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Regularity of solutions to a one dimensional plasticity model

REGULARITY OF SOLUTIONS TO A ONE DIMENSIONAL PLASTICITY MODEL (*)

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Abstract — *A quasi-static one dimensional plasticity model subject to multi-linear kinematic law is formulated as a system of variational inequalities H^2 regularity in the space variable is proved for the displacement and H^1 regularity is proved for the stress*

Résumé — *On étudie un modèle uni-dimensionnel quasi-statique de plasticité basé sur une loi cinématique multi-linéaire, et formulé comme un système d'inégalités variationnelles On montre la régularité H^2 dans la variable d'espace pour la déformation, et la régularité H^1 pour la contrainte*

1. INTRODUCTION

In this paper, we consider a cyclic plasticity model that is subject to a multi-linear kinematic hardening law. Our model is more general than the classical bilinear kinematic law but it does not include isotropic hardening. Besides certain new features in the modeling aspect, the main purpose is to prove higher regularity of the solution than those that appear naturally in the weak formulation. We also obtain higher order norm estimates of the solution in terms of the prescribed load. Our result is the first of this kind in the mathematical literature of cyclic plasticity. In the present paper, we restrict our attention to the quasi-static one dimensional case only in order not be obscured by the technicalities. We plan to address the same topic for the higher dimensional case in a separate article.

It is known that solutions to cyclic plasticity models suffer threshold of regularity due to the interface between the elastic and plastic regions. In particular, second order time derivatives of the displacement and the stress do not exist in general even for analytic load. This has incurred significant difficulties in establishing convergence rate for numerical solutions of the continuous problem. Previous work have either assumed higher regularity of the solution or certain variants of the same assumption in order to achieve a convergence rate. In a recent work by Li an Babuška [12], an h-version high order finite element method is considered for a large class of two dimensional models, but only weak convergence of the approximate solutions is obtained, the reason of which is partially due to the lack of certain regularity estimates. Our regularity result so obtained is sufficient to give rise to a first order convergence rate for a continuous Galerkin method. For detail, we refer the reader to the authors forthcoming paper [2].

There is a sizable literature devoted to the study of cyclic plasticity. For the constitutive theory, we reader to [13], [15], [14] and [17] for an overview. We refer to Babuška *et al.* [1] for a discussion on the reliability questions of various models.

On the mathematical side, the first existence result is given by Duvalt and Lions [4] for the dynamic problem of elasto-perfect plasticity. Johnson [8, 9] extend the analysis to the quasi-static case, including also the hardening effect. For related work we also refer the reader to the book by Hlavačk *et al.* [7] and the references therein. These approach carries a common feature of using the penalty method. They also resort to a special technique that enables to eliminate the velocity field from the equations. No regularity results for weak solutions have been obtained in these work.

Two monographs which symbolize recent advances of the mathematic theory of cyclic plasticity have appeared. The first is by Han and Reddy [6], by large devoted to the numerical analysis of the subject. They used the displacement and certain internal parameters as the primal variables in their formulation. The authors have developed an abstract theory of evolutional variational inequalities of the second kind which can be used to model the cyclic plasticity with a combined kinematic and isotropic hardening law. The existence and uniqueness of the solution has been proved by use of the Rothe's time discretization method. No regularity of the solution is discussed. The approach in [6] is quite different from the previous work in that a definite view has been taken in using the displacement as a primal variable, invoking in a sysmatic way the theory of evolutional variational inequalities. On the modeling aspect, the authors assumed *a priori* the existence of a so-called free energy functional and introduced a related notion of generalized stress tensor in the description of their constitutive laws. The second monograph is by Krejči [11], mainly devoted to the dynamic problems of cyclic plasticity although the technique introduced there can well be applied to the quasistatic case also. The method in [11] is based on the extensive use of convex analysis and the theory of stop-play operators introduced by Krasnosel'skii and Pokrovskii [10]. The basic idea is to represent various constitutive laws in a unified form $\sigma = F(\epsilon)$, where F is a functional characterized by certain abstract properties. The relation $\sigma = F(\epsilon)$ is in turn substituted into the equilibrium equation to obtain a system of governing equations for the displacement. In this way, the internal parameters do not appear explicitly in the formulation since they are embedded in the abstract properties of the functional F.

