M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 32, n° 6 (1998), p. 773-788
<http://www.numdam.org/item?id=M2AN_1998__32_6_773_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol 32, n° 6, 1998, p 773 à 788)

NESTED SEQUENCES OF CHEBYSHEV SPACES AND SHAPE PARAMETERS (*)

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Abstract — *Through a geometrical approach of the blossoming principle, we achieve a dimension elevation process for extended Chebyshev spaces Applied to a nested sequence of such spaces included in a polynomial one, this allows to compute the Bézier points from the initial Chebyshev-Bézier points This method leads to interesting shape effects © Elsevier, Paris*

Key words Chebyshev splines, shape parameters, shape effects, geometrical design
AMS subject classification: 65D17

Résumé — *Au moyen d'une approche géométrique du principe de floraison (« blossoming »), nous obtenons une procédure d'élévation de la dimension pour les espaces de Chebyshev généralisés Appliqué à une suite emboîtée de tels espaces inclus dans un espace de polynômes, ceci permet de calculer les points de Bézier à partir des points de Bézier-Chebyshev Cette méthode conduit à des effets de forme intéressants © Elsevier, Paris*

1. INTRODUCTION

Let us recall the classical *degree elevation* process for polynomials. Consider a polynomial function F of degree less than or equal to n with values in $\mathbb{R}^d$, and denote by $P_0, ..., P_n$ its Bézier points with respect to (0, 1). It is well known that, when considering F as a polynomial function of degree less than or equal to $n+1$, its new Bézier points $Q_0, ..., Q_{n+1}$ can be computed as follows:

$$Q_0 = P_0, \quad Q_i = \left(1 - \frac{i}{n+1}\right) P_{i-1} + \frac{i}{n+1} P_i \quad 1 \leqslant i \leqslant n, \quad Q_{n+1} = P_n . \tag{1.1}$$

As pointed out by L. Ramshaw [27], the blossoming principle allows a very elegant proof of this result. Denoting by f the blossom of F, that is to say, the only function of n variables to be symmetric and affine with respect to each variable, and to give F when restricted to the diagonal of $\mathbb{R}^n$, the Bézier points of F are given by

$$P_i = f(0^{n-i}\, 1^i) \quad i = 0, ..., n . \tag{1.2}$$

Here, the multiplicative notation a^k means that the point a is repeated exactly k times. Considered as a polynomial function of degree less than or equal to $n+1$, F has a blossom that we shall denote by $\tilde{f}$, which is clearly the following function of $n+1$ variables:

$$\tilde{f}(x_1, ..., x_{n+1}) := \frac{1}{n+1} \sum_{i=1}^{n+1} f(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}) . \tag{1.3}$$

Similarly to (1.2), the new Bézier points of F are defined by

$$Q_i = \tilde{f}(0^{n+1-i}\, 1^i) \quad i = 0, ..., n+1 .$$

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Thus, the degree elevation relations (1.1) are nothing but (1.3) applied to the $(n+1)$-tuples $(0^{n+1-i}\, 1^i)$.

In this paper, we shall extend the degree elevation process to the case of *extended Chebyshev spaces,* and show how this can yield very nice and efficient shape parameters. Observe that, in this more general framework, the expression "degree elevation", particular to the polynomial case, is now to be replaced by *dimension elevation.*

The blossoming principle has recently been developed for extended Chebyshev spaces. This can be done for instance through a geometrical approach [25, 14, 16, 17] which consists in defining blossoms by means of intersections of osculating flats, according to an idea introduced in [29] for geometrically continuous polynomial splines. Let us mention the existence of other approaches of a more algebraic nature (see [1, 3, 20, 21]).

Section 2 recalls the geometrical definition of Chebyshev blossoming (for more details see [14, 16, 17]). In Section 3, this approach proves to be particularly efficient to develop the dimension elevation process. In Section 4, by means of a detailed example, it is shown how this process can lead to shape parameters the efficiency of which is pointed out in Section 5 by studying the corresponding splines.

2. BLOSSOMING IN CHEBYSHEV SPACES

Consider a closed bounded interval $I = [t_0, t_1]$, with $t_0 < t_1$, and an $(n+1)$-dimensional subspace $\mathcal{U}$ of $C^\infty(I)$ containing the constant functions. Assume that $D\mathcal{U} := \{U' \mid U \in \mathcal{U}\}$ is an *extended Chebyshev space on* I (in short EC space), i.e., that each nonzero element of $D\mathcal{U}$ has at most $n-1$ zeros (counted with multiplicities) in I. This assumption clearly implies that each nonzero element of $\mathcal{U}$ has at most n zeros in I, which means that U itself is an EC space on I.

Choose n functions $\Phi_1, \ldots, \Phi_n \in \mathcal{U}$ such that $(\mathbb{I}, \Phi_1, \ldots, \Phi_n)$ form a basis of $\mathcal{U}$ and set $\Phi := (\Phi_1, \ldots, \Phi_n)^T$. Recall that *the osculating flat of order* i of Φ at a point $x \in I$ is the affine flat passing through $\Phi(x)$ and the direction of which is spanned by the first i derivatives $\Phi'(x), \ldots, \Phi^{(i)}(x)$. In the following, it will be denoted by $\mathrm{Osc}_i\, \Phi(x)$. Due to the assumption on the space $D\mathcal{U}$, it can be proved (see [14, 16, 17, 23]) that, for any distinct $\tau_1, \ldots, \tau_r \in I$, and any positive integers $\mu_1, \ldots, \mu_r$ satisfying $\sum_{i=1}^{r} \mu_i = n$, the intersection $\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i)$ consists of a single point.

Now, given an n-tuple $(x_1, \ldots, x_n) \in I^n$, denote by $(x_1, \ldots, x_n)^{ord}$ the n-tuple composed of the same elements arranged in increasing order. Using a multiplicative notation, we can always write

$$(x_1, \ldots, x_n)^{ord} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r}) ,$$

with $\tau_1 < \tau_2 < \ldots < \tau_r$, the notation τ^μ meaning that the point μ is repeated exactly μ times. Then, since $\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i)$ consists of a single point, this point can be labelled as follows:

$$\{\varphi(x_1, \ldots, x_n)\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i) . \tag{2.1}$$

This provides a function φ defined on I^n which will be called *the blossom of* Φ. Clearly, the blossom φ of Φ is *symmetric* and satisfies

$$\varphi(x^n) = \Phi(x) \quad \text{for all } x \in I . \tag{2.2}$$

Let us fix $n-1$ points $x_1, \ldots, x_{n-1} \in I$, with $(x_1, \ldots, x_{n-1})^{ord} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r})$. Then, by the very definition of the blossom, the function $\varphi(x_1, \ldots, x_{n-1}, \centerdot\,)$ has values in the affine flat $\mathcal{D} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i)$ which can be proved to be an affine line ([14, 16]). In fact, $\varphi(x_1, \ldots, x_{n-1}, \centerdot\,)$ is C^∞ and strictly monotone on I (see [1, 16, 17, 20, 23]).

This result is essential to develop the classical numerical algorithms. It can be stated equivalently as follows: given two distinct points $a, b \in I$, there exists a function α (depending on a, b and also on $x_1, ..., x_{n-1}$), C^∞ and strictly monotone on I, and such that, for all $x \in I$,

$$\varphi(x_1, ..., x_{n-1}, x) = (1 - \alpha(x))\, \varphi(x_1, ..., x_{n-1}, a) + \alpha(x)\, \varphi(x_1, ..., x_{n-1}, b)\,. \tag{2.3}$$

In particular, $\alpha(a) = 0$, $\alpha(b) = 1$.

