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INCREMENTAL UNKNOWNS ON NONUNIFORM MESHES (*)

J.-P. CHEHAB [1] A. MIRANVILLE [2]

Résumé — *On propose ici une extension de la méthode des inconnues incrémentales (II) aux maillages non uniformes, pour lesquels le pas de discrétisation en espace n'est pas constant, en vue de la simulation numérique de problèmes de couche limite On définit ces II à partir de la discrétisation du problème de Poisson sur ce type de maillages, et, en établissant la coercivité des opérateurs discrets intervenant dans ce contexte, on obtient sur les II des estimations a priori de type énergie qui généralisent celles obtenues dans le cas uniforme Les II ainsi définies sont alors utilisées pour construire des préconditionneurs hiérarchiques, non nécessairement symétriques, pour le problème de Poisson Ces préconditionneurs permettent d'obtenir une réduction importante du nombre de condition des matrices en présence Cette amélioration est illustrée par la résolution numérique de tels problèmes elliptiques sur des maillages utilisés en dynamique des fluides* © *Elsevier, Paris*

Abstract — *This article is devoted to the numerical analysis of the Incremental Unknowns method (IU) when applied to nonuniform meshes The extension of the IU we propose here is devoted to the numerical solution of boundary value problems e g in the presence of boundary layers which necessitate the use of refined grids near the boundary We define the incremental unknowns in this context and we introduce the corresponding hierarchical preconditioners in space dimensions one and two for the Poisson problem We establish the coercivity of the linear operator using the incremental unknowns We also obtain numerical results on the asymptotic behaviour of the condition number of the underlying matrices that are comparable to the ones derived in the uniform case in space dimension one In space dimension two we do not recover the same asymptotic results but the condition number is considerably reduced with our preconditioner The numerical examples we give concern the solution of elliptic problems on particular meshes used for boundary layer problems in Computational Fluid Dynamics Futhermore, we construct high order IUs in the nonuniform case by a generalization of the interpolation compact schemes* © *Elsevier, Paris*

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Key words Incremental Unknowns, Nonuniform Meshes, Hierarchical Methods, Preconditioning, Coercivity

1. INTRODUCTION

The Incremental Unknowns method (IU), that stems from the dynamical systems theory, was introduced in [20] for the approximation of inertial manifolds when finite differences are used. This new approach also provides a link between hierarchical methods and Nonlinear Galerkin methods (see [17] and [18] for instance). Incremental Unknowns can be defined when multilevel discretizations are used. For instance, if two levels of discretization are used, the IUs consist of the usual nodal values at the coarse grid points, and of an increment to the values of suitable neighboring points at the fine grid points that do not belong to the coarse grid.

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The Incremental Unknowns have been applied to many problems and situations and have led to subsequent improvements when compared to methods using the usual nodal unknowns (see [5],[6], [7], [8], [10], [11] and the references therein). These results concern the construction of both preconditioners and new multiresolution schemes for solving nonlinear problems, and for which the IUs were introduced. Thus it was shown in [11] that the condition number of the matrix associated to the discretization of selfadjoint elliptic operators with IUs is considerably reduced. In [5] the extension of this method to a shifted mesh of MAC type gave an efficient hierarchical preconditioner for the Uzawa operator associated to a Generalized Stokes problem. In [7] and [8] the use of the IU method gave efficient generalizations of the Marder and Weitzner scheme for solving nonlinear eigenvalue and bifurcation problems.

Up to now the IU method was applied only to uniform grids (i.e. when the spatial mesh size h is constant in each direction of the discretized domain). As it is well known, it is preferable to use meshes that are refined in the boundary layers in order to solve boundary layer problems. These are special cases of nonuniform meshes. Futhermore efficient solutions of the Navier-Stokes equations were built using a Chebyshev-like mesh (see [2] and [19]) for the driven cavity, and using an a tanh-type grid for the channel flow problem (see [15]). Numerical studies of these problems using the techniques introduced in this article will be developed elsewhere.

Our aim in this article is to develop the numerical analysis of the IU method for general grids, with an emphasis on the compression of data and on the construction of preconditioners for elliptic operators. In that way we introduce several tools for the implementation of this method for the solution of boundary layer problems.

This article is organized as follows. In Section 2 we introduce the second order incremental unknowns for nonuniform meshes in space dimensions one and two. Then, in Section 3, we derive *a priori* estimates based on energy methods. To obtain these estimates, we make a change of variable by writing $u = y + z$, where y is the coarse grid component (i.e. the value of u at the coarse grid points) and z the complementary grid component (i.e. the incremental quantity). We thus establish the coercivity of the linear operator using the Incremental Unknowns. Futhermore, these estimates enable us to prove that the incremental unknowns are small as expected.

In Section 4, we propose some hierarchical preconditioners for the solution of the Dirichlet problem. In particular, in space dimension one, we recover numerically the same asymptotic behaviour of the condition number of the underlying matrices as in the uniform case. In space dimension two, we observe, for Chebyshev-like meshes, that the condition number of the preconditioned matrix is asymptotically in $\frac{N}{2}$ while that of the usual matrix, written in the nodal basis is asymptotically in N^4. Finally in Section 5 we present some numerical experiments concerning the solution of the Dirichlet problem for specific meshes used in Computational Fluid Dynamics.

2. SECOND ORDER INCREMENTAL UNKNOWNS ON A NONUNIFORM MESH

In this section, we introduce the second order Incremental Unknowns (IU2) in space dimensions one and two when the mesh is nonuniform i.e. when the mesh size h is not a constant. Throughout this article G_f and G_c will denote the fine grid and the coarse grid respectively.

2.1. The one dimensional case

Let u_j, $j = 0, \dots, 2N-1$ be the nodal unknowns on G_f and let x_j, $j = 0, \dots, 2N-1$ be the discretization points (usually, when the mesh size h is constant $x_j = j \cdot h$). Here the x_j enjoy the following (obvious) property:

$$0 < x_1 < x_2 < \cdots < x_j < x_{j+1} < \cdots < x_{2N-1} < 1 ,$$

and we set

$$x_0 = 0 \quad \text{and} \quad x_{2N} = 1 .$$

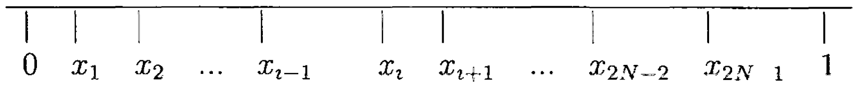

Figure1. — Space dimension 1: a nonuniform grid.

We now follow the usual scheme in order to define the incremental unknowns (see [20]).

Hierarchization

We distinguish the coarse grid components, which are on G_c and whose indices are even, and the complementary grid components, which are on $G_f \backslash G_c$ and whose indices are odd (see *fig.* 2).

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Figure 2. — Space dimension 1, $\Omega =]0, 1[$, ×: points in G_c, ○ points in $G_f \backslash G_c$

Incremental unknowns

We now introduce the IUs. Applying Taylor expansions to two consecutive unknowns of the coarse grid G_c, we obtain:

$$(a)\ u_{2i+2} = u_{2i+1} + (x_{2i+2} - x_{2i+1}) \left.\frac{\partial u}{\partial x}\right|_{x = x_{2i+1}} + \frac{(x_{2i+2} - x_{2i+1})^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x = x_{2i+1}}$$
$$+ \frac{(x_{2i+2} - x_{2i+1})^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{x = x_{2i+1}} + O((\Delta x)^4)\,,$$

$$(b)\ u_{2i} = u_{2i+1} - (x_{2i+1} - x_{2i}) \left.\frac{\partial u}{\partial x}\right|_{x = x_{2i+1}} + \frac{(x_{2i+1} - x_{2i})^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x = x_{2i+1}}$$
$$- \frac{(x_{2i+1} - x_{2i})^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{x = x_{2i+1}} + O((\Delta x)^4)\,,$$

where $\Delta x = Sup_{\{i \in \{1, \ldots, 2N-1\}\}} |x_{i+1} - x_i|$.

Taking $(x_{2i+1} - x_{2i})\,(a) + (x_{2i+2} - x_{2i+1})\,(b)$, we find

$$u_{2i+1} = \frac{(x_{2i+1} - x_{2i})\, u_{2i+2} + (x_{2i+2} - x_{2i+1})\, u_{2i}}{x_{2i+2} - x_{2i}} + O((\Delta x)^2)\,.$$

We can now define the second order IUs as the numbers Z_{2i+1}:

$$Z_{2i+1} = u_{2i+1} - \frac{(x_{2i+1} - x_{2i})\, u_{2i+2} + (x_{2i+2} - x_{2i+1})\, u_{2i}}{x_{2i+2} - x_{2i}}\,.$$

We note that if $x_{i+1} - x_i = h\ \forall i$, we recover the usual IUs (see [20]).

Remark 1: According to Taylor's formula these IUs are of order $O((\Delta x)^2)$. We shall verify this property in the next section using energy type estimates as in [10] for the uniform case.

2.2. The two dimensional case

We proceed as in space dimension one, starting with a hierarchization of the unknowns. For the sake of simplicity we only consider two grid levels (see *fig.* 3 below).

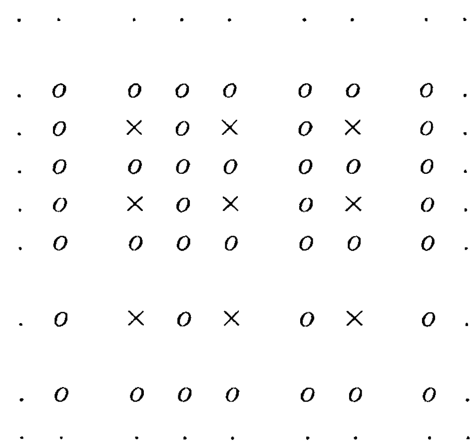


Figure 3. — Space dimension 2, $\Omega = (\,]0,1[\,)^2$, $\times$: points in G_c, $\circ$ points in $G_f \backslash G_c$

In space dimension one the unknowns of the complementary grid ($G_f \backslash G_c$) have the same geometric characteristics (see *fig.* 2). In space dimension two, we distinguish in fact three kinds of points in $G_f \backslash G_c$: points of type f1, f2 and f3 (see *fig.* 4).

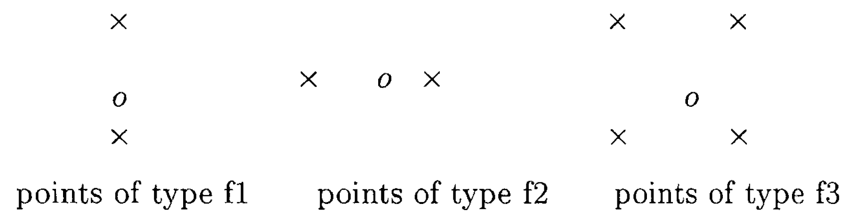


Figure 4. — The different types of points in $G_f \backslash G_c$

We extend the construction of the IUs given in the previous subsection for the one dimensional case as follows.

Let x_i and y_j be the sequences defining the mesh on the x-direction and the y-direction respectively:

$$0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots < x_{2N-1} < 1\,,$$

$$0 < y_1 < y_2 < \cdots < y_j < y_{j+1} < \cdots < y_{2N-1} < 1\,,$$

$$x_0 = y_0 = 0\,;\, x_{2N} = y_{2N} = 1\,.$$

As in the one dimensional case we define Δx and Δy by:

$$\Delta x = Sup_{\{i \in \{0, \dots, 2N-1\}\}} |x_{i+1} - x_i|\,,$$

$$\Delta y = Sup_{\{j \in \{0, \dots, 2N-1\}\}} |y_{j+1} - y_j|\,.$$

We set

$$Z_{2i,2j+1} = u_{2i,2j+1} - \frac{(y_{2j+1} - y_{2j})\, u_{2i,2j+2} + (y_{2j+2} - y_{2j+1})\, u_{2i,2j}}{y_{2j+2} - y_{2j}}, \tag{2.1}$$

at the points of type f1;

$$Z_{2i+1,2j} = u_{2i+1,2j} - \frac{(x_{2i+1} - x_{2i})\, u_{2i+2,2j} + (x_{2i+2} - x_{2i+1})\, u_{2i,2j}}{x_{2i+2} - x_{2i}}, \tag{2.2}$$

at the points of type f2;

$$\begin{aligned} Z_{2i+1,2j+1} &= u_{2i+1,2j+1} \\ &\quad - \left(\frac{x_{2i+2} - x_{2i+1}}{x_{2i+2} - x_{2i}}\right) \frac{(y_{2j+2} - y_{2j+1})\, u_{2i,2j} + (y_{2j+1} - y_{2j})\, u_{2i,2j+2}}{y_{2j+2} - y_{2j}} \\ &\quad - \left(\frac{y_{2i+2} - y_{2i+1}}{y_{2j+2} - y_{2j}}\right) \frac{(x_{2j+2} - x_{2j+1})\, u_{2i+2,2j} + (x_{2i+1} - x_{2i})\, u_{2i+2,2j+2}}{x_{2i+2} - x_{2i}}, \end{aligned} \tag{2.3}$$

at the points of type f3; $i, j = 0, \dots, N-1$.

Remark 2: As in the one dimensional case, according to Taylor's formula, the 2-D IU are expected to be small and of order $O((\Delta x)^2 + (\Delta y)^2)$. This will be confirmed in the next section by deriving *a priori* estimates. We also note that if $x_{i+1} - x_i = y_{j+1} - y_j = \text{Const} = h$, we recover the usual IUs on a uniform grid.

3. INCREMENTAL UNKNOWNS AND THE DIRICHLET PROBLEM

As in the uniform case, we relate the incremental unknowns to the solution of the Dirichlet problem. In Section 4 we shall construct appropriate hierarchical preconditioners.

3.1. The one dimensional case

3.1.1. *Discretization of the Laplacian*

We consider the Dirichlet problem:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} = f \quad \text{in } \Omega = \,]0,1[\,, \\ u(0) = u(1) = 0\,. \end{cases} \tag{3.4}$$

In order to solve (3.4), we use the following symmetric discretization which is as usual obtained using directely Taylor expansions. We have

$$\begin{aligned} (a)\ u_{i+1} &= u_i + (x_{i+1} - x_i) \left.\frac{\partial u}{\partial x}\right|_{x = x_i} + \frac{(x_{i+1} - x_i)^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x = x_i} \\ &\quad + \frac{(x_{i+1} - x_i)^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{x = x_i} + O((\Delta x)^4)\,, \\ (b)\ u_{i-1} &= u_i - (x_i - x_{i-1}) \left.\frac{\partial u}{\partial x}\right|_{x = x_i} + \frac{(x_i - x_{i-1})^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x = x_i} \\ &\quad - \frac{(x_i - x_{i-1})^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{x = x_i} + O((\Delta x)^4)\,. \end{aligned}$$

Taking $(a)\dfrac{1}{x_{i+1}-x_i}+(b)\dfrac{1}{x_i-x_{i-1}}$, we obtain

$$\left(\frac{1}{x_i-x_{i-1}}+\frac{1}{x_{i+1}-x_i}\right)u_i-\frac{1}{x_i-x_{i-1}}u_{i-1}-\frac{1}{x_{i+1}-x_i}u_{i+1}=-\frac{x_{i+1}-x_{i-1}}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i}+O((\Delta x)^3).$$

We set for convenience:

$$\alpha_i=\frac{1}{x_i-x_{i-1}}+\frac{1}{x_{i+1}-x_i},\quad \beta_i=\frac{1}{x_i-x_{i-1}},\quad \gamma_i=\frac{1}{x_{i+1}-x_i}.$$

Incremental unknowns

Using the present discretization of the Laplacian, we can define the incremental unknowns. We set

$$Z_{2i+1}=u_{2i+1}-\frac{1}{\alpha_{2i+1}}(\beta_{2i+1}u_{2i}+\gamma_{2i+1}u_{2i+2}).$$

These IUs are obviously the same as those introduced in Section 2.

