Omega^e)^3$ for $k > \frac{1}{2}$. In this last situation, the linear system of equations (i) and (ii) given above is not square. Thus

If g *verifies the hypothesis (102), then the boundary value problem (97) admits an infinity of solutions in* $W_k(\Omega^e)^3$ *for any* $k > \frac{1}{2}$.

In order to get well posedness of (97) one must add a supplementary condition. Here we propose to prescribe the normal derivative of the radial component at the boundary:

$$\frac{\partial B_r}{\partial r} = \rho \text{ at } |\mathbf{x}| = 1. \tag{113}$$

Remark 11. Of course, others conditions at $r = 1$ or at infinity may exist. Durant [21] in a discussion about extrapolation of coronal magnetic field proposed to minimize the L^2-norm of the transverse component $\|\mathbf{B} \times \mathbf{n}\|_{0,S}$ at $r = 1$. One can prove easily that such a constraint leads to a condition of type (113), which is more general. Note also that, although the choice of a boundary data of type (113) has, as far as we know, never been considered in solar physics, these data can today be provided by the new generation of telescopes.

Our aim here is to prove that the new boundary value problem (97)-(113) is well posed in $W_k(\Omega^e)$.
For any real number $s \geq 0$, we introduce the following space:

$$\mathcal{H}_\alpha^s(S) = \left\{u \in \mathcal{D}'(S);\ \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+1)^{2s}|h_l(\alpha)|^2|u_l^m|^2 < +\infty\right\},$$

where $u_l^m = \langle u, Y_l^m\rangle_S$. This is a Hilbert space equipped with the hermitian product:

$$((u,v))_{\mathcal{H}_\alpha^s(S)} = \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+1)^{2s}|h_l(\alpha)|^2 u_l^m \bar{v}_l^m.$$

Remark 12. One can remark that $\mathcal{H}^s_\alpha(S) \hookrightarrow H^s(S)$ for any s, where $H^s(S)$ is the classical Sobolev space defined by

$$H^s(S) = \left\{ u \in \mathcal{D}'(S);\ \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1)^{2s} |u_l^m|^2. < +\infty \right\}.$$

The definition of $\mathcal{H}_s(S)$ given above is the only one we have and no other characterization is available for the moment.

The remaining of this section is devoted to the proof of the following result

Theorem 6. *For any function* g *in* $\mathcal{H}^0_\alpha(S)$ *verifying (102) and any function* ρ *in* $\mathcal{H}^{-1}_\alpha(S)$*, the problem*

$$\mathbf{curl}\,\mathbf{B} = \alpha \mathbf{B},\ \operatorname{div}\mathbf{B} = 0 \text{ in } \Omega^e,\ \ \mathbf{B}.\mathbf{e}_r = \mathrm{g} \text{ and } \frac{\partial B_r}{\partial r} = \rho \text{ at } |\mathbf{x}| = 1, \tag{114}$$

admits one and only one solution $\mathbf{B} \in W_1(\Omega^e)^3$*. Furthermore* $\mathbf{B}$ *belongs to* $W_k(\Omega^e)^3$ *for any* $k > \dfrac{1}{2}$ *and verifies the estimates:*

$$\|\frac{\mathbf{B}}{r^k}\|_{0,\Omega^e} \leq c_{k,\alpha}(\|\mathrm{g}\|_{\mathcal{H}^0_\alpha(S)} + \|\rho\|_{\mathcal{H}^{-1}_\alpha(S)}), \tag{115}$$

$$\|\frac{\nabla\mathbf{B}}{r^k}\|_{0,\Omega^e} \leq C_{k,\alpha}(\|\mathrm{g}\|_{\mathcal{H}^0_\alpha(S)} + \|\rho\|_{\mathcal{H}^{-1}_\alpha(S)}). \tag{116}$$

Proof. The additional condition (113) is equivalent to

$$h_l^m(1)' = \rho_l^m + \mathrm{g}_l^m, \quad \text{for } l \geq 0 \text{ and } m \in \{0, ..., l\}, \tag{117}$$

where $\rho_l^m = (\rho, Y_l^m)_S$. Thus, for any $l \geq 0$ and $m \in \{0, ..., l\}$, λ_l^m and μ_l^m are solutions of the system:

$$\begin{cases} \lambda_l^m h_l(\alpha) + \mu_l^m \overline{h}_l(\alpha) = \mathrm{g}_l^m, \\ \alpha\lambda_l^m h_l'(\alpha) + \alpha\mu_l^m \overline{h}'_l(\alpha) = \mathrm{g}_l^m + \rho_l^m. \end{cases} \tag{118}$$

