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A PROBLEM OF MAGNETOSTATICS RELATED TO THIN PLATES (*)

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Abstract. — *Motivated by an industrial project concerning the electrolysis cells for the production of aluminium, we study the asymptotic behaviour of ferromagnetic plates first when the magnetic susceptibility is large and then when the thickness of the plates is small.* © *Elsevier, Paris*

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Résumé. — *Ce travail est motivé par un projet industriel relatif à l'électrolyse pour la production de l'aluminium. Nous étudions le comportement asymptotique de plaques ferromagnétiques d'abord lorsque la susceptibilité magnétique est grande, puis lorsque l'épaisseur des plaques tend vers zéro.* © *Elsevier, Paris*

1. INTRODUCTION

This study is motivated by an industrial project which concerns the electrolysis cells for the production of aluminium. Schematically a cell is a rectangular parallelipedon; typical sizes can be 10 meters for the length, 4 meters for the width and 2 meters for the height. To insure the strength of the device the lower and the lateral faces are covered by plates of steel which are about 3 centimeters thick. The continuous electric current enters into the cell through the anode bloc in the upper face and leaves it through conductors perpendicular to the two lateral faces of largest length; since there is no current in the ferromagnetic parts, those two faces have holes.

The problem is to find the magnetic field. Let us specify a bit the data: **a)** the current density field $\vec{j}$ is supposed to be given everywhere, i.e. in $\mathbb{R}^3$; **b)** the ferromagnetic domain is the union of the five plates; it is bounded, connected, but not simply connected; **c)** at any point of Λ, the induction $\vec{B}$ and the magnetic field $\vec{H}$ are bound by the relation $\vec{B} = \mu\vec{H}$ where μ where μ is a function of $|\vec{H}|$.

In the following the symbol Λ will denote the ferromagnetic domain which can be different from the one of an electrolysis cell.

In fact, this paper is essentially devoted to a double asymptotic analysis, Let μ, μ_0 and μ_R be respectively the magnetic susceptiblity, the magnetic susceptiblity of the vacuum and the relative susceptiblity i.e. the ratio μ/μ_0. In the steel parts of a cell, even in the situation of saturation, μ_R is large since one estimates that it can vary between 100 and 4000. It is consequently natural to be interested in studying the limit case where μ tends to infinity. This will be the subject of Section 2 where we suppose that μ is of the form $\mu(\vec{x}) = t\bar{\mu}(\vec{x})$ for $\vec{x}$ in Λ ; $\bar{\mu}(\,.\,)$ is here a given function and we obtain for large t a series expansion for the magnetic field of the form

$$\vec{H}(\vec{x}) = \sum_{k=0}^{\infty} \frac{t}{t^k} \vec{Q}_k(\vec{x}), \quad \vec{x} \in \mathbb{R}^3 . \tag{1.1}$$

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Relation (1.1) will precise and complete results already contained in the litterature, see [3]. Among them, one can quote the following striking one: Let $\vec{H}_*$ be the magnetic field generated by $\vec{j}$ in absence of ferromagnetic material; then at the limit situation $t=\infty$, the magnetic energy in Λ is finite if and only if there exists a potential φ such that $\vec{H}=\vec{\nabla}\varphi$ in Λ ; this property is equivalent to $Q_0=0$ in Λ and it holds if Λ is simply connected; see Remark 2.4.

In Section 3, we suppose that the ferromagnetic domain Λ is just a single plate of thickness δ ; we start from the situation $t=\infty$, i.e. by (1.1) $\vec{H}=\vec{Q}_0$, and then let δ tend to zero. We show that $\vec{H}$ converges, outside Λ, to a well defined limit. In this process we first let $t \to \infty$ and then $\delta \to 0$; one cannot permute the limits since for a fixed finite value of t, $\vec{H}$ will converge, as $\delta \to 0$, to $\vec{H}_*$ which is the magnetic field in absence of ferromagnetic material.

We found that the most appropriate numerical scheme for our original cell problem is a Galerkin method with piecewise constant elements in connection with the most standard integral equation modelling the magnetostatic problem; it is essentially a variant of the one used in the program GFUN, see [1]. In Section 4, we describe numerical algorithms corresponding to the analytic approaches of Sections 2 and 3. Numerical examples show the relevance of the asymptotic studies.

In [11], Rogier studies the same magnetostatic problem for large magnetic susceptiblity and thin ferromagnetic structures. However his approach differs from ours on several respects among which we quote the following points: **a)** in his paper μ and δ are connected by a relation of the form $\delta\mu(\delta,\vec{x})=\hat{\mu}(\vec{x})$ where $\hat{\mu}$ is a given function; **b)** as δ tends to zero and consequently μ tends to infinity, he obtains a limit problem which depends on $\hat{\mu}$ and which is more complicated than the ones we shall consider in Sections 2 and 3; **c)** he proves weak convergences whereas we show strong convergences.

We conclude this introduction by general bibliographic comments. This work belongs to the family of researches dealing with problems involving small (large) data, in particular thin domains. Among a huge litterature, let us quote some relevant publications. In 1973, J.-L. Lions published a basic book [8]. The theory of thin and multi-structures is treated in detail by P. G. Ciarlet in [4] and by H. Le Dret in [7]. Hale and Raugel propose in [6] a general approach to time dependent problems on thin domains. G. Raugel and G. Sell in [10] and more recently R. Temam and M. Ziane in [12] study the Navier-Stokes equations on thin domains. In this last paper, as in the present one, the authors avoid the use of a reference domain independent of the small parameter.

2. ASYMPTOTIC EXPANSION FOR LARGE MAGNETIC SUSCEPTIBILITY

We begin by recalling some basic equations of magnetostatics. The following fields are defined in $\mathbb{R}^3$: $\vec{H}$ is the magnetic field, $\vec{B}$ is the induction, μ is the magnetic susceptiblity, $\vec{j}$ is the current density. Λ is the ferromagnetic domain so that $\mu=\mu_0$ outside Λ where μ_0 is the magnetic susceptibility in the vacuum. The vector $\vec{H}_*$ will denote the magnetic field generated by $\vec{j}$ in absence of ferromagnetic material.

As a basic hypothesis, we suppose that $\vec{j}=\vec{0}$ in Λ.

The different fields are related by the fundamental equations which must be understood in the distribution sense in $\mathbb{R}^3$

$$\overrightarrow{\mathrm{rot}}\,\vec{H}=\vec{j},\quad \operatorname{div}\vec{B}=0,\quad \vec{B}=\mu\vec{H},\quad \vec{H}=\vec{\mathcal{O}}\left(\frac{1}{|\vec{x}|^2}\right), \tag{2.1}$$

$$\overrightarrow{\mathrm{rot}}\,\vec{H}_*=\vec{j},\quad \operatorname{div}\vec{H}_*=0,\quad \vec{H}_*=\vec{\mathcal{O}}\left(\frac{1}{|\vec{x}|^2}\right). \tag{2.2}$$

From (2.1), (2.2), we deduce that $\overrightarrow{\text{rot}}\,(\vec{H}-\vec{H}_*)=0$ and consequently there exists a potential ψ such that

$$\vec{H}=\vec{H}_*-\vec{\nabla}\psi \quad \text{in } \mathbb{R}^3 . \tag{2.3}$$

Using (2.1), (2.2), (2.3), we get

$$\vec{\nabla}\psi=\vec{0}\left(\frac{1}{|\vec{x}|^2}\right), \tag{2.4}$$

$$0=\operatorname{div}\vec{B}=\operatorname{div}\,(\mu\vec{H}_*-\mu\,\vec{\nabla}\psi)=\operatorname{div}\,((\mu-\mu_0)\,\vec{H}_*-\mu\,\vec{\nabla}\psi)\,. \tag{2.5}$$

Multiplying (2.5) by a function v with compact support and integrating on $\mathbb{R}^3$, we obtain by Gauss' theorem since $\mu=\mu_0$ outside Λ

$$\int_{\mathbb{R}^3}\mu\,\vec{\nabla}\psi\cdot\vec{\nabla}v=\int_\Lambda(\mu-\mu_0)\,\vec{H}_*\cdot\vec{\nabla}v\,. \tag{2.6}$$

We now specify the mathematical model we shall work on. First we assume

H1. Λ is a bounded open Lipschitzian subset of $\mathbb{R}^3$ with boundary $\partial\Lambda$. The complement $\bar{\Lambda}^c$ of the closure $\bar{\Lambda}$ is connected.

H2. There is an open set containing $\bar{\Lambda}$ on which $\vec{H}_*$ is of class C^∞.

H3. $\mu\in L^\infty(\mathbb{R}^3)$, $\mu\geqslant\mu_0$ in $\mathbb{R}^3$, $\mu=\mu_0$ on $\bar{\Lambda}^c$.