The rest of the paper is organized as follows. In Section 2, we discuss the constitutive model considered by the authors. We emphasis on its connection with the standard gauge function approach. In Section 3 we give the mathematical formulation of the problem and state the main result of the paper. The proof of the main result is contained Sections 4, 5.

2. THE CONSTITUTIVE MODEL

The formulation to be given in the present paper is based on an idea of representing the stress as a sum of suitable substresses $\{\sigma_j; j = 1, ..., N + 1\}$, the physical meaning of which will be made clear later. Let $\Omega = (0, \ell)$ denote the reference configuration of a one dimensional bar, and fix a time interval $(0, T)$. We denote the displacement and the stress field at the time t by $u(x, t)$ and $\sigma(x, t)$ respectively, and denote by $\epsilon(x, t) = u_x(x, t)$ the linearized strain. Our proposed model for a $N + 1$ piecewise linear kinematic hardening law is given as follows: for almost all $(x, t) \in (0, \ell) \times (0, T)$,

$$\sigma = \sum_{j=1}^{N+1} \sigma_j \tag{2.1}$$

$$\sigma_{jt} + \mathrm{sgn}^{-1}\left(\frac{\sigma_j}{\gamma_j}\right) \ni \alpha_j \epsilon_t, \quad 1 \leq j \leq N, \tag{2.2}$$

$$\sigma_{N+1,t} = \mu\epsilon_t, \tag{2.3}$$

where α_j, γ_j and μ are positive constants; $\mathrm{sgn}^{-1}(\,.\,)$ is the maximum monotone graph defined by

$$\mathrm{sgn}^{-1}(s) = \begin{cases} 0 & \text{if } |s| < 1 \\ [0, +\infty) & \text{if } s = 1 \\ (-\infty, 0] & \text{if } s = -1. \end{cases}$$

The inclusion (2.2) is understood as follows: there exist a function $\lambda(x, t)$ such that

$$\lambda(x, t) \in \operatorname{sgn}^{-1}\left(\frac{\sigma_j(x, t)}{\gamma_j}\right), \quad \sigma_{jt} + \lambda = \alpha_j \, \epsilon_t \,. \tag{2.4}$$

The constitutive relation in the form (2.1)-(2.3) has the origin from the general theological models (see [17]). It has also been used in Visintin [16] in a slightly different version. Since direct use of monotone graphs in the construction of multi-linear constitutive models in plasticity is not widely understood, it is desirable to make the connection between our proposed model with the the more popular gauge function approach before we proceed further.

Kinematic hardening is known as the phenomenological behavior of a material for which the center of the yield surface translates with the strain history while the size of the yield surface remains fixed. The yield surface is most commonly described in terms of a convex function $g(\sigma, \alpha)$, called the gauge function, where σ is the stress tensor and α is usually a function taking values in R^m, $m \geq 1$, referred as the internal parameters. An admissible state is characterized by the the pair (σ, α) for which

$$g(\sigma, \alpha) \leq 0 \,. \tag{2.5}$$

The set

$$\mathbb{E} = \left\{ (\sigma, \alpha) \, ; \, g(\sigma, \alpha) < 0 \right\} \tag{2.6}$$

is called the elastic region and the set

$$P = \left\{ (\sigma, \alpha) \, ; \, g(\sigma, \alpha) = 0 \right\} \tag{2.7}$$

is called the plastic region. The linearized strain tensor ϵ allows an additive decomposition

$$\epsilon = \epsilon^e + \epsilon^P, \tag{2.8}$$

where ϵ^e and ϵ^P are called the elastic and plastic parts of the strain respectively. It is further postulated that

$$\dot{\sigma}_{ij} = a_{ijkl} \dot{\epsilon}^e_{kl} \,, \tag{2.9}$$

where a_{ijkl} is a forth order, symmetric, and positive definite tensor, describing the elastic response, the dot meaning the time derivative. ϵ^P satisfies the normality principle

$$\dot{\epsilon}^P = \begin{cases} 0 & \text{in } \mathbb{E}, \\ \lambda \dfrac{\partial g}{\partial \sigma} & \text{in } \mathbb{P}, \text{ for some } \lambda \geq 0 \,. \end{cases} \tag{2.10}$$