This is a non-singular linear square system whose determinant is:

$$\alpha\left(h_l(\alpha)\overline{h}_l'(\alpha) - \overline{h}_l(\alpha)h_l'(\alpha)\right) = -\frac{2i}{\alpha},$$

where the two following properties of Hankel functions were used:

$$h_l'(r) = \frac{l}{r}h_l(r) - h_{l+1}(r), \quad \mathrm{Im}(h_{l+1}(r)\overline{h_l}(r)) = -\frac{1}{r^2}.$$

Thus, λ_l^m and μ_l^m are given by

$$\lambda_l^m = i\frac{\alpha}{2}\left((\alpha\overline{h}'_l(\alpha) - \overline{h}_l(\alpha))\mathrm{g}_l^m - \overline{h}_l(\alpha)\rho_l^m\right), \tag{119}$$

$$\mu_l^m = -i\frac{\alpha}{2}\left((\alpha h_l'(\alpha) - h_l(\alpha))g_l^m - h_l(\alpha)\rho_l^m\right). \tag{120}$$

Let us now prove estimates (115) and (116) and convergence of the sum (106). First, given two integers $l \geq 0$ and $m \in \{-l, l\}$, we set

$$\mathbf{b}_{l,m} = \frac{1}{2}(\mathbf{b}_l^m + \mathbf{b}_l^{-m}).$$

$\mathbf{b}_{l,m}$ is a real vector field verifying $\mathbf{curl}\,\mathbf{b} = \alpha\mathbf{b}$ (the indexes l and m will be dropped for simplicity). Let $R > 1$ and k be two real numbers and Ω_R the domain defined by

$$\Omega_R = \{\mathbf{r} \in \mathbb{R}^3, 1 < |\mathbf{x}| < R\}.$$

Multiplying the equation

$$\mathbf{curl}\,\mathbf{b} \times \mathbf{b} = \mathbf{b}.\nabla\mathbf{b} - \nabla\frac{|\mathbf{b}|^2}{2} = \mathbf{0},$$

by $\frac{\mathbf{r}}{r^{2k}}$ and integrating by parts over Ω_R, one gets easily after few calculus

$$2(2k-1)\int_{\Omega_R} \frac{|\mathbf{b}|^2}{r^{2k}} d\Omega - 8k\int_{\Omega_R} \frac{|\mathbf{b}.\mathbf{e}_r|^2}{r^{2k}} d\Omega = F_k(R) - F_k(1), \tag{121}$$

where the function F_k is given by

$$F_k(r) = \frac{2}{r^{2k-3}} \int_S [2|\mathbf{b}.\mathbf{e}_r|^2 - |\mathbf{b}|^2](\sigma, r)d\sigma.$$

At $r = 1$, one has by (108) and boundary conditions

$$F_k(1) = [l^2(l+1)^2 - \alpha^2 l(l+1)]|g_l^m|^2 - l(l+1)|2\mathrm{g}_l^m + \rho_l^m|^2,$$

while at infinity the function $F_k(r)$ looks like

$$F_k(r) \sim -2l(l+1)\frac{|\lambda_l^m|^2 + |\mu_l^m|^2}{r^{2k-1}}\,. \tag{122}$$

Indeed, orthogonality relation (108) gives

$$(\mathbf{b}, \mathbf{b})_S = \frac{1}{2}|\mathbf{b}_l^m|^2_{L^2(S)}.$$

Then, (122) stems from (111) and (112). Now, taking the limit in (121) when R tends to infinity with $k = \frac{1}{2}$, we obtain

$$\begin{aligned} 4\int_{\Omega^e} \frac{|\mathbf{b}.\mathbf{e}_r|^2}{r} d\Omega \;&=\; 2l(l+1)(|\lambda_l^m|^2 + |\mu_l^m|^2) \\ &\quad + [l^2(l+1)^2 - \alpha^2 l(l+1)]|g_l^m|^2 - l(l+1)|2\mathrm{g}_l^m + \rho_l^m|^2. \end{aligned} \tag{123}$$