Recalling Definition A1 of $W^1(\mathbb{R}^3)$ in the Annex we consider the problem to find ψ such that

$$\psi\in W^1(\mathbb{R}^3),\quad \int_{\mathbb{R}^3}\mu\,\vec{\nabla}\psi\cdot\vec{\nabla}v=\int_\Lambda(\mu-\mu_0)\,\vec{H}_*\cdot\vec{\nabla}v,\quad \forall v\in W^1(\mathbb{R}^3)\,. \tag{2.7}$$

Remark 2.1:

a) H2 is based on (2.2) and the hypothesis that $\vec{j}=\vec{0}$ in Λ.

b) If ψ satisfies (2.7), it is harmonic in $\bar{\Lambda}^c$; the choice of the functional space $W^1(\mathbb{R}^3)$ for the variational formulation (2.6) is a consequence of (2.4): indeed by Proposition A3 of the Annex, one can modify ψ in (2.3) by an additive constant in such a way that $\psi\in W^1(C)$ where C is the exterior of a ball containing $\bar{\Lambda}$.

PROPOSITION 2.1:

a) *Problem (2.7) has one and only one solution.*

b) *Let B be a ball containing $\bar{\Lambda}$ and ψ be the solution of (2.7); then*

$$\int_{\partial B}\frac{d\psi}{dn}=0 \quad \textit{where } \frac{d\psi}{dn} \textit{ is the normal derivative}\,.$$

Proof: Part a) is a consequence of Proposition A1 c). To prove Part b), we can suppose that $\vec{0}$ is the center of B; let $f_\varepsilon\in C^\infty([0,\infty))$ be such that $f_\varepsilon'(\xi)\leqslant 0$ for $\xi\in[0,\infty)$, $f_\varepsilon(\xi)=1$ for $\xi\leqslant a$, $f_\varepsilon(\xi)=0$ for $\xi\geqslant a+\varepsilon$ where a is the radius of B and $\varepsilon>0$. We set $v_\varepsilon(\vec{x})=f_\varepsilon(|\vec{x}|)$ in (2.7; the right hand-side member vanishes whereas the left hand-side member converges, as ε tends to zero, to $-\mu_0\int_{\partial B}\frac{d\psi}{dn}$.

□

Remark 2.2: As a consequence of Propositions A2, A3 and 2.1, the asymptotic behaviour (2.4) can be improved and one has $\vec{\nabla}\psi = \vec{0}(|\vec{x}|^{-3})$ as $|\vec{x}|$ tends to infinity.

Let $\bar{\mu}$ be a given function satisfying

H4. $\bar{\mu} \in L^{\infty}(\Lambda)$, $\bar{\mu} \geq \mu_0$ in Λ.

We introduce the parameter t and from now on suppose that

$$\mu = t\bar{\mu} \text{ in } \Lambda, \quad 1 \leq t < \infty ; \tag{2.8}$$

to express the dependence with respect to t, we shall furthermore note the solution of (2.7) by $\psi(t)$.

Using Remark A1, we define

$$z \in W^1(\mathbb{R}^3), \quad z = 1 \text{ on } \bar{\Lambda}, \quad z \text{ harmonic on } \bar{\Lambda}^c . \tag{2.9}$$

For what follows, we consider a fixed ball B containing $\bar{\Lambda}$ with exterior normal $\vec{n}$. By Proposition A4 we have

$$\int_{\partial B} \frac{dz}{dn} < 0 . \tag{2.10}$$

LEMMA 2.1: *Let $p \in H^1(\Lambda)$. Then there exist unique $u \in W^1(\mathbb{R}^3)$ and $d \in \mathbb{R}$ such that*

$$u \in W^1(\mathbb{R}^3), \quad u = p + d \text{ on } \Lambda, \quad \Delta u = 0 \text{ in } \bar{\Lambda}^c, \quad \int_{\partial\varphi} \frac{du}{dn} = 0 ;$$

furthermore, there exists a constant c independent of p such that

$$\int_{\mathbb{R}^3} |\vec{\nabla} u|^2 \leq c \int_{\Lambda} |\vec{\nabla} p|^2 .$$

Proof: Without loss of generality, we can suppose that $\int_{\Lambda} p = 0$ so that the H^1 norm of p is bounded by its Dirichlet seminorm. By Remark A1 p admits a unique extension to $\mathbb{R}^3$ such that $p \in W^1(\mathbb{R}^3)$ and $\Delta p = 0$ in $\bar{\Lambda}^c$. By (2.9), (2.10) there is a unique $d \in \mathbb{R}$ such that $u = p + dz$ satisfies the requirements of Lemma 2.1. □

We introduce the functional space

$$V = \left\{ v \in W^1(\mathbb{R}^3) \mid \Delta v = 0 \text{ in } \bar{\Lambda}^c, \int_{\partial B} \frac{dv}{dn} = 0 \right\} . \tag{2.11}$$

By Lemma 2.1, V is a Hilbert space for the equivalent norms

$$\left(\int_{\Lambda} |\vec{\nabla} v|^2 \right)^{1/2} \quad \text{and} \quad \left(\int_{\mathbb{R}^3} |\vec{\nabla} v|^2 \right)^{1/2} . \tag{2.12}$$

This allows to define recursively $q_k \in V$, $k = 0, 1, 2, \ldots$, by the variational relations

$$\int_A \bar{\mu}\, \vec{\nabla} q_0 \cdot \vec{\nabla} v = \int_A \bar{\mu} \vec{H}_* \cdot \vec{\nabla} v, \quad \forall v \in V, \tag{2.13}$$

$$\int_A \bar{\mu}\, \vec{\nabla} q_1 \cdot \vec{\nabla} v = -\mu_0 \left\{ \int_A \vec{H}_* \cdot \vec{\nabla} v + \int_{A^c} \vec{\nabla} q_0 \cdot \vec{\nabla} v \right\}, \quad \forall v \in V, \tag{2.14}$$

$$\int_A \bar{\mu}\, \vec{\nabla} q_{k+1} \cdot \vec{\nabla} v = -\mu_0 \int_{A^c} \vec{\nabla} q_k \cdot \vec{\nabla} v, \quad \forall v \in V, \quad k = 1, 2, \ldots . \tag{2.15}$$

We state and prove the main result of this section.

PROPOSITION 2.2: *There exists* $t_0 > 1$ *such that*

$$\psi(t) = \sum_{k=0}^{\infty} \frac{1}{t^k} q_k, \quad t \geq t_0 \tag{2.16}$$

where the convergence holds in space V, uniformly with respect to t.

Proof: Setting $\psi_0(t) = \psi(t)$ we define recursively the elements of V

$$\psi_{k+1}(t) = t(\psi_k(t) - q_k), \quad k = 0, 1, 2, \ldots . \tag{2.17}$$

Dividing (2.17) by t^{k+1} and summing for k from 0 to N, we get

$$\psi(t) = \sum_{k=0}^{N} \frac{1}{t^k} q_k + \frac{1}{t^{N+1}} \psi_{N+1}(t) . \tag{2.18}$$

Using the notations (2.8), (2.17), we substract (2.13) multiplied by t from (2.7):

$$\int_A \bar{\mu}\, \vec{\nabla} \psi_1(t) \cdot \vec{\nabla} v = -\mu_0 \left\{ \int_A \vec{H}_* \cdot \vec{\nabla} v + \int_{A^c} \vec{\nabla} \psi(t) \cdot \vec{\nabla} v \right\}, \quad \forall v \in V. \tag{2.19}$$

Substracting (2.14) from (2.19), we obtain for $k = 1$

$$\int_A \bar{\mu}\, \vec{\nabla} \psi_{k+1}(t) \cdot \vec{\nabla} v = -\mu_0 \int_{A^c} \vec{\nabla} \psi_k(t) \cdot \vec{\nabla} v, \quad \forall v \in V, \quad k = 1, 2, \ldots ; \tag{2.20}$$

one easily verifies with the help of (2.15) that (2.20) is true for all $k \geq 1$.

From H4, (2.12), (2.19), (2.20) we deduce the existence of two constants c and d independent of t and k such that

$$\int_A |\vec{\nabla} \psi_k(t)|^2 \leq d c^k \left(\int_A |\vec{H}_*|^2 + \int_A |\vec{\nabla} \psi(t)|^2 \right), \quad t \geq 1, \quad k = 1, 2, \ldots . \tag{2.21}$$

From (2.7) it easily follows that $\int_A |\vec{\nabla} \psi(t)|^2$ is uniformly bounded with respect to $t \geq 1$; Relations (2.18), (2.21) allow to conclude the proof by setting $t_0 > \max(1, \sqrt{c})$.

□

Remark 2.3: By (2.3) and Proposition 2.2 we set $\vec{Q}_0 = \vec{H}_* - \vec{\nabla} q_0$, $\vec{Q}_k = -\vec{\nabla} q_k$, $k \geq 1$, and obtain the series expansion (1.1) valid in $(L^2(\mathbb{R}^3))^3$ for $t > t_0$.