A function $F : I \to \mathbb{R}^d$ will be said to be a $\mathscr{U}$-function when each of its components belongs to $\mathscr{U}$. Observe that, given a $\mathscr{U}$-function F, there exists a unique affine map $h : \mathbb{R}^n \to \mathbb{R}^d$ such that $F = h \circ \Phi$. Then, the blossom f of F is defined by

$$f := h \circ \varphi\,. \tag{2.4}$$

Accordingly, if $x_1, ..., x_n \in I$, with $(x_1, ..., x_n)^{ord} = (\tau_1^{\mu_1} ... \tau_r^{\mu_r})$, we still have

$$f(x_1, ..., x_n) \in \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} F(\tau_i)\,, \tag{2.5}$$

but, without any additional assumption on h, there is no reason why the intersection appearing in (2.5) should be reduced to a single point. On the other hand, from (2.4), it can easily be deduced that the blossom f is symmetric and gives F when restricted to the diagonal of I^n. Finally, for $x_1, ..., x_{n-1}, a, b \in I$, $a \neq b$, the affinity of h allows us to write $f(x_1, ..., x_{n-1}, x)$ as an affine combination of $f(x_1, ..., x_{n-1}, a)$ and $f(x_1, ..., x_{n-1}, b)$. More precisely,

$$f(x_1, ..., x_{n-1}, x) = (1 - \alpha(x))\, f(x_1, ..., x_{n-1}, a) + \alpha(x)\, f(x_1, ..., x_{n-1}, b)\,, \tag{2.6}$$

α being the same function as in (2.3).

As a direct consequence, given $x \in I$, we can deduce the existence of real numbers $\gamma_{ij}(x) \in [0, 1]$, $j = 1, ..., n$, $i = 0, ..., n - j$, such that, for all $\mathscr{U}$-functions F,

$$\begin{aligned} f(t_0^{n-j-i}\, t_1^i\, x^j) &= (1 - \gamma_{ij}(x))\, f(t_0^{n-(j-1)-i}\, t_1^i\, x^{j-1}) \\ &\quad + \gamma_{ij}(x)\, f(t_0^{n-(j-1)-(i+1)}\, t_1^{i+1}\, x^{j-1})\,. \end{aligned} \tag{2.7}$$

When j goes from 0 to n, (2.7) describes a *de Casteljau type algorithm.* The affine combinations involved in (2.7) being in fact convex ones, this algorithm allows the computation of $f(x^n)$ as a convex combination of the $n + 1$ points $f(t_0^{n-i}\, t_1^i)$, $i = 0, ..., n$, which we shall refer to as *the Chebyshev-Bézier points* of the $\mathscr{U}$-function F. In other words,

$$F(x) = \sum_{i=0}^{n} \mathscr{B}_i(x)\, f(t_0^{n-i}\, t_1^i) \quad \text{for all } x \in I\,, \tag{2.8}$$

where $\mathscr{B}_0(x), ..., \mathscr{B}_n(x)$ do not depend on F and satisfy

$$\sum_{i=0}^{n} \mathscr{B}_i(x) = 1, \quad 0 \leqslant \mathscr{B}_i(x) \leqslant 1 \quad x \in I\,. \tag{2.9}$$

When applying (2.8) to any element of $\mathscr{U}$, it clearly follows that $(\mathscr{B}_0, ..., \mathscr{B}_n)$ is a basis of $\mathscr{U}$, which will be called its *Chebyshev-Bernstein basis.*

Let us consider the Chebyshev-Bézier points of function Φ, i.e., $\Pi_i := \varphi(t_0^{n-i} t_1^i)$, $i = 0, \dots, n$. According to (2.8), we have

$$\Phi(x) = \sum_{i=0}^{n} \mathcal{B}_i(x)\, \Pi_i \quad \text{for all } x \in I. \tag{2.10}$$

Due to the fact that functions $\mathbb{I}$, $\Phi_1, \dots, \Phi_n$ are linearly independent, it easily follows that the affine space spanned by the image of Φ, (denoted by $(\operatorname{aff}(\operatorname{Im} \Phi))$ is of dimension n (hence, is equal to $\mathbb{R}^n$). On the other hand, (2.10) shows that

$$\operatorname{aff}(\operatorname{Im} \Phi) \subset \operatorname{aff}(\Pi_0, \dots, \Pi_n), \tag{2.11}$$

which proves both the linear independence of the Chebyshev-Bézier points $\Pi_0, \dots, \Pi_n$ of Φ and the equality $\operatorname{aff}(\operatorname{Im} \Phi) = \operatorname{aff}(\Pi_0, \dots, \Pi_n)$.

For all $x \in I$ and all $i \leqslant n$, the osculating flat $\operatorname{Osc}_i \Phi(x)$ is i-dimensional: indeed, $(\Phi_1', \dots, \Phi_n')$ being a basis of the EC space $D\mathcal{U}$, the n vectors $\Phi'(x), \dots, \Phi^{(n)}(x)$ are linearly independent. Now, by the very geometrical definition of the blossom φ of Φ, for all $i = 0, \dots, n$, the $(i+1)$ points $\Pi_0, \dots, \Pi_i$ belong to $\operatorname{Osc}_i \Phi(t_0)$. Thus, according to the affine independence of the Chebyshev-Bézier points of Φ, we have

$$\operatorname{Osc}_i \Phi(t_0) = \operatorname{aff}(\Pi_0, \dots, \Pi_i) \quad i = 0, \dots, n. \tag{2.12}$$

Symmetrically, we obtain

$$\operatorname{Osc}_i \Phi(t_1) = \operatorname{aff}(\Pi_{n-i}, \dots, \Pi_n) \quad i = 0, \dots, n. \tag{2.13}$$

One could give explicit expressions of the Chebyshev-Bernstein basis (*cf.* [14, 16]), but the interesting thing is that, in addition to (2.9), it satisfies other properties similar to the polynomial Bernstein basis, which can directly be derived from the geometrical definition of the blossom, as shown in the proof of the next theorem.

THEOREM 1.1: *The Chebyshev-Bernstein basis satisfies*

$$\mathcal{B}_i^{(j)}(t_0) = 0 \quad 0 \leqslant j < i \leqslant n, \quad \mathcal{B}_{n-i}^{(j)}(t_1) = 0 \quad 0 \leqslant j < i \leqslant n, \tag{2.14}$$

$$\mathcal{B}_i^{(i)}(t_0) > 0, \quad (-1)^i \mathcal{B}_{n-i}^{(i)}(t_1) > 0 \quad i = 0, \dots, n. \tag{2.15}$$

Proof: The Chebyshev-Bézier points Π_i are in fact defined by

$$\{\Pi_i\} := \operatorname{Osc}_i \Phi(t_0) \cap \operatorname{Osc}_{n-i} \Phi(t_1) \quad i = 0, \dots, n. \tag{2.16}$$

Therefore, there exist real numbers $\lambda_{i,1}, \dots, \lambda_{i,i}, \mu_{i,1}, \dots, \mu_{i,n-i}$, such that

$$\Pi_i = \Pi_0 + \sum_{\ell=1}^{i} \lambda_{i,\ell}\, \Phi^{(\ell)}(t_0) = \Pi_n + \sum_{j=1}^{n-i} \mu_{i,j}\, \Phi^{(j)}(t_1) \quad i = 0, \dots, n. \tag{2.17}$$

Consider a fixed integer i, $1 \leqslant i \leqslant n$. The corresponding equality (2.17) gives

$$\Phi(t_1) - \Phi(t_0) = \sum_{\ell=1}^{i} \lambda_{i,\ell}\, \Phi^{(\ell)}(t_0) - \sum_{j=1}^{n-i} \mu_{i,j}\, \Phi^{(j)}(t_1). \tag{2.18}$$

As an immediate consequence of (2.18), we can derive in particular that

$$\lambda_{i,i} = \frac{g(t_1)}{g^{(i)}(t_0)}, \tag{2.19}$$

where g denotes the C^∞ function defined on I by:

$$g(x) := \det\left(\Phi'(t_0), \ldots, \Phi^{(i-1)}(t_0), \Phi(x) - \Phi(t_0), \Phi'(t_1), \ldots, \Phi^{(n-i)}(t_1)\right) \quad x \in I. \tag{2.20}$$

Clearly, $g(t_0) = g'(t_0) = \ldots = g^{(i-1)}(t_0) = 0$, and the ith derivative

$$g^{(i)}(t_0) = \det\left(\Phi'(t_0), \ldots, \Phi^{(i)}(t_0), \Phi'(t_1), \ldots, \Phi^{(n-i)}(t_1)\right)$$

is not equal to 0 since $D\mathscr{U}$ is an EC on I. Consequently,

$$g(x) \sim \frac{(x-t_0)^i}{i!} g^{(i)}(t_0) \quad \text{when } x \to t_0. \tag{2.21}$$