The evolution of the yield surface is governed by an ordinary differential equation for the internal parameter, given by

$$\dot{\alpha} = \begin{cases} 0 & \text{in } \mathbb{E}, \\ -\lambda \dfrac{\partial g}{\partial \alpha} & \text{in } \mathbb{P}, \text{ for some } \lambda \geq 0 \,. \end{cases} \tag{2.11}$$

We thus refer the general methodology described above as the gauge function theory. It turns out that in the special case of bilinear kinematic hardening, the gauge function theory coincides with the constitutive approach taken by the present paper. In order to justify this statement, we reason by aid of *figure 1*. In the situation we are concerned with, the gauge function is given by

$$g(\sigma, \alpha) = |\sigma - \alpha| - \frac{\gamma}{2}$$

where α is the internal parameter denoting the center of the yield surface (yield points in the one dimensional case here), and γ denotes the diameter of the yield surface (see [13, 17]) Because of the hardening effect, the yield points depend on the current position of (ϵ, σ), which are denoted by σ^+ and σ^- In view of *figure 1*, straightforward calculation shows that

$$\gamma = \sigma^+ - \sigma^- = \frac{2\,\tau E_2}{E_2 - E_1}, \quad \alpha = \frac{\tau E_2 E_1}{E_2 - E_1}\,\epsilon^P = \frac{\tau E_2 E_1}{E_2 - E_1}\,(\epsilon - E_2^{-1}\,\sigma)\,,$$

where E_1 and E_2 denote the plastic and elastic Young's modulus respectively, which coincide with the respective slopes of the lines L_1 and L_ϵ Therefore,
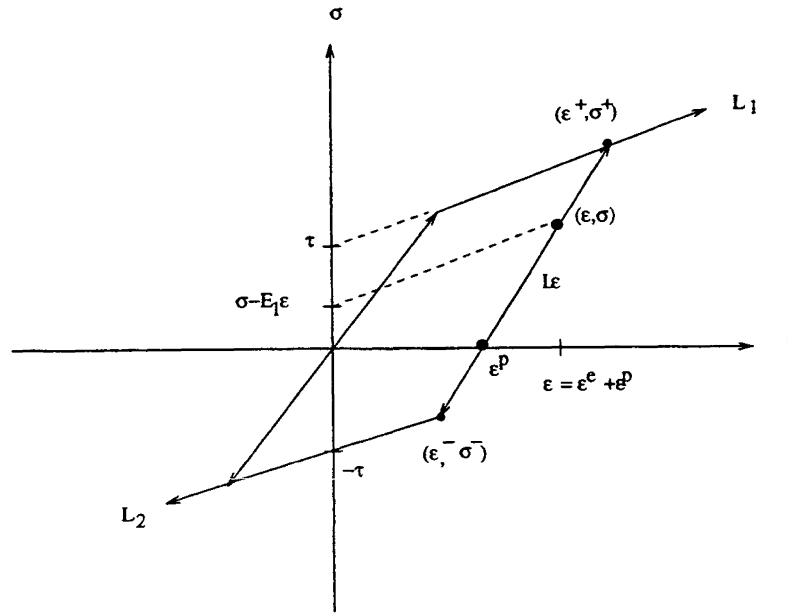
Figure 1. — The hysteresis diagram for bilinear kinematic hardening law.