Remark 2.4: By (2.13) and Lemma 2.1 we have $\vec{H}_* = \vec{\nabla} q_0$ in Λ if and only if the restriction of $\vec{H}_*$ to Λ is a gradient; this will be the case in particular if Λ is simply connected. Let in Λ $\vec{H}(t) = \vec{H}_* - \vec{\nabla}\psi(t)$, $\vec{B}(t) = t\bar{\mu}\vec{H}(t)$, $E(t) = \frac{1}{2}\int_\Lambda \mu|\vec{H}(t)|^2$ be the magnetic field, the induction field and the magnetic energy. We use Proposition 2.2 and consider two cases. First suppose that $\vec{H}_*$ is a gradient in Λ; then, in Λ, $\lim_{t\to\infty} \vec{H}(t) = \vec{0}$, $\lim_{t\to\infty} \vec{B}(t) = -\bar{\mu}\,\vec{\nabla} q_1$, $\lim_{t\to\infty} E(t) = \frac{1}{2}\int_\Lambda |\bar{\mu}\vec{\nabla} q_1|^2$. Next suppose that $\vec{H}_*$ is not a gradient in Λ; then, in Λ, $\lim_{t\to\infty} \vec{H}(t) = \vec{H}_* - \vec{\nabla} q_0 \neq \vec{0}$ and $\lim_{t\to\infty} E(t) = \infty$.

3. THE LIMIT CASE OF AN INFINITELY THIN PLATE

We consider the frame studied in Section 2 for the limit case $t = \infty$, i.e. by (2.3) and Proposition 2.2, the magnetic field is given by

$$\vec{H} = \vec{H}_* - \vec{\nabla} q_0 \quad \text{in } \mathbb{R}^3 . \tag{3.1}$$

We particularize the geometry by assuming that the ferromagnetic domain is a plate of the form

$$\Lambda = \Lambda(\delta) = \{\vec{x} = (x_1, x_2, x_3) \,|\, (x_1, x_2) \in G, 0 < x_3 < \delta\} \tag{3.2}$$

where $0 < \delta \leq \delta_0$ is a parameter. We complete Hypotheses H1-H4 of Section 2 by assuming.

H5. G is a bounded Lipschitzian subset of $\mathbb{R}^2$.

H6. There exists an open bounded set containing $\bar{\Lambda}(\delta_0)$ on which $\vec{H}_*$ is of class C^∞.

H7. $\bar{\mu}$ is independent of x_3, i.e. there exists $\tilde{\mu} \in L^\infty(G)$ such that $\bar{\mu}(x_1, x_2, x_3) = \tilde{\mu}(x_1, x_2)$.

Since we shall not use q_k for $k \geq 1$ and study the dependence with respect to δ, it is convenient to adopt from now on the notation $q(\delta)$ instead of q_0. For the limit case $\delta \to 0$, define

$$\tilde{r} \in H^1(G), \quad \int_G \tilde{\mu}\, \tilde{\vec{\nabla}} \tilde{r} \cdot \tilde{\vec{\nabla}} v = \int_G \tilde{\mu} \tilde{\vec{H}}_* \cdot \tilde{\vec{\nabla}} v, \quad \forall v \in H^1(G), \quad \int_G \tilde{r} = 0 , \tag{3.3}$$

where $\vec{H}_* = (H_{*1}, H_{*2}, H_{*3})$, $\tilde{\vec{H}}_*(x_1, x_2) = (H_{*1}(x_1, x_2, 0), H_{*2}(x_1, x_2, 0))$ and where $\tilde{\vec{\nabla}} = (\partial_1, \partial_2)$ is the twodimensional gradient. Although in H5 G is a subset of $\mathbb{R}^2$, we shall consider it in the following as a subset of the x_1, x_2 plane in $\mathbb{R}^3$; $\bar{G}$ is its closure and $\bar{G}^c = \mathbb{R}^3 - \bar{G}$. In the rest of this section, ∂B will denote the boundary of a ball containing $\bar{\Lambda}(\delta_0)$. Furthermore, as in the Annex, see Proposition A1, we shall use in $W^1(\mathbb{R}^3)$ the norm $|v| = \left(\int_{\mathbb{R}^3} |\vec{\nabla} v|^2\right)^{1/2}$.

We now state the main result of this section.

PROPOSITION 3.1: *There exist $\tilde{q} \in W^1(\mathbb{R}^3)$ and $\tilde{d} \in \mathbb{R}$ satisfying*

a)

$$\lim_{\delta\to 0} q(\delta) = \tilde{q} \quad \text{in } W^1(\mathbb{R}^3) ; \tag{3.4}$$

b)

$$\Delta \tilde{q} = 0 \text{ in } \bar{G}^c, \quad \tilde{q} = \tilde{r} + \tilde{d} \text{ on } \bar{G}, \quad \int_{\partial B} \frac{d\tilde{q}}{dn} = 0\,, \tag{3.5}$$

where the relation $\tilde{q} = \tilde{r} + \tilde{d}$ *must be understood in the sense of traces.*

Remark 3.1: One can show that the properties (3.5) completely characterize $\tilde{q} \in W^1(\mathbb{R}^3)$.

Remark 3.2: Proposition 3.1 has the following physical consequence: for a large magnetic susceptibility, the influence of the plate on the magnetic field does not depend on its thickness.

Remark 3.3: One can show that $\frac{1}{\delta}\int_{\Lambda(\delta)} |H_{*3} - \partial_3 q(\delta)|^2 = O(\delta)$ which is likely not an optimal estimate. This result, together with (3.1) and Proposition 3.1, can be interpreted physically by saying that in G the magnetic field $\vec{H}$ is given by the vector $(H_{*1}(x_1, x_2, 0) - \partial_1 \tilde{r}(x_1, x_2), H_{*2}(x_1, x_2, 0) - \partial_2 \tilde{r}(x_1, x_2), 0)$.

Before proving Proposition 3.1 we first establish.

LEMMA 3.1: *For* $p \in W^1(\mathbb{R}^3)$, *let* $w(\delta)$, $0 < \delta < \delta_0$ *and* $\tilde{w}$ *satisfy*

$$w(\delta) \in W_0^1(\bar{\Lambda}^c(\delta)), \quad \int_{\mathbb{R}^3} \vec{\nabla}(p + w(\delta)) \cdot \vec{\nabla} v = 0, \quad \forall v \in W_0^1(\bar{\Lambda}^c(\delta))\,, \tag{3.6}$$

$$\tilde{w} \in W_0^1(\bar{G}^c), \quad \int_{\mathbb{R}^3} \vec{\nabla}(p + \tilde{w}) \cdot \vec{\nabla} v = 0, \quad \forall v \in W_0^1(\bar{G}^c)\,, \tag{3.7}$$

where, by using extensions by zero, $W_0^1(\bar{\Lambda}^c(\delta))$ *and* $W_0^1(\bar{G}^c)$ *are considered as subsets of* $W^1(\mathbb{R}^3)$. *Then* $\lim_{\delta \to 0} w(\delta) = \tilde{w}$ *in* $W^1(\mathbb{R}^3)$.

Proof: We first show that $|w(\delta) - \tilde{w}|^2$ is an increasing function of δ. Let $0 < a < b < \delta_0$; since $W_0^1(\bar{\Lambda}^c(b)) \subset W_0^1(\bar{\Lambda}^c(a)) \subset W_0^1(\bar{G}^c)$, we have by (3.6), (3.7):

$$|p + w(a)|^2 = |p + \tilde{w}|^2 + |w(a) - \tilde{w}|^2, \tag{3.8}$$

$$|p + w(b)|^2 = |p + \tilde{w}|^2 + |w(b) - \tilde{w}|^2, \tag{3.9}$$

$$|p + w(b)|^2 = |p + w(a)|^2 + |w(b) - w(a)|^2\,. \tag{3.10}$$

Eliminating $|p + w(a)|^2$ and $|p + w(b)|^2$ in (3.8)-(3.10) we get the relation $|w(b) - \tilde{w}|^2 = |w(a) - \tilde{w}|^2 + |w(b) - w(a)|^2$, i.e. $|w(b) - \tilde{w}|^2 \geq |w(a) - \tilde{w}|^2$. In order to complete the proof, it suffices to find a sequence δ_k converging to zero such that $w(\delta_k)$ converges to $\tilde{w}$. To this end let $v_k \in C_0^\infty(\bar{G}^c)$ be such that $\lim_{k \to \infty} |v_k - \tilde{w}| = 0$; then there exists $0 < \delta_k < \delta_0$ such that $v_k \in W_0^1(\bar{\Lambda}^c(\delta_k))$ and from (3.6), (3.7) we deduce

$$|p + v_k|^2 = |p + \tilde{w}|^2 + |v_k - \tilde{w}|^2\,, \tag{3.11}$$

$$|p + v_k|^2 = |p + w(\delta_k)|^2 + |v_k - w(\delta_k)|^2\,, \tag{3.12}$$

$$|p + w(\delta_k)|^2 = |p + \tilde{w}|^2 + |w(\delta_k) - \tilde{w}|^2\,. \tag{3.13}$$