3.1.2. *A priori estimates*

The linear system can be expressed as

$$\alpha_i u_i-\beta_i u_{i-1}-\gamma_i u_{i+1}=\frac{x_{i+1}-x_{i-1}}{2}f_i, \tag{3.5}$$

where $f_i=f(x_i)$.

We multiply (3.5) by u_i, and summing these expressions on all indices i, we obtain

$$A=\sum_{i=1}^{2N-1}(\alpha_i u_i-\beta_i u_{i-1}-\gamma_i u_{i+1})u_i=\sum_{i=1}^{2N-1}\left(\frac{x_{i+1}-x_{i-1}}{2}f_i\right)u_i.$$

Using the relations $\alpha_i=\beta_i+\gamma_i$ and $\beta_{i+1}=\gamma_i$ (due to the symmetry), we have (we set for convenience $x_{-1}=0$)

$$A=\sum_{i=1}^{2N-1}(\beta_i u_i-\beta_i u_{i-1})u_i=\sum_{i=0}^{2N-1}\gamma_i(u_i-u_{i+1})u_i,$$

and thus

$$A=\sum_{i=0}^{2N-1}\gamma_i(u_{i+1}-u_i)^2=\sum_{i=1}^{2N-1}\left(\frac{x_{i+1}-x_{i-1}}{2}f_i\right)u_i.$$

At this point, we introduce the following discrete Poincaré inequality:

LEMMA 1: *Let u_i, $i=0,\dots,2N$ be a sequence of real numbers such that $u_0=u_{2N}=0$. Then we have*

$$\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})u_i^2\leqslant 2\,.\sum_{i=1}^{2N-1}\frac{1}{x_{i+1}-x_i}(u_{i+1}-u_i)^2,$$

where x_i is the sequence defined in the previous section.

Proof:
We have

$$\begin{aligned}
\sqrt{x_{i+1}-x_{i-1}}\,|u_i| &= \sqrt{x_{i+1}-x_{i-1}}\sum_{j=0}^{i-1}(|u_{j+1}|-|u_j|)\,,\\
&\leqslant \sqrt{x_{i+1}-x_{i-1}}\sum_{j=0}^{i-1}(|u_{j+1}-u_j|)\,,\\
&\leqslant \sqrt{x_{i+1}-x_{i-1}}\sum_{i=0}^{2N-1}\frac{|u_{i+1}-u_i|}{\sqrt{x_{i+1}-x_i}}\sqrt{x_{i+1}-x_i}\\
&\leqslant (\text{thanks to Cauchy} - \text{Schwarz inequality})\,,\\
&\leqslant \sqrt{x_{i+1}-x_{i-1}}\sqrt{\sum_{i=0}^{2N-1}\frac{(u_{i+1}-u_i)^2}{x_{i+1}-x_i}}\sqrt{\sum_{i=0}^{2N-1}(x_{i+1}-x_i)}\,.
\end{aligned}$$

We then obtain

$$\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})\,u_i^2 \leqslant \sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})\sum_{i=0}^{2N-1}\frac{(u_{i+1}-u_i)^2}{x_{i+1}-x_i}\,,$$

and finally

$$\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})\,u_i^2 \leqslant 2\sum_{i=0}^{2N-1}\frac{(u_{i+1}-u_i)^2}{x_{i+1}-x_i}\,.$$

■

Using Lemma 1 and Cauchy-Schwarz inequality we find

$$\sum_{i=0}^{2N-1}\gamma_i(u_{i+1}-u_i)^2 \leqslant \sqrt{\frac{1}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2}\sqrt{\sum_{i=0}^{2N-1}\gamma_i(u_{i+1}-u_i)^2}\,.$$

Therefore

$$\sum_{i=0}^{2N-1}\gamma_i(u_{i+1}-u_i)^2 \leqslant \frac{1}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2\,,$$

and, using the inequality $\gamma_i = \dfrac{1}{x_{i+1}-x_i} \geqslant \dfrac{1}{\Delta x}$, we have

$$A = \sum_{i=0}^{2N-1}(u_{i+1}-u_i)^2 \leqslant \frac{\Delta x}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2\,.$$

Now

$$A = \sum_{i=0}^{N-1}(u_{2i+1}-u_{2i})^2 + \sum_{i=0}^{N-1}(u_{2i+2}-u_{2i+1})^2 \leqslant \frac{\Delta x}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2\,,$$

which yields, setting $Y_{2i} = u_{2i}$

$$\sum_{i=0}^{N-1}\left(Z_{2i+1}+\frac{1}{\alpha_{2i+1}}((\beta_{2i+1}-\alpha_{2i+1})Y_{2i}+\gamma_{2i+1}Y_{2i+2})\right)^2+\sum_{i=0}^{N-1}\left(Z_{2i+1}-\frac{\beta_{2i+1}}{\alpha_{2i+1}}(Y_{2i+2}-Y_{2i})\right)^2$$
$$\leqslant \frac{\Delta x}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2,$$

and after simplification, we obtain

$$2\sum_{i=0}^{N-1}Z_{2i+1}^2+\sum_{i=0}^{N-1}\left(\frac{\gamma_{2i+1}^2}{\alpha_{2i+1}^2}+\frac{\beta_{2i+1}^2}{\alpha_{2i+1}^2}\right)(Y_{2i+2}-Y_{2i})^2-2\sum_{i=0}^{N-1}\left(\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right)Z_{2i+1}(Y_{2i+2}+Y_{2i})$$
$$\leqslant \frac{\Delta x}{2}\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})f_i^2.$$

Using Young inequality, we find

$$-2\sum_{i=0}^{N-1}\left(\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right)Z_{2i+1}(Y_{2i+2}-Y_{2i})\geqslant -\epsilon\sum_{i=0}^{N-1}\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|Z_{2i+1}^2$$
$$-\frac{1}{\epsilon}\sum_{i=0}^{N-1}\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|(Y_{2i+2}-Y_{2i})^2,$$

where ϵ is a strictly positive real number which will be fixed later.

If we replace $f_i = f(x_i)$ by $f_i = \dfrac{1}{x_{i+1}-x_{i-1}}\displaystyle\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx$, we have:

$$\sum_{i=0}^{N-1}\left(2-\epsilon\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|\right)Z_{2i+1}^2+\sum_{i=0}^{N-1}\left(\frac{\gamma_{2i+1}^2}{\alpha_{2i+1}^2}+\frac{\beta_{2i+1}^2}{\alpha_{2i+1}^2}-\frac{1}{\epsilon}\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|\right)(Y_{2i+2}-Y_{2i})^2$$

$$\leqslant \frac{\Delta x}{2}\sum_{i=1}^{2N-1}\int_{\alpha_i-1}^{\alpha_i+1}f^2(t)\,dt\leqslant \Delta x\|f\|^2,$$

where $\|f\| = |f|_{L^2(0,1)}$. Setting $\xi = \dfrac{\beta_{2i+1}}{\gamma_{2i+1}}$, we have

$$\left(2-\epsilon\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|\right)=2-\epsilon\left|\frac{1-\xi}{1+\xi}\right|=g_1(\xi),$$

$$\left(\frac{\gamma_{2i+1}^2}{\alpha_{2i+1}^2}+\frac{\beta_{2i+1}^2}{\alpha_{2i+1}^2}-\frac{1}{\epsilon}\left|\frac{\beta_{2i+1}-\gamma_{2i+1}}{\alpha_{2i+1}}\right|\right)=\frac{1+\xi^2}{(1+\xi)^2}-\frac{1}{\epsilon}\left|\frac{1-\xi}{1+\xi}\right|=g_2(\xi).$$

It is easy to prove that for $\epsilon > 1$ there exists a constant $C_1(\epsilon)$ such that

$$g_2(\xi)\geqslant C_1(\epsilon)>0,\quad \forall \xi\in[0,\infty[.$$

On the other hand, we note that $g_1(\xi) \geqslant 2 - \epsilon$, $\forall \xi \in [0, \infty[$, since

$$\left|\frac{1-\xi}{1+\xi}\right| \leqslant 1, \quad \forall \xi \in [0, \infty[\, .$$

Taking $\epsilon = \frac{3}{2}$ we see that g_1 and g_2 are both bounded from below by a strictly positive constant $\kappa = \text{Min}\left(\frac{1}{2}, C_1\left(\frac{3}{2}\right)\right)$ and we finally obtain the following result:

PROPOSITION 1: *The second order Incremental Unknowns associated to the discretization of (3.4) satisfy the following a priori estimates:*

$$\sum_{i=0}^{N-1} Z_{2i+1}^2 \leqslant C \, . \, \Delta x \, ,$$

$$\sum_{i=0}^{N-1} (Y_{2i+2} - Y_{2i})^2 \leqslant C \, . \, \Delta x \, ,$$

where Δx is defined above and C is a constant independent of the mesh.

In particular, we find

$$\sum_{i=0}^{N-1} (x_{2i+2} - x_{2i+1}) \, Z_{2i+1}^2 \leqslant C \cdot (\Delta x)^2 \, .$$

We thus obtain the same estimates as in the uniform case and we conclude that the Incremental Unknowns are small, as expected. We also note that this result is obtained without any assumption on the mesh.

3.2. The two dimensional case

We consider the Dirichlet problem:

$$\begin{cases} -\Delta u = f \ \text{in} \ \Omega = \,]0, 1[^2, \\ u = 0 \qquad \text{on} \ \partial\Omega \, . \end{cases} \tag{3.6}$$

We discretize (3.6) with finite differences on a mesh defined by its discrete coordinates x_i and y_j, and obtain the following discrete system:

$$\frac{y_{j+1} - y_{j-1}}{2} (\alpha_i u_{i,j} - \beta_i u_{i-1,j} - \gamma_i u_{i+1,j}) + \frac{x_{i+1} - x_{i-1}}{2} (\tilde{\alpha}_j u_{i,j} - \tilde{\beta}_j u_{i,j-1} - \tilde{\gamma}_j u_{i,j+1})$$
$$= \frac{y_{j+1} - y_{j-1}}{2} \frac{x_{i+1} - x_{i-1}}{2} f_{i,j} \, , \tag{3.7}$$

where

$$\alpha_i = \frac{1}{x_{i+1} - x_i} + \frac{1}{x_i - x_{i-1}} \, ; \quad \tilde{\alpha}_j = \frac{1}{y_{j+1} - y_j} + \frac{1}{y_j - y_{j-1}}$$
$$\beta_i = \frac{1}{x_i - x_{i-1}} \qquad \tilde{\beta}_j = \frac{1}{y_j - y_{j-1}}$$
$$\gamma_i = \frac{1}{x_{i+1} - x_i} \qquad \tilde{\gamma}_j = \frac{1}{y_{j+1} - y_j} \, ,$$

and $f_{i,j} = f(x_i, y_j)$. We recall the definition of the incremental unknowns:

DEFINITION 1: *The incremental unknowns consist of the nodal values $Y_{2i,2j} = u_{2i,2j}$ at the coarse grid points and of appropriate incremental quantities $Z_{\omega,\psi}$ (ω or ψ odd) at the points that are on the complementary grid, where*

$$Z_{2i,2j+1} = u_{2i,2j+1} - \frac{1}{\tilde{\alpha}_{2j+1}} (\tilde{\beta}_{2j+1} u_{2i,2j} + \tilde{\gamma}_{2j+1} u_{2i,2j+2})$$

at the points of type f1 ;

$$Z_{2i+1,2j} = u_{2i+1,2j} - \frac{1}{\alpha_{2i+1}} (\beta_{2i+1} u_{2i,2j} + \gamma_{2i+1} u_{2i+2,2j})$$

at the points of type f2;

$$Z_{2i,1+2j+1} = u_{2i+1,2j+1} - \frac{1}{\alpha_{2i+1} \cdot \tilde{\alpha}_{2j+1}} (\tilde{\beta}_{2j+1} \beta_{2i+1} u_{2i,2j} + \tilde{\beta}_{2j+1} \gamma_{2i+1} u_{2i+2,2j}$$
$$+ \tilde{\gamma}_{2j+1} \beta_{2i+1} u_{2i,2j+2} + \tilde{\gamma}_{2j+1} \gamma_{2i+1} u_{2i+2,2j+2})$$

at the points of type f3.