Moreover, function g does not vanish on $]t_0, t_1]$: otherwise, there would exist a point $\xi \in\ ]t_0, t_1[$ such that

$$g'(\xi) = \det\left(\Phi'(t_0), \ldots, \Phi^{(i-1)}(t_0), \Phi'(\xi), \Phi'(t_1), \ldots, \Phi^{(n-i)}(t_1)\right) = 0,$$

which would contradict the fact that $D\mathscr{U}$ is an EC space on I. Therefore, it follows from (2.21) that $g(x)$ keeps the sign of $g^{(i)}(t_0)$ all over $]t_0, t_1]$. Consequently, (2.19) proves that

$$\lambda_{i,i} > 0 \quad i = 1, \ldots, n. \tag{2.22}$$

On the other hand, differentiating (2.10) and the first part of (2.9) up to order j yields

$$\Phi^{(j)}(t_0) = \sum_{i=1}^{n} \mathscr{B}_i^{(j)}(t_0)\,(\Pi_i - \Pi_0) \quad 1 \leq j \leq n. \tag{2.23}$$

On account of the affine independence of the points $\Pi_0, \ldots, \Pi_n$, comparing these latter relations with the left equalities in (2.17) proves that the $n \times n$ matrix $(\mathscr{B}_i^{(j)}(t_0))_{j,i=1,\ldots,n}$ is the inverse of the lower triangular matrix defined by the $\lambda_{i,\ell}$'s, $1 \leq \ell \leq i \leq n$. Accordingly, we have:

$$\mathscr{B}_i^{(j)}(t_0) = 0 \quad 0 \leq j < i \leq n, \quad \mathscr{B}_i^{(i)}(t_0) = \frac{1}{\lambda_{i,i}} \quad i = 0, \ldots, n.$$

Thus, due to (2.22), $\mathscr{B}_i^{(i)}(t_0) > 0$. The corresponding properties at t_1 can be proved in a similar way. ■

Theorem 1.1 generalizes classical properties of the polynomial Bernstein basis. For instance, $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ satisfies $B_i^{n(i)}(0) = (-1)^i B_{n-i}^{n}{}^{(i)}(1) = n(n-1)\ldots(n-i+1)$.

3. DIMENSION ELEVATION

In this section, we suppose that $\mathscr{U} \subset \tilde{\mathscr{U}}$, where $\tilde{\mathscr{U}}$ is an $(n+2)$-dimensional EC space on the same interval $I = [t_0, t_1]$. Then, let us first observe that any $\mathscr{U}$-function is a $\tilde{\mathscr{U}}$-function. We additionally assume $D\tilde{\mathscr{U}}$ itself to be an EC space on I. Thus, with a given $\mathscr{U}$-function F, we can now associate two blossoms, first f, which is the function of n variables introduced in the previous section, but also the blossom $\tilde{f}$ of F viewed as a $\tilde{\mathscr{U}}$-function, which means in particular that $\tilde{f}$ is a function of $(n+1)$ variables. The following result was given in [25]. We here give another proof, based on the properties of the Chebyshev-Bernstein basis.

THEOREM 3.1: *Let F be a $\mathcal{U}$-function, and $P_i = f(t_0^{n-i} t_1^i)$, $i = 0, \ldots, n$ its Chebyshev-Bézier points. Considered as a $\tilde{\mathcal{U}}$-function, F has $n+2$ Chebyshev-Bézier points, namely $\tilde{P}_i := \tilde{f}(t_0^{n+1-i} t_1^i)$, $i = 0, \ldots, n+1$, where $\tilde{f}$ is the blossom of F viewed as a $\tilde{\mathcal{U}}$-function. Then, these new Chebyshev-Bézier points can be computed from the original ones as follows:*

$$\tilde{P}_0 = P_0, \quad \tilde{P}_i = (1-\alpha_i) P_{i-1} + \alpha_i P_i \quad i = 1, \ldots, n, \quad \tilde{P}_{n+1} = P_n, \tag{3.1}$$

where $\alpha_1, \ldots, \alpha_n$ belong to $]0, 1[$ and do not depend on F.

Proof: Clearly, $F(t_0) = P_0 = \tilde{P}_0$ and $F(t_1) = P_n = \tilde{P}_{n+1}$. Let us introduce the new Chebyshev-Bézier points of function Φ:

$$\tilde{\Pi}_i := \tilde{\varphi}(t_0^{n+1-i} t_1^i) \quad i = 0, \ldots, n+1. \tag{3.2}$$

Then, denoting by h the affine map such that $F = h \circ \Phi$, we have $P_i = h(\Pi_i)$, for all $i = 0, \ldots, n$. Moreover, one can also check that $\tilde{P}_i = h(\tilde{\Pi}_i)$ for all $i = 0, \ldots, n+1$. Therefore, it is sufficient to prove the result for Φ.

For a fixed integer i, $1 \leqslant i \leqslant n$, by applying (2.5), we have:

$$\tilde{\Pi}_i \in \operatorname{Osc}_i \Phi(t_0) \cap \operatorname{Osc} \Phi_{n+1-i}(t_1). \tag{3.3}$$

Now, $\operatorname{Osc} \Phi_{n+1-i}(t_1) = \operatorname{Osc} \Phi_{n-(i-1)}(t_1)$. Consequently, according to (2.12) and (2.13),

$$\tilde{\Pi}_i \in \operatorname{aff}(\Pi_0, \ldots, \Pi_i) \cap \operatorname{aff}(\Pi_{i-1}, \ldots, \Pi_n). \tag{3.4}$$

Due to the linear independence of the Chebyshev-Bézier points $(\Pi_0, \ldots, \Pi_n)$, the intersection involved in (3.4) consists in the affine line passing through Π_{i-1} and Π_i. Hence, there exists a real number α_i, such that

$$\tilde{\Pi}_i = (1-\alpha_i) \Pi_{i-1} + \alpha_i \Pi_i. \tag{3.5}$$

It only remains to prove that $\alpha_i \in]0, 1[$.

Let us denote by $(\tilde{\mathcal{B}}_0, \ldots, \tilde{\mathcal{B}}_{n+1})$ the Chebyshev-Bernstein basis of the space $\tilde{\mathcal{U}}$, so that, in particular,

$$\Phi(x) = \sum_{i=0}^{n+1} \tilde{\mathcal{B}}_i(x) \tilde{\Pi}_i. \tag{3.6}$$

Using (3.5), we can rewrite (3.6) as follows:

$$\begin{aligned} \Phi(x) &= \tilde{\mathcal{B}}_0(x) \tilde{\Pi}_0 + \sum_{i=1}^{n} \tilde{\mathcal{B}}_i(x) [(1-\alpha_i) \Pi_{i-1} + \alpha_i \Pi_i] + \tilde{\mathcal{B}}_{n+1}(x) \tilde{\Pi}_n, \\ &= [\tilde{\mathcal{B}}_0(x) + (1-\alpha_1) \tilde{\mathcal{B}}_1(x)] \Pi_0 + \sum_{i=1}^{n-1} [\alpha_i \tilde{\mathcal{B}}_i(x) + (1-\alpha_{i+1}) \tilde{\mathcal{B}}_{i+1}(x)] \Pi_i \\ &\quad + [\alpha_n \tilde{\mathcal{B}}_n(x) + \tilde{\mathcal{B}}_{n+1}(x)] \Pi_n. \end{aligned} \tag{3.7}$$

The Chebyshev-Bézier points $\Pi_0, \ldots, \Pi_n$ being affinely independent, by comparison of (2.10) and (3.7), we can conclude that the Chebyshev-Bernstein bases of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are linked by the following relations:

$$\mathcal{B}_i = \alpha_i \tilde{\mathcal{B}}_i + (1-\alpha_{i+1}) \tilde{\mathcal{B}}_{i+1} \quad i = 0, \ldots, n, \tag{3.8}$$

with $\alpha_0 := 1$ and $\alpha_{n+1} := 0$. By differentiation of the latter equalities, we obtain

$$\mathscr{B}_i^{(i)}(t_0) = \alpha_i \bar{\mathscr{B}}_i^{(i)}(t_0) + (1 - \alpha_{i+1})\, \bar{\mathscr{B}}_{i+1}^{(i)}(t_0) \quad i = 0, ..., n . \tag{3.9}$$

