M2AN - Modélisation mathématique et analyse numérique

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M2AN - Modélisation mathématique et analyse numérique, tome 32, n° 7 (1998), p. 897-915

<http://www.numdam.org/item?id=M2AN_1998__32_7_897_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 32, n° 7, 1998, p. 897 à 915)

ON PROPAGATION IN AN ELECTROMAGNETIC WAVEGUIDE WITH CONCENTRATED DISSIPATION (*)

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Abstract. — *We are interested in the propagation of an electromagnetic wave in a closed waveguide containing a dissipative wire, the diameter of which is of order $O(\eta)$ and conductivity $O(\eta^{-m})$ $(m>0)$. We study the perturbation to the propagation brought by the presence of the wire. In high frequencies, existence of a global and local TEM wave is established. © Elsevier, Paris*

Résumé. — *Nous nous intéressons à la propagation d'une onde électromagnétique dans un guide d'onde fermé, parcouru par un fil dissipatif de diamètre $O(\eta)$ et de conductivité $O(\eta^{-m})$ $(m>0)$. Nous étudions la perturbation à la propagation apportée par la présence du fil. En hautes fréquences, l'existence d'une onde TEM, globale et locale, est mise en évidence. © Elsevier, Paris*

INTRODUCTION

By their various applications, electromagnetic waves have been already plentifully studied by both physicians and mathematicians. This paper is devoted to a problem of applied mathematics related to a closed waveguide. In the domain of numerical analysis, the references of [5] give a lot of studies concerning electromagnetic open waveguides.

Dispersion relation and guided modes are easily determined by computational methods. Now Nédelec's elements [7] are of classical use in electromagnetism. But when the size of one or several waveguide constituants is small, compared to the diameter of the section, computational time can become long. Then it can be useful to develop theoretical methods of approximation (asymptotic methods, homogenization...).

We consider a waveguide enclosing dissipative small wires. In this work only one of them is present. Its diameter is $O(\eta)$ and the conductivity is large, of order η^{-m} $(m>0)$. We study some different properties of the propagation that arise, depending on the parameter m. In the low frequencies case, for $m<2$, the dissipation appears to be weak and the propagation is a $O(\eta^{2-m})$ perturbation of the non dissipative case. If $m \geqslant 2$, the guided modes of a global propagation are of order $O(1)$ except in a region near the concentrated dissipative part where they are small, of order $O(1/\ln \eta)$. In the high frequencies case, for $m>1$, a global and local propagation of a TEM wave can occur.

In sections 1 and 2, problem, notations and general considerations are presented. Sections 3 and 4 are devoted to the study of the propagation in the low frequencies case, for $m<2$ and $m \geqslant 2$. The high frequencies is approached in the last sections, for $m>1$.

1. STATEMENT OF THE PROBLEM

1.1. The equations

Let us consider the electromagnetic cylindrical waveguide $\Omega \times \mathbf{R}$. The bounded section Ω, open domain of $\mathbf{R}^2$ is divided into two parts: ηD and $\Omega \backslash \eta \bar{D}$. The set ηD, homothetic of D with ratio η, is a connected domain with a *regular boundary* $\eta \Gamma$. ηD is the section of a dissipative dielectric medium and $\Omega \backslash \eta \bar{D}$ of a non dissipative one. We suppose that ηD contains the origin, as shown on *figure 1*

(*) Manuscript received July 7, 1997. Accepted November 9, 1997.
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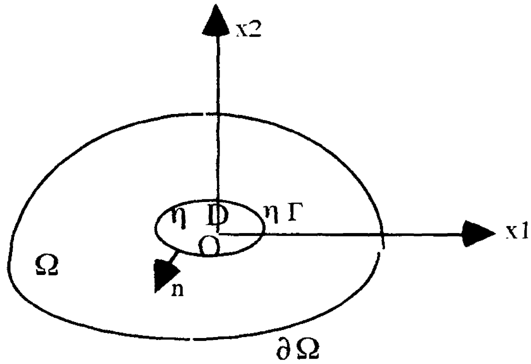

Figure 1

On the boundary $\partial\Omega$, the waveguide is in contact with a perfect infinite conductor.

We denote by (x, x_3), with $x = (x_1, x_2)$ in $\mathbf{R}^2$, the generic point of $\mathbf{R}^3$. The propagation in the guide of the electric field $\vec{E}(x, t)\, e^{-i\beta x_3}$ and the magnetic field $\vec{H}(x, t)\, e^{-i\beta x_3}$, with β wave number in the x_3 direction, β *real positive given parameter,* is governed by Maxwell's equations ([3], [4], [8]):

$$\varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} - \mathrm{Rot}_\beta \vec{H} = \vec{0} \qquad \text{in } \Omega \tag{1.1}$$

$$\mu \frac{\partial \vec{H}}{\partial t} + \mathrm{Rot}_\beta \vec{E} = \vec{0} \tag{1.2}$$

$$\mathrm{Div}_\beta (\varepsilon \vec{E}) = 0 \tag{1.3}$$

$$\mathrm{Div}_\beta (\mu \vec{H}) = 0 \tag{1.4}$$

$$\vec{E} \wedge \vec{n} = \vec{0} \quad \vec{H} \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega \tag{1.5}$$

with the initial conditions (1.6), that satisfy the boundary conditions (1.5):

$$\vec{E}(x, 0) = \vec{E}_0(x) \qquad \vec{H}(x, 0) = \vec{H}_0(x)\,. \tag{1.6}$$

In these equations, ε is the dielectric constant of the medium, σ the conductivity, μ the magnetic permeability. We assume that ε, σ, μ are *real, positive constants* and

$$\sigma = 0 \quad \text{in} \quad \Omega \backslash \eta \bar{D}\,.$$

In (1.1) - (1.4), operators Rot_β and Div_β are deduced from the classical rotational and divergence operators:

$$\mathrm{Rot}_\beta\, \vec{u} = \begin{pmatrix} u_{3,2} + i\beta u_2 \\ -u_{3,1} - i\beta u_1 \\ u_{2,1} - u_{1,2} \end{pmatrix} \qquad \mathrm{Div}_\beta\, \vec{u} = u_{1,1} + u_{2,2} - i\beta u_3 \tag{1.7}$$

with the notation: $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.

We must mention that equations (1.3), (1.4) result from the particular initial data:

$$\mathrm{Div}_\beta(\varepsilon\vec{E}_0) = 0 \qquad \mathrm{Div}_\beta(\mu\vec{H}_0) = 0 .$$

Equations (1.1)-(1.4) are to be considered in the sense of distributions on Ω and involve the usual transmission conditions on the boundary $\eta\Gamma$ of the domain ηD :

$$[\vec{E}\wedge\vec{n}] = \vec{0} \quad [\vec{H}\wedge\vec{n}] = \vec{0} \quad [\varepsilon\vec{E}\cdot\vec{n}] = 0 \quad [\mu\vec{H}\cdot\vec{n}] = 0 \tag{1.8}$$

where the brackets denote the jump of the enclosed quantities and $\vec{n}$ the unit outer normal vector to $\eta\Gamma$.

1.2. Existence of an unique solution

Problem (1.1)-(1.6) can be formulated with one unknown function only, $\vec{E} = \vec{E}(x, t)$ for instance:

$$\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \sigma\frac{\partial\vec{E}}{\partial t} + \mathrm{Rot}_\beta(\mu^{-1}\,\mathrm{Rot}_\beta\,\vec{E}) = \vec{0} \qquad \text{in } \Omega \tag{1.9}$$

$$\mathrm{Div}_\beta(\varepsilon\vec{E}) = 0 \tag{1.10}$$

$$\vec{E}\wedge\vec{n} = \vec{0} \qquad \text{on } \partial\Omega \tag{1.11}$$

and

$$\vec{E}(x,0) = \vec{E}_0(x) \qquad \frac{\partial\vec{E}}{\partial t}(x,0) = \vec{E}_1(x) \tag{1.12}$$

with: $\quad \vec{E}_1(x) = -\sigma\varepsilon^{-1}\,\vec{E}_0(x) + \varepsilon^{-1}\,\mathrm{Rot}_\beta\,\vec{H}_0(x)$.

Existence of a solution of such a problem is obtained in a suitable framework.

Let us denote by $H_\varepsilon(\Omega)$, or shortly H_ε, the space $L^2(\Omega)^3$ equipped with the inner product with weight ε :

$$(\vec{u}, \vec{v}) = \int_\Omega \varepsilon u_i \bar{v}_i \, dx$$

(the superimposed bar indicates complex conjugacy).