3.2.1. *A Priori estimates*

We multiply (3.7) by $u_{i,j}$ and sum on all indices. Using the relations $\alpha_i = \beta_i + \gamma_i$ and $\tilde{\alpha}_j = \tilde{\beta}_j + \tilde{\gamma}_j$, we obtain

$$A = \sum_{i,j=1}^{2N-1} \gamma_i \left(\frac{y_{j+1} - y_{j-1}}{2}\right) (u_{i+1,j} - u_{i,j}) u_{i,j} - \sum_{i,j=1}^{2N-1} \beta_i \left(\frac{y_{j+1} - y_{j-1}}{2}\right) (u_{i,j} - u_{i-1,j}) u_{i,j}$$
$$+ \sum_{i,j=1}^{2N-1} \tilde{\gamma}_j \left(\frac{x_{i+1} - x_{i-1}}{2}\right) (u_{i,j+1} - u_{i,j}) u_{i,j} - \sum_{i,j=1}^{2N-1} \tilde{\beta}_j \left(\frac{x_{i+1} - x_{i-1}}{2}\right) (u_{i,j} - u_{i,j-1}) u_{i,j}$$
$$= \sum_{i,j=1}^{2N-1} \frac{(y_{j+1} - y_{j-1})}{2} \frac{(x_{i+1} - x_{i-1})}{2} f_{i,j} u_{i,j} .$$

We deduce, setting for convenience $x_{-1} = y_{-1} = 0$

$$A = \sum_{i=0}^{2N-1} \sum_{j=1}^{2N-1} \left(\frac{y_{j+1} - y_{j-1}}{2}\right) (\gamma_i u_{i+1,j} - \beta_{i+1} u_{i,j}) (u_{i+1,j} - u_{i,j})$$
$$+ \sum_{i=1}^{2N-1} \sum_{j=0}^{2N-1} \left(\frac{x_{i+1} - x_{i-1}}{2}\right) (\tilde{\gamma}_j u_{i,j+1} - \tilde{\beta}_{j+1} u_{i,j}) (u_{i,j+1} - u_{i,j})$$
$$= \sum_{i,j=1}^{2N-1} \frac{(y_{j+1} - y_{j-1})}{2} \frac{(x_{i+1} - x_{i-1})}{2} f_{i,j} u_{i,j} .$$

Since $\gamma_i = \beta_{i+1}$ and $\tilde{\gamma}_j = \tilde{\beta}_{j+1}$, the previous expression is reduced to

$$A = \sum_{i=0}^{2N-1} \sum_{j=1}^{2N-1} \gamma_i \left(\frac{y_{j+1} - y_{j-1}}{2} \right) (u_{i+1,j} - u_{i,j})^2$$

$$+ \sum_{i=1}^{2N-1} \sum_{j=0}^{2N-1} \tilde{\gamma}_j \left(\frac{x_{i+1} - x_{i-1}}{2} \right) (u_{i,j+1} - u_{i,j})^2$$

$$= \sum_{i,j=1}^{2N-1} \frac{(y_{j+1} - y_{j-1})}{2} \frac{(x_{i+1} - x_{i-1})}{2} f_{i,j} u_{i,j} .$$

We set

$$h = \max_{1 \leqslant i,j \leqslant 2N-1} \sqrt{(x_{i+1} - x_{i-1})(y_{j+1} - y_{j-1})} ,$$

$$\delta h = \min \left(\frac{\delta x}{\Delta y}, \frac{\delta y}{\Delta x} \right) ,$$

where $\delta x = \min_{0 \leqslant i \leqslant 2N-1} (x_{i+1} - x_i)$ and $\Delta x = \max_{0 \leqslant i \leqslant 2N-1} (x_{i+1} - x_i)$ (δy and Δy are defined similarly). We have

$$\sum_{i=0}^{2N-1} \sum_{j=1}^{2N-1} \gamma_i (y_{j+1} - y_{j-1})(u_{i+1,j} - u_{i,j})^2 + \sum_{i=1}^{2N-1} \sum_{j=0}^{2N-1} \tilde{\gamma}_j (x_{i+1} - x_{i-1})(u_{i,j+1} - u_{i,j})^2$$

$$\leqslant \sum_{i,j=1}^{2N-1} (y_{j+1} - y_{j-1})(x_{i+1} - x_{i-1}) f_{i,j} \cdot u_{i,j} ,$$

$$\leqslant (\text{thanks to Cauchy} - \text{Schwarz inequality}) ,$$

$$\leqslant \sqrt{\frac{1}{4} \sum_{i,j=1}^{2N-1} (y_{j+1} - y_{j-1})(x_{i+1} - x_{i-1}) f_{i,j}^2} \cdot \sqrt{\sum_{i,j=1}^{2N-1} (y_{j+1} - y_{j-1})(x_{i+1} - x_{i-1}) u_{i,j}^2} .$$

At this point we introduce the following Poincaré inequalities:

LEMMA 2: *Let $u_{i,j}$, $i, j = 0, \ldots, 2N$ be a sequence of real numbers such that $u_{\phi,\psi} = 0$ if ϕ or ψ takes the values 0 or $2N$. Then we have*

$$\sum_{i,j=1}^{2N-1} (x_{i+1} - x_{i-1})(y_{j+1} - y_{j-1}) u_{i,j}^2 \leqslant 2 \sum_{i=0}^{2N-1} \sum_{j=1}^{2N-1} (y_{j+1} - y_{j-1}) \frac{(u_{i+1,j} - u_{i,j})^2}{x_{i+1} - x_i} ,$$

and

$$\sum_{i,j=1}^{2N-1} (x_{i+1} - x_{i-1})(y_{j+1} - y_{j-1}) u_{i,j}^2 \leqslant 2 \sum_{i=1}^{2N-1} \sum_{j=0}^{2N-1} (x_{i+1} - x_{i-1}) \frac{(u_{i,j+1} - u_{i,j})^2}{y_{j+1} - y_j} .$$

Here x_i and y_j are the sequences introduced in Section 2.

Proof:
We have

$$\sqrt{(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})}\,|u_{i,j}| = \sqrt{(x_{i+1}-x_{i-1})}\sum_{k=0}^{i-1}\left(|u_{k+1,j}|-|u_{k,j}|\right)\cdot\sqrt{(y_{j+1}-y_{j-1})}$$

$$\leqslant \sqrt{x_{i+1}-x_{i-1}}\sum_{i=0}^{2N-1}\sqrt{(y_{j+1}-y_{j-1})}\,\frac{(|u_{i+1,j}-u_{i,j}|)}{\sqrt{x_{i+1}-x_i}}\sqrt{x_{i+1}-x_i}$$

$$\leqslant \sqrt{(x_{i+1}-x_{i-1})}\sqrt{\sum_{i=0}^{2N-1}(y_{j+1}-y_{j-1})\frac{(u_{i+1,j}-u_{i,j})^2}{x_{i+1}-x_i}}\,.$$

Therefore

$$\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})\,u_{i,j}^2 \leqslant \left(\sum_{i=1}^{2N-1}(x_{i+1}-x_{i-1})\right)\sum_{i=0}^{2N-1}(y_{j+1}-y_{j-1})\frac{(u_{i+1,j}-u_{i,j})^2}{x_{i+1}-x_i}\,,$$

which finally yields

$$\sum_{i,j=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})\,u_{i,j}^2 \leqslant 2\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}(y_{j+1}-y_{j-1})\frac{(u_{i+1,j}-u_{i,j})^2}{x_{i+1}-x_i}\,.$$

Similarly we have

$$\sum_{i,j=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})\,u_{i,j}^2 \leqslant 2\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}(x_{i+1}-x_{i-1})\frac{(u_{i,j+1}-u_{i,j})^2}{y_{j+1}-y_j}\,.$$

∎

We deduce from Lemma 2 that

$$\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}\gamma_i(y_{j+1}-y_{j-1})(u_{i+1,j}-u_{i,j})^2+\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}\tilde{\gamma}_j(x_{i+1}-x_{i-1})(u_{i,j+1}-u_{i,j})^2$$

$$\leqslant \sqrt{\frac{1}{4}\sum_{i,j=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f_{i,j}^2}$$

$$\times\sqrt{\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}\gamma_i(y_{j+1}-y_{j-1})(u_{i+1,j}-u_{i,j})^2+\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}\tilde{\gamma}_j(x_{i+1}-x_{i-1})(u_{i,j+1}-u_{i,j})^2}\,,$$

which implies that

$$\frac{1}{2}\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}\gamma_i(y_{j+1}-y_{j-1})(u_{i+1,j}-u_{i,j})^2+\frac{1}{2}\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}\tilde{\gamma}_j(x_{i+1}-x_{i-1})(u_{i,j+1}-u_{i,j})^2$$

$$\leqslant \frac{1}{8}\sum_{i,j=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f_{i,j}^2\,,$$

and thus

$$\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}(u_{i+1,j}-u_{i,j})^2+\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}(u_{i,j+1}-u_{i,j})^2$$

$$\leqslant \frac{1}{4}\max\left(\frac{\Delta x}{\delta y},\frac{\Delta y}{\delta x}\right)\sum_{i,j=1}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f_{i,j}^2 .$$

We now set

$$\mathscr{L}=\sum_{i=0}^{2N-1}\sum_{j=1}^{2N-1}(u_{i+1,j}-u_{i,j})^2+\sum_{i=1}^{2N-1}\sum_{j=0}^{2N-1}(u_{i,j+1}-u_{i,j})^2 ,$$

and we split $\mathscr{L}$ as follows:

$$\mathscr{L}=\mathscr{L}_a+\mathscr{L}_b+\mathscr{L}_c+\mathscr{L}_d ,$$

where

$\mathscr{L}_a$ corresponds to the points of type $f0$,

$\mathscr{L}_b$ corresponds to the points of type $f1$,

$\mathscr{L}_c$ corresponds to the points of type $f2$,

$\mathscr{L}_d$ corresponds to the points of type $f3$.

We then introduce the Incremental Unknowns and we find:

$$\mathscr{L}_a=\sum_{i,j=0}^{N-1}\left\{Z_{2i+1,2j}-\frac{\gamma_{2i+1}}{\alpha_{2i+1}}(Y_{2i+2,2j}-Y_{2i,2j})\right\}^2+\sum_{i,j=0}^{N-1}\left\{Z_{2i,2j+1}-\frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}}(Y_{2i,2j+2}-Y_{2i,2j})\right\}^2 ,$$

$$\mathscr{L}_b=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\{Z_{2i+1,2j+1}-Z_{2i,2j+1}+g_{i,j}\}^2+\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{-Z_{2i,2j+1}-\frac{\tilde{\beta}_{2j+1}}{\tilde{\alpha}_{2j+1}}(Y_{2i,2j+2}-Y_{2i,2j})\right\}^2 ,$$

where

$$g_{i,j}=\frac{1}{\alpha_{2i+1}\tilde{\alpha}_{2j+1}}\{\gamma_{2i+1}\tilde{\gamma}_{2j+1}Y_{2i+2,2j+2}+\gamma_{2i+1}\tilde{\beta}_{2j+1}Y_{2i+2,2j}-\gamma_{2i+1}\tilde{\gamma}_{2j+1}Y_{2i,2j+2}-\gamma_{2i+1}\tilde{\beta}_{2j+1}Y_{2i,2j}\} ;$$

$$\mathscr{L}_c=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\{Z_{2i+1,2j+1}-Z_{2i+1,2j}+g'_{i,j}\}^2+\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{-Z_{2i+1,2j}-\frac{\beta_{2i+1}}{\alpha_{2i+1}}(Y_{2i+2,2j}-Y_{2i,2j})\right\}^2 ,$$

where

$$g'_{i,j} = \frac{1}{\alpha_{2i+1}\tilde{\alpha}_{2j+1}} \{\gamma_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i+2,2j+2} - \gamma_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i+2,2j} + \beta_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i,2j+2} - \beta_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i,2j}\};$$

$$\mathscr{L}_d = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \{Z_{2i+2,2j+1} - Z_{2i+1,2j+1} + h_{i,j}\}^2 + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \{Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + h'_{i,j}\}^2,$$

where

$$h_{i,j} = \frac{1}{\alpha_{2i+1}\tilde{\alpha}_{2j+1}} \{\beta_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i+2,2j+2} + \beta_{2i+1}\tilde{\beta}_{2j+1} Y_{2i+2,2j} - \beta_{2i+1}\tilde{\gamma}_{2j+1} Y_{2i,2j+2} - \beta_{2i+1}\tilde{\beta}_{2j+1} Y_{2i,2j}\},$$

and

$$h'_{i,j} = \frac{1}{\alpha_{2i+1}\tilde{\alpha}_{2j+1}} \{\gamma_{2i+1}\tilde{\beta}_{2j+1} Y_{2i+2,2j+2} - \gamma_{2i+1}\tilde{\beta}_{2j+1} Y_{2i+2,2j} + \beta_{2i+1}\tilde{\beta}_{2j+1} Y_{2i,2j+2} - \beta_{2i+1}\tilde{\beta}_{2j+1} Y_{2i,2j}\}.$$

We can write

$$g_{i,j} = \frac{\gamma_{2i+1}}{\alpha_{2i+1}} q_{i,j},\ h_{i,j} = \frac{\beta_{2i+1}}{\alpha_{2i+1}} q_{i,j},\ g'_{i,j} = \frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j},\ h'_{i,j} = \frac{\tilde{\beta}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j},$$

where

$$q'_{i,j} = \frac{1}{\tilde{\alpha}_{2j+1}} \{\tilde{\gamma}_{2j+1} Y_{2i+2,2j+2} + \tilde{\beta}_{2j+1} Y_{2i+2,2j} - \tilde{\gamma}_{2j+1} Y_{2i,2j+2} - \tilde{\beta}_{2j+1} Y_{2i,2j}\},$$

and

$$q_{i,j} = \frac{1}{\alpha_{2i+1}} \{\gamma_{2i+1} Y_{2i+2,2j+2} - \gamma_{2i+1} Y_{2i+2,2j} + \beta_{2i+1} Y_{2i,2j+2} - \beta_{2i+1} Y_{2i,2j}\}.$$

Developping these expressions and setting

$$\mathscr{L} = \mathscr{L}_a + \mathscr{L}_b + \mathscr{L}_c + \mathscr{L}_d = \sum_{i,j=0}^{N-1} \mathscr{L}_{i,j},$$

we obtain

$$\begin{aligned}
\mathscr{L}_{i,j} = {} & 2 \cdot Z^2_{2i+1,2j} + 2 \cdot Z^2_{2i,2j+1} \frac{\beta^2_{2i+1} + \gamma^2_{2i+1}}{\alpha^2_{2i+1}} (Y_{2i+2,2j} - Y_{2i,2j})^2 + \frac{\tilde{\beta}^2_{2j+1} + \tilde{\gamma}^2_{2j+1}}{\tilde{\alpha}^2_{2j+1}} (Y_{2i,2j+2} - Y_{2i,2j})^2 \\
& + \left(Z_{2i+1,2j+1} - Z_{2i,2j+1} + \frac{\gamma_{2i+1}}{\alpha_{2i+1}} q_{i,j}\right)^2 + \left(Z_{2i+2,2j+1} - Z_{2i+1,2j+1} + \frac{\beta_{2i+1}}{\alpha_{2i+1}} q_{i,j}\right)^2 \\
& + \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j}\right)^2 + \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\tilde{\beta}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j}\right)^2 \\
& + 2 \cdot \frac{\beta_{2i+1} - \gamma_{2i+1}}{\alpha_{2i+1}} Z_{2i+1,2j} \cdot (Y_{2i+2,2j} - Y_{2i,2j}) + 2 \cdot \frac{\tilde{\beta}_{2j+1} + \tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} Z_{2i,2j+1} \cdot (Y_{2i,2j+2} - Y_{2i,2j}).
\end{aligned}$$

We now consider the term

$$P_{i,j} = 2 \cdot \frac{\beta_{2i+1} - \gamma_{2i+1}}{\alpha_{2i+1}} Z_{2i+1,2j} \cdot (Y_{2i+2,2j} - Y_{2i,2j}) + 2 \cdot \frac{\tilde{\beta}_{2j+1} - \tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} Z_{2i,2j+1} \cdot (Y_{2i,2j+2} - Y_{2i,2j}) .$$