But,

$$|\lambda_l^m|^2 + |\mu_l^m|^2 \le C_\alpha |h_l(\alpha)|^2 \left(|z_l(\alpha) - 1|^2 |\mathrm{g}_l^m|^2 + |\rho_l^m|^2\right),$$

where C_α is a constant not depending on l neither on m and $z_l(r)$ is the function defined by

$$z_l(r) = r\frac{h_l'(r)}{h_l(r)}.$$

It is well-known that the function z_l verifies the estimate (see, *e.g.*, [40]):

$$|z_l(r)|^2 \le r^2 + (l+1)^2.$$

Hence,

$$|\lambda_l^m|^2 + |\mu_l^m|^2 \le c_\alpha |h_l(\alpha)|^2 \left[(l+1)^2 |\mathrm{g}_k^m|^2 + |\rho_l^m|^2\right],$$

Thus

$$2\int_{\Omega^e}\frac{|\mathbf{b}.\mathbf{e}_r|^2}{r}d\Omega \leq C_\alpha|h_l(\alpha)|^2\left((l+1)^4|\mathrm{g}_l^m|^2+(l+1)^2|\rho_l^m|^2\right). \tag{124}$$

Finally, taking the limit when R tends to infinity in (121) with $k > \frac{1}{2}$ yields

$$\begin{aligned}\int_{\Omega^e}\frac{|\mathbf{b}|^2}{r^{2k}}d\Omega &= \frac{4k}{2k-1}\int_{\Omega^e}\frac{|\mathbf{b}.\mathbf{e}_r|^2}{r^{2k}}d\Omega - \frac{1}{2k-1}F_k(1), \\ &\leq C_{\alpha,k}|h_l(\alpha)|^2\left((l+1)^4|\mathrm{g}_l^m|^2+(l+1)^2|\rho_l^m|^2\right).\end{aligned} \tag{125}$$

Thus, if g belongs to $\mathcal{H}^0_\alpha(S)$ and ρ belongs $\mathcal{H}^{-1}_\alpha(S)$, then the sum (106) converges and one has the estimate:

$$\|\frac{\mathbf{B}}{r^k}\|_{0,\Omega^e} \leq C_{k,\alpha}(\|\mathrm{g}\|_{\mathcal{H}^0_\alpha(S)} + \|\rho\|_{\mathcal{H}^{-1}_\alpha(S)}).$$

The estimate (116) is obtained by multiplying Helmoltz's equation

$$\Delta\mathbf{B} + \alpha^2\mathbf{B} = \mathbf{0},$$

by $\frac{\mathbf{B}}{r^{2k}}$ and integrating over Ω_R, before making R tending to infinity. □

Appendix

Proof of Lemma 12

(i) First, for any $\mathbf{j} \in X$, $K\mathbf{j}$ belongs to V and

$$\|K\mathbf{j}\|_{0,\Omega} \leq \frac{1}{\alpha_0}\|K\mathbf{j}\|_V = \frac{1}{\alpha_0}\|\mathbf{curl}\,(K\mathbf{j})\|_{0,\Omega} = \frac{1}{\alpha_0}\|\mathbf{j}\|_{0,\Omega}.$$

Hence, $\|K\| \leq \frac{1}{\alpha_0}$. Now, let $\tilde{\lambda}_2 > 0$ be the second eigenvalue of the Neumann problem

$$-\Delta u = \lambda u,\ \frac{\partial u}{\partial n} = 0.$$

Let u_2 be solution of this eigenvalue problem with $\tilde{\lambda} = \tilde{\lambda}_2$ and set $\mathbf{v}_2 = \nabla u_2$. Then $\mathbf{v}_2$ belongs to V and

$$\frac{\|\mathbf{v}_2\|^2_V}{\|\mathbf{v}_2\|^2_{0,\Omega}} = \frac{\|\Delta u_2\|^2_{0,\Omega}}{\|\nabla u_2\|^2_{0,\Omega}} = \lambda_2.$$

Hence, $\alpha_0^2 \leq \tilde{\lambda}_2$ and we get (29). Similarly, one gets (30).

(ii) Suppose that Γ is connected and let Φ be a function in $H^1_0(\Omega)^3$. Φ can be decomposed into the sum

$$\Phi = \nabla s + \Phi_1,$$

where $s \in H^1_0(\Omega)$ is solution of the Laplace equation $\Delta s = \operatorname{div}\Phi$, and Φ_1 is divergence-free and verifies $\Phi_1 \times \mathbf{n} = 0$ on Γ. It is clear that

$$\Phi_1 = K^*(\mathbf{curl}\,\Phi) \quad \text{and} \quad \|\nabla s\|^2_{0,\Omega} \leq \frac{1}{\lambda_1}\|\operatorname{div}\Phi\|^2_{0,\Omega}.$$