Let us define the space $H(\mathrm{Rot}_\beta, \Omega)$, or shortly $H(\mathrm{Rot}_\beta\,)$:

$$H(\mathrm{Rot}_\beta\,) = \{\vec{u}, \vec{u}\in L^2(\Omega)^3, \mathrm{Rot}_\beta \in L^2(\Omega)^3\} .$$

For $\vec{u}\in L^2(\Omega)^3$, we introduce the following notations:

$$\vec{u} = (u, u_3), \quad u\in L^2(\Omega)^2, \quad \text{and} \quad \mathrm{rot}\, u = u_{2,1} - u_{1,2} .$$

It results from definition (1.7) that $H(\mathrm{Rot}_\beta\,)$ is isomorphic to $H(\mathrm{rot}, \Omega)\times H^1(\Omega)$ with: $H(\mathrm{rot}, \Omega) = \{u, u\in L^2(\Omega)^2, \mathrm{rot}\, u\in L^2(\Omega)\}$.

On the space $H(\mathrm{Rot}_\beta)$ can be defined the tangential trace ([9]):

$$H(Rot_\beta) \to H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

$$\vec{u} \to (u_3|_{\partial\Omega}, (u_1 n_2 - u_2 n_1)|_{\partial\Omega})$$

that leads to a Green's formula and makes sense to $\vec{u} \wedge \vec{n}$ on $\eta\Gamma$ and $\partial\Omega$. The space of the vectors of $H(\mathrm{Rot}_\beta)$ which have a null trace on $\partial\Omega$ is denoted by $H_0(\mathrm{Rot}_\beta)$.

Finally, let us define the space H^ε_β:

$$H^\varepsilon_\beta = \{\vec{u} \in H_\varepsilon, \mathrm{Div}_\beta(\varepsilon\vec{u}) = 0\}.$$

In this functional framework, we give a weak formulation of problem (1.9)-(1.12).

Let A^η_β be the unbounded operator on H_ε defined by:

$$\begin{cases} A^\eta_\beta : D(A^\eta_\beta) \subset H^\varepsilon_\beta \to H^\varepsilon_\beta \\ D(A^\eta_\beta) = \{\vec{u}, \vec{u} \in H_0(\mathrm{Rot}_\beta), \mathrm{Rot}_\beta(\mu^{-1}\mathrm{Rot}_\beta \vec{u}) \in H_\varepsilon, \mathrm{Div}_\beta(\varepsilon\vec{u}) = 0\} \\ A^\eta_\beta(\vec{u}) = \varepsilon^{-1}\mathrm{Rot}_\beta(\mu^{-1}\mathrm{Rot}_\beta \vec{u}). \end{cases} \tag{1.13}$$

Let B^η be the bounded operator on H_ε defined by: $B^\eta\vec{u} = \sigma\varepsilon^{-1}\vec{u}$.

Then, problem (1.9)-(1.12) can be reformulated: find $\vec{E}$, function of t, with values in H^ε_β, such that:

$$\frac{d^2\vec{E}}{dt^2} + B^\eta \frac{d\vec{E}}{dt} + A^\eta_\beta \vec{E} = \vec{0} \tag{1.14}$$

$$\vec{E}(0) = \vec{E}_0 \qquad \frac{d\vec{E}}{dt}(0) = \vec{E}_1. \tag{1.15}$$

The hermitian form associated with the operator A^η_β is coercive on $H_0(\mathrm{Rot}_\beta) \cap H^\varepsilon_\beta$ [9].

Let us denote by $V_{0\beta}$ this space:

$$V_{0\beta} = H_0(\mathrm{Rot}_\beta) \cap H^\varepsilon_\beta.$$

We are in a standard situation ([10]) and owing to the semigroup theory, we prove:

THEOREM 1.1: *If* $\vec{E}_0 \in V_{0\beta}$, $\vec{E}_1 \in H^\varepsilon_\beta$, *then system (1.14), (1.15) has an unique solution* $\left(\vec{E}^\eta(t), \frac{d\vec{E}^\eta}{dt}(t)\right) \in V_{0\beta} \times H^\varepsilon_\beta$.

1.3. First considerations about eigenvalues

Looking for a solution of (1.14) such that:

$$\vec{E}^\eta = e^{-\lambda^\eta t}\vec{v}^\eta, \quad \vec{v}^\eta \in V_{0\beta}, \quad \vec{v}^\eta \neq \vec{0} \tag{1.16}$$

we must solve the following equation

$$\lambda^{\eta^2} \vec{v}^\eta - \lambda^\eta B^\eta \vec{v}^\eta + A_\beta^\eta \vec{v}^\eta = \vec{0} . \tag{1.17}$$

The inner product in H_ε of (1.17) by $\vec{v}^\eta$ leads to two real relations. From the imaginary part, we obtain: $2(\mathrm{Re}\,\lambda^\eta)(\vec{v}^\eta, \vec{v}^\eta) = (B^\eta \vec{v}^\eta, \vec{v}^\eta)$. Problem (1.14), (1.15) is related to the description of a physical dissipative phenomena (to be compared to a problem of viscoelasticity with short memory in mechanics).

In the following, η is a small parameter and we assume that the conductivity of the wire $\eta D \times \mathbf{R}$ is very large compared to the diameter of the wire:

$$\sigma = \eta^{-m} \tilde{\sigma}, \qquad m > 0 . \tag{1.18}$$

2. SOME RESULTS ABOUT CONVERGENCE

2.1. Limit of the solution $\vec{E}^\eta$, $\eta \to 0$

Notation: $\dfrac{d\vec{E}}{dt} = \dot{\vec{E}}$.

THEOREM 2.1: *Let $\vec{E}^\eta$ be the solution of the initial value problem (1.14), (1.15) (with (1.18)). Let the initial values $\vec{E}_0 \in V_{0\beta}$, $\vec{E}_1 \in H_\beta^\varepsilon$ be independant of η and let $\vec{E}_1$ vanish in a neighbourhood of $x = 0$. Then*

$$\vec{E}^\eta \to \vec{E}^0 \text{ in } L^\infty(0, +\infty\,; V_{0\beta} \text{ weakly } *), \quad \dot{\vec{E}}^\eta \to \dot{\vec{E}}^0 \text{ in } L^\infty(0, +\infty\,; H_\beta^\varepsilon \text{ weakly } *),$$

where $\vec{E}^0$ is the corresponding solution of the problem without concentrated dissipative inclusion:

$$\varepsilon \frac{\partial^2 \vec{E}^0}{\partial t^2} + \mathrm{Rot}_\beta (\mu^{-1} \mathrm{Rot}_\beta \vec{E}^0) = \vec{0} \qquad \text{in } \Omega \tag{2.1}$$

$$\vec{E}^0 \wedge \vec{n} = \vec{0} \qquad \text{on } \partial\Omega \tag{2.2}$$

$$\vec{E}^0(x, 0) = \vec{E}_0(x), \quad \frac{\partial \vec{E}^0}{\partial t}(x, 0) = \vec{E}_1(x) . \tag{2.3}$$

This result is similar to theorem VII.12.1 of [10] and arguments of the proof are taken from this reference. Two lemmas are needed.

Let us denote by a_β^η and b^η the hermitian forms associated with the operators A_β^η and B^η:

$$\vec{u}, \vec{v} \in H_0(\mathrm{Rot}_\beta), \quad a_\beta^\eta(\vec{u}, \vec{v}) = \int_\Omega \mu^{-1} \mathrm{Rot}_\beta \vec{u} \,.\, \overline{\mathrm{Rot}_\beta \vec{v}}\, dx$$

$$\vec{u}, \vec{v} \in H_\varepsilon, \quad b^\eta(\vec{u}, \vec{v}) = \int_\Omega \sigma \vec{u} \,.\, \bar{\vec{v}}\, dx .$$

LEMMA 2.1 (characterization of the solution $\vec{E}^\eta$).

The solution of (1.14), (1.15) is the unique function $\vec{E}^{\eta}(t)$ which satisfies for any fixed positive T:

$$\vec{E}^{\eta} \in L^{\infty}(0, T; V_{0\beta}), \quad \dot{\vec{E}}^{\eta} \in L^{\infty}(0, T; H^{\varepsilon}_{\beta})$$

$$\vec{E}^{\eta}(0) = \vec{E}_0$$

$$\int_0^T [a^{\eta}_{\beta}(\vec{E}^{\eta}(t), \vec{u})\, \varphi(t) - (\dot{\vec{E}}^{\eta}(t), \vec{u})\, \dot{\varphi}(t) + b^{\eta}(\dot{\vec{E}}^{\eta}(t), \vec{u})\, \varphi(t)]\, dt = (\vec{E}_1, \vec{u})\, \varphi(0)$$

for any $\vec{u}$ in a dense set of $V_{0\beta}$ and any φ in the set:

$$\Phi = \{\varphi, \varphi \in C^1(0, T), \varphi(T) = 0\}.$$

The proof is similar to the one of proposition I.12.1 of [10].