We obtain using Young's inequality

$$P_{i,j} \geqslant - \epsilon_1 Z^2_{2i+1,2j} \left| \frac{\beta_{2i+1} - \gamma_{2i+1}}{\alpha_{2i+1}} \right| - \frac{1}{\epsilon_1} (Y_{2i+2,2j} - Y_{2i,2j})^2$$
$$- \epsilon_2 Z^2_{2i,2j+1} - \frac{1}{\epsilon_2} \left| \frac{\tilde{\beta}_{2j+1} - \tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} \right| (Y_{2i,2j+2} - Y_{2i,2j})^2 ,$$

where, as in the one dimensional case, ϵ_1 and ϵ_2 are two positive constants which will be fixed later. We set $\xi(i) = \dfrac{\beta_{2i+1}}{\gamma_{2i+1}}$ and $\eta(j) = \dfrac{\tilde{\beta}_{2i+1}}{\tilde{\gamma}_{2j+1}}$ and we introduce the following functions:

$$g_1(x, \epsilon) = 2 - \epsilon \left| \frac{1-x}{1+x} \right| \quad \text{and} \quad g_2(x, \epsilon) = \frac{1+x^2}{(1+x)^2} - \frac{1}{\epsilon} \left| \frac{1-x}{1+x} \right| .$$

We thus have

$$\mathscr{L}_{i,j} \geqslant g_1(\xi(i), \epsilon_1) Z^2_{2i+1,2j} + g_1(\eta(j), \epsilon_2) Z^2_{2i,2j+1}$$
$$+ g_2(\xi(i), \epsilon_1) (Y_{2i+2,2j} - Y_{2i,2j})^2 + g_2(\eta(j), \epsilon_2) (Y_{2i,2j+2} - Y_{2i,2j})^2$$
$$+ \left(Z_{2i+1,2j+1} - Z_{2i,2j+1} + \frac{\gamma_{2i+1}}{\alpha_{2i+1}} q_{i,j} \right)^2 + \left(Z_{2i+2,2j+1} - Z_{2i+1,2j+1} + \frac{\beta_{2j+1}}{\alpha_{2i+1}} q_{i,j} \right)^2$$
$$+ \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j} \right)^2 + \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\beta_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j} \right)^2 .$$

As in space dimension one, we can chose ϵ_1 and ϵ_2 such that the functions g_1 and g_2 are bounded from below by a strictly positive constant which is independent of the mesh. Therefore

$$\mathscr{L}_{i,j} \geqslant C . Z^2_{2i+1,2j} + C . Z^2_{2i,2j+1} + C . (Y_{2i+2,2j} - Y_{2i,2j})^2 + C . (Y_{2i,2j+2} - Y_{2i,2j})^2$$
$$+ \left(Z_{2i+1,2j+1} - Z_{2i,2j+1} + \frac{\gamma_{2i+1}}{\alpha_{2i+1}} q_{i,j} \right)^2 + \left(Z_{2i+2,2j+1} - Z_{2i+1,2j+1} + \frac{\beta_{2i+1}}{\alpha_{2i+1}} q_{i,j} \right)^2$$
$$+ \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j} \right)^2 + \left(Z_{2i+1,2j+2} - Z_{2i+1,2j+1} + \frac{\beta_{2j+1}}{\tilde{\alpha}_{2j+1}} q'_{i,j} \right)^2 ,$$

where C is an absolute strictly positive constant. We now introduce the following technical lemma:

LEMMA 3: *Let $s, t \in \mathbb{R}$ be such that $s^2 + t^2 \neq 0$, $s + t \neq 0$. Then for every $a, b, c, d \in \mathbb{R}$ and for every $k \geqslant 1$:*

$$(a - b + s . c)^2 + (d - a + t . c)^2 \geqslant \frac{1}{k} \frac{(s+t)^2}{s^2 + t^2} \left(\frac{1}{2} a^2 - \frac{2 . t^2}{(s+t)^2} b^2 - \frac{2 . s^2}{(s+t)^2} d^2 \right) .$$

Proof:
We have

$$
\begin{aligned}
(a-b+sc)^2+(d-a+tc)^2 &= (a-b)^2+(d-a)^2+(s^2+t^2)\,c^2 \\
&\quad + 2\,sc(a-b)+2\,tc(d-a) \\
&= (a-b)^2+(d-a)^2 \\
&\quad + (s^2+t^2)\left(c^2+\frac{2\,s}{s^2+t^2}c(a-b)+\frac{2\,t}{s^2+t^2}c(d-a)\right) \\
&= (a-b)^2+(d-a)^2 \\
&\quad + (s^2+t^2)\left(c+\frac{s}{s^2+t^2}(a-b)+\frac{t}{s^2+t^2}(d-a)\right)^2 \\
&\quad - \frac{s^2}{s^2+t^2}(a-b)^2-\frac{t^2}{s^2+t^2}(d-a)^2-2\frac{st}{s^2+t^2}(a-b)(d-a) \\
&= \frac{t^2}{s^2+t^2}(a-b)^2+\frac{s^2}{s^2+t^2}(d-a)^2-2\frac{st}{s^2+t^2}(a-b)(d-a) \\
&\quad + (s^2+t^2)\left(c+\frac{s}{s^2+t^2}(a-b)+\frac{t}{s^2+t^2}(d-a)\right)^2 \\
&= \frac{1}{s^2+t^2}\left(t(a-b)-s(d-a)\right)^2 \\
&\quad + (s^2+t^2)\left(c+\frac{s}{s^2+t^2}(a-b)+\frac{t}{s^2+t^2}(d-a)\right)^2 .
\end{aligned}
$$

Thus

$$
(a-b+sc)^2+(d-a+tc)^2 \geqslant \frac{(s+t)^2}{s^2+t^2}\left(a-\frac{t}{s+t}b-\frac{s}{s+t}d\right)^2,
$$

and for all $k \geqslant 1$ we have

$$
(a-b+sc)^2+(d-a+tc)^2 \geqslant \frac{1}{k}\frac{(s+t)^2}{s^2+t^2}\left(a-\frac{t}{s+t}b-\frac{s}{s+t}d\right)^2 .
$$

Now

$$
\begin{aligned}
\left(a-\frac{t}{s+t}b-\frac{s}{s+t}d\right)^2 &= a^2+\left(\frac{t}{s+t}b+\frac{s}{s+t}d\right)^2-2\,a\left(\frac{t}{s+t}b+\frac{s}{s+t}c\right) \\
&\geqslant \frac{1}{2}a^2-\left(\frac{t}{s+t}b+\frac{s}{s+t}d\right)^2 \\
&\geqslant \frac{1}{2}a^2-2\frac{t^2}{(s+t)^2}b^2-2\frac{s^2}{(s+t)^2}d^2 ,
\end{aligned}
$$

hence the result. ■

We note here that the boundary conditions imply.

$$Z_{2i,2j+1} = Y_{2i,2j+1} = 0 \quad \text{for} \quad i = 0 \text{ or } N,$$
$$Z_{2i+1,2j} = Y_{2i+1,2j} = 0 \quad \text{for} \quad j = 0 \text{ or } N.$$

Consequently we can replace, in the expression of $\mathscr{L}_{i,j}$,

$$C \cdot Z^2_{2i+1,2j} + C \cdot Z^2_{2i,2j+1}$$

by

$$\frac{C}{2}\left(Z^2_{2i+1,2j} + Z^2_{2i+2,2j+1} + Z^2_{2i,2j+1} + Z^2_{2i+1,2j+2}\right).$$

We set

$$s_i = \frac{\gamma_{2i+1}}{\alpha_{2i+1}},\ t_i = \frac{\beta_{2i+1}}{\alpha_{2i+1}},\ \tilde{s}_j = \frac{\tilde{\gamma}_{2j+1}}{\tilde{\alpha}_{2j+1}},\ \tilde{t}_j = \frac{\tilde{\beta}_{2j+1}}{\tilde{\alpha}_{2j+1}},$$

and we obtain using Lemma 3 and the relations $s_i + t_i = 1$ and $\tilde{s}_j + \tilde{t}_j = 1$:

$$\begin{aligned}
\mathscr{L}_{i,j} \geq{} & C \cdot \left(Z^2_{2i+1,2j} + Z^2_{2i+2,2j} + Z^2_{2i,2j+1} + Z^2_{2i,2j+2}\right) \\
& + C \cdot \left(Y_{2i+2,2j} - Y_{2i,2j}\right)^2 + C \cdot \left(Y_{2i,2j+2} - Y_{2i,2j}\right)^2 \\
& + \frac{1}{2\,k(s_i^2 + t_i^2)} Z^2_{2i+1,2j+1} - \frac{2\,t_i^2}{k(s_i^2 + t_i^2)} Z^2_{2i,2j+1} - \frac{2\,s_i^2}{k(s_i^2 + t_i^2)} Z^2_{2i+2,2j+1} \\
& + \frac{1}{2\,k(\tilde{s}_j^2 + \tilde{t}_j^2)} Z^2_{2i+1,2j+1} - \frac{2\,\tilde{t}_j^2}{k(\tilde{s}_j^2 + \tilde{t}_j^2)} Z^2_{2i+1,2j} - \frac{2\,\tilde{s}_j^2}{k(\tilde{s}_j^2 + \tilde{t}_j^2)} Z^2_{2i+1,2j+2}.
\end{aligned}$$

We then choose k such that for all i, j:

$$\frac{2\,t_i^2}{k(s_i^2 + t_i^2)} \leq \frac{C}{2}, \quad \frac{2\,\tilde{t}_j^2}{k(\tilde{s}_j^2 + \tilde{t}_j^2)} \leq \frac{C}{2},$$

$$\frac{2\,s_i^2}{k(s_i^2 + t_i^2)} \leq \frac{C}{2}, \quad \frac{2\,\tilde{s}_j^2}{k(\tilde{s}_j^2 + \tilde{t}_j^2)} \leq \frac{C}{2}.$$

This is possible since the function $\dfrac{t^2}{t^2 + (1-t)^2}$ is bounded for $t \in [0, 1]$. Furthermore the function $\dfrac{(1+x)^2}{1+x^2}$ is bounded from below by 1 for $x \in [0, \infty[$. We deduce that there exists a constant c independent of Δx and Δy such that

$$\mathscr{L}_{i,j} \geq c \cdot \left(Z^2_{2i+1,2j} + Z^2_{2i,2j+1} + Z^2_{2i+1,2j+1} + \left(Y_{2i+2,2j} - Y_{2i,2j}\right)^2 + \left(Y_{2i,2j+2} - Y_{2i,2j}\right)^2\right).$$

Therefore

$$\sum_{i,j=0}^{N-1}\{Z^2_{2i+1,2j}+Z^2_{2i,2j+1}+Z^2_{2i+1,2j+1}\}+\sum_{i,j=0}^{N-1}\{(Y_{2i+2,2j}-Y_{2i,2j})^2+(Y_{2i,2j+2}-Y_{2i,2j})^2\}$$

$$\leqslant \frac{c}{\delta h}\sum_{i,j=0}^{2N-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f^2_{i,j}.$$

If we replace $f_{i,j}=f(x_i,y_j)$ by $f_{i,j}=\frac{1}{(x_{i+1}-x_{i-1})\cdot(y_{j+1}-y_{j-1})}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} f(x,y)\,dx\,dy$, we obtain

$$\sum_{i,j=0}^{N-1}\{Z^2_{2i+1,2j}+Z^2_{2i,2j+1}+Z^2_{2i+1,2j+1}\}+\sum_{i,j=0}^{N-1}\{(Y_{2i+2,2j}-Y_{2i,2j})^2+(Y_{2i,2j+2}-Y_{2i,2j})^2\}$$

$$\leqslant \frac{c}{\delta h}|f|^2_{L^2},$$

where c is again a constant independent of Δx and Δy. Finally we obtain the following result:

PROPOSITION 2: *The second order Incremental Unknowns associated to the discretization of (3.6) satisfy the following a priori estimates:*

$$\begin{aligned}\sum_{i,j=0}^{N-1}\{Z^2_{2i+1,2j}+Z^2_{2i,2j+1}+Z^2_{2i+1,2j+1}\} &\leqslant \frac{c}{\delta h}|f|^2_{L^2},\\ \sum_{i,j=0}^{N-1}(Y_{2i+2,2j}-Y_{2i,2j})^2 &\leqslant \frac{c}{\delta h}|f|^2_{L^2},\\ \sum_{i,j=0}^{N-1}(Y_{2i,2j+2}-Y_{2i,2j})^2 &\leqslant \frac{c}{\delta h}|f|^2_{L^2},\end{aligned}$$

where δh is defined above and c is a constant independent of the mesh

In particular, we find

$$\sum_{i,j=0}^{N-1}\{(x_{2i+2}-x_{2i+1})(y_{2j+1}-y_{2j})Z^2_{2i+1,2j}+(x_{2i+1}-x_{2i})(y_{2j+2}-y_{2j+1})Z^2_{2i,2j+1}\}$$

$$+\sum_{i,j=0}^{N-1}(x_{2i+2}-x_{2i+1})(y_{2j+2}-y_{2j+1})Z^2_{2i+1,2j+1}$$

$$\leqslant \frac{c\cdot h^2}{\delta h}|f|^2_{L^2}.$$

In that way, as in the one dimensional case, we obtain estimates that are similar to those established in the uniform case. Futhermore, if $\frac{h}{\sqrt{\delta h}}\to 0$, the incremental unknowns are small as expected. We note that this assumption is satisfied for the meshes that are usually used for boundary layer problems (Chebyshev and a tanh like meshes).

4. HIERARCHICAL PRECONDITIONING

In this section, we propose a construction of hierarchical preconditioners for the Laplacian associated to a nonuniform mesh. In particular, we try to recover the usual asymptotic condition number of the matrix obtained for uniform grids. This approach is typical of the finite differences method.

4.1. The one dimensional case

4.1.1. *Construction of the preconditioner*

For the sake of simplicity, we only consider the one dimensional case; the two dimensional case will follow by extension.

The basic idea is to construct a hierarchical preconditioner which, as in the regular mesh case, gives a H_0^1 orthogonality type property. From the point of view of the matricial framework, this construction can be summarized by the determination of two matrices S and ${}^tT$ under and upper triangular respectively such that

$$
{}^tT.A.S
$$

is bloc diagonal, A being the discretization matrix of the Laplacian. We note that $T = S$ when A is symmetric.

We consider the Dirichlet problem:

$$
\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} = f \quad \text{in } \Omega =]0, 1[, \\ u(0) = u(1) = 0 . \end{cases} \tag{4.8}
$$

We discretize (4.8) by finite differences with a three-points scheme. We obtain a system of the form:

$$
\begin{cases}
\alpha_1 u_1 - \gamma_2 u_2 & = f_1, \\
\alpha_2 u_2 - \beta_2 u_1 - \gamma_2 u_3 & = f_2, \\
. & \\
. & \\
. & \\
\alpha_i u_i - \beta_i u_{i-1} - \gamma_i u_{i+1} & = f_i, \\
. & \\
. & \\
. & \\
\alpha_{N-1} u_{N-1} - \beta_{N-1} u_{N-2} - \gamma_{N-1} u_N & = f_{N-1}, \\
\alpha_N u_N - \beta_{N-1} u_{N-1} & = f_N,
\end{cases} \tag{4.9}
$$

where α_i, β_i and γ_i can be different from the numbers introduced in Section 3.