According to (2.14), $\bar{\mathscr{B}}_{i+1}^{(i)}(t_0) = 0$, and (3.10) eventually reduces to

$$\alpha_i = \frac{\mathscr{B}_i^{(i)}(t_0)}{\bar{\mathscr{B}}_i^{(i)}(t_0)} \quad i = 1, ..., n . \tag{3.10}$$

Thus, (2.15) proves that $\alpha_i > 0$. Exchanging the rôles of t_0 and t_1 would similarly lead to $(1 - \alpha_i) > 0$ for all $i = 1, ..., n$. ■

In practice, the previous result will be used as follows. Consider a nested sequence

$$\mathscr{U}_n \subset \mathscr{U}_{n+1} \subset ... \subset \mathscr{U}_{n+m} ,$$

where, for $i = n, ..., n + m$, $\mathscr{U}_i$ is an $(i + 1)$-dimensional EC space on I containing the constant functions such that $D\mathscr{U}_i$ is also an EC space on I. Starting with a $\mathscr{U}_n$-function F, with Chebyshev-Bézier points $P_0, ..., P_n$, we can perform m consecutive dimension elevation processes, so as to obtain the $n + m + 1$ Chebyshev-Bézier points of F viewed as a $\mathscr{U}_{n+m}$-function. This is of special interest when the last space $\mathscr{U}_{n+m}$ is easy to deal with, the best example being of course the case when $\mathscr{U}_{n+m}$ is the polynomial space of degree $n + m$. In that case, the final $n + m + 1$ Chebyshev-Bézier points are simply the Bézier points of F viewed as a polynomial function of degree less than or equal to $n + m$, from which the curve defined by F can thus be obtained by applying the classical polynomial subdivision algorithm. Moreover, in case $\mathscr{U}_n$ depends on one or several parameters, these parameters will play the rôle of shape parameters. This will be illustrated by an example in the next section.

4. DIMENSION ELEVATION AND SHAPE PARAMETERS

It is well known that, since the interval $I = [t_0, t_1]$ is closed and bounded, each EC space on I is the kernel of a differential operator (see [7,15,28]). More precisely, given an $(n + 1)$-dimensional subspace $\mathscr{U}$ of $C^\infty(I)$, it is an EC space on I iff there exist $n + 1$ functions $w_0, ..., w_n$ (called *weight functions*), C^∞ and positive on I such that

$$\mathscr{U} = \operatorname{Ker} D \circ L_n , \tag{4.1}$$

where D is the ordinary differentiation and $L_0, ..., L_n$, are the differential operators defined on $C^\infty(I)$ by

$$L_0\, U := \frac{1}{w_0} U, \quad L_i\, U := \frac{1}{w_i} (L_{i-1}\, U)' \quad i = 1, ..., n . \tag{4.2}$$

The previous result provides an easy way of constructing EC spaces. Given positive weight functions $\omega_i \in C^\infty(I)$, $i = 0, ..., n$ and the corresponding operators L_i, the EC space $\operatorname{Ker} D \circ L_n$ will be denoted by $\mathrm{EC}(\omega_0, ..., \omega_n)$. Observe that an EC space can be associated with several different systems of weight functions.

If $\mathscr{U} \subset C^\infty(I)$ is an $(n + 1)$-dimensional space containing the constant functions, the space $D\mathscr{U}$ is an EC space on I iff there exists weight functions $\omega_0 = \mathbb{I}, \omega_1, ..., \omega_n$ such that $\mathscr{U} = \mathrm{EC}(w_0, ..., w_n)$ (see [15]). From now on, we shall always consider such spaces.

Let us denote by $\mathscr{P}_i$ the space of polynomial functions of degree less than or equal to i on $I = [t_0, t_1]$. Consider the 4-dimensional EC space on I associated with the following four weight functions

$$w_0 := \mathbb{I}, \quad w_1 := p^2, \quad w_2 := w_3 := \mathbb{I} , \tag{4.3}$$

where p is a given positive function belonging to $\mathscr{P}_1$, that is $\mathscr{U}_3 := \mathrm{EC}(\mathbb{1}, p^2, \mathbb{1}, \mathbb{1})$. Observe that, p being completely determined by the two positive numbers $p_0 := p(t_0)$ and $p_1 := p(t_1)$, the space $\mathscr{U}_3$ depends only on p_0, p_1, and in fact, more precisely, only on the ratio $\frac{p_1}{p_0}$.

Given $U \in C^\infty(I)$, by (4.1), one can check that

$$U \in \mathscr{U}_3 \Leftrightarrow \frac{1}{p^2} U' \in \mathscr{P}_2 . \tag{4.4}$$

Suppose first that $p \in \mathscr{P}_1 \backslash \mathscr{P}_0$, *i.e.*, that the parameter $\frac{p_1}{p_0}$ is not equal to 1. Then, since $\mathscr{P}_2 = \mathrm{span}\,(\mathbb{1}, p, p^2)$, (4.4) implies that

$$\mathscr{U}_3 = \mathrm{span}\,(\mathbb{1}, p^3, p^4, p^5) . \tag{4.5}$$

Using (4.1), it can similarly be proved that $\mathrm{EC}(\mathbb{1}, p, \mathbb{1}, \mathbb{1}, \mathbb{1}) = \mathrm{span}\,(\mathbb{1}, p^2, p^3, p^4, p^5)$. Therefore, we obtain the following nested sequence of EC spaces on I:

$$\mathscr{U}_3 \subset \mathscr{U}_4 := \mathrm{EC}(\mathbb{1}, p, \mathbb{1}, \mathbb{1}, \mathbb{1}) \subset \mathscr{U}_5 := \mathrm{EC}(\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}) = \mathscr{P}_5 . \tag{4.6}$$

Given a $\mathscr{U}_3$-function F, with Chebyshev-Bézier points P_0, P_1, P_2, P_3, denote by Q_i, $i = 0, \ldots, 4$, its five Chebyshev-Bézier points when considered as a $\mathscr{U}_4$-function, and by R_i, $i = 0, \ldots, 5$, its six Chebyshev-Bézier points when considered as a $\mathscr{U}_5$-function. We know that

$$Q_0 = P_0, \quad Q_i = (1 - \alpha_i) P_{i-1} + \alpha_i P_i \quad i = 1, 2, 3, \quad Q_4 = P_3, \tag{4.7}$$

and

$$R_0 = Q_0, \quad R_i = (1 - \beta_i) Q_{i-1} + \beta_i Q_i \quad i = 1, 2, 3, 4, \quad R_5 = Q_4 , \tag{4.8}$$

where the α_i ' s and the β_i ' s belong to $]0, 1[$ and do not depend on F. We intend to compute these coefficients.

Now, for any distinct a, b, $c \in I$, the value of the blossom $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ of $\Phi := (\Phi_1, \Phi_2, \Phi_3)^T := (p^3, p^4, p^5)^T$ is given by

$$\{\varphi(a, b, c)\} = \mathrm{Osc}_2\, \Phi(a) \cap \mathrm{Osc}_2\, \Phi(b) \cap \mathrm{Osc}_2\, \Phi(c) . \tag{4.9}$$

Let us set $A := p(a)$, $B := p(b)$, $C := p(c)$. A point $X = (x, y, z)^T \in \mathbb{R}^3$ belongs to $\mathrm{Osc}_2\, \Phi(a)$ iff $\det(X - \Phi(a), \Phi'(a), \Phi''(a)) = 0$, *i.e.*, iff

$$10 A^2 x - 15 A y + 6 z = A^5 .$$

Thus, solving the linear system corresponding to (4.9) eventually leads to:

$$\begin{aligned}
\varphi_1(a, b, c) &= \frac{1}{10} (A^3 + B^3 + C^3 + A^2(B + C) + B^2(C + A) + C^2(A + B) + ABC) , \\
\varphi_2(a, b, c) &= \frac{1}{15} (A^3(B + C) + B^3(C + A) + C^3(A + B) \\
&\qquad (+ A^2 B^2 + B^2 C^2 + C^2 A^2 + 2 ABC(A + B + C)) , \\
\varphi_3(a, b, c) &= \frac{1}{6} ABC(A^2 + B^2 + C^2 + AB + BC + CA) .
\end{aligned} \tag{4.10}$$

As for how to calculate the value of φ at (a, a, c), this should be done by solving

$$\{\varphi(a, a, b)\} = \operatorname{Osc}_1 \Phi(a) \cap \operatorname{Osc}_2 \Phi(c) .$$

However, all functions $\varphi(x_1, x_2, .)$ being continuous on I, it follows that, in fact, formula (4.10) is still valid even if a, b, c are not assumed to be distinct.