$$g(\sigma, \alpha) = \frac{E_2}{E_2 - E_1}\,|\sigma - E_1\,\epsilon| - \frac{E_2\,\tau}{E_2 - E_1} \qquad (2\ 12)$$

We now let

$$\sigma_1 = \sigma - E_1\,\epsilon, \quad \sigma_2 = -E_1\,\epsilon \qquad (2\ 13)$$

Then we arrive at the identity

$$\hat{g}\,(\sigma_1) \equiv g(\sigma, \alpha) = \frac{E_2}{E_2 - E_1}\,|\sigma_1| - \frac{E_2\,\tau}{E_2 - E_1} \qquad (2\ 14)$$

Moreover, (2 13) leads to the stress decomposition

$$\sigma = \sigma_1 + \sigma_2\,, \qquad (2\ 15)$$

where σ_2 is in linear relation to the strain. From *figure 1*, it is easy to see that the elastic region and plastic region can be characterized in terms of the single parameter σ_1, giving the simple form

$$\mathbb{E} = \left\{ \sigma_1; \ |\sigma_1| < \tau \right\} \quad \text{and} \quad \mathbb{P} = \left\{ \sigma_1 ; \ |\sigma_1| = \tau \right\}.$$

In the region $\mathbb{E}$, there holds an elastic relation $\sigma_{1\,t} = \epsilon_t$, and in the region $\mathbb{P}$,

$$\sigma_{1\,t} = \sigma_t - E_1 \epsilon_t = (E_2 - E_1) \epsilon_t.$$

Also observe that $\hat{g}$ defined in (2.14) has the same form of a gauge function for an elastic perfect-plastic material [15]; the normality principle (2.10) can be written equivalently as

$$\dot\epsilon^P = \begin{cases} 0 & \text{in } \mathbb{E}, \\ \lambda \dfrac{\partial \hat{g}}{\partial \sigma_1} & \text{in } \mathbb{P}, \text{ for some } \lambda \geq 0. \end{cases} \tag{2.16}$$

Writing (2.16) by use of monotone graphs, we immediately arrive at

$$\sigma_{1\,t} + \text{sgn}^{-1}\left(\frac{\sigma_1}{\tau}\right) \ni (E_2 - E_1)\,\epsilon_t,$$

which is identical to our proposed model.

The stress decomposition given in (2.1)-(2.3) is a postulate that the total stress is an additive sum of finite number of substresses, in which one of them is linear to the strain while others are subject to the elastic perfect-plastic responses. The coefficients $\{\alpha_j; j = 1, ..., N\}$ and μ control the slope of each linear segment in the multi-linear model. With $\mu = 0$ we recover a multi-linear elastic perfect-plastic material. This is the physical meaning behind our constitutive theory.

In concluding this section, we direct our attention to a concrete example. Let

$$\begin{cases} \sigma = \sigma_1 + \sigma_2 + \sigma_3, \\ \sigma_{jt} + \text{sgn}^{-1}\left(\dfrac{\sigma_j}{j}\right) \ni \epsilon_t, \quad 1 \leq j \leq 2, \\ \sigma_{3\,t} = \epsilon_t. \end{cases} \tag{2.17}$$

Suppose the input ϵ is prescribed by

$$\epsilon(x, t) = \begin{cases} t, & 0 \leq t \leq 6, \\ 12 - t, & t \geq 6. \end{cases}$$

We first establish a table that contains the information to produce the hysteresis diamgram.

t	$(0, 1)$	1	$(1, 2$	2	$(2, 6)$	6	$(6, 8)$	8	$(8, 10)$	10	> 10
$\dot\epsilon$	1	1	1	1	1	$[-1, 1]$	-1	-1	-1	-1	-1
σ_1	ϵ	1	1	1	1	1	$\epsilon - 5$	-1	-1	-1	-1
σ_2	ϵ	1	ϵ	2	2	2	$\epsilon - 4$	0	$\epsilon - 4$	-2	-2
σ_3	ϵ	1	ϵ	2	ϵ	6	ϵ	4	ϵ	2	ϵ
σ	3ϵ	3	$1 + 2\epsilon$	5	$3 + \epsilon$	9	$3\epsilon - 9$	3	$2\epsilon - 5$	-1	$\epsilon - 3$