Construction of ${}^tT$ and S

We proceed by a bloc procedure. We first consider two grid levels. The discretization matrix A is written with the hierarchical ordering in the form

$$
\bar{A} = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}, \tag{4.10}
$$

where A_i, $i = 1, 2$ are invertible diagonal matrices.

Construction of S

We want to construct a matrix S of the form:

$$
S = \begin{pmatrix} I & 0 \\ G_1 & I \end{pmatrix}, \tag{4.11}
$$

and such that $\tilde{A}S$ is upper triangular. We have

$$\tilde{A}S = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ G_1 & I \end{pmatrix} = \begin{pmatrix} A_1 + B_1 G_1 & B_1 \\ B_2 + A_2 G_1 & A_2 \end{pmatrix}. \tag{4.12}$$

Therefore the under-matrix G_1 satisfies

$$G_1 = - A_2^{-1} B_2 ,$$

hence

$$S = \begin{pmatrix} I & 0 \\ - A_2^{-1} B_2 & I \end{pmatrix}. \tag{4.13}$$

Construction of ${}^tT$

We now want to construct a matrix ${}^tT$ of the form:

$${}^tT = \begin{pmatrix} I & G_2 \\ 0 & I \end{pmatrix}, \tag{4.14}$$

and such that ${}^tT\tilde{A}\,.\,S$ is bloc diagonal. We have

$${}^tT\tilde{A}S = {}^tT(\tilde{A}S) = \begin{pmatrix} I & G_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 + B_1 G_1 & B_1 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 + B_1 G_1 & B_1 + G_2 \,.\, A_2 \\ 0 & A_2 \end{pmatrix}, \tag{4.15}$$

and then G_2 must satisfy

$$B_2 + G_2 A_2 = 0 .$$

Thus

$${}^tT = \begin{pmatrix} I & - B_2 A_2^{-1} \\ 0 & I \end{pmatrix}, \tag{4.16}$$

and $\tilde{A}$ can be written in the form

$$\hat{A} = {}^tT\tilde{A}S = {}^tT = \begin{pmatrix} A_1 + B_1 G_1 & 0 \\ 0 & A_2 \end{pmatrix}. \tag{4.17}$$

The first diagonal bloc of $\hat{A}$ is still tridiagonal and we can repeat recursively the reduction procedure described above by considering now $d+1$ grid levels. This procedure is easily accomplished when one knows the coefficients of $\hat{A}$.

Computation of the coefficients of ${}^tT.A.S$

We first consider two grid levels. We restart from (4.9), and we introduce the following Incremental Unknowns (that can differ from those introduced in Section 3: the problem considered here is more general)

$$Z_{2i+1} = u_{2i+1} - \frac{1}{\alpha_{2i+1}} (\beta_{2i+1} u_{2i} + \gamma_{2i+1} u_{2i+2}) .$$

Injecting this relation into (4.9), we obtain

$$\begin{cases} \textbf{For the fine grid points}: \\ \alpha_{2\iota+1} u_{2\iota+1} - \beta_{2\iota+1} u_{2\iota} - \gamma_{2\iota+1} u_{2\iota+2} = f_{2\iota+1}, \\ \text{that is to say} \\ \alpha_{2\iota+1} Z_{2\iota+1} = f_{2\iota+1}. \\ \textbf{For the coarse grid points}: \\ \alpha_{2\iota} u_{2\iota} - \beta_{2\iota} u_{2\iota-1} - \gamma_{2\iota} u_{2\iota+1} = f_{2\iota}. \end{cases}$$

Since

$$u_{2\iota+1} = Z_{2\iota+1} + \frac{1}{\alpha_{2\iota+1}} (\beta_{2\iota+1} u_{2\iota} + \gamma_{2\iota+1} u_{2\iota+2}),$$

and

$$u_{2\iota-1} = Z_{2\iota-1} + \frac{1}{\alpha_{2\iota-1}} (\beta_{2\iota-1} u_{2\iota-2} + \gamma_{2\iota-1} u_{2\iota}),$$

we find

$$\left(\alpha_{2\iota} - \frac{\beta_{2\iota}\gamma_{2\iota-1}}{\alpha_{2\iota-1}} - \frac{\beta_{2\iota+1}\gamma_{2\iota}}{\alpha_{2\iota+1}}\right) u_{2\iota} - \frac{\beta_{2\iota}\beta_{2\iota-1}}{\alpha_{2\iota-1}} u_{2\iota-2} - \frac{\gamma_{2\iota}\gamma_{2\iota+1}}{\alpha_{2\iota+1}} u_{2\iota+2} = \gamma_{2\iota} \cdot Z_{2\iota+1} + \beta_{2\iota} \cdot Z_{2\iota-1} + f_{2\iota},$$

By analogy with the form of the reduced matrix, we can say that, for d grid levels, the coefficients of the matrix $\hat{A}^d$ are

$$\hat{A}^d = \begin{pmatrix} P^d & 0 & . & . & 0 \\ 0 & K^1 & 0 & & . \\ . & & K^2 & & \\ . & & & & \\ . & & & & K^d \end{pmatrix}, \tag{4.19}$$

where

$$P^d = \begin{pmatrix} \alpha_1^d & -\gamma_1^d & & & \\ -\beta_2^d & \alpha_2^d & -\gamma_2^d & & \\ 0 & -\beta_3^d & \alpha_3^d & -\gamma_3^d & \\ . & & . & . & . \\ . & & -\beta_{N-2}^d & \alpha_{N-2}^d & -\gamma_{N-2}^d \\ 0 & \dots & 0 & -\beta_{N-1}^d & \alpha_{N-1}^d \end{pmatrix}. \tag{4.20}$$

Futhermore the coefficients α_i^d, β_i^d, γ_i^d satisfy the following recursive relations:

$$\begin{cases} \alpha_i^d = \alpha_{2i}^{d-1} - \dfrac{\beta_{2i}^{d-1}\gamma_{2i-1}^{d-1}}{\alpha_{2i-1}^{d-1}} - \dfrac{\beta_{2i+1}^{d-1}\gamma_{2i}^{d-1}}{\alpha_{2i+1}^{d-1}} & ; \quad i = 1, N-1\,, \\ \beta_i^d = \dfrac{\beta_{2i-1}^{d-1}\beta_{2i}^{d-1}}{\alpha_{2i-1}^{d-1}} & ; \quad i = 1, N-1\,, \\ \gamma_i^d = \dfrac{\gamma_{2i}^{d-1}\gamma_{2i+1}^{d-1}}{\alpha_{2i+1}^{d-1}} & ; \quad i = 1, N-1\,, \\ K_i^l = \alpha_i^{d-l} & ; \quad i = 1, N,\ l = 1, \dots, d-1\,. \end{cases} \tag{4.21}$$

Remark 3: Unlike in the uniform mesh case, the local change of variable depends here on the grid level via the coefficients of the under-matrix K^l.

Remark 4: The procedure described above is still valid when the discretization matrix A is not symmetric and it generalizes the hierarchical reduction for tridiagonal matrices.

5. NUMERICAL RESULTS

We introduce the following notation. We shall say that a grid has a $C_{k,l}$ configuration if it is obtained with l dyadic refinements of a grid composed of k points in each direction of the domain. The fine grid is thus composed of $2^l(k+1)-1$ points in each direction.

5.1. Hierarchical preconditioning: the one dimensional case

5.1.1. *Condition number of the matrix for some specific meshes*

a) *The Chebyshev mesh*

We consider here the hierarchical preconditioning of the discretization matrix of the Laplacian on a Chebyshev-like mesh. The corresponding sequence x_i is defined by

$$x_i = \frac{1}{2}(\sin(\pi(i\,.\,h - 0.5)) + 1), \quad i = 1, \dots, N-1\,,$$

where $h = \frac{1}{N}$.

The discretization matrix of the Laplace operator is constructed using the formulae given in Section 3. We compare in the tabular hereafter $Cond(A)$ and $Cond({}^t T.A.S)$ for several grid levels. The grids are all of type $C_{1,l}$.

Hierarchical preconditioning without diagonal preconditioning

Table 1. — Condition number of the matrix A.

l	Nodal basis	Second order I.U.
$l=1$ (N = 7)	51.45	23.19
$l=2$ (N = 15)	770.79	178.42
$l=3$ (N = 31)	12 135.05	1 413.62
$l=4$ (N = 63)	193 381.54	11 281.78
$l=5$ (N = 127)	3 090 998.67	90 199.93

Table 2. — Asymptotic behaviour of $C(A)$.

l	Nodal basis: $C(A)/N^4$	Second order IU $C({}^tT.A.S)/N^3$
$l=1$ (N = 7)	0.01256	0.0529
$l=2$ (N = 15)	0.0176	0.04359
$l=3$ (N = 31)	0.01572	0.04314
$l=4$ (N = 63)	0.01526	0.043036
$l=5$ (N = 127)	0.01514	0.043010

Hierarchical preconditioning with a diagonal preconditioning

As we can see $C({}^tT.A.S)$ is asymptitocally in N^3 whereas $C(A)$ is in N^4 (like the discretization matrix of the Laplacian associated to the Chebyshev polynomials). The improvement of that hierarchical preconditioner is not totally convincing. For this reason, we propose to introduce a diagonal inner preconditioning, that is to say we consider

$$ {}^tT.D.A.S \text{ rather } {}^tT.A.S\,, $$

where $D^{-1} = DIAG(A)$. We obtain the following results:

Table 3. — Condition number of the matrix D.A.

l	Nodal basis	Second order I.U.
$l=2$ (N = 7)	12.95	2.6131
$l=3$ (N = 15)	51.81	5.1258
$l=4$ (N = 31)	207.45	10.2022
$l=5$ (N = 63)	827.62	20.3800
$l=6$ (N = 127)	3 308.09	40.7477

Table 4. — Asymptotic behaviour of $C(D\,.\,A)$.

l	Nodal basis: $C(D\,.\,A)/N^2$	Second order IU $C({}^t\,T.D.A.S)/N$
$l = 2$ (N = 7)	0.2024	0.3266
$l = 3$ (N = 15)	0.2024	0.3203
$l = 4$ (N = 31)	0.2025	0.3188
$l = 5$ (N = 63)	0.2020	0.3184
$l = 6$ (N = 127)	0.2019	0.3183

We observe that the asymptotic behaviour of $C({}^t\,T.D.A.S)/N$ given in Table 4 is comparable to the uniform grid case.

b) *The A tanh-type mesh*

Hierarchical preconditioning without diagonal preconditioning

We consider an a tanh-like mesh. The sequence x_i is defined by

$$x_i = 0.5\left(1 + \frac{1}{2}\tanh\left(a\tanh\left(a\right)\xi_i\right)\right),$$

where $a\tanh(a) = 2.3934$, $\xi_i = -1 + 2\,.\,i\,.\,h$ and $h = \frac{1}{N}$.

This type of mesh is used, for example, for the numerical solution of the channel flow problem as in [15]. In the 2-D case, these authors study a flow with periodic boundary conditions in the x-direction and dirichlet boundary conditions in the y-direction. The mesh associated to this problem is then regular in the x-direction (periodicity) and refined around the boundary layer (an a tanh-type grid) in the y-direction. For further applications, it is then important to develop appropriate preconditioners for elliptic problems on the a tanh mesh. Indeed, elliptic operators (for instance the Laplacian) that appear in the formulation of the channel flow ([14]), are ill-conditioned on refined meshes of a tanh or Chebyshev type.

The discretization matrix of the Laplacian is constructed using the formulae given in Section 3. We compare in the tabular hereafter $Cond(A)$ and $Cond({}^t\,T.A.S)$ for several grid levels. The grids are all of type $C_{1,l}$.

Table 5. — Condition number of the matrix.

l	Nodal basis	Second order I.U.
$l = 1$ (N = 7)	145.18	8.94
$l = 2$ (N = 15)	5 432.87	139.32
$l = 3$ (N = 31)	88 314.07	2 136.43
$l = 4$ (N = 63)	879 489.14	10 281.72
$l = 5$ (N = 127)		31 959.34

Table 6. — Asymptotic behaviour of $C(A)$.

l		Nodal basis: $C(A)/N^3$	Second order IU $C({}^t TAS)/N^2$
$l=1$	(N = 7)	2.3938	1.0539
$l=2$	(N = 15)	3.1018	1.8987
$l=3$	(N = 31)	3.2860	2.2121
$l=4$	(N = 63)	3.2910	2.2212
$l=5$	(N = 127)		2.1377

Hierarchical preconditioning with a diagonal preconditioning

As we can see $C({}^t TAS)$ is asymptitocally in N^2 whereas $C(A)$ is in N^3 (like the discretization matrix of the Laplacian associated to the Chebyshev polynomials). The improvement of that hierarchical preconditioner is not totally convincing. For this reason we again introduce a diagonal inner preconditioning, that is to say we consider

$${}^t T.D.A.S \text{ rather than } {}^t T.A.S\,,$$

where $D^{-1} = DIAG(A)$. We obtain the following results:

Table 7. — Condition number of the matrix $C(D\,.\,A)$.

l		Nodal basis	Second order I.U.
$l=2$	(N = 7)	70.72	5.9140
$l=3$	(N = 15)	546.72	10.972
$l=4$	(N = 31)	3114.63	21.3438
$l=5$	(N = 63)	14 901.41	42.4511
$l=6$	(N = 127)	65 203.04	84.7839

Table 8. — Asymptotic behaviour of $C(D\,.\,A)$.

l		Nodal basis: $C(D\,.\,A)/N^2$	Second order I.U. $C({}^t T.D.AS)/N$
$l=2$	(N = 7)	1.10511	0.739253
$l=3$	(N = 15)	2.1355	0.681701
$l=4$	(N = 31)	3.0416	0.666996
$l=5$	(N = 63)	3.6380	0.663299
$l=6$	(N = 127)	3.9796	0.662374

As for the Chebyshev-like meshes, we observe that the asymptotic behaviour of $C({}^t T.D.A.S)/N$ given in Table 8 is comparable to the uniform grid case, that is to say $C({}^t T.D.A.S) \simeq C.N.$

5.2. Hierarchical preconditioning: the two dimensional case

We extend the construction process of the hierarchical preconditioner considered in the one dimensional case to the two dimensional case. We consider here a bidimensional Chebyshev-like grid. This type of mesh was used for the solution of the Navier-Stokes equations in a driven cavity (see [2, 19]) because of its refinement near the corners and the boundaries where several vertex and details appear as the Reynolds number increases. As in the one dimensional case, the grid considered are all of type $C_{1,l}$. We study the preconditioning of the matrix $D.A$. Here $h = 1/(N+1)$.