Hence,

$$\begin{aligned}\|\Phi\|_{0,\Omega}^2 &\leq \frac{1}{\lambda_1}\|\operatorname{div}\Phi\|_{0,\Omega}^2 + \|K^*\|^2\|\mathbf{curl}\,\Phi\|_{0,\Omega}^2 \\ &\leq \frac{1}{\lambda_1}(\|\operatorname{div}\Phi\|_{0,\Omega}^2 + \|\mathbf{curl}\,\Phi\|_{0,\Omega}^2) + (\|K^*\|^2 - \frac{1}{\lambda_1})\|\mathbf{curl}\,\Phi\|_{0,\Omega}^2.\end{aligned}$$

But, since $\Phi \in H_0^1(\Omega)^3$, one has

$$\|\operatorname{div}\Phi\|_{0,\Omega}^2 + \|\mathbf{curl}\,\Phi\|_{0,\Omega}^2 = \|\nabla\Phi\|_{0,\Omega}^2.$$

Thus,

$$\|\Phi\|_{0,\Omega}^2 \leq \frac{1}{\lambda_1}\|\nabla\Phi\|_{0,\Omega}^2 + (\|K^*\|^2 - \frac{1}{\lambda_1})\|\mathbf{curl}\,\Phi\|_{0,\Omega}^2. \tag{126}$$

Now, let ω_1 be a function in $H_0^1(\Omega)$ such that $\Delta\omega_1 = \lambda_1\omega_1$. There exists a constant vector $\mathbf{a}$ such that $\|\nabla\omega_1 \times \mathbf{a}\|_{0,\Omega} \neq 0$. We take $\Phi = \omega_1\mathbf{a}$. Then (126) becomes

$$(\|K^*\|^2 - \frac{1}{\lambda_1})\|\nabla\omega_1 \times \mathbf{a}\|_{0,\Omega}^2 \geq 0.$$

Hence

$$\|K^*\|^2 \geq \frac{1}{\lambda_1}. \tag{127}$$

Now, let $\mathbf{u} \in U$. Then $\mathbf{u}$ can be written into the form

$$\mathbf{u} = \nabla s + \mathbf{u}_1,$$

where $s \in H_0^1(\Omega)$ verifies $\Delta s = \operatorname{div}\mathbf{u}$. $\mathbf{u}_1$ satisfies

$$\operatorname{div}\mathbf{u}_1 = 0, \quad \mathbf{u}_1 \times \mathbf{n} = 0 \text{ on } \Gamma, \quad \langle \mathbf{u}_1.\mathbf{n}, 1\rangle_\Gamma = 0 \text{ (by Green's formula)}.$$

That means that $\mathbf{u}_1 = K^*(\mathbf{curl}\,\mathbf{u})$ since Γ is connected. Hence, for any $\mathbf{u}$ in U, we have

$$\begin{aligned}\|\mathbf{u}\|_{0,\Omega}^2 &= \|\nabla s\|_{0,\Omega}^2 + \|K^*(\mathbf{curl}\,\mathbf{u})\|_{0,\Omega}^2, \\ &\leq \frac{1}{\lambda_1}\|\operatorname{div}\mathbf{u}\|_{0,\Omega}^2 + \|K^*\|^2\|\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}^2, \\ &\leq \|K^*\|^2(\|\operatorname{div}\mathbf{u}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{u}\|_{0,\Omega}^2) \text{ [by (127)]}.\end{aligned}$$

Thus, necessarily α_1 verifies

$$\alpha_1 \geq \frac{1}{\|K\|}.$$

This inequality combined with (127) imply (31).

(iii) Assume now Ω to be simply-connected. Then any vector field $\mathbf{v}$ in V can be written into the form

$$\mathbf{v} = \nabla s + K(\mathbf{curl}\,\mathbf{v}),$$

where $s \in H^1(\Omega)/\mathbb{R}$ is solution of the Neumann problem $\Delta s = \operatorname{div}\mathbf{v}$, $\dfrac{\partial s}{\partial n} = 0$. It results that

$$\|\mathbf{v}\|_{0,\Omega}^2 = \|\nabla s\|_{0,\Omega}^2 + \|K(\mathbf{curl}\,\mathbf{v})\|_{0,\Omega}^2. \tag{128}$$

It is well-known that $C_0 = \frac{1}{\tilde{\lambda}_2}$ is the best constant in the Poincaré-Wirtinger inequality

$$\|u - \frac{1}{|\Omega|}\int_\Omega u(\mathbf{x})d\mathbf{x}\|_{0,\Omega}^2 \leq C_0\|\nabla u\|_{0,\Omega}^2, \quad \forall u \in H^1(\Omega). \tag{129}$$