On a small time interval begining from $t = 0$, $\sigma_1 = \sigma_2 = \epsilon = t$. This relation remains the same untill σ_1 reaches 1 to enter the plastic region. That moment is $t = 1$. As time advances, σ_1 remains as 1 while σ_2 maintains its previous relation to ϵ untill it reaches 2 when $t = 2$, also to enter the plastic region. On the time interval $(1, 2)$, $\sigma_{1,t} = 0$ while $\epsilon = t$, thus the inclusion in (2.17) forces $\mathbf{sgn}^{-1}(\sigma_1) = 1$. As the time continues to evolve, both σ_1 and σ_2 remains unchanged since they are all in the plastic region. As the time reaches $t = 6$, the unloading begins and the strain is given by $\epsilon = 12 - t$. Since $\epsilon_t = -1$ for $t > 6$ and $\sigma_{1,t} = \sigma_{2,t} = 0$ on the time interval $(2, 6)$, the relation in (2.17) forces both $\sigma_{1,t}$ and $\sigma_{2,t}$ to be negative immediately after $t = 6$, which in turn causses σ_1 and σ_2 fall into the elastic region, implying that $\mathbf{sgn}^{-1}(\sigma_1) = \mathbf{sgn}^{-1}(\sigma_2) = 0$, and the following equalities become valid: $\sigma_1 = \epsilon - 5$, $\sigma_2 = \epsilon - 4$. The constants -5 and -4 are chosen to ensure the continuity of the stresses in time. The remaining entries of the table can be obtained by similar arguments which we leave to the interested reader for details.

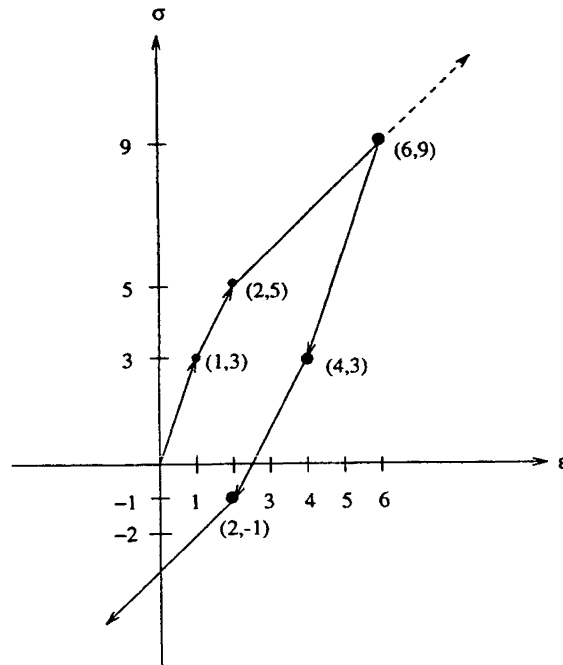Based on the table, the stress-strain relation for this example is depicted as follows.

Figure 2. — The hysteresis diagram for kinematic hardening law.

3. THE MATHEMATICAL PROBLEM

Let $Q_T = \Omega \times (0, T)$, $T > 0$. The quasi-static equilibrium equation coupled with (3.1)-(2.3) leads to the following initial boundary value problem

$$-\sum_{j=1}^{N} \sigma_{jx} - \mu u_{xx} = f \quad \text{in } Q_T, \tag{3.1}$$

$$\sigma_{jt} + \mathbf{sgn}_j^{-1}(\sigma_j) - \alpha_j u_{xt} \ni 0 \quad \text{in } Q_T, \tag{3.2}$$

$$1 \leqslant j \leqslant N,$$

$$u(0, t) = u(\ell, t) = 0 \quad t \in [0, T], \tag{3.3}$$

$$\sigma_j(x, 0) = \varphi_j(x) \quad x \in \Omega, \tag{3.4}$$

where f and φ_j are given functions,

$$\mathrm{sgn}_j^{-1}(\sigma_j) = \mathrm{sgn}^{-1}\left(\frac{\sigma_j}{\gamma_j}\right) \quad 1 \le j \le N.$$ (3.5)