Eliminating $|p+v_k|^2$ and $|p+w(\delta_k)|^2$ in (3.11)-3.13) we get the relation $|w(\delta_k)-\tilde{w}|^2+|v_k-w(\delta_k)|^2=|v_k-\tilde{w}|^2$ from which follows $\lim_{k\to\infty}|w(\delta_k)-\tilde{w}|=0$. □

Proof of Proposition 3.1: We introduce several auxiliary quantities. For $0<\delta\leq\delta_0$, $r(\delta)\in H^1(\Lambda(\delta))$ is defined by the variational relation

$$\int_{\Lambda(\delta)}\bar{\mu}\,\vec{\nabla}r(\delta)\cdot\vec{\nabla}v=\int_{\Lambda(\delta)}\bar{\mu}\vec{H}_*\cdot\vec{\nabla}v,\quad\forall v\in H^1(\Lambda(\delta)),\quad\int_{\Lambda(\delta)}r(\delta)=0\,. \tag{3.14}$$

Then we set

$$\phi:\Lambda(\delta_0)\to\mathbb{R},\quad\phi(x_1,x_2,x_3)=\int_0^{x_3}H_{*3}(x_1,x_2,\xi)\,d\xi\,, \tag{3.15}$$

$$r_1(\delta):\Lambda(\delta)\to\mathbb{R},\quad r_1(\delta)=r(\delta)-\phi\,. \tag{3.16}$$

$r_2(\delta)$ will denote an extension of $r_1(\delta)$ obtained by symmetry and periodicity and characterized by the relations

$$r_2(\delta):G\times\mathbb{R}\to\mathbb{R},\quad r_2(\delta)\,(x_1,x_2,-x_3)=r_2(\delta)\,(x_1,x_2,x_3)\,,$$

$$r_2(\delta)\,(x_1,x_2,x_3+2\,\delta)=r_2(\delta)\,(x_1,x_2,x_3),\quad r_2(\delta)=r_1(\delta)\text{ in }\Lambda(\delta)\,, \tag{3.17}$$

where $(x_1,x_2)\in G$, $x_3\in\mathbb{R}$. We construct $r_3(\delta)$ and $r_4(\delta)$ in the following way:

$$r_3(\delta)\in W_0^1(\mathbb{R}^3),\quad r_3(\delta)=r_2(\delta)\text{ on }\Lambda(\delta),\quad\Delta r_3(\delta)=0\text{ in }\bar{\Lambda}^c(\delta)\,, \tag{3.18}$$

$$r_4(\delta)\in W_0^1(\mathbb{R}^3),\quad r_4(\delta)=r_2(\delta)\text{ on }\Lambda(\delta_0),\quad\Delta r_4(\delta)=0\text{ in }\bar{\Lambda}^c(\delta_0)\,; \tag{3.19}$$

note that $r_1(\delta)$, $r_2(\delta)$, $r_3(\delta)$ and $r_4(\delta)$ coincide on $\Lambda(\delta)$. Corresponding to $r_2(\delta)$, $r_3(\delta)$, we define from (3.3):

$$\tilde{r}_2:G\times\mathbb{R}\to\mathbb{R},\quad\tilde{r}_2(x_1,x_2,x_3)=\tilde{r}(x_1,x_2)\,, \tag{3.20}$$

$$\tilde{r}_3(\delta)\in W_0^1(\mathbb{R}^3),\quad\tilde{r}_3(\delta)=\tilde{r}_2\text{ on }\Lambda(\delta),\quad\Delta\tilde{r}_3(\delta)=0\text{ in }\bar{\Lambda}^c(\delta)\,. \tag{3.21}$$

Next let

$$\vec{K}:\Lambda(\delta_0)\to\mathbb{R}^3,\quad\vec{K}=(H_{*1}-\partial_1\phi,H_{*2}-\partial_2\phi,0)\,, \tag{3.22}$$

$$\vec{K}_2(\delta):G\times\mathbb{R}\to\mathbb{R}^3,\quad\vec{K}_2(\delta)\,(x_1,x_2,-x_3)=\vec{K}_2(\delta)\,(x_1,x_2,x_3)\,, \tag{3.23}$$

$$\vec{K}_2(\delta)\,(x_1,x_2,x_3+2\,\delta)=\vec{K}_2(\delta)\,(x_1,x_2,x_3),\quad\vec{K}_2(\delta)=\vec{K}\text{ in }\Lambda(\delta)\,, \tag{3.24}$$

$$\tilde{\vec{K}}:G\times\mathbb{R}\to\mathbb{R}^3,\quad\tilde{\vec{K}}(x_1,x_2,x_3)=(H_{*1}(x_1,x_2,0),H_{*2}(x_1,x_2,0),0)\,, \tag{3.25}$$

where $(x_1,x_2)\in G$, $x_3\in\mathbb{R}$.

By (3.15), (3.22)-(3.25) and Hypothesis H6, we have

$$\lim_{\delta\to0}\int_{\Lambda(2\,\delta_0)}\bar{\mu}|\vec{K}_2(\delta)-\tilde{\vec{K}}|^2=0\,. \tag{3.26}$$

By (3.14), (3.15), (3.16), (3.22), we get

$$\int_{\Lambda(\delta)} \bar{\mu}\vec{\nabla} r_1(\delta) \cdot \vec{\nabla} v = \int_{\Lambda(\delta)} \bar{\mu}\vec{K} \cdot \vec{\nabla} v, \quad \forall v \in H^1(\Lambda(\delta)) . \tag{3.27}$$

By (3.17), (3.23), (3.24) and Hypothesis H7, we deduce from (3.27) for any positive integer n

$$\int_{\Lambda(\delta)} \bar{\mu}\vec{\nabla} r_2(\delta) \cdot \vec{\nabla} v = \int_{\Lambda(n\delta)} \bar{\mu}\vec{K}_2(\delta) \cdot \vec{\nabla} v, \quad \forall v \in H^1(\Lambda(n\delta)) ; \tag{3.28}$$

trivially (3.3), (3.20), (3.25) imply

$$\int_{\Lambda(n\delta)} \bar{\mu}\vec{\nabla} \tilde{r}_2 \cdot \vec{\nabla} v = \int_{\Lambda(n\delta)} \bar{\mu}\vec{\tilde{K}} \cdot \vec{\nabla} v, \quad \forall v \in H^1(\Lambda(n\delta)) ; \tag{3.29}$$

we choose n such that $\delta_0 < n\delta < 2\,\delta_0$ and deduce from (3.26), (3.28), (3.29)

$$\lim_{\delta \to 0} \int_{\Lambda(\delta_0)} \bar{\mu} | \vec{\nabla}(r_2(\delta) - \tilde{r}_2)|^2 = 0 . \tag{3.30}$$

By (3.3), (3.14)-(3.17), (3.20), we have

$$\int_{\Lambda(\delta_0)} \tilde{r}_2 = 0, \quad \lim_{\delta \to 0} \int_{\Lambda(\delta_0)} r_2(\delta) = 0 , \tag{3.31}$$

which, together with (3.30) implies

$$\lim_{\delta \to 0} r_2(\delta) = \tilde{r}_2 \quad \text{in } H^1(\Lambda(\delta_0)) , \tag{3.32}$$

and by (3.19), (3.21) and Remark A1 of the Annex

$$\lim_{\delta \to 0} |r_4(\delta) - \tilde{r}_3(\delta_0)| = 0 . \tag{3.33}$$

As in the first part of the proof of Lemma 3.1, one easily shows that $|r_3(\delta) - \tilde{r}_3(\delta)| \leqslant |r_4(\delta) - \tilde{r}_3(\delta_0)|$ so that we get by (3.33)

$$\lim_{\delta \to 0} |r_3(\delta) - \tilde{r}_3(\delta)| = 0 . \tag{3.34}$$

Let us set for $0 < \delta < \delta_0$

$$\phi_1(\delta) \in W^1(\mathbb{R}^3), \quad \phi_1(\delta) = \phi \text{ on } \Lambda(\delta), \quad \Delta\phi_1(\delta) = 0 \text{ in } \bar{\Lambda}^c(\delta) . \tag{3.35}$$

By (3.15) and Hypothesis H6 one can easily construct a function $\phi_2(\delta) \in W^1(\mathbb{R}^3)$, equal to ϕ on $\Lambda(\delta)$, such that $\lim_{\delta \to 0} |\phi_2(\delta)| = 0$. Since, see the proof of Lemma 3.1, $|\phi_1(\delta)| \leqslant |\phi_2(\delta)|$ we have

$$\lim_{\delta \to 0} |\phi_1(\delta)| = 0 . \tag{3.36}$$

We finally introduce

$$z(\delta) \in W^1(\mathbb{R}^3), \quad z(\delta) = 1 \text{ on } \Lambda(\delta), \quad \Delta z(\delta) = 0 \text{ in } \bar{\Lambda}^c(\delta) . \tag{3.37}$$

Using Lemma 3.1, we obtain the existence of functions $\tilde{r}_4$ and $\tilde{z} \in W^1(\mathbb{R}^3)$ satisfying

$$\lim_{\delta \to 0} |\tilde{r}_3(\delta) - \tilde{r}_4| = 0, \quad \lim_{\delta \to 0} |z(\delta) - \tilde{z}| = 0 . \tag{3.38}$$