LEMMA 2.2: *Let Ω be an open bounded domain of $\mathbf{R}^2$, containing the origin. The set of the functions of $V_{0\beta}$ vanishing in a neighbourhood of the origin is dense in $H_0(\mathrm{Rot}_\beta)$.*

It suffices to prove that any function $\vec{u} \in D(\Omega)^3$, that fulfils $\mathrm{Div}_\beta(\varepsilon\vec{u}) = 0$, can be approximated, in the norm $H(\mathrm{Rot}_\beta)$, by functions of $V_{0\beta}$ vanishing for sufficiently small $|x|$.

Taking up the ideas of [10], for small positive α, we define the approximation $\vec{u}^{\alpha}$ of the vector $\vec{u}$, the components of which beeing denoted by (u_r, u_θ, u_3) in the cylindric basis.

$$u_r^{\alpha} = \begin{cases} u_r(r, \theta) + \dfrac{b^{\alpha}(\theta)}{r} & \text{for } r > \alpha \\ \dfrac{a^{\alpha}(\theta)}{\ln \alpha/\alpha^2}\left(\ln \dfrac{r}{\alpha^2} - 1 + \dfrac{\alpha^2}{r}\right) & \text{for } \alpha^2 \leqslant r \leqslant \alpha \\ 0 & \text{for } r < \alpha^2 \end{cases}$$

$$u_\theta^{\alpha} = \begin{cases} u_\theta(r, \theta) & \text{for } r > \alpha \\ \dfrac{u_\theta(\alpha, \theta)}{\ln \alpha/\alpha^2} \ln \dfrac{r}{\alpha^2} & \text{for } \alpha^2 \leqslant r \leqslant \alpha \\ 0 & \text{for } r < \alpha^2 \end{cases}$$

$$i\beta u_3^{\alpha} = \begin{cases} \dfrac{\partial u_r(r, \theta)}{\partial r} + \dfrac{1}{r} u_r(r, \theta) + \dfrac{1}{r}\dfrac{\partial u_\theta(r, \theta)}{\partial \theta} & \text{for } r > \alpha \\ \dfrac{1}{\alpha \ln \alpha/\alpha^2}\left[a^{\alpha}(\theta) + \dfrac{\partial u_\theta}{\partial \theta}(\alpha, \theta)\right] \ln \dfrac{r}{\alpha^2} & \text{for } \alpha^2 \leqslant r \leqslant \alpha \\ 0 & \text{for } r < \alpha^2 \end{cases}$$

with

$$a^{\alpha}(\theta) = \alpha \frac{\partial u_r}{\partial r}(\alpha, \theta) + u_r(\alpha, \theta)$$

$$b^{\alpha}(\theta) = \alpha \frac{a^{\alpha}(\theta)}{\ln \alpha/\alpha^2}\left(\ln \frac{\alpha}{\alpha^2} - 1 + \alpha\right) - \alpha u_r(\alpha, \theta).$$

These functions $a^{\alpha}(\theta)$ and $b^{\alpha}(\theta)$ are such that they provide the continuity of u_r^{α}, u_{θ}^{α}, $\operatorname{div} u^{\alpha}$ (and therefore u_3^{α}) for $r = \alpha$ and $r = \alpha^2$.

As $\vec{u} \in D(\Omega)^3 \cap H_{\beta}^{\varepsilon}$, then for small α, we have:

$$|u_{\theta}(\alpha, \theta)| < C\alpha, \quad \left|\frac{\partial u_{\theta}}{\partial \theta}(\alpha, \theta)\right| < C\alpha, \quad |a^{\alpha}(\theta)| < C\alpha$$

$$\left|\frac{da^{\alpha}}{d\theta}(\theta)\right| < C\alpha^2, \quad |b^{\alpha}(\theta)| < C\alpha^2, \quad \left|\frac{db^{\alpha}}{d\theta}(\theta)\right| < C\alpha^3 .$$

These properties lead to: $\vec{u}^{\alpha} \to \vec{u}$ for $\alpha \to 0$ in $H(\mathrm{Rot}_{\beta})$ strong.

Let us come back to the proof of theorem 2.1.

The system is dissipative, so the energy decreases. From (1.14), we have:

$$\frac{1}{2}\frac{d}{dt}[(\dot{\vec{E}}^{\eta}, \dot{\vec{E}}^{\eta}) + a_{\beta}^{\eta}(\vec{E}^{\eta}, \vec{E}^{\eta})] + b^{\eta}(\dot{\vec{E}}^{\eta}, \dot{\vec{E}}^{\eta}) = 0 .$$

It results that:

$$\|\dot{\vec{E}}^{\eta}(t)\|_{H_{\varepsilon}}^2 + a_{\beta}^{\eta}(\vec{E}^{\eta}(t), \vec{E}^{\eta}(t)) \leq \|\vec{E}_1\|_{H_{\varepsilon}}^2 + C a_{\beta}^0(\vec{E}_0, \vec{E}_0) \quad (C \text{ cst.}) \tag{2.4}$$

where a_{β}^0 is the form without dissipative wire. The right-hand side of (2.4) is independant of η, owing to the assumptions made on the initial data. So (2.4) becomes an *a priori* estimate for the solution $\vec{E}^{\eta}$. Then, for a sub-sequence (it will be clear later that is the whole sequence), we have:

$$\vec{E}^{\eta} \to \vec{E}^0 \text{ in } L^{\infty}(0, +\infty\,; V_{0\beta} \text{ weakly } *) \quad \dot{\vec{E}}^{\eta} \to \dot{\vec{E}}^0 \text{ in } L^{\infty}(0, +\infty\,; H_{\beta}^{\varepsilon} \text{ weakly } *) .$$

According to lemma 2.1, with a test function vanishing in a neighbourhood of the origin (lemma 2.2), we have, for sufficiently small η:

$$\int_0^T [a_{\beta}^0(\vec{E}^{\eta}(t), \vec{u})\,\varphi(t) - (\dot{\vec{E}}(t), \vec{u})\,\dot{\varphi}(t)]\,dt = (\vec{E}_1, \vec{u})\,\varphi(0) \quad \forall \varphi \in \Phi . \tag{2.5}$$

We pass to the limit in (2.5), and again using lemma 2.1, we see that $(\vec{E}^0, \dot{\vec{E}}^0)$ is the solution of the problem without concentrated dissipation.

2.2. Considerations about the limit of eigenelements

In the problem without a dissipative part, the coercivity of the form α_{β}^0 on $V_{0\beta}$ leads to the fact that the operator A_{β}^0 has a compact inverse. So, A_{β}^0 possesses a countable infinity of positive eigenvalues $\omega_n^{0^2}$, $n \in \mathbf{N}$, and the associated eigenvectors $\vec{v}_n^0$ form a basis of $V_{0\beta}$, orthonormal in H_{ε}.

Let $\vec{E}^\eta_n(t)$ be the solution of (1.14), (1.15), with $\vec{E}_0 = \vec{v}^0_n$, $\vec{E}_1 = \vec{0}$. With these initial data, the solution of the problem without concentrated dissipation is $\vec{E}^0_n(t) = \cos \omega^0_n t\, \vec{v}^0_n$ (without summation). If we take the Laplace transform $E^\eta_n(t) \to \hat{E}^\eta_n(p)$, according to theorem 2.1, we obtain:

$$a^\eta_\beta(\hat{E}^\eta_n(p), \vec{v}^0_k) \to \frac{p\omega^{0^2}_n}{p^2 + \omega^{0^2}_n}\delta_{nk} \quad \forall k \in \mathbf{N} \tag{2.6}$$

$$(\hat{E}^\eta_n(p), \vec{v}^0_k) \to \frac{p}{p^2 + \omega^{0^2}_n}\delta_{nk} \quad \forall k \in \mathbf{N}\,. \tag{2.7}$$

But the Laplace transform $\hat{E}^\eta_n(p)$ of $E(t)$ satisfies:

$$p^2(\hat{E}^\eta_n, \vec{v}^0_k) + pb^\eta(\hat{E}^\eta_n, \vec{v}^0_k) + a^\eta_\beta(\hat{E}^\eta_n, \vec{v}^0_k) - p(\vec{v}^0_n, \vec{v}^0_k) - b^\eta(\vec{v}^0_n, \vec{v}^0_k) = 0 \quad \forall k$$

From (2.6), (2.7), it results:

$$\lim_{\eta \to 0}\, [pb^\eta(\hat{E}^\eta_n, \vec{v}^0_k) - b^\eta(\vec{v}^0_n, \vec{v}^0_k)] = 0 \quad \forall k\,.$$

An expansion of $\hat{E}^\eta_n(p)$ on the $\vec{v}^0_k$ basis and property (2.7) lead to:

$$\lim_{\eta \to 0} b^\eta(\vec{v}^0_n, \vec{v}^0_k) = 0 \quad \forall n, k \in \mathbf{N}\,. \tag{2.8}$$

Let us introduce $\vec{v}^\eta$, eigenvector of the dissipative problem, $\vec{v}^\eta$ solution of (1.17).