The value of the blossom $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ of $\Psi := (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T := (p^2, p^3, p^4, p^5)^T$ at $(a, b, c, d) \in I^4$ can be similarly computed from the equality

$$\{\psi(a, b, c, d)\} = \operatorname{Osc}_3 \Psi(a) \cap \operatorname{Osc}_3 \Psi(b) \cap \operatorname{Osc}_3 \Psi(c) \cap \operatorname{Osc}_3 \Psi(d) ,$$

which is valid only when the four points are distinct. One can eventually prove that, whatever the points $a, b, c, d \in I$ may be, if $D := p(d)$,

$$\begin{aligned}
\psi_1(a, b, c, d) &= \frac{1}{10}(A^2 + B^2 + C^2 + D^2 + AB + AC + AD + BC + BD + CD) ,\\
\psi_2(a, b, c, d) &= \frac{1}{20}(A^2(B + C + D) + B^2(A + C + D) + C^2(A + B + D)\\
&\quad + D^2(A + B + C) + 2(ABC + ABD + ACD + BCD)) ,\\
\psi_3(a, b, c, d) &= \frac{1}{15}(A^2(BC + CD + DB) + B^2(AC + CD + DA)\\
&\quad + C^2(AB + BD + DA) + D^2(AB + BC + CA) + 3\,ABCD) ,\\
\psi_4(a, b, c, d) &= \frac{1}{4} ABCD(A + B + C + D) .
\end{aligned} \tag{4.11}$$

In order to calculate the 7 coefficients α_i and β_i appearing in (4.7) and (4.8), let us focus on function $F = \Phi_3 = \Psi_4 = p^5$. Using (4.10), we can calculate its Chebyshev-Bézier points $P_1 = \varphi_3(t_0, t_0, t_1)$ and $P_2 = \varphi_3(t_0, t_1, t_1)$, which gives

$$P_1 = \frac{1}{6} p_0^2 p_1 (3 p_0^2 + 2 p_0 p_1 + p_1^2), \quad P_2 = \frac{1}{6} p_0 p_1^2 (p_0^2 + 2 p_0 p_1 + 3 p_1^2) . \tag{4.12}$$

Its new Chebyshev-Bézier points are given by $Q_i = \psi_4(t_0^{4-i} t_1^i)$, and (4.11), leads to

$$Q_1 = \frac{1}{4} p_0^3 p_1 (3 p_0 + p_1), \quad Q_2 = \frac{1}{2} p_0^2 p_1^2 (p_0 + p_1) , \quad Q_3 = \frac{1}{4} p_0 p_1^3 (p_0 + 3 p_1) . \tag{4.13}$$

From (4.12) and (4.13), it results that

$$\alpha_1 = \frac{3 p_0(4 p_0 + p_1)}{2(6 p_0^2 + 3 p_0 p_1 + p_1^2)}, \quad \alpha_2 = \frac{p_0(3 p_0 + 2 p_1)}{3 p_0^2 + 4 p_0 p_1 + 3 p_1^2}, \quad \alpha_3 = \frac{p_0(2 p_0 + 3 p_1)}{2(p_0^2 + 3 p_0 p_1 + 6 p_1^2)}. \tag{4.14}$$

Furthermore, the points $R_0, \ldots, R_5$ are in fact the Bézier points of the monomial x^5 with respect to (p_0, p_1), namely $R_i = p_0^{5-i} p_1^i$, $i = 0, \ldots, 5$. Accordingly, from (4.13), one can check that

$$\beta_1 = \frac{4 p_0}{4 p_0 + p_1}, \quad \beta_2 = \frac{3 p_0}{3 p_0 + 2 p_1}, \quad \beta_3 = \frac{2 p_0}{2 p_0 + 3 p_1}, \quad \beta_4 = \frac{p_0}{p_0 + 4 p_1} . \tag{4.15}$$

All these results are summarized in *figure 1*. The ratio p_1/p_0 acts as a shape parameter. In particular, in the limit case $p_1/p_0 \to +\infty$, we obtain the polynomial curve of degree less than or equald to 5 whose Bézier points R_i are given by $R_0 = R_1 = R_2 = P_0$, $R_3 = P_1$, $R_4 = P_2$, $R_5 = P_3$, and the symmetric situation when $p_1/p_0 \to 0^+$: $R_0 = P_0$, $R_1 = P_1$, $R_2 = P_2$, $R_3 = R_4 = R_5 = P_3$.

Suppose now that $p \in \mathscr{P}_0$. Then, we clearly have $\mathscr{U}_i = \mathscr{P}_i$ for $i = 3, 4, 5$, which proves that (4.6) still holds. Moreover, since this case corresponds to $p_0 = p_1$, the ratios indicated in *figure 1* are still valid: they describe the classical degree elevation, which thus appears as a particular case of the dimension elevation process described above.

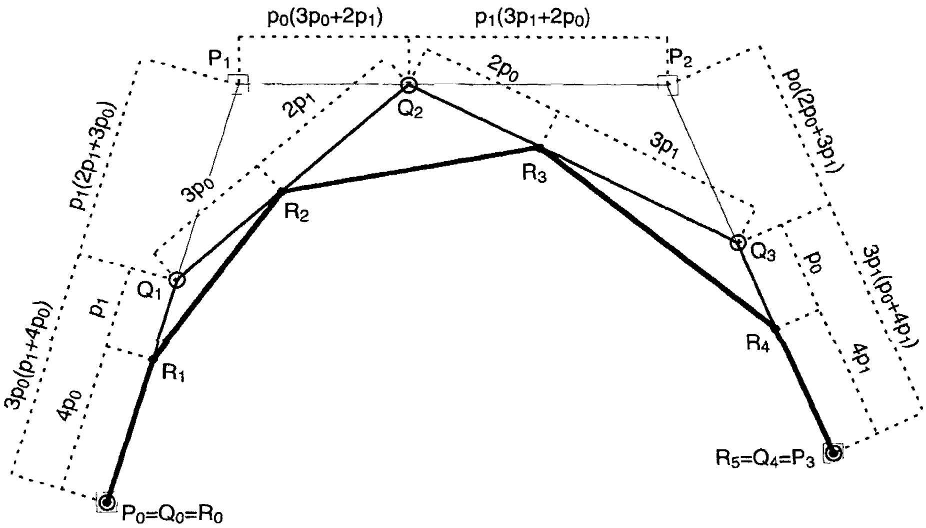

Figure 1. — Chebyshev-Bézier points P_i, Q_i and Bézier points R_i

5. CHEBYSHEV SPLINES

In this section, we suppose that w_1, w_2, w_3, are three continuous positive functions, assumed to be piecewise C^∞ with respect to a given subdivision $t_0 < t_1 < \cdots < t_s < t_{s+1}$. For all $i = 0, \ldots, s$, consider the 4-dimensional EC space associated with $\mathbb{I}$ and the restrictions of these functions to the interval $[t_i, t_{i+1}]$,

$$\mathscr{U}_i := \mathrm{EC}(\mathbb{I}, w_1|_{[t_i, t_{i+1}]}, w_2|_{[t_i, t_{i+1}]}, w_3|_{[t_i, t_{i+1}]}), \tag{5.1}$$

i.e., $\mathscr{U}_i = \operatorname{Ker} D \circ L_n^i$, where the L_j^i's are the differential operators defined on $C^\infty([t_i, t_{i+1}])$ similarly to (4.2). Consider the $(s+4)$-dimensional spline space $\mathscr{S}$ composed of all continuous functions $S : [t_0, t_{s+1}] \to \mathbb{R}$ such that:

(i) for all $i = 0, \ldots, s$, the restriction of S to $[t_i, t_{i+1}]$ belongs to $\mathscr{U}_i$,

(ii) for all $i = 1, \ldots, s$, S satisfies the following connection condition at t_i:

$$\begin{pmatrix} L_1^i S(t_i^+) \\ L_2^i S(t_i^+) \end{pmatrix} = \begin{pmatrix} \beta_1^i & 0 \\ \beta_2^i & \beta_1^{i\,2} \end{pmatrix} \cdot \begin{pmatrix} L_1^{i-1} S(t_i^-) \\ L_2^{i-1} S(t_i^-) \end{pmatrix}, \tag{5.2}$$

where $\beta_1^i > 0$ and $\beta_2^i \geqslant 0$.