By Proposition A4, we get

$$\int_{\partial B} \frac{dz(\delta)}{dn} < 0, \quad 0 < \delta < \delta_0, \quad \int_{\partial B} \frac{d\tilde{z}}{dn} < 0 . \tag{3.39}$$

We recall that we use here the notation $q(\delta)$ instead of q_0 instead of q_0 which is defined in (2.13). Taking (2.11), (3.14), (3.16) and (3.18) into account, we can write

$$q(\delta) = r_3(\delta) + \phi_1(\delta) + d(\delta)\, z(\delta), \quad 0 < \delta < \delta_0 , \tag{3.40}$$

where by (3.39) $d(\delta) \in \mathbb{R}$ is determined uniquely by the condition $\int_{\partial B} \frac{dq(\delta)}{dn} = 0.$

We let δ tend to zero; by (3.34) and (3.38), $r_3(\delta)$ converges to $\tilde{r}_4$, $z(\delta)$ converges to $\tilde{z}$ and as a consequence $d(\delta)$ converges to a number $\tilde{d}$. It follows by (3.40) and (3.36) that $q(\delta)$ converges in $W^1(\mathbb{R}^3)$ towards an element $\tilde{q} = \tilde{r}_4 + \tilde{d}\tilde{z}$; we notice by (3.20), (3.21) that the trace of $\tilde{r}_4$ on $\bar{G}$ is equal to $\tilde{r}$; this concludes the proof of Proposition 3.1.

□

4. NUMERICAL ALGORITHMS AND RESULTS

The purpose of this section is to present numerical tests illustrating the results of Sections 2 and 3 and to sketch without mathematical justification the description of the algorithmes we have used to obtain them.

All quantities will be expressed in the system of units MKSA, i.e. lengths in meters, electric currents in Amperes, intensity of magnetic fields in Amperes/meter.

We first fix the geometry. To this end, see *figure 1,* let $0 < \delta < 0.1$ and

$$D = [1/3, 2/3] \times [1/3, 2/3] \times \{0\} ,$$

$$G_1 = (0, 1) \times (0, 1) \times \{0\} \backslash D, \quad G_2 = \{0\} \times (0, 1) \times (0, 1) ,$$

$$G_3 = G_1 \cup G_2 \cup \{(0, x_2, 0) \mid x_2 \in (0, 1)\} ,$$

$$\Lambda_1 = \{(x_1, x_2, x_3) \mid (x_1, x_2, 0) \in G_1, x_3 \in (0, \delta)\} ,$$

$$\Lambda_2 = (0, \delta) \times (0, 1) \times (0, 1), \quad \Lambda_3 = \Lambda_1 \cup \Lambda_2 .$$

Let A_k, $1 \leqslant k \leqslant 4$, be four points with coordinates $A_1(0.4, 0.6, 0)$, $A_2(2.4, -1.4, 10)$, $A_3(-0.4, 0.6, 0)$, $A_4(-2.4, -1.4, 10)$. d_1 and d_2 are the oriented straight lines passing from A_1 to A_2 and from A_3 to A_4 respectively. They represent wires in which run electric currents of 500 Amperes; let $\vec{J}_k$ be the current density defined by d_k, $1 \leqslant k \leqslant 2$ and set $\vec{J}_3 = \vec{J}_1 + \vec{J}_2$; $\vec{J}_k$ is in fact a vector distribution, $1 \leqslant k \leqslant 3$.

In the tests, the relative susceptibility is either a given constant or a nonlinear function of $|\vec{H}|$, $\mu_R = \kappa(|\vec{H}|)$, where $\kappa \in C^1([0, \infty))$ satisfies the standard properties $\kappa'(\xi) \leqslant 0$, $(\xi\kappa(\xi))' \geqslant \kappa_0 > 1$, $\xi \in [0, \infty)$, see *figure 2.*

We need three algorithms A1, A2, A3. Algorithm A1 is the "reference algorithm" which numerically solves the original problem (2.3), (2.7). Algorithm A2 is based on the results on Section 2 and allows to compute the

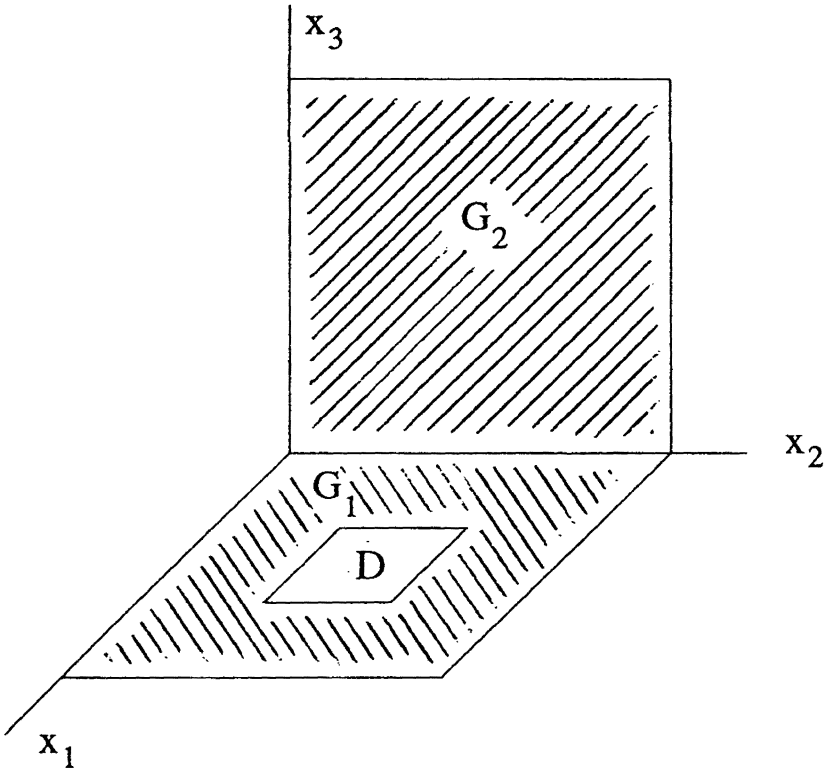

Figure 1. — Geometry of the plates and coordinates system.

approximation $\vec{Q}_0$ of $\vec{H}$, see (1.1), corresponding to a large susceptibility. Algorithm A3 exploits the results of Section 3 for the situation of large susceptibility and thin ferromagnetic plates. We describe these algorithms in the frame of the geometry Λ_1 (for A1, A2), of G_1 (for A3) and the current density $\vec{J}_1$.

To begin with A1, let $\vec{R} \in \mathscr{L}((L^2(\Lambda_1))^3, (L^2(\mathbb{R}^3))^3)$ be the operator defined by the relations $\vec{R}(\vec{N}) = \vec{\nabla}\varphi$ where $\vec{N} \in (L^2(\Lambda_1))^3$,

$$\varphi \in W^1(\mathbb{R}^3), \quad \mu_0 \int_{\mathbb{R}^3} \vec{\nabla}\varphi \cdot \vec{\nabla} v = \int_{\Lambda_1} \vec{N} \cdot \vec{\nabla} v, \quad \forall v \in W^1(\mathbb{R}^3) . \tag{4.1}$$

The operator $\vec{R}$ is studied in detail in [5]; it admits the integral representation which is used for numerical purposes

$$\vec{R}(\vec{N})(\vec{x}) = \frac{1}{4\pi\mu_0} \vec{\nabla} \int_{\Lambda_1} \vec{N}(\xi) \cdot \vec{\nabla}_\xi \left(\frac{1}{|\vec{x} - \vec{\xi}|} \right) d\xi, \quad \vec{x} \in \mathbb{R}^3 . \tag{4.2}$$

From (2.33), (2.6) and (4.1), we get the following relation for the unknown $\vec{H}$:

$$\vec{H} + \vec{R}((\mu - \mu_0)\vec{H}) = \vec{H}_* \quad \text{in } \mathbb{R}^3 . \tag{4.3}$$