Using this inequality, one can proves easily the estimate $\|\nabla s\|_{0,\Omega}^2 \leq \frac{1}{\tilde{\lambda}_2}\|\mathrm{div}\,\mathbf{v}\|_{0,\Omega}^2$. Substituting in (128) yields

$$\|\mathbf{v}\|_{0,\Omega}^2 \leq \frac{1}{\tilde{\lambda}_2}\|\mathrm{div}\,\mathbf{v}\|_{0,\Omega}^2 + \|K\|^2\|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2 \leq \sup(\frac{1}{\tilde{\lambda}_2}, \|K\|)\|\mathbf{v}\|_V^2.$$

Hence, $\alpha_0 \geq \inf(\|K\|^{-1}, \tilde{\lambda}_2)$ and (32) holds.

The proof of estimate (33) is dropped here for similicity (the reader interesting in that proof can consult reference [13], Chapter VI).

The authors wish to thank the unknown referee for helpful comments and suggestions about bibliography.

References

[1] S. Agmon, On the eigenfuntions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15** (1962) 119–147.

[2] C.E. Allissandrakis, On the computation of constant α force-free fields. *Astron. Astrophys.* **100** (1981) 197–200.

[3] J.J. Aly, A model for magnetic storage and Taylor's relaxation in the solar corona: I. Helicity-constrained minimum energy state in a half-cylinder. *Phys. Fluids B* **5** (1993) 151–163.

[4] J.J. Aly, On some properties of force-free magnetic fields in infinite regions of space. *Astrophys. J.* **283** (1984) 349–362.

[5] T. Amari, J.J. Aly, J.F. Luciani, T.Z. Boulmezaoud and Z. Mikic, Reconstructing the solar coronal magnetic field as a force free magnetic field. *Solar Physics* **174** (1997) 129–149.

[6] P.R. Baldwin and G.M. Townsend, Complex Trkalian fields and solutions to Euler's Equations for ideal fluid. *Phys. Rev. E* **51** (1995) 2059–2068.

[7] F. Beltrami, Considerazioni idrodinamiche. *Rend. Reale Ist. Lombardo* **22** (1889) 121–130.

[8] M.A. Berger and G.B. Field, The topological properties of magnetic helicity. *J. Fluid Mech.* **147** (1984) 133–148.

[9] O. Bjorgum, *On Beltrami vector fields and flows, Part I*, Univ. Bergen Årbok, Naturv. rekke. No. 1 (1951).

[10] O. Bjorgum, *On Beltrami vector fields and flows, Part II*, Univ. Bergen Årbok, Naturv. rekke. No. 13 (1952).

[11] T.Z. Boulmezaoud, On the linear force-free fields in a bounded domain and in a half-cylinder of R^3. *C. R. Acad. Sci.* **325** (1997) 1235–1240.

[12] T.Z. Boulmezaoud, Y. Maday and T. Amari, On the existence of non-linear force-free fields in a bounded domain. *SIAM J. Appl. Math.* (submitted).

[13] T.Z. Boulmezaoud, *Étude des champs de Beltrami dans des domaines de $\mathbb{R}^3$ bornés et non-bornés et applications en astrophysique.* Ph.D. thesis, Laboratoire d'Analyse Numérique, University of Pierre et Marie Curie, France (1998).

[14] S. Chandrasekhar and P.C. Kendal, On Force-free magnetic fields. *Astrophys. J.* **126** (1957) 457–460.

[15] P. Constantin and A. Majda, The Beltrami Spectrum for incompressible fluid flows. *Comm. Math. Phys.* **115** (1988) 435–456.

[16] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique*, Tome 5. Masson, Paris (1988).

[17] P. Démoulin, E.R. Priest and U. Anzer, A three-dimensional model for solar prominences. *Astron. Astrophys.* **221** (1989) 326–337.

[18] J.-M. Dominguez, *Étude des équations de la Magnétohydrodynamique stationnaires et leur approximation par élements finis*, thèse de Docteur Ingénieur. University of Pierre et Marie Curie, France (1982).

[19] D. Dritshel, Generalized helical Beltrami flows in hydrodynamics and magnetohydrodynamics. *J. Fluid Mech.* **222** (1991) 525–541.

[20] G. Duvaut and J.-L. Lions, *Les inéquations en mécanique et en physique.* Dunod, Paris (1972).

[21] C.J. Durant, Linear Force-free fields and coronal models. *Aust. J. Phys.* **42** (1989) 317–329.