With the test function $\vec{v}^0_n$ in (1.17) and property (2.8), we have:

$$\lim_{\eta \to 0}\, (\lambda^{\eta^2} + \omega^{0^2}_n)\,(\vec{v}^\eta, \vec{v}^0_n) = 0 \quad \forall n \in \mathbf{N}$$

(without sommation on n).

This result involves some kind of global convergence of eigenvalues and eigenvectors.

The propagation in the waveguide depends on the parameter m in (1.18). Two cases arise: low and high frequencies. For low frequencies, if $m < 2$, we have weak dissipation and the propagation is a perturbation of the non dissipative case, of order η^{2-m}; if $m \geqslant 2$, the amplitude of global propagation is very small in ηD. For high frequencies, global and local propagations exist for $m > 1$.

3. LOW FREQUENCIES PROBLEM. WEAK DISSIPATIVE CASE $(m < 2)$

3.1. Asymptotic expansions of the eigenelements

For $m < 2$, the operator B^η introduces a perturbation, which order is η^{2-m}.

We take the eigenvalue problem under the following form: find $\omega^\eta \in \mathbf{C}$, $(\lambda^\eta = -\mathrm{i}\omega^\eta)$, and $\vec{v}^\eta \in V_{0\beta}$, $\vec{v}^\eta \neq \vec{0}$, such that:

$$-\omega^{\eta^2}(\vec{v}^\eta, \vec{w}) + \mathrm{i}\omega^\eta\, b^\eta(\vec{v}^\eta, \vec{w}) + a^\eta_\beta(\vec{v}^\eta, \vec{w}) = 0 \quad \forall \vec{w} \in V_{0\beta}\,. \tag{3.1}$$

The three hermitian forms of (3.1) depend on η in the following way:

$$b^{\eta}(\vec{v}, w) = \eta^{2-m}\, b(\vec{v}, \vec{w}) \tag{3.2}$$

$$(\vec{v}, \vec{w}) = (\vec{v}, \vec{w})^{0} + \eta^{2}\, c(\vec{v}, \vec{w}) \tag{3.3}$$

$$a^{\eta}_{\beta}(\vec{v}, \vec{w}) = a^{0}_{\beta}(\vec{v}, \vec{w}) + \eta^{2}\, a(\vec{v}, \vec{w})\,. \tag{3.4}$$

As in previous sections, the notation ^{0} indicates quantities related to the problem without dissipation. The forms b, c, a of (3.2) - (3.4) are respectively defined by:

$$b(\vec{v}, \vec{w}) = \int_{D} \tilde{\sigma}\vec{v}\,.\,\bar{\vec{w}}\, dy, \quad c(\vec{v}, \vec{w}) = \int_{D} (\varepsilon' - \varepsilon)\, \vec{v}\,.\,\bar{\vec{w}}\, dy$$

$$a(\vec{v}, \vec{w}) = \int_{D} (\mu'^{-1} - \mu^{-1})\, \mathrm{Rot}_{\beta}\, \vec{v}\,.\, \overline{\mathrm{Rot}_{\beta}\, \vec{w}}\, dy$$

ε', μ' (resp. ε, μ) are the values of the coefficients in ηD (resp. $\Omega \backslash \eta \bar{D}$). We search for ω^{η} and $\vec{v}^{\eta}$ through asymptotic expansions of the form:

$$\omega^{\eta} = \omega^{0} + \eta^{2-m}\, \omega^{1} + \cdots \tag{3.5}$$

$$\vec{v}^{\eta} = \vec{v}^{0} + \eta^{2-m} \vec{v}^{1} + \cdots \quad \vec{v}^{0}, \vec{v}^{1} \in V_{0\beta}\,. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.1), we obtain at order η^{0} that $\vec{v}^{0}$ is one of the eigenvector $\vec{v}^{0}_{n}$, $(n \in \mathbf{N})$, of the problem without dissipation associated with the eigenvalue $\omega^{0} = \omega^{0}_{n}$, $\omega^{0}_{n} \in \mathbf{R}$.

At order η^{2-m}, we obtain:

$$-2\, \omega^{0}_{n}\, \omega^{1}(\vec{v}^{0}_{n}, \vec{w}) - \omega^{0^{2}}_{n}(\vec{v}^{1}, \vec{w}) + \mathrm{i}\omega^{0}_{n}\, b(\vec{v}^{0}_{n}, \vec{w}) + a^{0}_{\beta}(\vec{v}^{1}, \vec{w}) = 0 \quad \forall \vec{w} \in V_{0\beta}\,. \tag{3.7}$$

The choice $\vec{w} = \vec{v}^{0}_{n}$ for the test function in (3.7) gives:

$$2\, \omega^{1} = \mathrm{i} b(\vec{v}^{0}_{n}, \vec{v}^{0}_{n}) \tag{3.8}$$

(3.8) shows the damping property, with regard to the variable t, of the solution $e^{-\mathrm{i}\omega^{\eta} t}\vec{v}^{\eta}$. The choice $\vec{w} = \vec{v}^{0}_{p}$ $(p \neq n)$ in (3.7), leads to the projections of $\vec{v}^{1}$ on the $\vec{v}^{0}_{p}$ basis:

$$(\vec{v}^{1}, \vec{v}^{0}_{p}) = \frac{\mathrm{i}\omega^{0}_{n}}{\omega^{0^{2}}_{p} - \omega^{0^{2}}_{n}}\, b(\vec{v}^{0}_{n}, \vec{v}^{0}_{p}) \quad (p \neq n) \tag{3.9}$$

(no summation on n)

The vector $\vec{v}^{1}$ is known, after a normalization in H_{ε}.

3.2. Justification of the preceeding expansions

With expansions (3.2), (3.3), (3.4), the eigenvalue problem can be written:

$$a(z,\zeta\,;\vec{v},\vec{w}) = -\zeta^2(\vec{v},\vec{w})^0 \quad \forall \vec{w} \in V_{0\beta}$$

with $a(z,\zeta\,;\vec{v},\vec{w}) = a^0_\beta(\vec{v},\vec{w}) - \zeta z b(\vec{v},\vec{w}) + z^{2/(2-m)}[a(\vec{v},\vec{w}) + \zeta^2 c(\vec{v},\vec{w})]$.

The hermitian form $a(z,\zeta\,;\vec{v},\vec{w})$ is holomorphic in ζ and z. It is a perturbation of the coercive form $a^0_\beta(\vec{v},\vec{w})$ for sufficiently small $|z|$ and ζ given in a connected domain of the complex plane. In the general framework of holomorphic perturbation theory ([10]), we conclude that $\zeta(z)$ is holomorphic for small $|z|$.