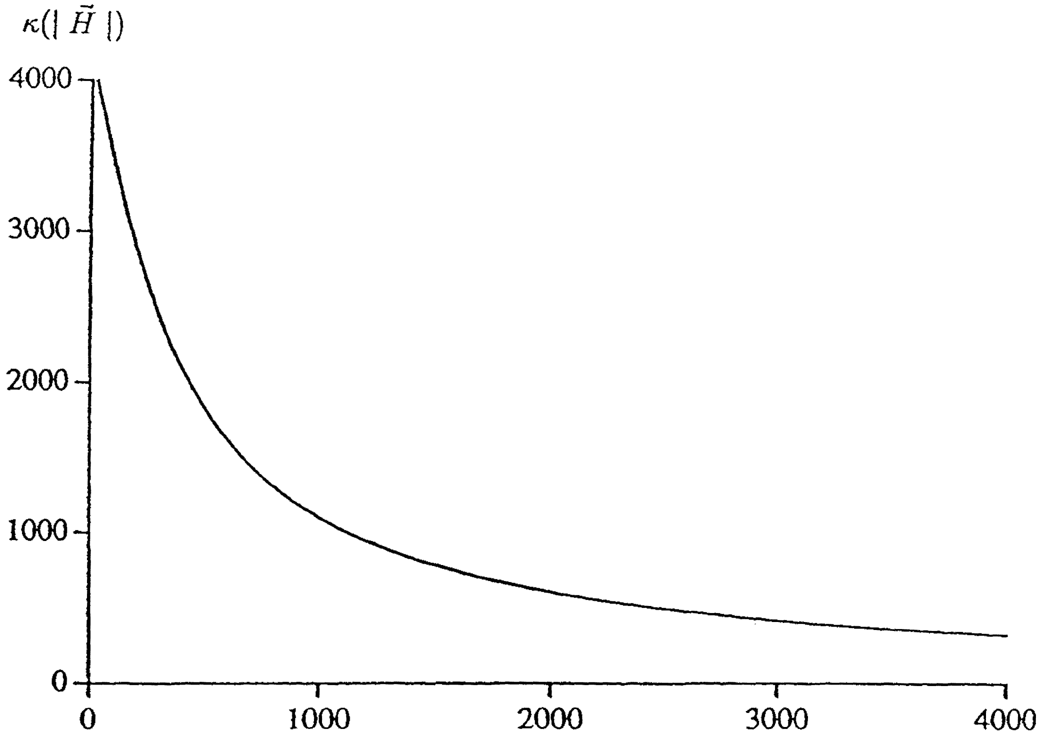

Figure 2. — Graph of the function $\mu_R = \kappa(|\vec{H}|)$.

The basic principle of GFUN, see [1], is to apply to (4.3), restricted to Λ_1, the Galerkin method with a piecewise constant functions finite element subspace V_h of $(L^2(\Lambda_1))^3$, so that $\vec{H}$ is approximated by $\vec{H}_h \in V_h$ satisfying

$$\int_{\Lambda_1} \vec{H}_h \cdot \vec{N} + \int_{\Lambda_1} \vec{R}((\mu - \mu_0)\vec{H}_h) \cdot \vec{N} = \int_{\Lambda_1} \vec{H}_* \cdot \vec{N}, \quad \forall \vec{N} \in V_h . \tag{4.4}$$

However, (4.4) is not adapted to the data Λ_1, $\vec{J}_1$, because it induces a "locking type phenomenon". To see this, we suppose for the sake of simplicity that μ is constant in Λ_1 ; in accordance with (2.8), we set $\mu = t$ with $\bar{\mu} = 1$. By Remark 2.4, for large values of t, $\vec{H}$ will be different from zero on Λ. On another side, one easily verifies the following properties: $\vec{R}$ is positive semidefinite for the scalar product of $(L^2(\Lambda_1))^3$ and the kernel of $\vec{R}$ restricted to V_h contains only the nul element. Setting $\vec{N} = \vec{H}_h$ in (4.4), we obtain, for a fixed V_h, the existence of a constant c independent of t such that

$$\|\vec{H}_h\|_{(L^2(\Lambda_1))^3} \leq \frac{c}{t - \mu_0} \|\vec{H}_*\|_{(L^2(\Lambda_1))^3} . \tag{4.5}$$

(4.5) shows that for large values of t, $\vec{H}_h$ is small and cannot be a realistic approximation of $\vec{H}$.

Coming back to the general situation of a non constant magnetic susceptiblity μ, we see that Proposition 2.2 supplies an easy remedy to the locking problem. Relation (2.13) suggests to define u by the relations

$$u \in H^1(\Lambda_1), \quad \int_{\Lambda_1} u = 0, \quad \int_{\Lambda_1} (\mu - \mu_0)(\vec{\nabla} u - \vec{H}_*) \cdot \vec{\nabla} v = 0, \quad \forall v \in H^1(\Lambda_1). \tag{4.6}$$

By (4.1) we have $\vec{R}((\mu - \mu_0)(\vec{\nabla} u - \vec{H}_*)) = \vec{0}$ so that by setting

$$\vec{H} = \vec{H}_* - \vec{\nabla} u + \vec{S} \quad \text{in } \Lambda_1, \tag{4.7}$$

we obtain by (4.3) the following equation for the new unknown $\vec{S}$

$$\vec{S} + \vec{R}((\mu - \mu_0)\vec{S}) = \vec{\nabla} u \quad \text{in } \Lambda_1. \tag{4.8}$$

If μ is a given function, Relations (4.6), (4.8) form an uncoupled linear system for u and $\vec{S}$*;* however in the nonlinear case where $\mu = \kappa(|\vec{H}|)$, the system is coupled. We notice that once $\vec{S}$ and $\vec{\nabla} u$ are known in Λ_1, one obtains $\vec{H} = \vec{H}_* - \vec{R}((\mu - \mu_0)\vec{S})$ in $\mathbb{R}^3$. The reference method A1 is defined as the finite element Galerkin approximation of (4.6), (4.8) (Q_1 elements for (4.6), piecewise constant elements as in (4.4) for (4.8)).

Remark 4.1: The locking problem we have pointed out, and dramatically verified by numerical calculations, is related but nevertheless different from the one mentioned in [9] page 436; indeed this paper is restricted to the situation where $\vec{H}_*$ is a gradient on the ferromagnetic domain.

The purpose of the algorithm A2 is to exploit the approximation, valid for large values of μ, obtained by restricting the series (1.1) to its first term $\vec{Q}_0$. By (2.3) and Proposition 2.2 one has

$$\vec{Q}_0(\vec{x}) = \vec{H}_*(\vec{x}) - \vec{\nabla} q_0(\vec{x}) \tag{4.9}$$

where q_0 is defined in (2.13). Because of (2.8), we can replace $\bar{\mu}$ by μ in (2.13). We introduce the auxiliary quantities $\tilde{q} \in H^1(\Lambda_1)$ and $d \in \mathbb{R}$ and by Lemma 2.1 characterize q_0 by the relations

$$\int_{\Lambda_1} \mu(\vec{\nabla}\tilde{q} - \vec{H}_*) \cdot \vec{\nabla} v = 0, \quad \forall v \in H^1(\Lambda_1), \quad \int_{\Lambda_1} \tilde{q} = 0, \tag{4.10}$$

$$q_0 \in W^1(\mathbb{R}^3), \quad q_0 = \tilde{q} + d \text{ in } \Lambda_1, \quad \Delta q_0 = 0 \text{ in } \mathbb{R}^3 - \bar{\Lambda}_1, \quad \int_{\partial B} \frac{dq_0}{dn} = 0, \tag{4.11}$$

where B is a ball containing $\bar{\Lambda}_1$. We represent q_0 in $\mathbb{R}^3 - \Lambda_1$ with the help of a single layer potential $\gamma \in H^{-1/2}(\partial \Lambda_1)$

$$q_0(\vec{x}) = \int_{\partial \Lambda_1} \frac{\gamma(\vec{\xi})}{|\vec{x} - \vec{\xi}|} d\sigma_\xi, \quad \vec{x} \in \mathbb{R}^3 - \Lambda_1. \tag{4.12}$$

The last condition of (4.11) is equivalent to the requirement

$$\int_{\partial \Lambda_1} \gamma = 0. \tag{4.13}$$

The relation $q_0 = \tilde{q} + d$ on Λ_1 implies the boundary integral equation

$$\int_{\partial \Lambda_1} \frac{\gamma(\vec{\xi})}{|\vec{x} - \vec{\xi}|} d\sigma_{\xi} = \tilde{q}(\vec{x}) + d, \quad \vec{x} \in \partial \Lambda_1 . \tag{4.14}$$

The algorithm A2 consists in approximating the separate problems (4.10) on one side, (4.13), (4.14) on the other side, by standard finite elements procedures. Note that (4.10) is a linear problem if μ is a given function. It is a nonlinear one if $\mu = \kappa(|\vec{H}_* - \vec{\nabla}\tilde{q}|)$; classical monotony arguments, see [13], allow to show that, with the properties we have assumed for κ, it possesses one and only one solution.