[22] C. Foias and R. Temam, Remarques sur les équations de Navier-Stockes et phénomènes successifs de bifurcation. *Ann. Sc. Norm. Sup. Pisa. S. 4* **5** (1978) 29–63.

[23] K.O. Friedrichs, Differential forms on Reimannian manifolds. *Comm. Pure Appl. Math.* **8** (1955) 551–590.

[24] H.P. Furth, S.C. Jardin and D.B. Montgomery, Force-free Coil principles Applied to High-Temperature Superconducting Materials. *IEEE Trans. Magnet.* **24** (1998) 1467–1468.

[25] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations.* Springer-Verlag (1986).
[26] J. Gobert, Sur une inégalité de coercivité. *J. Math. Anal. Appl.* **36** (1971) 518–528.
[27] H. Grad and H. Rubin, Hydromagnetic equilibria and force force-free fields, in *Proc. 2nd Intern. Conf. on Peaceful Uses of Atomic Energy*, Geneva, United Nations **31** (1958) 190–197.
[28] N.M. Günter, *Potential Theory.* Ungar Ed. (1967).
[29] T.H. Jensen and M.S. Chu, Current drive and helicity injection, *Phys. Fluids* **27** (1994) 2881–2885.
[30] R. Kress, Ein Neumannsches Randwertproblem bei kraftfreien Feldern. *Meth. Verf. Math. Phys.* **7** (1972) 81–97.
[31] R. Kress, The treatment of a Neumann boundary value problem for force-free fields by an integral equation method. *Proceedings of the Royal Society of Edimburgh* **82** (1978) 71–86.
[32] R. Kress, A remark on a boundary value problem for force-free fields. *J. Appl. Math. Phys.* **28** (1977) 715–722.
[33] P. Laurence and M. Avellaneda, On Woltjer's variational principle for force-free fields. *J. Math. Phys.* **32** (1991) 1240–1253.
[34] F. Marqués, On boundary conditions for velocity potentials in confined flows, application to couette flow. *Phys. Fluids A* **2** (1990) 729–737.
[35] G.E. Marsh, Magnetic energy, multiply connected domains and force-free fields. *Phys. Rev. A* **46** (1992) 2117–2123.
[36] G.E. Marsh, Helicity, topology and Force-free fields. *Phys. Rev. E.* **47** (1992) 3607–3611.
[37] H.K. Moffatt, The degree of knottedness of tangled vortes lines. *J. Fluid Mech.* **35** (1969) 117–129.
[38] D. McLaughlin and O. Pironneau, Some notes on periodic Beltrami fields in Cartesian geometry. *J. Math. Phys.* **32** (1991) 797–804.
[39] H.E. Moses, Eigenfunctions of the curl operator, rotationally invariant Helmoltz theorem, and applications to electromagnetic theory and fluid mechanics. *SIAM J. Appl. Math.* **21** (1971) 114–144.
[40] C. Nedelec, *Équations intégrales.* Cours de D.E.A. d'Analyse Numérique, University of Pierre et Marie Curie, France (1995).
[41] R. Picard, Ein Randwertproblem in der Theorie kraftfreier Magnetfelfer. *Zeit. Angew. Math. Phys.* **27** (1976) 196–180.
[42] E.R. Priest, *Solar Magnetohydrodynamics.* Reidel, Dordrecht (1982).
[43] T. Sakurai, Computational modeling of magnetic fields in solar active regions. *Space Sci. Rev.* **51** (1989) 11–48.
[44] N. Seehafer, Determination of constant α force-free fields from magnetograph data. *Solar Phys.* **58** (1978) 215–223.
[45] V.D. Shafranov, On magnetohydrodynamical Equilibrium configurations. *Soviet. Phys. JETP* **6** (1958) 545–554.
[46] J.B. Taylor, Relaxation of toroidal plasma and generation of reverse magnetic fields. *Phys. Rev. Lett.* **33** (1974) 1139–1141.
[47] J.B. Taylor, Relaxation of magnetic reconnection in plasmas. *Rev. Mod. Phys.* **58** (1986) 741–763.
[48] Z. Yoshida and Y. Giga, Remarks on spectra of operator Rot. *Math. Z.* **204** (1990) 235–245.
[49] C.Y. Wang, Exact solutions of the steady-state Navier-Stokes equations. *Annu. Rev. Fluid Mech.* **23** (1991) 159–177.
[50] L. Woltjer, *Proc. Nat. Acad. Sci.* **44** (1958) 489.