4. LOW FREQUENCIES PROBLEM. STRONG DISSIPATIVE CASE $(m \geq 2)$

In this case, the eigenelements converge to those of the problem without dissipation in a weak way (section 2). We search for ω^η and $\vec{v}^\eta$ asymptotic expansions of the form:

$$\omega^\eta = \omega^0 + o(1), \quad \omega^0 = \omega^0_n \,(n \in \mathbf{N}) \tag{4.1}$$

$$\vec{v}^\eta = \vec{v}^0(x) + o(1), \quad \vec{v}^0 = \vec{v}^0_n. \tag{4.2}$$

With the *x-variables*, the eigenvalues problem reads:

$$\operatorname{rot}_x \operatorname{rot}_x v^\eta - i\beta \nabla_x v^\eta_3 + (\beta^2 - \omega^{\eta^2} \mu\varepsilon + i\omega^\eta \mu\bar{\sigma}\eta^{-m})\, v^\eta = 0 \quad \text{in} \quad \eta D \quad \text{and} \quad \Omega \backslash \eta \bar{D} \tag{4.3}$$

$$-\Delta_x v^\eta_3 - i\beta \operatorname{div}_x v^\eta + (-\omega^{\eta^2}\mu\varepsilon + i\omega^\eta \mu\bar{\sigma}\eta^{-m})\, v^\eta_3 = 0 \tag{4.4}$$

$$\varepsilon(\operatorname{div}_x v^\eta - i\beta v^\eta_3) = 0 \tag{4.5}$$

$$v^\eta_3 = 0, \quad v^\eta_\tau = 0 \quad \text{on} \quad \partial\Omega \tag{4.6}$$

where v_τ is the tangential component of v, and v_n the normal one:

$$v = v_\tau\, \tau + v_n\, n\,. \tag{4.7}$$

In (4.3), (4.4), ∇_x represents the gradient operator, in the x-variables and $\operatorname{rot}_x \operatorname{rot}_x v$ the vector:

$$\operatorname{rot}_x \operatorname{rot}_x v = \begin{pmatrix} v_{2,12} - v_{1,22} \\ v_{1,21} - v_{2,11} \\ 0 \end{pmatrix}. \tag{4.8}$$

On $\eta\Gamma$, we have the transmission conditions:

$$[v^\eta_3] = 0, \quad [v^\eta_\tau] = 0, \quad [\mu^{-1} \operatorname{rot}_x v^\eta] = 0, \quad \left[\mu^{-1}\left(\frac{\partial v^\eta_3}{\partial n} + i\beta v^\eta_n\right)\right] = 0, \quad [\varepsilon v^\eta_n] = 0\,. \tag{4.9}$$

We substitute (4.1), (4.2) into (4.3) - (4.6). For $\eta \to 0$, owing to the divergence free condition, we obtain the limit problem:

$$- \Delta_x v_3^0 + (\beta^2 - \omega^{0^2} \mu\varepsilon)\, v_3^0 = 0 \quad \text{in} \quad \Omega\backslash\{0\} \tag{4.10}$$

$$\text{rot}_x\, \text{rot}_x\, v^0 - i\beta\, \nabla_x v_3^0 + (\beta^2 - \omega^{0^2} \mu\varepsilon)\, v^0 = 0 \quad \text{div}_x\, v^0 - i\beta v_3^0 = 0 \tag{4.11}$$

$$v_3^0 = 0, \quad v_\tau^0 = 0 \quad \text{on} \quad \partial\Omega\,. \tag{4.12}$$

The solution is a guided mode $\vec{v}_n^0$ of the non dispersive waveguide. The dispersion relation and the component v_3^0 are given by (4.10), (4.12-1) and then v^0 by (4.11), (4.12-2).

In the spirit of the method of matched asymptotic expansions ([10]), we perform the dilatation $y = x\eta^{-1}$ in order to study the solution in a vicinity of the origin. In this region, we search for $\vec{v}^\eta$ an asymptotic expansion of the form:

$$\vec{v}^\eta = \delta(\eta)\, \vec{w}^0(y) + \cdots \tag{4.13}$$

With the *y-variables,* we obtain the eigenvalue problem:

$$\text{rot}_y\, \text{rot}_y\, v^\eta - i\beta\eta\, \nabla_y v_3^\eta + (\beta^2 \eta^2 - \omega^{\eta^2} \mu\varepsilon\eta^2 + i\omega^\eta \mu\tilde{\sigma}\eta^{2-m})\, v^\eta = 0 \text{ in } D \text{ and } \eta^{-1}\, \Omega\backslash\bar{D} \tag{4.14}$$

$$- \Delta_y v_3^\eta - i\beta\eta\, \text{div}_y\, v^\eta + (- \omega^{\eta^2} \mu\varepsilon\eta^2 + i\omega^\eta \mu\tilde{\sigma}\eta^{2-m})\, v_3^\eta = 0 \tag{4.15}$$

$$\varepsilon(\text{div}_y\, v^\eta - i\beta\eta v_3^\eta) = 0 \tag{4.16}$$

$$[v_3^\eta] = 0, \quad [v_\tau^\eta] = 0, \quad [\mu^{-1}\, \text{rot}\, v^\eta] = 0, \quad \left[\mu^{-1}\left(\frac{\partial v_3^\eta}{\partial n} + i\beta\eta v_n^\eta\right)\right] = 0, \quad [\varepsilon v_n^\eta] = 0 \quad \text{on } \Gamma \tag{4.17}$$

$$v_3^\eta = 0, \quad v_\tau^\eta = 0 \quad \text{on} \quad \partial\eta^{-1}\, \Omega\,. \tag{4.18}$$

Associated with the y_3 variable, the wave number is here $\beta\eta$.

We substitute (4.1), (4.13) into (4.14) - (4.17).

For $m > 2$, we obtain, for $\eta \to 0$, the following problem with the unknown function w_3^0:

$$w_3^0 = 0 \qquad \text{in } D \tag{4.19}$$

$$- \Delta_y w_3^0 = 0 \qquad \text{in } \mathbf{R}^2 \backslash\bar{D} \tag{4.20}$$

$$[w_3^0] = 0 \qquad \text{on } \Gamma \tag{4.21}$$

$$\text{inn.lim}\, v_3^0(x) = \text{out.lim}\, \delta(\eta)\, w_3^0(y)\,. \tag{4.22}$$

Inner (resp. outer) limit means limit as $\eta \to 0$, for fixed y (resp. fixed x).

The second transmission condition on v_3^η in (4.17) may be disregarded because of the wave length of the propagation in D. A similar situation occurs in [6].

$v_3^0(x)$ is uniquely defined by (4.10), (4.12-1) and $\text{inn.lim}\, v_3^0(x) = \lim_{\eta \to 0} v_3^0(x) = v_3^0(0)$.

After normalization, $v_3^0(0)$ is different from zero and consequently, the problem (4.19) - (4.22) has no solution without singularity as $|y| \to +\infty$. We thus look for the weakest singular solution of:

$$-\Delta_y w_3^0 = 0 \qquad \text{in } \mathbf{R}^2 \backslash \bar{D} \tag{4.23}$$

$$w_3^0 = 0 \qquad \text{on } \Gamma \tag{4.24}$$

$$\lim_{\eta \to 0 \text{ fixed } x} \delta(\eta)\, w_3^0(y) = v_3^0(0)\,. \tag{4.25}$$

We know ([3]) that the weakest singularity at infinity of the laplacian in $\mathbf{R}^2$ is logarithmic and, consequently, the solution has the following behaviour at infinity:

$$w_3^0(y) \approx c \ln |y| + f(y)$$

where $f(y)$ is a nonsingular function as $|y| \to +\infty$. For $\eta \to 0$, fixed x, $\delta(\eta)\, w_3^0(y) \approx c\delta(\eta)\,(\ln |x| - \ln \eta)$ so the condition (4.25) leads to:

$$\delta(\eta) = (\ln \eta)^{-1} \tag{4.26}$$

and $c = -\, v_3^0(0)$, and:

$$v_3^\eta(y) = -\, v_3^0(0)\,(\ln \eta)^{-1} \ln |y| + f(y) + o(1)\,.$$

The regular function $f(y)$ is the solution of the well posed problem:

$$-\Delta_y f(y) = 0 \qquad \text{in } \mathbf{R}^2 \backslash \bar{D}$$

$$f = -\, c \ln |y| \qquad \text{on } \Gamma$$

$$f \to cst. \qquad \text{for } |y| \to +\infty\,.$$

The constant at infinity is determined in point of fact that this problem is well posed. The value (4.26) of the gauge function $\delta(\eta)$ justifies the global character of the propagation.

For $m = 2$, the limit problem in the y-variables has the same character though the equation in D is different. (4.19) has to be substituted by:

$$-\, \Delta_y w_3^0 + \mathrm{i}\omega^0 \mu\tilde{\sigma} w_3^0 = 0\,.$$

But the matching condition leads to the same gauge function.