Table 9. — Condition number of the matrix.

l	Nodal basis (DA)	Second order I.U. $({}^tT.D.A.S)$
$l=2$ (N = 7)	13.26354	5.85544
$l=3$ (N = 15)	52.8609	10.03395
$l=4$ (N = 31)	211.321659	17.49790
$l=5$ (N = 63)	845.21288	33.42486
$l=6$ (N = 127)	3 380.53602	68.053628

Table 10. — Asymptotic behaviour of $C(D.A)$.

l	Nodal basis $C(DA).h^2$	Second order I.U. $C({}^tT.D.A.S).h$
$l=2$ (N = 7)	0.20724	0.731930
$l=3$ (N = 15)	0.20648	0.62712
$l=4$ (N = 31)	0.20636	0.546809
$l=5$ (N = 63)	0.20635	0.52226
$l=6$ (N = 127)	0.206331	0.5316689

Unlike in the one dimensional case, we do not recover the same asymptotic behaviour of the condition number of the preconditioned matrix as the one obtained for uniform grids. However, we observe in Table 10 that the condition number asymptotically behaves like $\frac{N}{2}$ while that of the usual matrix written in the nodal basis is asymptotically in N^2 (using a diagonal preconditioner) and in N^4 (without using the diagonal preconditioner).

5.3. Numerical solution of the 2D-Poisson problem

In this section, we present some numerical results concerning the solution of the two-dimensional Dirichlet problem on the unit square. The grid considered here is of Chebyshev type. The discrete system to be solved is written in the form

$$A.U = F. \tag{5.22}$$

We introduce the nonsymmetric hierarchical preconditioner presented above and we obtain the following equivalent linear system:

$${}^tT.D.A.S.\hat{U} = {}^tT.D.F, \tag{5.23}$$

where $U = S \, . \, \hat{U}$. Since the matrix $\hat{A} = {}^t T.A.S$ is nonsymmetric, we use the Bi-Cgstab method [22] in order to solve (5.23).

For our experiments we have taken $F = 0$; there is indeed no loss of generality in taking a null source term. The initial guess is $U^0_{i,j} = \sin(16(1 - x_i)(1 - y_j) x_i y_j) e^{x_i + y_j}$, $i, j = 1, ..., 2N - 1$. The computations were realized on the CRAY YMP of the Université de Paris XI, Orsay.

Figure 5 shows the evolution of the euclidian norm of the residual *versus* iterations (a) and *versus* the CPU time (b) for a 255×255 fine grid. We note that the efficiency of the hierarchical preconditioner increases with the number of grids (this phenomenon was already observed in the case of a uniform grid (see [10])).

Figure 5 shows the evolution of the residual *versus* iterations and the CPU time but here the finer grid is composed 511×511 points of discretization. We observe that the efficiency of the nonsymmetric IU hierarchical preconditioner is accentued when compared with the previous case.

5.4. Compact schemes on nonuniform meshes

In this section we present briefly the construction of compact schemes (CS) in the nonuniform case. This construction is realized using compact schemes which were introduced for the simulation of turbulence, the solution of hyperbolic systems (see e.g. [3]) and the calculation of shocks ([4]) but mainly in order to obtain a high level accuracy in finite differences. This accuracy is close to the spectral accuracy and one of the advantages of the Compact Schemes is that they can be used for non periodic boundary conditions (see [16] and the references therein). The link between IUs and compact schemes established in [6] is aimed at generating high order IUs (their order is $O(h^p)$, $p = 2, ..., 10$ instead of $(O(h^2))$ but also at preconditioning high order accurate discretization matrices of elliptic selfadjoint operators. We describe here the construction of interpolation compact schemes and of compact schemes that are associated to the discretization of PDE's on a nonuniform meshe. For that purpose, we first propose a general formulation of the compact scheme with which we recover the CS presented in [16] in the uniform case. Then, we extend this formulation for general meshes and we define fourth order IUs. Numerical results on compression of the data are also presented at the end of this section.

5.4.1. *The uniform mesh case*

For the sake of simplicity, we consider the one dimensional case and we restrict ourselves to compact schemes that are at most of tenth order of accuracy.

Let $\mathscr{D}$ be a linear operator; $\mathscr{D}$ can be a partial differential operator as well as an interpolation operator.

Let f be a regular function. We introduce the following discrete operators:

- $D_{kh}(f_i)$ is a second order accurate centered difference approximation of $\mathscr{D}f(x)|_{x = i.h}$ associated to the step size kh.
- $I_{kh}(f(i \, . \, h))$ is a second order accurate difference interpolation of U_i by $f((i \pm k) \, . \, h)$.

Here h is the spatial mesh size. One can express the compact scheme associated to the discretization of $\mathscr{D}$ at f_i as

$$\mathscr{D}(f(ih)) + \alpha \mathscr{D}(I_h f(ih)) + \beta \mathscr{D}(I_{2h} f(ih)) = a \, . \, D_h(f_i) + b \, . \, D_{2h}(f_i) + c \, . \, D_{3h}(f_i) \, , \tag{5.24}$$

this formula being valid for internal points only.

We recover with (5.24) the compact schemes proposed by Lele in [16] as shown in the following examples.

a) *Approximation of* $\frac{\partial}{\partial x}$

We have

$$I_{kh}(f(i \, . \, h)) = \frac{f((i+k) \, . \, h) + f((i-k) \, . \, h)}{2}$$

$$D_{kh}(f(i \, . \, h)) = \frac{f((i+k) \, . \, h) - f((i-k) \, . \, h)}{2 \, . \, k \, . \, h}$$

$$\mathscr{D}f(ih) = \frac{\partial f}{\partial x} \Big|_{x = i.h} \, .$$

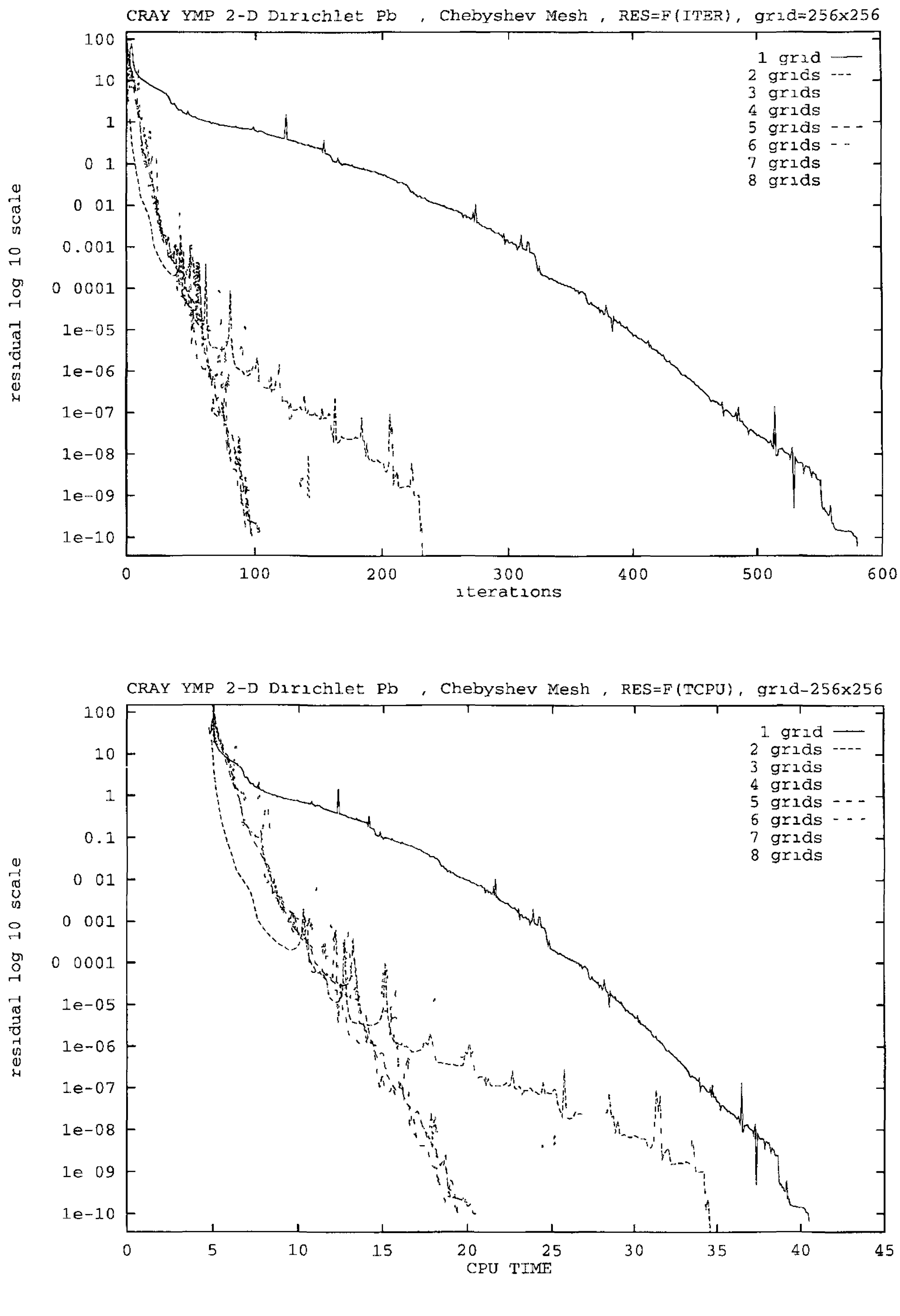
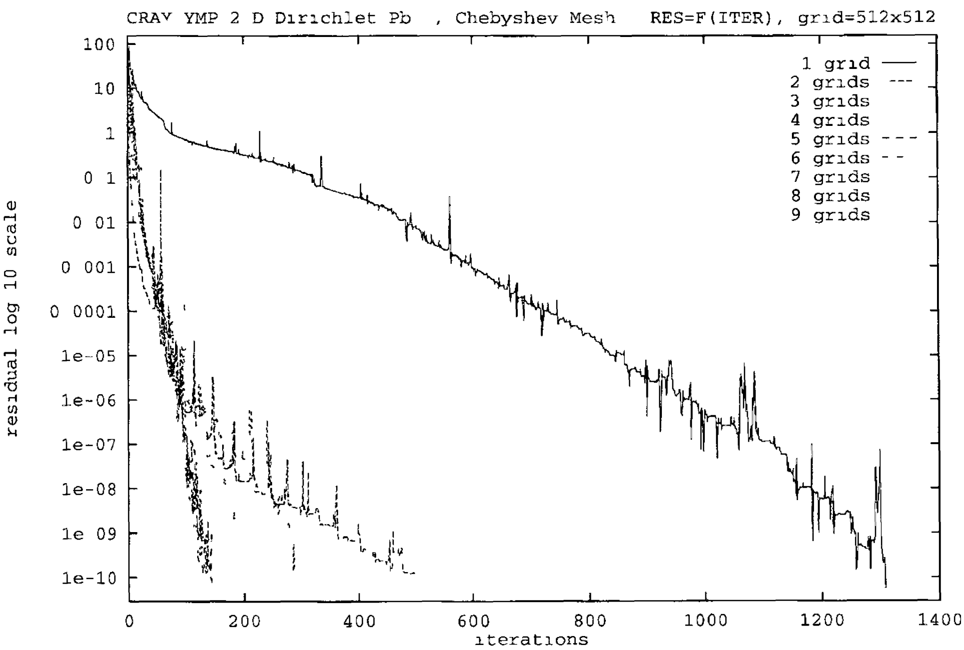


Figure 5. — Bi-Cgstab method with nonsymmetric hierarchical preconditioner. The fine grid is $C_{1,7}$ (i.e.) composed of 255×255 points.



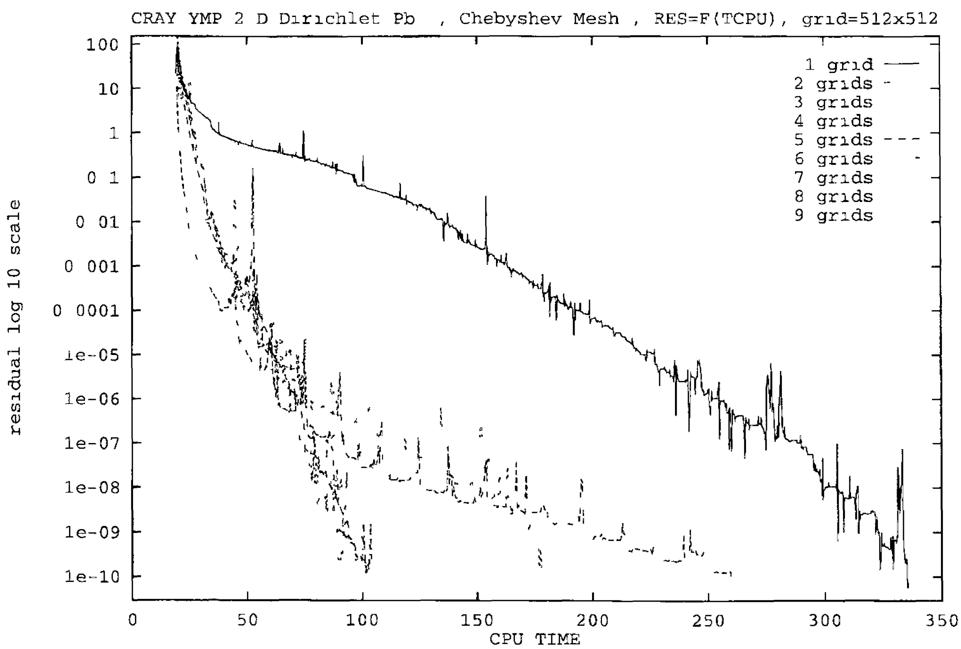


Figure 6. — Bi-Cgstab method with nonsymmetric hierarchical preconditioner. The fine grid is $C_{1\ 8}$ (i.e.) composed of 511×511 points.

The general form of the associated compact scheme is then

$$f'_i + \alpha \frac{f'_{i+1} + f'_{i-1}}{2} + \beta \frac{f'_{i+2} + f'_{i-2}}{2} = a \frac{f_{i+1} - f_{i-1}}{2h} + b \frac{f_{i+2} - f_{i-2}}{4h} + c \frac{f_{i+3} - f_{i-3}}{6h} \tag{5.25}$$

where $f_i \simeq f(i.h)$ and $f'_i \simeq \frac{\partial f}{\partial x}\big|_{x=i.h}$.

b) *Approximation of* $\frac{\partial^2}{\partial x^2}$

We have here

$$I_{kh}(f(i.h)) = \frac{f((i+k).h) + f((i-k).h)}{2},$$

$$D_{kh}(f(i.h)) = \frac{f((i+k).h) + f((i-k).h) - 2.f(i.h)}{(k.h)^2}$$

$$\mathscr{D}f(ih) = \frac{\partial^2 f}{\partial x^2}\big|_{x=i.h}.$$

The general form of the associated compact scheme is then

$$f''_i + \alpha \frac{f''_{i+1} + f''_{i-1}}{2} + \beta \frac{f''_{i+2} + f''_{i-2}}{2} = a \frac{f_{i+1} + f_{i-1} - 2.f_i}{h^2} + b \frac{f_{i+2} + f_{i-2} - 2.f_i}{4h^2} + c \frac{f_{i+3} + f_{i-3} - 2.f_i}{9h^2}, \tag{5.26}$$

with obvious notations.

c) *High order interpolation schemes*

One of the main step of the construction of Incremental Unknowns is the use of an appropriate interpolation scheme: the complementary grid components are interpolated by the coarse grid components. If we set

$$D_{kh}(f(i.h)) = \frac{f((i+k).h) + f((i-k).h)}{2},$$

$$I_{kh}(f(i.h)) = \frac{f((i+2k).h) + f((i-2k).h)}{2},$$

$$\mathscr{D}f(ih) = f(ih),$$

we obtain the following scheme:

$$f_{2i+1} + \alpha \frac{f_{2i+3} + f_{2i-1}}{2} + \beta \frac{f_{2i+5} + f_{2i-3}}{2} = a \frac{f_{2i} + f_{2i+2}}{2} + b \frac{f_{2i-2} + f_{2i+4}}{2} + c \frac{f_{2i-4} + f_{2i+6}}{2}. \tag{5.27}$$

Hence, we recover in cases *a)* and *b)* the stencil of the compact scheme proposed by Lele [16] and in case *c)* that of the interpolation scheme used in [6] for defining high order IUs.

d) *Boundary schemes*

Of course all these formulae must be completed with boundary schemes (or closure formulae) for the points that are near the boundary, when the boundary conditions are not periodic.