In terms of the ordinary derivatives, condition (5.2) can be written as follows:

$$\begin{pmatrix} S'(t_i^+) \\ S''(t_i^+) \end{pmatrix} = \begin{pmatrix} \beta_1^i & 0 \\ \tilde{\beta}_2^i & {\beta_1^i}^2 \end{pmatrix} \cdot \begin{pmatrix} S'(t_i^-) \\ S''(t_i^-) \end{pmatrix}, \tag{5.3}$$

where

$$\tilde{\beta}_2^i := \beta_2^i + \frac{w_1'(t_i^+)\,\beta_1^i - w_1'(t_i^-)\,{\beta_1^i}^2}{w_1(t_i)}. \tag{5.4}$$

Therefore, it is a geometric continuity connection. However, the corresponding matrix is not necessarily totally positive since it may happen that $\tilde{\beta}_2^i < 0$: indeed, the crucial assumption of total positivity must be made on the connection matrix appearing in (5.2).

Using a result of P. J. Barry [1], it has been proved in [17] that any spline $S \in \mathcal{S}^d$ is uniquely determined by $s+4$ *poles* $P_{-3}, \ldots, P_s \in \mathbb{R}^d$ which have a geometrical definition in terms of osculating flats that we shall now recall.

First observe that, for a given i, $1 \leqslant i \leqslant s$, since the first two derivatives of S at t_i^- and t_i^+ are linked by (5.3), we can define the osculating flats $\mathrm{Osc}_1\, S(t_i)$ and $\mathrm{Osc}_2\, S(t_i)$ as that of either its $(i-1)$th section or its ith one.

Now, assume the Chebyshev-Bézier points of each section to be affinely independent. Then, we have ([17]):

$$\{P_{i-2}\} := \mathrm{Osc}_2\, S(t_{i-1}) \cap \mathrm{Osc}_2\, S(t_i) \cap \mathrm{Osc}_2\, S(t_{i+1}) \quad 1 \leqslant i \leqslant s, \tag{5.5}$$

and the remaining poles can be defined for instance by

$$\begin{gathered} P_{-3} := S(t_0), \quad \{P_{-2}\} := \mathrm{Osc}_1\, S(t_0) \cap \mathrm{Osc}_2\, S(t_1), \\ \{P_{s-1}\} := \mathrm{Osc}_2\, S(t_s) \cap \mathrm{Osc}_1\, S(t_{s+1}), \quad P_s := S(t_{s+1}). \end{gathered} \tag{5.6}$$

For $i = 0, \ldots, s$, denote by P_j^i, $j = 0, \ldots, 3$, the Chebyshev-Bézier points of the ith section $S_{|[t_i, t_{i+1}]}$ of the spline S. From (2.12) and (2.13), we can deduce that

$$\mathrm{Osc}_2\, S(t_{i-1}) \cap \mathrm{Osc}_2\, S(t_i) = \mathrm{aff}(P_1^{i-1}, P_2^{i-1}), \tag{5.7}$$

$$\mathrm{Osc}_2\, S(t_i) \cap \mathrm{Osc}_2\, S(t_{i+1}) = \mathrm{aff}(P_1^i, P_2^i). \tag{5.8}$$

Consequently, (5.5) means that, for $1 \leqslant i \leqslant s$, the pole P_{i-2} is obtained by intersecting the two affine lines $\mathrm{aff}(P_1^{i-1}, P_2^{i-1})$ and $\mathrm{aff}(P_1^i, P_2^i)$. Similarly, (5.6) means that $P_{-3} = P_0^0$, $P_{-2} := P_1^0$, $P_{s-1} := P_2^s$, and $P_s := P_3^s$. It is well known that it is then possible to get rid of the assumption of affine independence by using affine maps.

From the definition of the poles, it follows that each spline S can be written as

$$S(t) = \sum_{\ell=-3}^{s} \mathcal{N}_\ell(t)\, P_\ell,$$

where $(\mathcal{N}_\ell)_{\ell=-3, \ldots, s}$, called the *Chebyshev B-spline basis* of $\mathcal{S}$, satisfies

$$\sum_{i=-3}^{s} \mathcal{N}_\ell(x) = 1, \quad 0 \leqslant \mathcal{N}_\ell(x) \leqslant 1 \quad x \in [t_0, t_{s+1}].$$

Furthermore, the support of each $\mathcal{N}_\ell$, *i.e.*, the domain of influence of the pole P_ℓ, is equal to $[t_\ell, t_{\ell+4}]$, with $t_{-3}=t_{-2}=t_{-1}:=t_0$, and $t_{s+2}=t_{s+3}=t_{s+4}:=t_{s+1}$.

Let us now develop an explicit example based on the particular EC space studied in section 4. Given positive real numbers p_i, $i=0, \dots, s+1$, denote by $p:[t_0, t_{s+1}] \to \mathbb{R}$ the continuous piecewise affine function such that $p(t_i)=p_i$ for all i, and consider $\mathcal{U}_i := \mathrm{EC}(\mathbb{I}, p^2_{|[t_i, t_{i+1}]}, \mathbb{I}, \mathbb{I})$. From now on, assume that $t_{i+1}-t_i=1$ for all $i=0, \dots, s$, and choose $\beta_1^i=1$ and $\beta_2^i=0$ for $i=1, \dots, s$. Then, it follows from (5.3) and (5.4) that the space $\mathcal{S}$ described above is the space of all C^1 functions $S:[t_0, t_{s+1}] \to \mathbb{R}$ such that $S_{|[t_i, t_{i+1}]} \in \mathcal{U}_i$, $i=0, \dots, s$, and

$$S''(t_i^+) = S''(t_i^-) + 2\frac{p_{i+1} - 2p_i + p_{i-1}}{p_i} S'(t_i) \quad \text{for all } i = 1, \dots, s \,. \tag{5.9}$$

Consider a spline function $S \in \mathcal{S}^d$. Assuming the Chebyshev-Bézier points of each of its sections to be affinely independent, it is possible to compute the poles by using their geometrical definition recalled in (5.5) and (5.6). We have thus to intersect the two lines $\mathrm{aff}(P_1^{i-1}, P_2^{i-1})$ and $\mathrm{aff}(P_1^i, P_2^i)$. Now, from the very definition of the Chebyshev-Bézier points, we know that, for all $i=1, \dots, s$, there exist two regular lower triangular matrices such that

$$\begin{pmatrix} P_1^i - P_0^i \\ P_2^i - P_0^i \end{pmatrix} = \begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix} \cdot \begin{pmatrix} S'(t_i) \\ S''(t_i^+) \end{pmatrix}, \begin{pmatrix} P_2^{i-1} - P_3^{i-1} \\ P_1^{i-1} - P_3^{i-1} \end{pmatrix} = \begin{pmatrix} D_i & 0 \\ E_i & F_i \end{pmatrix} \cdot \begin{pmatrix} S'(t_i) \\ S''(t_i^-) \end{pmatrix}, \tag{5.10}$$

where, in fact, $P_3^{i-1} = P_0^i$ because of the continuity of S. One can check that

$$\begin{aligned} A_i &= \frac{p_{i+1}^2 + 3p_i p_{i+1} + 6p_i^2}{30 p_i^2}, \\ B_i &= \frac{-3p_{i+1}^3 + 3p_i p_{i+1}^2 + 8p_i^2 p_{i+1} + 12 p_i^3}{30 p_i^3}, \\ C_i &= \frac{3p_{i+1}^2 + 4p_i p_{i+1} + 3p_i^2}{60 p_i^2}, \end{aligned} \tag{5.11}$$

and similarly,

$$\begin{aligned} D_i &= \frac{p_{i-1}^2 + 3p_i p_{i-1} + 6p_i^2}{30 p_i^2}, \\ E_i &= \frac{-3p_{i-1}^3 + 3p_i p_{i-1}^2 + 8p_i^2 p_{i-1} + 12 p_i^3}{30 p_i^3}, \\ F_i &= \frac{3p_{i-1}^2 + 4p_i p_{i-1} + 3p_i^2}{60 p_i^2}. \end{aligned} \tag{5.12}$$