5. HIGH FREQUENCIES PROBLEM. GLOBAL PROPAGATION $(m > 1)$

Global and local propagations only can occur if β and ω are of order η^{-m}, as σ is:

$$\sigma = \eta^{-m}\tilde{\sigma}, \quad \beta = \eta^{-m}\tilde{\beta}, \quad \omega = \eta^{-m}\tilde{\omega}\,.$$

With the *x-variables,* the eigenvalue problem reads:

$$\mathrm{rot}_x\,\mathrm{rot}_x\, v^\eta - \mathrm{i}\tilde{\beta}\eta^{-m}\nabla_x v_3^0 + (\tilde{\beta}^2 - \tilde{\omega}^{\eta^2}\,\mu\varepsilon + \mathrm{i}\tilde{\omega}^\eta\,\mu\tilde{\sigma})\,\eta^{-2m}\, v^\eta = 0 \text{ in } \eta D \text{ and } \Omega\backslash\eta\bar{D} \tag{5.1}$$

$$-\Delta_x v_3^\eta - \mathrm{i}\tilde{\beta}\eta^{-m}\,\mathrm{div}_x\, v^\eta + (-\tilde{\omega}^{\eta^2}\,\mu\varepsilon + \mathrm{i}\tilde{\omega}^\eta\,\mu\tilde{\sigma})\,\eta^{-2m}\, v_3^\eta = 0 \tag{5.2}$$

$$\varepsilon(\mathrm{div}_x\, v^\eta - \mathrm{i}\tilde{\beta}\eta^{-m}\, v_3^\eta) = 0 \tag{5.3}$$

$$v_3^\eta = 0, \quad v_\tau^\eta = 0 \quad \text{on} \quad \partial\Omega\,. \tag{5.4}$$

On Γ, the usual transmission conditions hold:

$$[v_3^\eta] = 0, \quad [v_\tau^\eta] = 0, \quad [\mu^{-1}\,\mathrm{rot}_x\, v^\eta] = 0, \quad \left[\mu^{-1}\left(\frac{\partial v_3^\eta}{\partial n} + \mathrm{i}\tilde{\beta}\eta^{-m}\, v_n^\eta\right)\right] = 0, \quad [\varepsilon v_n^\eta] = 0\,. \tag{5.5}$$

For small values of the parameter η, we have propagation in the guide of a perturbed TEM wave ([8]). We seach for $\vec{v}^\eta$, solution of (5.1) - (5.5), an asymptotic expansion of the form:

$$v^\eta(x) = v^0(x) + \cdots, \quad v_3^\eta = \eta^m\, v_3^1 + \cdots \tag{5.6}$$

associated with the dispersion relation:

$$\tilde{\beta}^2 - \tilde{\omega}^{\eta^2}\,\mu\varepsilon = -\eta^{2m}\,\Omega^2 + \cdots \tag{5.7}$$

We substitute (5.6), (5.7) into (5.1) - (5.5). With the divergence condition (5.3), we obtain the limiting problem:

$$-\Delta_x v_3^1 = \Omega^2\, v_3^1 \qquad \text{in } \Omega\backslash\{0\} \tag{5.8}$$

$$\mathrm{rot}_x\,\mathrm{rot}_x\, v^0 - \mathrm{i}\tilde{\beta}\,\nabla_x v_3^1 = \Omega^2\, v^0, \quad \mathrm{div}_x\, v^0 - \mathrm{i}\tilde{\beta} v_3^1 = 0 \tag{5.9}$$

$$v_3^0 = 0, \quad v_\tau^0 = 0 \qquad \text{on } \partial\Omega\,. \tag{5.10}$$

Problem (5.8), (5.10-1) leads to the values Ω_n^2, $n \in \mathbf{N}$, of the perburbation Ω^2 of the dispersion relation.

Near the origin, we use the variable y, $y = x\eta^{-1}$, to study the solution. In the y-variables, the problem in D and $\eta^{-1}\,\Omega\backslash\bar{D}$ has the same form as (5.1) - (5.5), with the change of $-m$ into $1-m$. We approach the solution by:

$$v^\eta = \delta(\eta)\, w^0(y) + \cdots, \quad v_3^\eta = \eta^m\,\delta(\eta)\, w_3^1(y) + \cdots \tag{5.11}$$

We substitute (5.11) and (5.7) in the equations. The problem splits up in one problem in w_3^1 and then one in w^0. For $m > 1$, we obtain the equations (4.19)-(4.22) of the previous section. The same conclusion holds.

6. HIGH FREQUENCIES PROBLEM. LOCAL PROPAGATION $(m > 1)$

6.1. Decomposition of the eigenvalue problem

To study a local propagation, in the *y-variables,* we introduce the new parameter α :

$$\alpha = \eta^{m-1}\,.$$

In the sequel, the derivatives are related to the y-variables, but the notation y will be omitted. The eigenvalue problem reads:

$$\alpha^2 \operatorname{rot} \operatorname{rot} v - i\tilde{\beta}\alpha \nabla v_3 + (\tilde{\beta}^2 - \tilde{\omega}^2 \mu\varepsilon + i\tilde{\omega}\mu\tilde{\sigma})\, v = 0 \quad \text{in } D \quad \text{and} \quad \eta^{-1}\,\Omega\backslash\bar{D} \tag{6.1}$$

$$-\alpha^2 \Delta v_3 - i\tilde{\beta}\alpha \operatorname{div} v + (-\tilde{\omega}^2 \mu\varepsilon + i\tilde{\omega}\mu\tilde{\sigma})\, v_3 = 0 \tag{6.2}$$

$$\varepsilon(\alpha \operatorname{div} v - i\tilde{\beta} v_3) = 0 \tag{6.3}$$

$$v_3 = 0, \quad v_\tau = 0 \quad \text{on} \quad \partial\eta^{-1}\,\Omega \tag{6.4}$$

and the transmission conditions on Γ:

$$[v_3] = 0, \quad [v_\tau] = 0, \quad [\mu^{-1}\alpha^2 \operatorname{rot} v] = 0 = \left[\mu^{-1}\left(\alpha^2 \frac{\partial v_3}{\partial n} + i\tilde{\beta}\alpha v_n\right)\right], \quad [\varepsilon\alpha v_n] = 0\,. \tag{6.5}$$

When $\eta \to 0$, the section $\eta^{-1}\,\Omega$ tends to $\mathbf{R}^2$. So we proceed in two phases to solve the problem (6.1) - (6.5):

— study of an exterior problem, settled on $\eta^{-1}\,\Omega\backslash\bar{D}$, and introduction of an operator, similar to the Steklov-Poincaré's operator related to the laplacian,

— use of the transmission conditions on Γ to obtain an eigenvalue problem on D.

6.2. Outer problem (P_e)

We consider equations (6.1) - (6.3) in $\eta^{-1}\,\Omega\backslash\bar{D}$ (where $\tilde{\sigma} = 0$) and the boundary conditions (6.4) on $\partial\eta^{-1}\,\Omega$ and (6.6) on Γ (with given functions φ_τ and φ_3):

$$v_3 = \varphi_3 \quad v_\tau = \varphi_\tau \qquad \text{on } \Gamma\,. \tag{6.6}$$

Let us denote by (P_e) the problem (6.1), (6.2), (6.3), (6.4), (6.6).

PROPOSITION 6.1: *With* η *fixed parameter and* $\operatorname{Re}(\tilde{\beta}^2 - \tilde{\omega}^2 \mu\varepsilon) > 0$, *for* $(\varphi_\tau, \varphi_3) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, *the problem* (P_e) *has an unique solution.*

For the proof, see [9].

This result allows us to define the dual quantities of the data $(\varphi_\tau, \varphi_3)$. Therefore two operators T and S are introduced. With $\vec{v}$ solution of problem (P_e) and n outer unit normal to D on Γ, we define:

$$T : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$$

$$(\varphi_\tau, \varphi_3) \to \left((\alpha^2 \mu^{-1} \operatorname{rot} v)|_\Gamma, \left(\alpha^2 \mu^{-1} \frac{\partial v_3}{\partial n} + i\tilde{\beta}\alpha\mu^{-1} v_n\right)|_\Gamma\right) \tag{6.7}$$

$$S : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad S(\varphi_\tau, \varphi_3) = (\varepsilon\alpha v_n)|_\Gamma\,. \tag{6.8}$$

In the sequel, we shall use the notation:

$$T_1(\varphi_\tau, \varphi_3) = (\alpha^2 \mu^{-1} \operatorname{rot} v)|_\Gamma, \quad T_2(\varphi_\tau, \varphi_3) = \left(\alpha^2 \mu^{-1} \frac{\partial v_3}{\partial n} + i\tilde{\beta}\alpha\mu^{-1} v_n\right)|_\Gamma\,. \tag{6.9}$$

6.3. Asymptotic expansion in boundary layer terms

The structure of equations (6.1), (6.2) shows that problem (P_e) is truly a boundary layer problem and the "skin effect" will be produced by a boundary layer expansion of the eigenvector $\vec{v} = \vec{v}^\alpha$ in a neighbourhood of Γ ([1] and [2]).