Following the same approach, we can recover the boundary schemes given by Lele in [16]. We define the following operators:

- I^b_{kh} is a second order interpolation scheme which interpolates f_1 with f_0 and f_{k+1}.
- $D^b_{kh}(f_1)$ corresponds to a first or second order accurate discretization of $\mathscr{D}f(h)$ in which the boundary term f_0 appears; the first order is used for the approximation of a PDE's operator and the second order for an interpolation scheme.

We can now write the boundary scheme (or the closure formula) in the general following form:

$$\mathscr{D}(f(h)) + \alpha \mathscr{D}(I^b_h f(h)) + \beta \mathscr{D}(I^b_{2h} f(h)) = \sum_{k=1}^{p} a_k D^b_{kh}(f_1) . \tag{5.28}$$

The real numbers α, β and a_k, $k = 1, \dots, p$ are computed in order to obtain the same accuracy as in (5.24).

Let us now apply this formula to the construction of compact schemes associated to the discretization of the operators that we have considered in *a)*, *b)* and *c)*, where homogeneous Dirichlet boundary conditions are considered, (i.e.) $f_0 = 0$. For the sake of simplicity, we restric ourselves to the fourth accuracy case.

1. Closure formula for $\frac{\partial}{\partial x}$.

We have

$$f'_1 + \alpha \frac{f'_2 + 0}{2} = a_1 \frac{f_1 - f_0}{h} + a_2 \frac{f_2 - f_0}{2\,h} + a_3 \frac{f_3 - f_0}{3\,h} + a_4 \frac{f_4 - f_0}{4\,h} .$$

We recover the boundary scheme given by Lele by taking $\frac{\alpha}{2} = \frac{1}{4}$.

2. Closure formula for $\frac{\partial^2}{\partial x^2}$

We have

$$f''_1 + \alpha \frac{f''_2 + 0}{2} = a_1 \frac{f_2 + f_0 - 2 \,.\, f_1}{h^2} + a_2 \frac{f_3 + 2 \,.\, f_0 - 3 \,.\, f_1}{3\,h^2}$$
$$+ a_3 \frac{f_4 + 3 \,.\, f_0 - 4 \,.\, f_1}{6\,h^2} + a_4 \frac{f_5 + 4 \,.\, f_0 - 5 \,.\, f_1}{10\,h^2} .$$

We recover the boundary scheme given by Lele by taking $\frac{\alpha}{2} = \frac{1}{10}$.

3. High Order Interpolation Boundary scheme

We have

$$f_1 + \alpha \frac{f_3 + 2 \,.\, f_0}{3} = a_1 \frac{f_2 + f_0}{2} + a_2 \frac{f_4 + 3 \,.\, f_0}{4} + a_3 \frac{f_6 + 5 \,.\, f_0}{6} .$$

Taking $\frac{\alpha}{3} = \frac{1}{6}$, we recover exactly the coefficients of the closure formula given in [6].

There are many advantages of such a formulation: firstly, it can be extended to nonuniform meshes as we shall see hereafter and secondly, the size of the linear system to be satisfied by the boundary coefficients is generally lower than that obtained when we express the right hand side as $\sum_{k=0}^{p} a_k f_k$ as, e.g., in [16] for the operators $\frac{\partial^k}{\partial x^k}$, $k = 1, 2$ and in [6] for the interpolation.

5.4.2. *The nonuniform mesh case*

We now consider a nonuniform mesh caracterized by the sequence x_i, $i = 1, \ldots, 2N - 1$. In this case, the use of centered schemes for a second accurate approximation of an operator is not appropriate. Futhermore the odd terms of the Taylor's extension of a given scheme are present because of the nonuniformity of the mesh. Consequently, we can not use directly the formalism presented above for the construction of the compact scheme and the order of accuracy is expected to be lower. However we shall see that few modifications of formula (5.24) can give compact schemes with an indeed compact stencil in the fourth accurate case to which we focus. We concentrate on the interpolation schemes; compact schemes for the approximation of PDE's operators will be studied and exploited in a forthcoming work.

a) *Definition of I_{hk} and of D_{hk}*

We set $f_i \simeq f(x_i)$.

Let $k \in \mathbb{N}^*$. The Taylor expansions give, setting for convenience $\partial^p = \frac{\partial^p u}{\partial x^p}\big|_{x = x_i}$:

$$f_{i+k} = f_i + (x_{i+k} - x_i)\,\partial + \frac{1}{2}(x_{i+k} - x_i)^2\,\partial^2 + \frac{1}{6}(x_{i+k} - x_i)^3\,\partial^3 + \frac{1}{24}(x_{i+k} - x_i)^4\,\partial^4 + O((\Delta x)^5)\,,$$

$$f_{i-k} = f_i - (x_i - x_{i-k})\,\partial + \frac{1}{2}(x_i - x_{i-k})^2\,\partial^2 + \frac{1}{6}(x_i - x_{i-k})^3\,\partial^3 + \frac{1}{24}(x_i - x_{i-k})^4\,\partial^4 + O((\Delta x)^5)\,.$$

Hence, setting $a = x_i - x_{i-k}$ and $b = x_{i+k} - x_i$, we have

$$I_{kh}^2(f_i) = \frac{a \cdot f_{i+k} + b \cdot f_{i-k}}{a + b} = f_i + \frac{1}{2}ab\partial^2 + \frac{1}{6}ab(b - a)\,\partial^3 + \frac{1}{24}ab(b^2 - ab + a^2)\,\partial^4 + O((\Delta x)^5)\,,$$

for the second accuracy case, and

$$I_{kh}^+(f_i) = f_{i+k} \quad \text{and} \quad I_{kh}^-(f_i) = f_{i-k}$$

for the first accuracy case.

For the construction of the closure formulae, we introduce the second order discrete boundary interpolation operator:

$$I_{kh}^b(f_1) = \frac{(x_1 - x_0) \cdot f_k + (x_k - x_1) \cdot f_0}{x_k - x_0}\,.$$

The operator D_h^1, which corresponds to the second order accurate discretization of $\frac{\partial}{\partial x}$ is given by

$$D_{kh}^1(f_i) = \frac{a^2 f_{i+k} - b^2 f_{i-k} + (b^2 - a^2)\,U_j}{ab(b + a)} = \partial + \frac{1}{3!}ab\partial^3 + \frac{1}{4!}ab(b - a)\,\partial^4 + O((\Delta x)^5)\,.$$

Finally, the operator D_h^2, which corresponds to the second order accurate discretization of $\frac{\partial^2}{\partial x^2}$ is given by

$$D_{kh}^2(f_i) = \frac{a \cdot f_{i+k} + b \cdot f_{i-k} - (a+b) f_i}{\frac{1}{2} ab(a+b)} = \partial^2 + \frac{1}{3}(b-a)\,\partial^3 + \frac{1}{12}(b^2 - ab + a^2)\,\partial^4 + O((\Delta x)^4)\,.$$

Remark 5: The above scheme are indeed second order schemes in pratical cases. If we assume that $x_i = \omega(i \cdot h)$, where $\omega \in \mathscr{C}^2[0,1]$, h being the spatial step size, $h = \frac{1}{N+1}$ when N discretization points are considered, then $b - a = \mathcal{O}(h^2)$. Indeed, we have

$$b - a = x_{i+k} + x_{i-k} - 2 \cdot x_i = \omega(i \cdot h + kh) + \omega(i \cdot h - kh) - 2 \cdot \omega(i \cdot h) = (kh)^2 \frac{\partial^2 \omega}{\partial x^2} + o(h^2)\,.$$

We define the compact scheme associated to $\mathscr{D} = \frac{\partial}{\partial x}$ or $\mathscr{D} = \frac{\partial^2}{\partial x^2}$ by

$$\mathscr{D}(f(ih)) + \alpha(i)\,\mathscr{D}(I_h^+ f(ih)) + \beta(i)\,\mathscr{D}(I_h^- f(ih)) = A(i) \cdot D_h(f_i) + B(i) \cdot D_{2h}(f_i)\,. \tag{5.29}$$

The scheme (5.29) is at most third order accurate if we set $B(i) = 0\ \forall i$ and at most fourth order accurate otherwise. The boundary schemes are defined as in (5.28).

Remark 6: We observe that the coefficients of the interpolation compact scheme depend on the indice *i*. This is not the case when the mesh is uniform.

b) *Fourth order IUs*

b.1 The internal points case

We can now define the fourth order compact schemes aimed at defining the IUs. We have

$$f_{2i+1} + \alpha(i)\, I_{2h}^2(f_{2i+1}) = A(i)\, I_h^2(f_{2i+1})\,. \tag{5.30}$$

If we set $x = x_{2i+2} - x_{2i+1}$, $y = x_{2i+1} - x_{2i}$, $z = x_{2i+1} - x_{2i-1}$ and $t = x_{2i+3} - x_{2i+1}$ then the above scheme is fourth order accurate if $A(i)$ and $\alpha(i)$ satisfy the linear system:

$$\begin{cases} 1 + \alpha(i) = A(i)\,, \\ z \cdot t \cdot \alpha(i) = x \cdot y \cdot A(i)\,. \end{cases} \tag{5.31}$$

We find $A(i) = \frac{z \cdot t}{z \cdot t - x \cdot y}$ and $\alpha(i) = \frac{x \cdot y}{z \cdot t - x \cdot y}$. Both $A(i)$ and $\alpha(i)$ are well defined since $zt > xy$.

Remark 7: We observe that, as for the approximation of PDE's operators, the coefficients of the interpolation compact scheme depend on the indice *i*. Moreover if this process is repeated recursively, then these coefficients will also depend on the grid level.

B.2 The boundary points case

If the boundary conditions are not periodic, we must define an appropriate scheme for the interpolation near the boundaries. The general form of such a scheme is

$$f_1 + \alpha(1)\, I_{2h}^b(f_1) = A(1) \cdot I_{2h}^b(f_1) + B(1) \cdot I_{4h}^b(f_1) + C(1) \cdot I_{6h}^b(f_1)\,, \tag{5.32}$$

where $A(1)$, $B(1)$, $C(1)$ satisfy the system ($\alpha(1)$ is fixed):

$$\begin{cases} 1+\alpha(1)=A(1)+B(1)+C(1)\,, \\ a\,.\,\alpha(1)=c\,.\,A(1)+d\,.\,B(1)+e\,.\,C(1)\,, \\ a\,.\,(a-b)\,\alpha(1)=c\,.\,(c-b)\,.\,A(1)+d\,.\,(d-b)\,.\,B(1)+e\,.\,(e-b)\,.\,C(1)\,, \end{cases} \tag{5.33}$$

where we have set $a=x_3-x_1$, $b=x_1-x_0$, $c=x_2-x_1$, $d=x_4-x_1$ and $d=x_6-x_1$.

Of course an analogous system is satisfied for the points that are near the boundary $x=1$; we do not explicit this system here and in the sequel. We can now define the fourth order IUs on a nonuniform mesh by:

DEFINITION 2: *The Incremental Unknowns are the numbers Z_{2i+1} defined by*

$$Z_{2i+1}=u_{2i+1}-\bar{u}_{2i+1}\,,$$

where the numbers $\bar{u}_{2i+1}$ satisfy the linear system

$$\begin{cases} \bar{u}_1+\alpha(1)\,I^b_{2h}(\bar{u}_1)=A(1)\,.\,I^b_{2h}(u_1)+B(1)\,.\,I^b_{4h}(u_1)+C(1)\,.\,I^b_{6h}(u_1)\,, \\ \bar{u}_{2i+1}+\alpha(i)\,I^2_{2h}(\bar{u}_{2i+1})=A(i)\,I^2_h(u_{2i+1}),\quad i=2,\dots,N-2 \\ \bar{u}_{2N-1}+\alpha(N)\,I^b_{2h}(\bar{u}_{2N-1})=A(N)\,.\,I^b_{2h}(u_{2N-1})+B(N)\,.\,I^b_{4h}(u_{2N-1})+C(N)\,.\,I^b_{6h}(u_{2N-1})\,, \end{cases} \tag{5.34}$$

and where $A(i)$ and $\alpha(i)$ are defined by (5.31)-(5.33).

Remark 8: It is possible that the problems (5.31)-(5.33) are very ill-conditioned for meshes that are locally very anisotropic (this is not the case of the Chebyshev-like mesh). The preconditioning of such problems will be addressed in a future work.

5.4.3. *Numerical results: compression of the data*

We present here some numerical results that illustrate the improvement of the compression of the data when fourth order IUs are used instead of second order IUs which were defined in the previous sections. We have fixed $\alpha=\frac{1}{6}$ in the boundary scheme; this is the value of α in the uniform mesh case. The mesh that we consider in the following examples is of Chebyshev type. We consider the functions:

- example 1: $f(x)=\sin(27\,\pi x)$,
- example 2: $f(x)=\sin(3\,.\,\pi x)\,e^{6\sin(6\pi x)}$,
- example 3: $f(x)=\sin(144\,.\,x\,.\,(1-x))$.

In examples 2 and 3 the finer grid is composed of 4095 discretization points and of 511 points in example 1.