For $i=1, \dots, s$, the common point of the two affine lines $\mathrm{aff}(P_1^{i-1}, P_2^{i-1})$ and $\mathrm{aff}(P_1^i, P_2^i)$, *i.e.*, the pole P_{i-2}, will be obtained by solving the equation

$$P_2^{i-1} + \lambda(P_2^{i-1} - P_1^{i-1}) = P_1^i + \mu(P_1^i - P_2^i)\,. \tag{5.13}$$

Taking (5.10) into account and replacing $S''(t_i^+)$ by its expression resulting from (5.9), equality (5.13) leads to

$$\left[(1+\mu)-\mu B_i - 2\mu C_i \frac{p_{i+1}-2p_i+p_{i-1}}{p_i}\right] S'(t_i) - \mu C_i S''(t_i^-)$$
$$= [(1+\lambda) D_i - \lambda E_i] S'(t_i) - \lambda F_i S''(t_i^-). \quad (5.14)$$

Due to the linear independence of the two derivatives, equality (5.14) is equivalent to

$$\mu C_i = \lambda F_i, \quad \left[(1+\mu)-\mu B_i - 2\mu C_i \frac{p_{i+1}-2p_i+p_{i-1}}{p_i}\right] = [(1+\lambda) D_i - \lambda E_i]. \quad (5.15)$$

Solving this system in (λ, μ) finally gives:

$$\lambda = \frac{u_{i-1}+v_i}{c_{i-1}}, \quad \mu = \frac{u_{i-1}+v_i}{c_i}, \quad (5.16)$$

where

$$u_i := p_i^2 + 3p_i p_{i+1} + 6p_{i+1}^2, \quad (5.17)$$
$$v_i := 6p_i^2 + 3p_i p_{i+1} + p_{i+1}^2,$$
$$c_i := 6p_i^2 + 8p_i p_{i+1} + 6p_{i+1}^2.$$

Therefore, conversely, starting from given poles $P_i \in \mathbb{R}^d$, $i = -3, \ldots, s$, the Chebyshev-Bézier points of each section are obtained as described in *figures 2* and *3*. Then, after two dimension elevations, we shall obtain the Bézier points of each section, from which the corresponding curve can be drawn by polynomial subdivision.

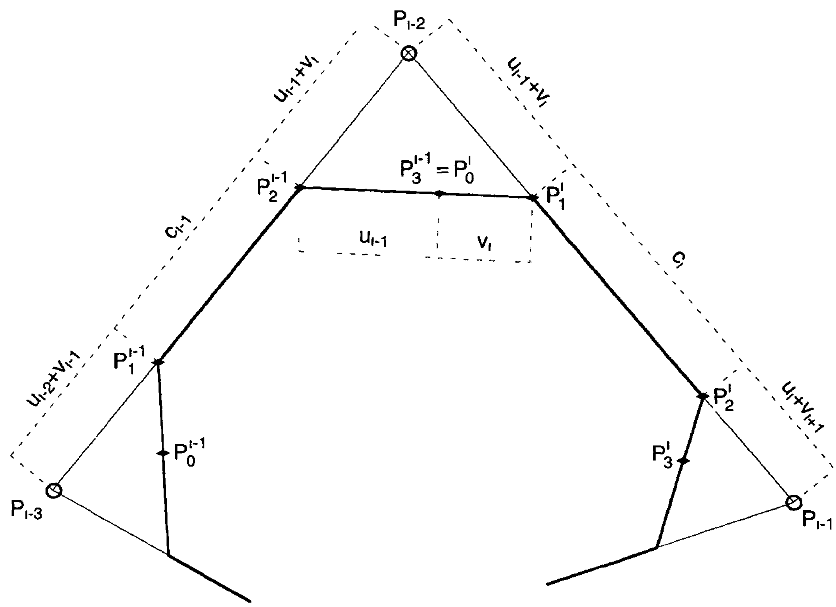

Figure 2. — Computing the Chebyshev-Bézier points from the poles ($i = 2, \ldots, s-1$).

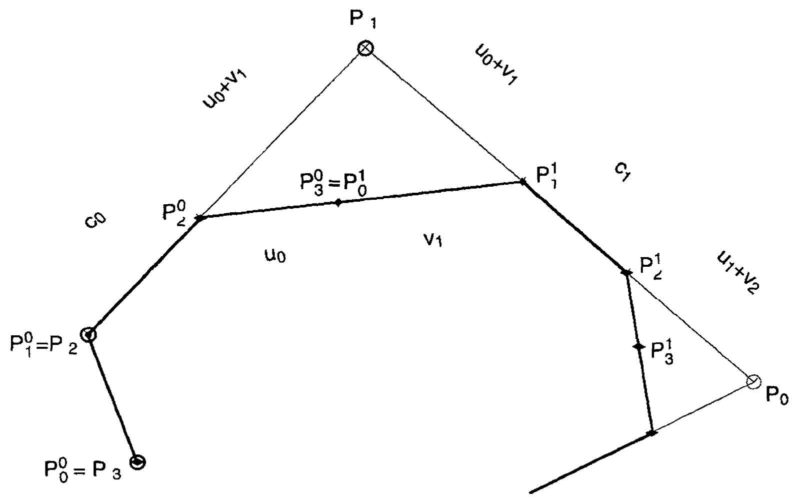

Figure 3. — Computing the Chebyshev-Bézier points of the first section from the poles.

For fixed poles P_i, $i = -3, \ldots, s$, the positive numbers p_i act as shape parameters. The efficiency of these shape parameters is clearly illustrated by *figures 4* and *5*. Here, we deal with a closed curve defined by 11 poles. Therefore, the computation of the Chebyshev-Bézier points from the poles will be done only according to *figure 2*, all the data being now defined modulo 11. A small circle (resp. a star) at a pole means that $p_i = 100$ (resp. $p_i = 1/100$) if t_i is the center of the support of the corresponding Chebyshev B-spline, all the other p_i's being equal to 1.

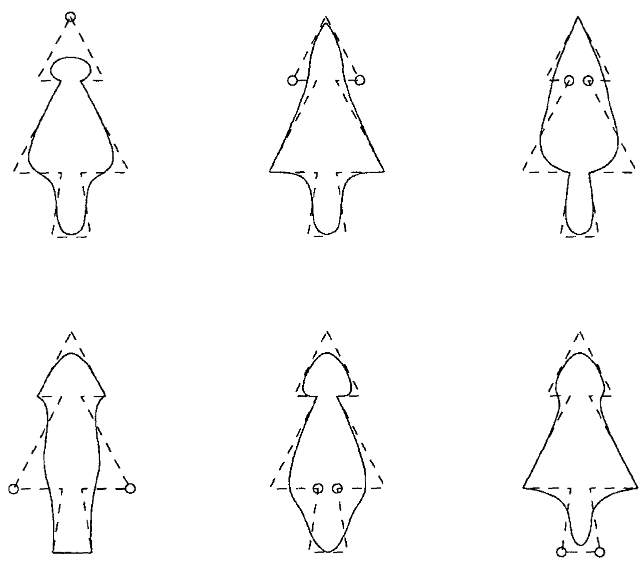

Figure 4. — $p_i = 100$ for each "o", otherwise $p_i = 1$.

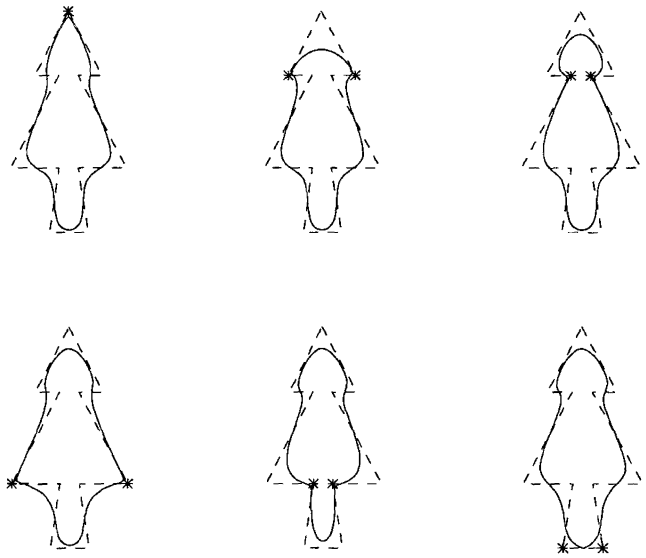

Figure 5. — $p_i = 1/100$ for each "$*$", otherwise $p_i = 1$.