A point P on Γ is located by its coordinates $(\gamma_1(\xi_2), \gamma_2(\xi_2))$, where ξ_2 is the arc length parameter on Γ. With $n = (n_1(\xi_2), n_2(\xi_2))$ denoting the unit normal, exterior to D, a point M in $\eta^{-1}\,\Omega\backslash\bar{D}$ is well defined, in a neighbourhood of Γ by:

$$y_1 = \gamma_1(\xi_2) + \xi_1\, n_1(\xi_2) \quad y_2 = \gamma_2(\xi_2) + \xi_1\, n_2(\xi_2)$$

(ξ_1, ξ_2) defines a local orthogonal curvilinear system of coordinates, the metric of which is given by the quadratic form:

$$dM^2 = d\xi_1^2 + s^2\, d\xi_2^2, \quad s = 1 + \xi_1\, R^{-1}(\xi_2), \quad R(\xi_2) \text{ radius of curvature of } \Gamma \text{ in } P\,.$$

To solve problem (P_e), we introduce a fast normal variable, $\zeta_1 = \xi_1\, \alpha^{-1}$ and expand the eigenvector $\vec{v} = \vec{v}^\alpha$:

$$\vec{v}^\alpha = \vec{v}^0(\xi_1, \xi_2, \zeta_1) + \alpha \vec{v}^1(\xi_1, \xi_2, \zeta_1) + \cdots \tag{6.10}$$

Owing to condition (6.3), problem (P_e) can be split in two problems: one with the unknown function v_3^α and, then, one with the function v^α.

Problem in v_3^α

$$-\alpha^2\, \Delta v_3^\alpha + k^2\, v_3^\alpha = 0 \qquad \text{in } \eta^{-1}\,\Omega\backslash\bar{D} \tag{6.11}$$

$$v_3^\alpha = \varphi_3 \quad \text{on } \Gamma \qquad v_3^\alpha = 0 \quad \text{on } \partial\eta^{-1}\,\Omega \tag{6.12}$$

where k is defined by: $k^2 = \tilde{\beta}^2 - \tilde{\omega}^2\, \varepsilon\mu$, $\operatorname{Re} k > 0$.

Let us denote D_1^n and D_2^n the n^{th} partial derivates with respect to ξ_1 and ξ_2. With $u(\xi_1, \xi_2)$ scalar function, we have:

$$\Delta u = D_1^2\, u + (sR)^{-1}\, D_1\, u + s^{-1}\, D_2(s^{-1}\, D_2\, u)\,. \tag{6.13}$$

We substitute expansion (6.10) into (6.11), with (6.13). We obtain from the terms at order α^0 and α^1, and the boundary conditions (6.12), the following approximation of v_3^α in a neighbourhood of Γ in $\eta^{-1}\,\Omega\backslash\bar{D}$ ([2]):

$$v_3^\alpha = \varphi_3(\xi_2)\, \frac{1}{\sqrt{s(\xi_1, \xi_2)}}\, e^{-k\xi_1\,\alpha^{-1}} \tag{6.14}$$

and the consequence:

$$\left.\frac{\partial v_3^\alpha}{\partial n}\right|_\Gamma = -\left(\frac{k}{\alpha} + \frac{1}{2R}\right) \varphi_3 + O(\alpha^2)\,. \tag{6.15}$$

Problem in v^{α}

$$\alpha^2 \operatorname{rot} \operatorname{rot} v^{\alpha} + k^2 v^{\alpha} = i\tilde{\beta}\alpha \nabla v_3^{\alpha} \quad \text{in } \eta^{-1} \Omega \backslash \bar{D} \tag{6.16}$$

$$v_{\tau}^{\alpha} = \varphi_{\tau} \quad \text{on } \Gamma \qquad v_{\tau}^{\alpha} = 0 \quad \text{on } \partial\eta^{-1} \Omega . \tag{6.17}$$

With $\vec{v}(\xi_1, \xi_2)$ vector in the basis $(\vec{n}, \vec{\tau}, \vec{e}_3)$, we have:

$$\operatorname{Rot} \vec{v} = \begin{pmatrix} s^{-1} D_2 v_3 \\ -D_1 v_3 \\ D_1 v_{\tau} + (sR)^{-1} v_{\tau} - s^{-1} D_2 v_n \end{pmatrix}$$

and in the basis n, τ :

$$\nabla v = \begin{pmatrix} D_1 v \\ s^{-1} D_2 v \end{pmatrix} .$$

We apply for v_{τ}^{α} and v_n^{α} the boundary layer expansions:

$$v_n^{\alpha} = v_n^0(\xi_1, \xi_2, \zeta_1) + \alpha v_n^1(\xi_1, \xi_2, \zeta_1) + \cdots \tag{6.18}$$

$$v_{\tau}^{\alpha} = v_{\tau}^0(\xi_1, \xi_2, \zeta_1) + \alpha v_{\tau}^1(\xi_1, \xi_2, \zeta_1) + \cdots \tag{6.19}$$

Substituting (6.18), (6.19) into (6.16), we obtain from the terms at order α^0 and α^1 and the boundary conditions (6.17), the following approximations of v_n^{α} and v_{τ}^{α} in a neighbourhood of Γ in $\eta^{-1} \Omega \backslash \bar{D}$:

$$v_n^0 = -i\tilde{\beta}k^{-1} \frac{\varphi_3(\xi_2)}{\sqrt{s(\xi_1, \xi_2)}} e^{-k\xi_1 \alpha^{-1}} \tag{6.20}$$

$$v_{\tau}^0 = \frac{\varphi_{\tau}(\xi_2)}{\sqrt{s(\xi_1, \xi_2)}} e^{-k\xi_1 \alpha^{-1}} . \tag{6.21}$$

Consequently, at order α^0, we have the approximation:

$$(\operatorname{rot} v^0)|_{\Gamma} = -\left(\frac{k}{\alpha} - \frac{1}{2R}\right) \varphi_{\tau} + i \frac{\tilde{\beta}}{k} \left(D_2 \varphi_3 + \frac{D_2 R}{2R^2} \varphi_3 \right) . \tag{6.22}$$

Expressions (6.15), (6.20), (6.22) give approximations of expressions (6.7) and (6.8) of $T(\varphi_{\tau}, \varphi_3)$ and $S(\varphi_{\tau}, \varphi_3)$.

6.4. Justification of the approximations

We use the results obtained in [2] by M. Crouzeix and J. Descloux.
Let u_{α} be solution of the equation $k^2 u_{\alpha} - \alpha^2 \Delta u_{\alpha} = 0$ in $\eta^{-1} \Omega \backslash \bar{D}$.
In a neighbourhood V of Γ in $\eta^{-1} \Omega \backslash \bar{D}$, the function $u_{\alpha}^0(\xi_1, \xi_2)$:

$$u_{\alpha}^0(\xi_1, \xi_2) = \frac{e^{-k\xi_1 \alpha^{-1}}}{\sqrt{s(\xi_1, \xi_2)}} u_{\alpha}(0, \xi_2)$$

is an approximation of u_α and moreover, for $n = 1, 2, \ldots$

$$\| D_2^n (u_\alpha - \overset{0}{u}_\alpha) \|_{L^\infty(\bar V)} = O(\alpha^3) \quad \| D_2^n D_1 (u_\alpha - \overset{0}{u}_\alpha) \|_{L^\infty(\bar V)} = O(\alpha^2) \,. \tag{6.23}$$

We apply estimations (6.23) to functions v_3^α and v^α, after the change v^α into $v^{*\alpha}$, $v^{*\alpha} = v^\alpha - \mathrm{i}\tilde\beta\alpha k^{-2}\, \nabla v_3^\alpha$. In this way, we obtain:

PROPOSITION 6.2: *Let* v_3^α *(resp.* v^α) be the solution of (6.11), (6.12) (resp. (6.16), (6.17). Assume that Γ *is of class* C^∞. *Then:*

$$\left\| \frac{\partial v_3^\alpha}{\partial n} + \left(\frac{k}{\alpha} + \frac{1}{2R} \right) \varphi_3 \right\|_{L^\infty(\Gamma)} = O(\alpha^2) \tag{6.24}$$

$$\| v_n^\alpha + \mathrm{i}\tilde\beta k^{-1}\, \varphi_3 \|_{L^\infty(\Gamma)} = O(\alpha^2) \tag{6.25}$$

$$\left\| \mathrm{rot}\, v^\alpha + \left(\frac{k}{\alpha} - \frac{1}{2R} \right) \varphi_\tau - i \frac{\tilde\beta}{k} \left(D_2\, \varphi_3 + \frac{D_2 R}{2R^2}\, \varphi_3 \right) \right\|_{L^\infty(\Gamma)} = O(\alpha^2) \,. \tag{6.26}$$