We remark that no initial conditions are necessary for the displacement u. Indeed, $u(\,.\,, 0)$ is uniquely determined by the equilibrium equation (3.1) and the initial conditions for $\{\sigma_j; \ j = 1, ..., N\}$.

DEFINITION 3.1: *We say that a pair of vector-valued functions*

$$(\sigma_1, \sigma_2, ... \sigma_N, u) \quad \text{and} \quad (\lambda_1, \lambda_2, ... \lambda_N)$$

is strong solution to the problem (3.1)-(3.5) if u, u_{xt}, $u_{xx} \in L^2(Q_T)$, and for every $1 \le j \le N$,

$$\sigma_j \in H^1(Q_T),$$

$$\lambda_j(x, t) \in \mathrm{sgn}_j^{-1}(\sigma_j(x, t)) \quad \text{a.e.} \quad \text{in } Q_T,$$

the equilibrium equation (3.1) and the initial boundary conditions (3.3)-(3.4) are satisfied almost everywhere, and the inclusion (3.2) is satisfied in the sense

$$\sigma_{jt} + \lambda_j - \alpha_j u_{xt} = 0 \quad \text{a.e.} \quad \text{in } Q_T.$$

An alternative definition which is equivalent to Definition 3.1 can be given in terms of variational inequalities where $\mathrm{sgn}_j^{-1}(\sigma_j(x, t))$ and λ_j do not appear explicitly in the formulation.

DEFINITION 3.2: *We say that a vector-valued function $\vec{\sigma} = (\sigma_1, \sigma_2, ... \sigma_N, u)$ is strong solution to the problem (3.1)-(3.5) if u, u_{xt}, $u_{xx} \in L^2(Q_T)$, and $\sigma_j \in H^1(Q_T)$ for every $1 \le j \le N$, the equilibrium equation (3.1) and the initial boundary conditions (3.3) -(3.4) are satisfied almost everywhere, and the following variational inequality holds: for all $t \in (0, T)$*

$$\int_\Omega (\sigma_{jt} - \alpha u_{xt})\,(\xi_j - \sigma_j)\, dx \ge 0, \quad 1 \le j \le N.$$

The proof the equivalence between the above two definitions is a common practice in the theory of variational inequalities by using the fact that sgn_j^{-1} is the subdifferential of the indicator function of the set $[-\gamma_j, \gamma_j]$, for which we leave the detail for the interested reader (see Ekeland and Temam [5]).

The main results of the paper are the following three theorems.

THEOREM 3.3: *Assume that for each $1 \le j \le N$,*

$$\varphi_j \in H^1(\Omega), \quad f \in H^2(0, T; L^2(\Omega)).$$ (3.6)

Then the problem (3.1)-(3.5) has a unique strong solution. Moreover, the following estimate holds: with $\vec{\alpha} = \{\alpha_1, ..., \alpha_N\}$, *there exists a constant* $C = C(\ell, N, \vec{\alpha})$ *such that*

$$\left(\int_0^\ell \sum_{j=1}^N \sigma_{jt}^2(x, t)\, dx \right)^{1/2} \leq C \left\{ \int_0^t \left(\int_0^\ell |f_{tt}(x, \tau)|^2\, dx \right)^{1/2} d\tau \right\} \tag{3.7}$$

$$\left[\sum_{j=1}^N \sigma_{jx}^2(x, t) \right]^{1/2} \leq C \left\{ \left(\sum_{j=1}^N \varphi_{jx}^2(x) \right)^{1/2} + \int_0^t |f_t(x, \tau)|\, d\tau \right\} \tag{3.8}$$