These shape effects can also be pointed out by observing the Chebyshev B-splines. Suppose that $t_i = i$ and that $p_5 = 100$ (resp. $1/100$), all the other p_i's being equal to one. Then, only five Chebyshev B-splines are different from the usual B-splines of degree 3, namely those whose support intersect $]4, 6[$. These Chebyshev B-splines are shown in *figure 6.*

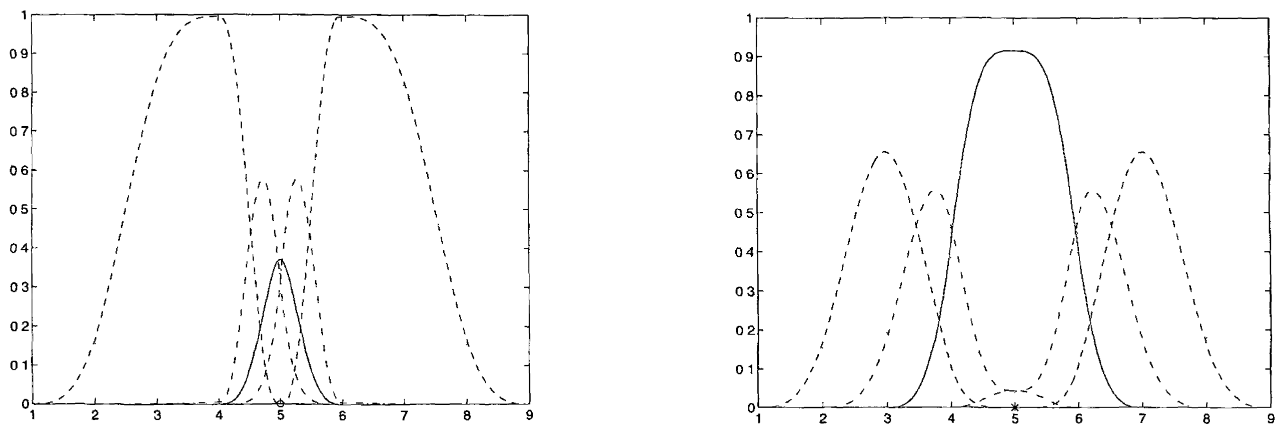

Figure 6. — Chebyshev B-splines $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$ when $p_5 = 100$ (left) or $p_5 = 1/100$ (right), $p_i = 1$ for $i \neq 5$.

REFERENCES

[1] P. J. BARRY, de Boor-Fix dual functionals and algorithms for Tchebycheffian B-spline curves, *Constructive Approximation* **12** (1996), 385-408.

[2] P. J. BARRY, N. DYN, R. N. GOLDMAN and C. A. MICCHELLI, Identities for piecewise polynomial spaces determined by connection matrices, *Aequationes Mathematicae* **42** (1991), 123-136.

[3] D. BISTER and H. PRAUTZSCH, A new approach to Tchebycheffian B-splines, in *Curves and Surfaces with Applications to CAGD,* Vanderbilt University Press (1997), 35-42.

[4] N. DYN, A. EDELMAN and C. A. MICHELLI, On locally supported basis functions for the representation of geometrically continuous curves, *Analysis* **7** (1987), 313-341.

[5] N. DYN and A. RON, Recurrence relations for Tchebycheffian B-splines, *Journal d'Analyse Mathématique* **51** (1988), 118-138.

[6] S. KARLIN, *Total Positivity,* Stanford University Press, Stanford, 1968.

[7] S. KARLIN and W. J. STUDDEN, *Tchebycheff Systems,* Wiley Interscience, New York, 1966.

[8] S. KARLIN and Z. ZIEGLER, Chebyshevian spline functions, *SIAM Journal Numerical Analysis* **3** (1966), 514-543.

[9] R. KULKARNI and P.-J. LAURENT, Q-splines, *Numerical Algorithms* **1** (1991), 45-74.

[10] R. KULKARNI, P.-J. LAURENT and M.-L. MAZURE, Non affine blossoms and subdivision for Q-splines, in *Math. Methods in Computer Aided Geometric Design II,* Academic Press, New York, 1992, 367-380.

[11] P.-J. LAURENT, M.-L. MAZURE and V. T. MAXIM, Chebyshev splines and shape parameters, RR 980M IMAG, Université Joseph Fourier, Grenoble, September 1997, *Numerical Algorithms* **15** (1997), 373-383.

[12] P.-J. LAURENT, M.-L. MAZURE and G. MORIN, Shape effects with polynomial Chebyshev splines, in *Curves and Surfaces with Applications in CAGD,* Vanderbilt University Press, 1997, 255-262.

[13] T. LYCHE, A recurrence relation for Chebyshevian B-splines, *Constructive Approximation* **1** (1985), 155-173.

[14] M.-L. MAZURE, Blossoming of Chebyshev splines, In *Mathematical Methods for Curves and Surfaces,* Vanderbilt University Press, 1995, 355-364.

[15] M.-L. MAZURE, Chebyshev spaces, RR 952M IMAG, Université Joseph Fourier, Grenoble, January 1996.

[16] M.-L. MAZURE, Chebyshev blossoming, RR 953M IMAG, Université Joseph Fourier, Grenoble, January 1996.

[17] M.-L. MAZURE, Blossoming: a geometric approach, RR 968M IMAG, Université Joseph Fourier, Grenoble, January 1997, to appear in *Constructive Approximation.*

[18] M.-L. MAZURE, Vandermonde type determinants and blossoming, The Fourth International Conference on Mathematical Methods for Curves and Surfaces, Lillehammer, Norway, July 3-8, 1997, RR 979M IMAG, Université Joseph Fourier, Grenoble, September 1997, *Advances in Computational Math.* **8** (1998), 291-315.

[19] M.-L. MAZURE and P.-J. LAURENT, Affine and non affine blossoms, in *Computational Geometry,* World Scientific, 1993, 201-230.

[20] M.-L. MAZURE and P.-J. LAURENT, Marsden identities, blossoming and de Boor-Fix formula, in *Advanced Topics in Multivariate Approximation,* World Scientific Pub., 1996, 227-242.

[21] M.-L. MAZURE and P.-J. LAURENT, Piecewise smooth spaces in duality: application to blossoming, RR969-M, IMAG, Université Joseph Fourier, Grenoble, January 1997, to appear in *Journal of Approximation Theory.*

[22] M.-L. MAZURE and P.-J. LAURENT, Polynomial Chebyshev Splines, to appear.

[23] M.-L. MAZURE and H. POTTMANN, Tchebycheff curves, in *Total Positivity and its Applications,* Kluwer Academic Pub. (1996), 187-218.

[24] C. A. MICCHELLI, *Mathematical Aspects of Geometric Modeling,* CBMS-NSF Regional Conference Series in Applied Math. **65**, SIAM, Philadelphia, 1995.

[25] H. POTTMANN, The geometry of Tchebycheffian splines, Computer Aided Geometric Design **10** (1993), 181-210.

[26] H. POTTMANN and M. G. WAGNER, Helix splines as an example of affine Tchebycheffian splines, *Advances in Computational Math.* **2** (1994), 123-142.

[27] L. RAMSHAW, Blossoms are polar forms, *Computer Aided Geometric Design* **6** (1989), 323-358.

[28] L. L. SCHUMAKER, *Spline Functions: Basic Theory,* Wiley Interscience, New York, 1981.

[29] H.-P. SEIDEL, New algorithms and techniques for computing with geometrically continuous spline curves of arbitrary degree, *Math. Modelling and Numerical Analysis* **26** (1992), 149-176.

[30] M. G. WAGNER and H. POTTMANN, Symmetric Tchebycheffian B-splines schemes, in *Curves and Surfaces in Geometric Design,* A. K. Peters, Wellesley, MA, 1994, 483-490.