6.5. Inner problem (P_i)

The inner problem on D arises from the transmission conditions (6.4), (6.5). Owing to condition (6.3), we define the new spectral parameter:

$$\tilde\beta^2 - \tilde\omega^2\, \varepsilon'\mu' + \mathrm{i}\tilde\omega\tilde\sigma\mu' = -\,\alpha^2\,\Omega^2 + \cdots \tag{6.27}$$

and we obtain the inner problem (P_i) :

$$\mathrm{rot}\ \mathrm{rot}\ v^\alpha - \mathrm{i}\tilde\beta\alpha^{-1}\, \nabla v_3 = \Omega^2\, v^\alpha \quad \text{in } D \tag{6.28}$$

$$-\,\Delta v_3^\alpha = \Omega^2\, v_3^\alpha \quad \text{in } D \tag{6.29}$$

$$\alpha\, \mathrm{div}\ v^\alpha - \mathrm{i}\tilde\beta\, v_3^\alpha = 0 \quad \text{in } D \tag{6.30}$$

$$\alpha^2\, \mu'^{-1}\, rot\ v^\alpha = T_1(v_\tau^\alpha|_\Gamma, v_3^\alpha|_\Gamma) \quad \text{on } \Gamma \tag{6.31}$$

$$\alpha^2\, \mu'^{-1}\, \frac{\partial v_3^\alpha}{\partial n} + \mathrm{i}\tilde\beta\alpha\mu'^{-1}\, v_n^\alpha = T_2(v_\tau^\alpha|_\Gamma, v_3^\alpha|_\Gamma) \quad \text{on } \Gamma \tag{6.32}$$

$$\alpha\varepsilon' v_n^\alpha = S(v_\tau^\alpha|_\Gamma, v_3^\alpha|_\Gamma) \quad \text{on } \Gamma \tag{6.33}$$

with the operators S, T_1, T_2 defined by (6.8) and (6.9).

We search for the solution of (6.28) - (6.33) the following expansions:

$$v^\alpha = v^0 + \alpha v^1 + \cdots \qquad v_3^\alpha = \alpha v_3^1 + \cdots \tag{6.34}$$

$$\Omega^2 = \Omega^{0^2} + \alpha\Omega^1 + \cdots \tag{6.35}$$

Sections 6.3 and 6.4 bring asymptotic behaviour for the terms on the boundary Γ.

Approximations (6.15), (6.20), (6.22) with $\varphi_3 = (\alpha v_3^1 + \cdots)|_\Gamma$ and $\varphi_\tau = (v_\tau^0 + \cdots)|_\Gamma$. Give:

$$v_\tau^0 = 0 \quad v_3^1 = 0 \quad \text{on } \Gamma .$$

So the approximation (v_3^1, Ω^0) is deduced from the problem:

$$-\Delta v_3^1 = \Omega^{0^2} v_3^1 \qquad \text{in } D$$

$$v_3^1 = 0 \qquad \text{on } \Gamma .$$

Existence of a countable infinity of positive eigenvalues $\Omega_n^{0^2}$ $(n \in \mathbf{N})$ results. From (6.27), we deduce the expansion of $\tilde{\omega} : \tilde{\omega} = \tilde{\omega}_0 + \alpha^2 \tilde{\omega}_1 + \cdots$ with:

$$-\tilde{\omega}_0^2 \varepsilon' \mu' + \mathrm{i}\tilde{\omega}_0 \mu' \tilde{\sigma} + \tilde{\beta}^2 = 0 \quad \text{and} \quad -2\,\tilde{\omega}_0 \tilde{\omega}_1 \varepsilon' \mu' + \mathrm{i}\tilde{\omega}_1 \tilde{\sigma} \mu' + \tilde{\beta}^2 = -\Omega_n^{0^2} .$$

And coming back, to the pulsation $\omega : \omega = i \dfrac{\sigma}{2\,\varepsilon'} \pm \sqrt{\dfrac{\beta^2}{\mu' \varepsilon'} - \dfrac{\sigma^2}{4\,\varepsilon'}} + \eta^{m-2} \tilde{\omega}_1 + \cdots$

Sections (6.3) and (6.4) provide the local character of this propagation. In the outer part $\eta^{-1}\,\Omega \backslash \bar{D}$, the electric field has an exponentional decreasing behaviour (the factor α making weaker the amplitude).

APPENDIX

Dispersion relation for a circular guide with circular dissipative inclusion.

In order to develop easier computations, without a lost of generality, we suppose that ε and μ are constants, all over the section. A problem with the function v_3 only arises. Considering the geometry of the section, the boundary and transmission conditions, we search for the eigenvector a solution $\vec{v}(r)\, e^{in\theta}$, $n \in \mathbf{N}$, with (r, θ) polar coordinates in $\mathbf{R}^2$. The third component of this vector, denoted by $v(r)\, e^{in\theta}$, satisfies:

$$-\Delta v + (\beta^2 - \omega^2 \mu\varepsilon + \mathrm{i}\omega\mu\sigma)\, v = 0 \quad \text{for} \quad 0 < r < R \tag{A.1}$$

$$v = 0 \qquad \text{on } r = R \tag{A.2}$$

$$[v] = 0 = \left[\frac{dv}{dr}\right] \qquad \text{on } r = \eta . \tag{A.3}$$

Let us introduce $h^2 = \beta^2 - \omega^2 \mu\varepsilon$ and $h'^2 = \beta^2 - \omega^2 \mu\varepsilon + \mathrm{i}\omega\mu\tilde{\sigma}\eta^{-m}$. Equation (A.1) is a Bessel equation with the variable $\rho = h'r$ for $0 < r < \eta$ and $\rho = hr$ for $\eta < r < R$. The solution of (A.1) is expressed in terms of Bessel functions J_n and Y_n:

$$v = AJ_n(h'r) \quad \text{for} \quad 0 < r < \eta \quad \text{and} \quad v = BJ_n(hr) + CY_n(hr) \quad \text{for} \quad \eta < r < R .$$

The boundary conditions (A.2) and the transmission conditions (A.3) lead to the dispersion equation:

$$\begin{aligned} &J_n(hR)\,[h' Y_n(h\eta)\, J_n'(h'\eta) - hJ_n(h'\eta)\, Y_n'(h\eta)] \\ &= Y_n(hR)\,[h' J_n(h\eta)\, J_n'(h'\eta) - hJ_n(h'\eta)\, J_n'(h\eta)] \end{aligned} \tag{A.4}$$

with $J_n' = \dfrac{dJ_n}{d\rho}$ and $Y_n' = \dfrac{dY_n}{d\rho}$.

From (A.4), we can obtain, in this special case, the approximations of sections 4, 5, 6.

For the low frequencies, we have $h = O(1)$ and $h' = O(\eta^{-m/2})$, or

$$h\eta = O(\eta) \quad \text{and} \quad h'\eta = O(\eta^{\frac{2-m}{2}}) . \tag{A.5}$$

So as to use the behaviour of the Bessel functions, with $m > 2$, we transform (A.4):

$$J_n(hR) = Y_n(hR) \left[J_n(h\eta) - \frac{h}{h'} \frac{J_n(h'\eta)}{J_n'(h'\eta)} J_n'(h\eta) \right] \left[Y_n(h_\eta) - \frac{h}{h'} \frac{Y_n'(h\eta)}{J_n'(h'\eta)} J_n(h'\eta) \right]^{-1} . \tag{A.6}$$

From the asymptotic expansions of $J_n(z)$ and $Y_n(z)$ ([11]), if we pass to the limit in (A.6), with (A.5), we obtain $J_n(hR) = 0$, dispersion relation without dissipative inclusion.

For the high frequencies, two cases arise. For $h = O(1)$ and $h' = O(\eta^{-m})$, or

$$h\eta = O(\eta) \quad \text{and} \quad h'\eta = O(\eta^{1-m}) . \tag{A.7}$$

For $m > 1$, (A.6), with assumptions (A.7), gives the same result as before: $J_n(hR) = 0$.

For $h = O(\eta^{-m})$ and $h' = O(\eta^{-1})$, or

$$h\eta = O(\eta^{1-m}) \quad \text{and} \quad h'\eta = O(1) \tag{A.8}$$

we transform (A.4) in:

$$J_n(h'\eta) = \frac{h'}{h} J_n'(h'\eta) \, [Y_n(h\eta) \, J_n(hR) - Y_n(hR) \, J_n(h\eta)] \, [Y_n'(h\eta) \, J_n(hR) - Y_n(hR) \, J_n'(h\eta)]^{-1}$$

(A.8) implicates that the second member is $O(\eta^{m-1})$. The limit $\mathrm{i}\Omega^0$ of $h'\eta$ is determined by $J_n(\mathrm{i}\Omega^0) = 0$, in harmony with section 6.

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