The algorithm A3 is the limite case of A2 when the thickness δ of the plate tends to zero. (4.6)-(4.14), (3.3), Proposition 3.1 and Remark 3.3 suggest to define:

$$\tilde{\vec{H}}_*(x_1, x_2) = (H_{*1}(x_1, x_2, 0), H_{*2}(x_1, x_2, 0)) , \tag{4.15}$$

$$\tilde{r} \in H^1(G_1), \quad \int_{G_1} \mu(\vec{\nabla}\tilde{r} - \tilde{\vec{H}}_*) \cdot \vec{\nabla} v = 0, \quad \forall v \in H^1(G_1), \quad \int_{G_1} \tilde{r} = 0 , \tag{4.16}$$

$\gamma \in H^{-1/2}(G_1)$ and $d \in \mathbb{R}$ such that

$$\left.\begin{aligned} &\int_{G_1} \frac{\gamma(\vec{\xi})}{|\vec{x} - \vec{\xi}|} d\sigma_{\xi} = \tilde{r}(\vec{x}) + d, \quad \vec{x} \in G_1, \\ &\int_{G_1} \gamma = 0, \end{aligned}\right\} \tag{4.17}$$

where $H^{-1/2}(G_1)$ is the dual of the space of the traces on G_1 of $W^1(\mathbb{R}^3)$ functions,

$$\tilde{q}(\vec{x}) = \int_{G_1} \frac{\gamma(\vec{\xi})}{|\vec{x} - \vec{\xi}|} d\sigma_{\xi}, \quad \vec{x} \in \mathbb{R}^3 , \tag{4.18}$$

$$\vec{H} = (H_{*1} - \partial_1 \tilde{r}, H_{*2} - \partial_2 \tilde{r}, 0) \quad \text{on } G_1, \tag{4.19}$$

$$\vec{H} = \vec{H}_* - \vec{\nabla}\tilde{q} \quad \text{in } \mathbb{R}^3 - \bar{G}_1 . \tag{4.20}$$

The algorithm A3 results from the discretization of (4.16), (4.17) by the finite element method (Q_1 for (4.16), piecewise constant functions for (4.17)). As for A2, for the nonlinear situation, one can prove existence and uniqueness.

The extension of the three algorithms to the other domains G_2, Λ_2, G_3, Λ_3 are obvious except for A3 on G_3 ; here, the function $\tilde{r}$ corresponding to (4.16) will be defined on G_3 with H^1 restrictions to G_1 and G_2 and equal traces on $\bar{G}_1 \cap \bar{G}_2$.

We now turn to the numerical results. Our first tests concern the geometries Λ_1, G_1 in connection with the current density $\vec{J}_1$. The symbol Σ_1 will denote the piece of surface obtained by translating G_1 by (-0.2) in the x_3 direction. Let $\vec{H}_1$, $\vec{H}_2$, $\vec{H}_3$ be the magnetic fields computed by the algorithms A1, A2, A3 respectively, whereas $\vec{H}_*$, as before, is the magnetic field in absence of ferromagnetic parts. We use the symbol $\| \cdot \|$ for the L^2-norm on Σ_1. We remark that the scattering effect of Λ_1 is relatively weak since for $\delta = 0.015$ (thickness of

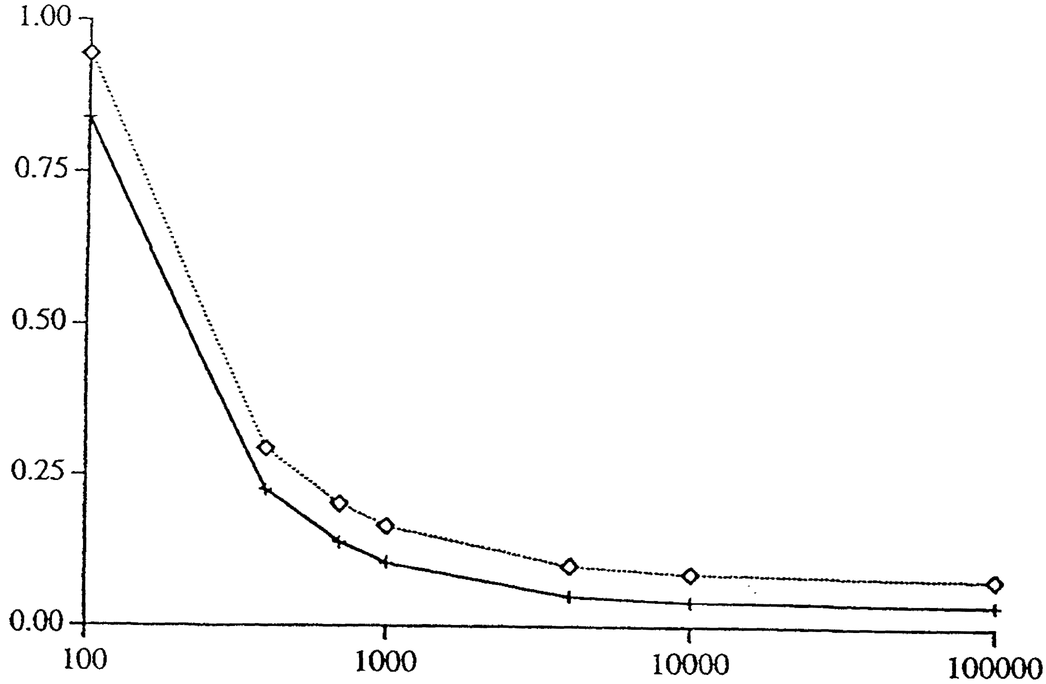

Figure 3. — Graph of $\|\vec{H}_2 - \vec{H}_1\| \,/\, \|\vec{H}_1 - \vec{H}_*\|$ (◇) and of $\|\vec{H}_3 - \vec{H}_1\| \,/\, \|\vec{H}_1 - \vec{H}_*\|$ (+), as functions of μ_R, on Σ_I.

Λ_1) and $\mu_R = 1\,000$ we have $\|\vec{H}_*\| = 305$ and $\|\vec{H}_1 - \vec{H}_*\| = 40.8$. *Figure 3* represents $\|\vec{H}_2 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\|$ and $\|\vec{H}_3 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\|$ as functions of μ_R for $\delta = 0.015$; it shows that the first quantity is smaller than the second one for large values of μ_R which can be expected from the theory developped in Sections 2 and 3. We check Remark 3.2: for $\mu_R = 10^4$ let $w(\delta) = \|\vec{H}_3 - \vec{H}_*\|$; we have that $w(0.03)/w(0.015) = w(0.015)/w(0.0075) = 0.88$. Here are some results concerning the nonlinear situation: the minimum and maximum values of μ_R in Λ_1 or G_1 relative to $\vec{H}_1$ or $\vec{H}_3$ are respectively 1345, 4087, 1416, 4090; the relative error of μ_R is 0.005 in L^2-norm; finally $\|\vec{H}_3 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\| = 0.16$.

The second series of tests are similar to the preceding one; they have been realized for the geometries Λ_2, G_2 with density $\vec{J}_2$; Σ_2 is obtained by translating G_2 by (-0.2) in the x_1 direction. $\|\ .\ \|$ is now the L^2-norm on Σ_2. If $\vec{H}_1$, $\vec{H}_2$, $\vec{H}_3$ denote the magnetic field computed with A1, A2, A3, *figure 4* represents $\|\vec{H}_2 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\|$ and $\|\vec{H}_3 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\|$ as functions of μ_R for $\delta = 0.015$. In the nonlinear situation, as expected from Remark 2.4, μ_R is close to the constant function with values 4100; the error $\|\vec{H}_3 - \vec{H}_1\|/\|\vec{H}_1 - \vec{H}_*\|$ is equal to 0.05.

Our last numerical experiment concerns the systems of twoplates Λ_3 with $\delta = 0.015$ and G_3 submitted to the action of $\vec{J}_3$. We restrict ourselves to the nonlinear situation and use the following notations: $\|\ .\ \|$ is the L^2-norm on G_3, $\vec{H}_1$ and $\vec{H}_3$ are the magnetic fields computed by A1 and A3 respectively, μ_{R1} and μ_{R3} are the corresponding relative susceptibility. We have $1306 \leqslant \mu_{R1} \leqslant 4137$, $1419 \leqslant \mu_{R3} \leqslant 4136$, $\|\mu_{R3} - \mu_{R1}\|/\|\mu_{R1}\| = 0.008$, $\|\vec{H}_3 - \vec{H}_1\|/\|\vec{H}_1\| = 0.05$.