In the numerical results we give hereafter, we focus on the one hand, on the decay of the euclidian norm of the IUs according to the associated grid level, and on the other hand to the asymptotic behaviour of the ratio

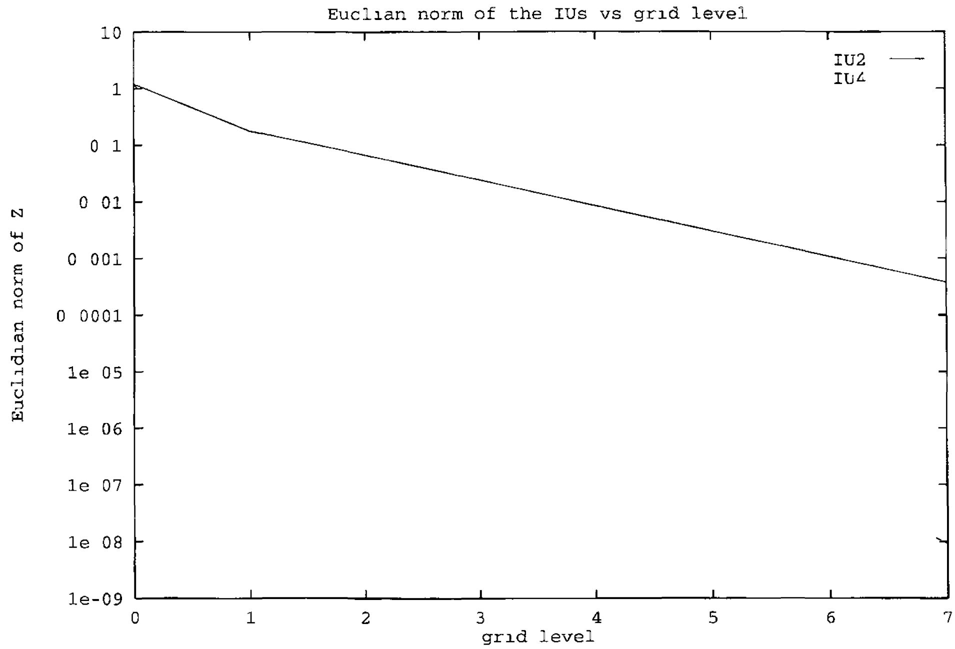
$$r=\frac{\|Z\|}{\|Y\|\,h^k}=\frac{\sqrt{\sum_{p=0}^{N-1}Z^2_{2i+1}}}{h^k\sqrt{\sum_{p=1}^{N-1}Y^2_{2i}}}\,,$$

where $Y_{2i}=u_{2i}$, and Z_{2i+1} is the second order IU associated to u_{2i+1} if $k=2$ and the fourth order IU if $k=4$.

When $\lim_{h\to 0} r=C$, where C is a constant which depends only on the function u, we have

$$\|Z\|=\mathcal{O}(h^k\|Y\|)\,,$$

for h small enough. This means that the numbers Z are indeed k-th order IUs.



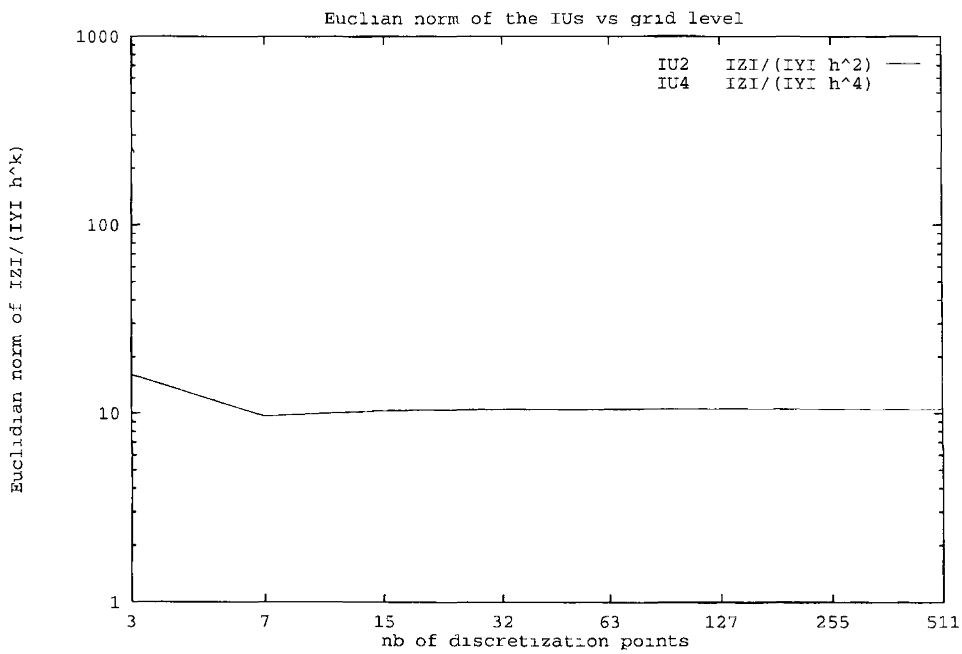


Figure 7 — **Data compression in space dimension one (a) Decay of the structures according to the grid level to which they belong Comparison between the second and the fourth order IUs (b) Asymptotic behaviour of $\frac{\|Z\|}{\|Y\| h^k}$ vs the number of discretization points, $k = 2$ for the second order IUs and $k = 4$ for the fourth order ones The grid is of type $C_{3\ 7}$**

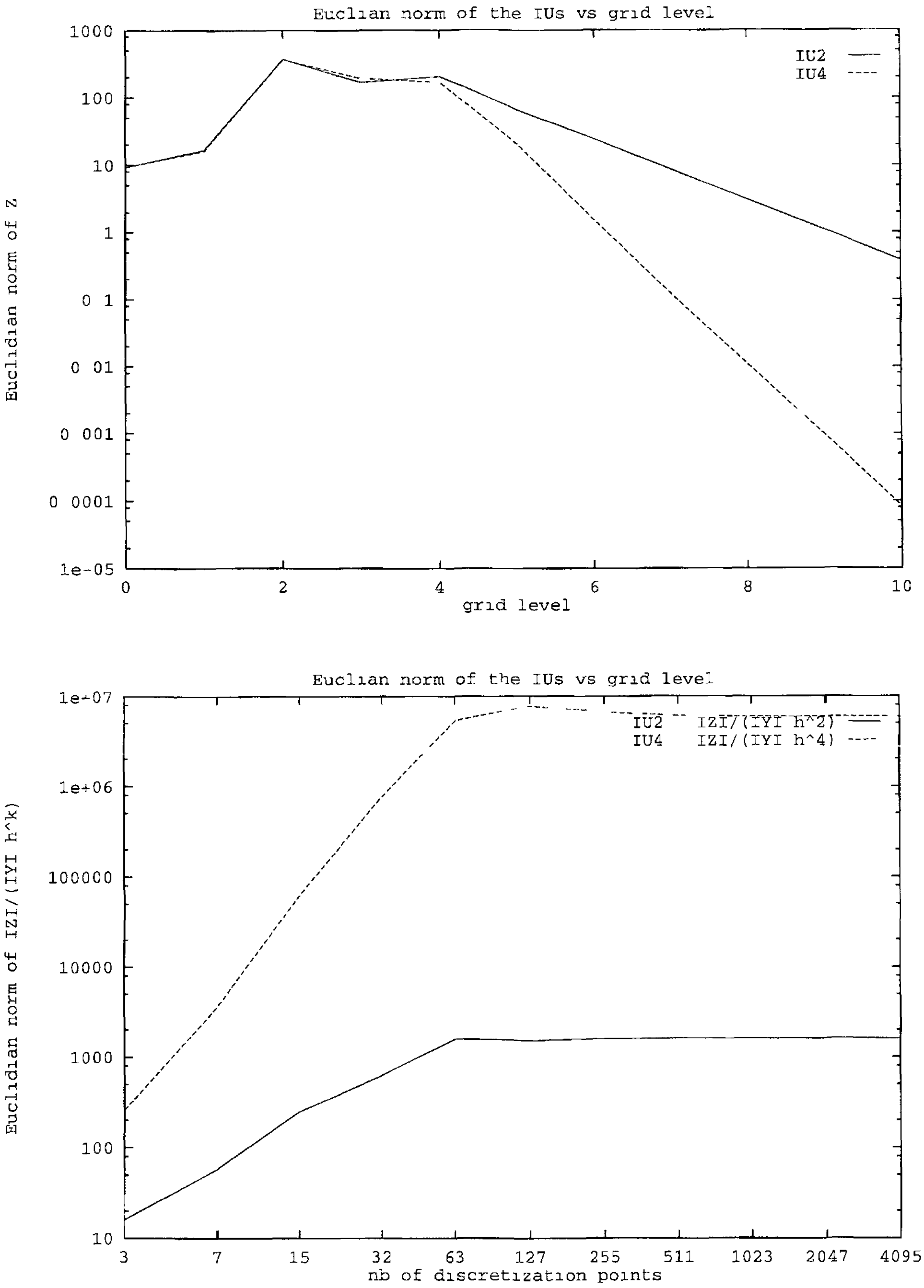


Figure 8. — Data compression in space dimension one. (a) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. (b) Asymptotic behaviour of $\frac{\|Z\|}{\|Y\|\, h^k}$ vs the number of discretization points; $k = 2$ for the second order IUs and $k = 4$ for the fourth order ones. The grid is of type $C_{3,10}$

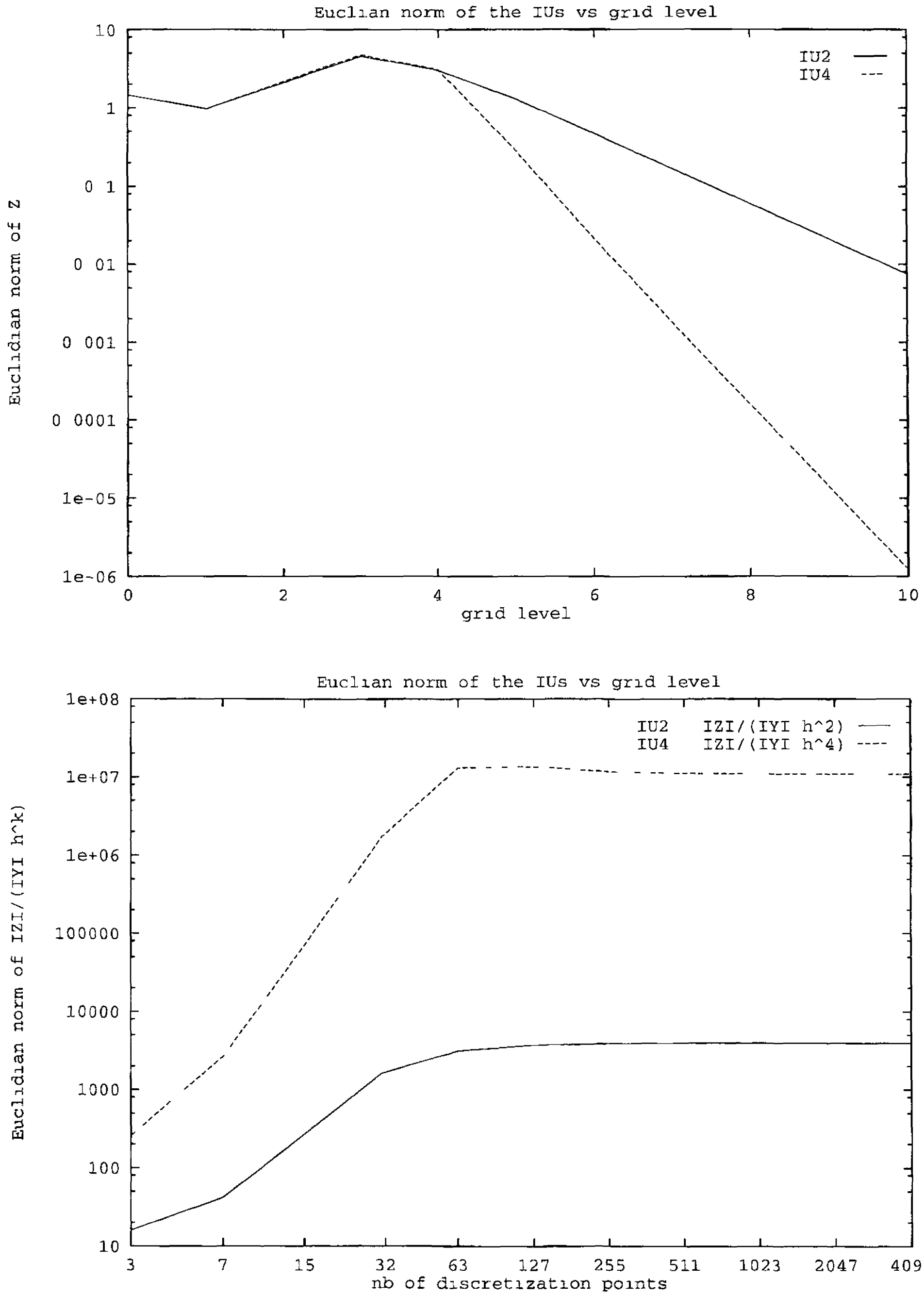


Figure 9. — Data compression. (a) Decay of the structures according to the grid level to which they belong. Comparison between the second and the fourth order IUs. (b) Asymptotic behaviour of $\frac{\|Z\|}{\|Y\|\, h^k}$ vs the number of discretization points; $k = 2$ for the second order IUs and $k = 4$ for the fourth order ones. The grid is of type $C_{3\;10}$

In *figure 7a* we consider the discretization of the oscillating function $f(x) = \sin(27\pi x)$. The associated vector is written in both the IU2 and the IU4 base. We observe that the decay of the IU4 is much more accentued than the one of the IU2. Indeed in the finest grid we can observe that there is about a factor 1000 between the magnitude of this two types of IUs. Futhermore we can conjecture that $\lim_{h \to 0} r = C$ (*fig. 7b*).

In *figure 8a* the function considered, example 2, has strong gradients near the points $x = 0.09$, 0.42, 0.76. As in the previous example, we observe that, starting from a certain grid level, the decay of the magnitude of the IU4 is greather than that of the IU2. We find about a factor 1000. Here again we conjecture that $\lim_{h \to 0} r = C$ (*fig. 8b*).

A comparable phenomenon is observed in *figure 9a* for the function given in example 3; this function, which has $x = 1/2$ as symmetric axe, has strong gradients near the points $x = 0.01$, 0.034, 0.057, 0.082, 0.11, 0.14, 0.17, 0.20, 0.24, 0.29, 0.354. There is about a factor 10 000 between the euclidian norm of the IU2 and that of the IU4 on the finer grid. As in the previous cases we can expect that $\lim_{h \to 0} r = C$ (*fig. 9b*).

In all these 3 cases, we observe that, numerically, and starting from a certain grid level j we have $\| Z^4 \|_l = \mathcal{O}(\| Z^2 \|_l^2)$, where $\| Z^k \|_l$ is the euclidian norm of the IUs of order k on the l-th grid level.

6. CONCLUSION

In this article we have extended the IU method to the nonuniform case with an amphasis on the compression of data and on the preconditioning (for particular meshes, but we hope in the near future to generalize these hierarchical preconditioners).

We think that the techniques introduced here are new tools for the development and the implementation of Multiresolution methods for solving boundary layer problems in Computational Fluid Dynamics. Moreover, with the extension of the compact schemes, it is thinkable to solve these problems with an accuracy comparable to the spectral one but on more general meshes for which no spectral techniques can be used (typically on an a tanh-type grid). This is also a new illustration of the versatility of the IU method.

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