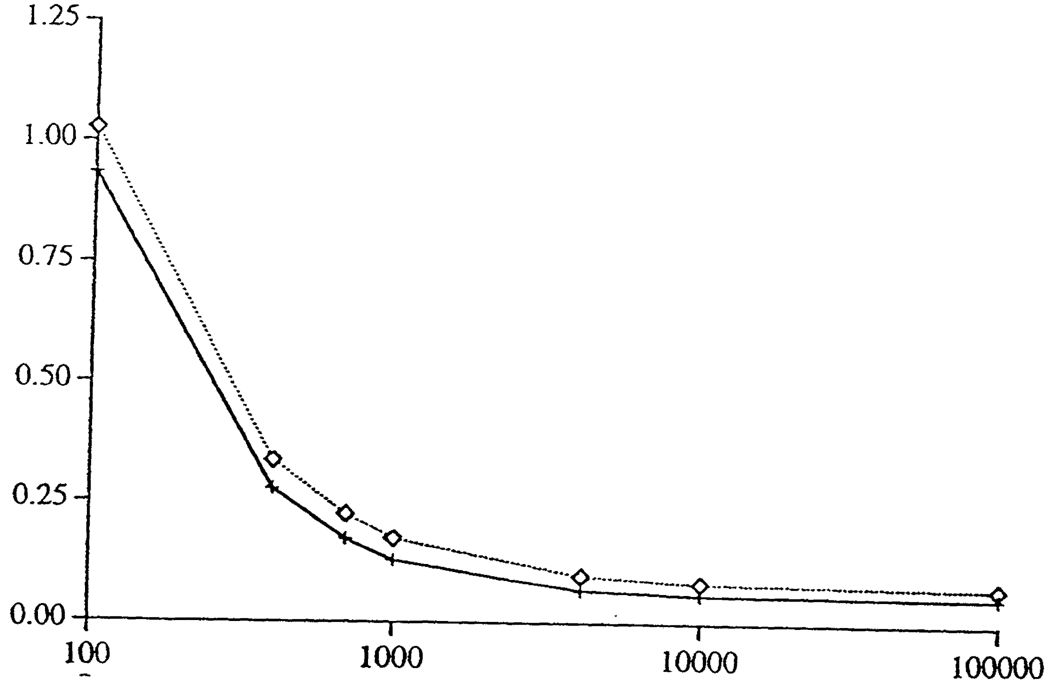

Figure 4. — Graph of $\|\vec{H}_2-\vec{H}_1\| \,/\, \|\vec{H}_1-\vec{H}_*\|$ **(◇) and of** $\|\vec{H}_3-\vec{H}_1\| \,/\, \|\vec{H}_1-\vec{H}_*\|$ **(+), as functions of** μ_R**, on** Σ_2.

We conclude this section by adding two remarks:

a) The 3D domains have been decomposed in parallelepipeds with edges of approximate lengths 0.04, 0.04, δ. The 2D domains have been decomposed in squares with sides of approximate length 0.04.

b) For discretizing (4.8) we have used the Galerkin finite element method characterized by piecewise constant functions on parallelepipeds; $\vec{R}$ is the integral operator (4.2). As analyzed in [5], for large values of μ the stiffness matrix is ill conditionned; we found it important to compute accurately its elements which is possible since they can be expressed by explicit formulae.

A. ANNEX

In this annex we collect some mathematical results. We begin with

DEFINITION A1: *For $\Omega \subset \mathbb{R}^3$, an open connected set the complement of which is bounded, possibly $\Omega = \mathbb{R}^3$, we set*

$$W^1(\Omega) = \{v : \Omega \to \mathbb{R} \mid (1+|\vec{x}|^2)^{-1/2}\, v \in L^2(\Omega),\ |\vec{\nabla} v| \in L^2(\Omega)\} \tag{A1}$$

equipped with norm and seminorm

$$\|v\| = \left(\int_\Omega (1+|\vec{x}|^2)^{-1} v^2 + |\vec{\nabla} v|^2\right)^{1/2}, \quad |v| = \left(\int_\Omega |\vec{\nabla} v|^2\right)^{1/2} \text{for } v \in W^1(\Omega), \tag{A2}$$

$$W_0^1(\Omega) = \text{closure of } C_0^\infty(\Omega) \quad \text{in } W^1(\Omega) \text{ for the norm } \|\ .\ \|. \tag{A3}$$

Functions belonging to $C_0^\infty(\Omega)$ *or to* $W_0^1(\Omega)$ *can be extended by zero outside* Ω *so that* $C_0^\infty(\Omega) \subset C^\infty(\mathbb{R}^3)$, $W_0^1(\Omega) \subset W^1(\mathbb{R}^3)$. *The following result can be found in [14], Chapter XI B.*

PROPOSITION A1: **a)** $W^1(\Omega)$ *and* $W_0^1(\Omega)$ *are Banach spaces for the norm* $\|\,.\,\|$; **b)** $W^1(\mathbb{R}^3) = W_0^1(\mathbb{R}^3)$; **c)** $\|\,.\,\|$ *and* $|\,.\,|$ *are equivalent norms in* $W^1(\Omega)$.

□

We next quote classical properties of potential theory, see for example [2].

For $0 \leqslant a < b$ we use the notations $C_a = \{\vec{x} \mid |\vec{x}| > a\}$, $C_{a,b} = \{\vec{x} \mid |\vec{x}| < b\}$.

PROPOSITION A2: *Let* $r_0 > 0$, $u : C_{r_0} \to \mathbb{R}$ *be a harmonic function,* $\tilde{u}(r, \theta, \varphi)$ *be its representation in spherical coordinates. Then*

a) There exists spherical harmonics $p_m(\theta, \varphi)$, $q_m(\theta, \varphi)$ of degree m such that

$$\tilde{u}(r, \theta, \varphi) = \sum_{m=0}^{\infty} (r^m p_m(\theta, \varphi) + r^{-m-1} q_m(\theta, \varphi)) \quad \text{for } r > r_0 . \tag{A4}$$

b) The series (A4) and the series obtained by differentiating (A4) term by term at any order converge absolutely and uniformly in $C_{a,b}$ for $r_0 < a < b$.

c) $\lim_{|\vec{x}| \to \infty} u(\vec{x}) = 0$ if and only if $p_m = 0$, $0 \leqslant m < \infty$.

□

Recalling that spherical harmonics of different degrees are orthogonal on the unit sphere we deduce from Definition A1 and Propositions A1, A2:

PROPOSITION A3: *Let* u *and* $\tilde{u}$ *be defined as in Proposition A2,* Γ *be the spherical surface centered in* $\vec{0}$ *with radius* $a > r_0$. *Then*

a) $\lim_{|\vec{x}| \to \infty} u(\vec{x}) = 0$ if and only if $u \in W^1(C_a)$ for all $a > r_0$; in this case we have

$$q_0 = \frac{1}{4\pi a} \int_\Gamma u = -\frac{1}{4\pi} \int_\Gamma \frac{du}{dn}$$

where $\frac{du}{dn}$ is the normal derivative corresponding to increasing values of r.

b) $|\vec{\nabla} u(\vec{x})| = O(|\vec{x}|^{-2})$ as $|\vec{x}| \to \infty$ if and only if $p_m = 0, 1 \leqslant m < \infty$.

□

Consider the following situation. Ω is an open connected subset of $\mathbb{R}^3$ the complement Ω^c of which is bounded. For $p \in W^1(\mathbb{R}^3)$ we state the problem to find w satisfying

$$w \in W_0^1(\Omega), \quad \int_\Omega \vec{\nabla}(p + w) \cdot \vec{\nabla} v = 0 \quad \forall v \in W_0^1(\Omega) . \tag{A5}$$

By Proposition A1, this problem has one and only one solution.

PROPOSITION A4: *Let* Ω, p, w *be as in the situation described above and set* $u = p + w$. *Let* $a > 0$ *be such that* $\bar{C}_a \subset \Omega$. *We suppose that* p *is non negative in* $\mathbb{R}^3$ *and that its restriction to* Ω *does not belong to* $W_0^1(\Omega)$. Then

a) $u(\vec{x}) > 0$, $\vec{x} \in \Omega$;

b) $\int_{\partial C_a} \frac{du}{dn} < 0$ where the normal is interior to C_a.

Proof:

a) From (A5) follows the Dirichlet principle: $\int_{\Omega} |\vec{\nabla} u|^2 \leqslant \int_{\Omega} |\vec{\nabla}(u+v)|^2$ for $v \in W_0^1(\Omega)$. Set $u = u^+ - u^-$ in Ω where $u^+ = \max(u, 0)$. We first show that $u^- = 0$. Since $\int_{\Omega} |\vec{\nabla} u|^2 = \int_{\Omega} |\vec{\nabla} u^+|^2 + \int_{\Omega} |\vec{\nabla} u^-|^2$, it suffices to show that $u^- \in W_0^1(\Omega)$. To this end we remark that there exists a sequence $w_n \in C_0^{\infty}(\Omega)$ converging to w; set $u_n = p + w_n$. Then support $(u_n^-) \subset$ support $(w_n) \subset \Omega$ and consequently $u_n^- \in W_0^1(\Omega)$; since u_n^- converges to u^-, we obtain $u^- \in W_0^1(\Omega)$. We have shown that $u \geqslant 0$ in Ω; since u is harmonic in Ω, in order to see that $u > 0$ in Ω, it suffices to remark that the restriction of u to Ω does not vanish which is a consequence of the fact that it does not belong to $W_0^1(\Omega)$.

b) Part b) is a consequence of Part a) and of Proposition A3 a).

□

Remark A1: Suppose in (A5) that Ω^c is the closure of a Lipschitzian domain and set $u = p + w$. u has be following properties: **a)** $u \in W^1(\mathbb{R}^3)$; **b)** $u = p$ on $\bar{\Omega}^c$; **c)** u is harmonic in Ω. These three properties completely characterize u; indeed if $\tilde{p} \in W^1(\mathbb{R}^3)$ with $p = \tilde{p}$ on $\bar{\Omega}^c$, then, because $\bar{\Omega}^c$ is Lipschitzian, we have $p - \tilde{p} \in W_0^1(\Omega)$.

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