$$|u_t(x, t)| \leq C \left\{ \int_0^\ell |f_t(x, t)|\, dx + \int_0^t \left(\int_0^\ell |f_{tt}(x, \tau)|^2\, dx \right)^{1/2} d\tau \right\} \tag{3.9}$$

$$|u_x(x, t)| \leq C \left(\int_0^\ell f^2(x, t)\, dx \right)^{1/2} + C \tag{3.10}$$

$$\left(\int_0^\ell u_{xt}^2(x, t)\, dx \right)^{1/2} \leq C \left\{ \left(\int_0^\ell f_t^2(x, t)\, dx \right)^{1/2} + \int_0^t \left(\int_0^\ell |f_{tt}(x, \tau)|^2\, dx \right)^{1/2} d\tau \right\} \tag{3.11}$$

$$|u_{xx}(x, t)| \leq C \left\{ \left(\sum_{j=1}^N \varphi_{jx}^2(x) \right)^{1/2} + \int_0^t |f_t(x, \tau)|\, d\tau \right\} . \tag{3.12}$$

Moreover, the stress $\sigma = \sum_{j=1}^N \sigma_j + \mu u_x$ *is estimated by*

$$|\sigma(x, t)| \leq \frac{1}{\mu} \left| \int_x^1 f(y, t)\, dy - \int_0^\ell y f(y, t)\, dy \right| + \sum_{j=1}^N \gamma_j . \tag{3.13}$$

The proof of this theorem will be given in the next three sections.

4. REDUCTION TO INTEGRO-DIFFERENTIAL EQUATIONS

We first reduce the problem (3.1)-(3.5) equivalently to a system of integro-differential inclusions. We introduce the Green's function

$$G(x, y) = \begin{cases} \frac{1}{\ell} y(\ell - x), & 0 \leq y \leq x, \\ \frac{1}{\ell} x(\ell - y), & x \leq y \leq \ell. \end{cases}$$

Assume that $(\sigma_1, \sigma_2, ... \sigma_N, u)$ is a strong solution to the problem (3.1)-(3.5). Then the equilibrium equation (3.1) is equivalent to

$$u(x, t) = \frac{1}{\mu} \int_0^1 G(x, y) \left[\sum_{j=1}^N \sigma_{jy}(y, t) + f(y, t) \right] dy .$$

An integration by parts implies that

$$u(x, t) = F(x, t) + \frac{1}{\mu} \frac{1}{\ell} \sum_{j=1}^{N} x \int_0^\ell \sigma_j(y, t) \, dy - \frac{1}{\mu} \sum_{j=1}^{N} \int_0^x \sigma_j(y, t) \, dy, \tag{4.1}$$

where

$$F(x, t) = \frac{1}{\mu} \int_0^1 G(x, y) f(y, t) \, dy. \tag{4.2}$$

Hence

$$u_x(x, t) = F_x(x, t) + \frac{1}{\mu} \frac{1}{\ell} \sum_{j=1}^{N} \int_0^\ell \sigma_j(y, t) \, dy - \frac{1}{\mu} \sum_{j=1}^{N} \sigma_j(x, t). \tag{4.3}$$

We substitute (4.3) into (3.2) to obtain

$$\sigma_{jt} + \text{sgn}_j^{-1}(\sigma_j) - \alpha_j \left\{ F_{xt} + \frac{1}{\mu} \frac{1}{\ell} \sum_{k=1}^{N} \int_0^\ell \sigma_{kt}(y, t) \, dy - \frac{1}{\mu} \sum_{k=1}^{N} \sigma_{kt} \right\} \ni 0, \tag{4.4}$$

that is,

$$\sum_{k=1}^{N} \left(\delta_{jk} \alpha_j^{-1} + \frac{1}{\mu} \right) \sigma_{kt} - \sum_{k=1}^{N} \frac{1}{\mu \ell} \int_0^\ell \sigma_{kt} \, dy + \alpha_j^{-1} \text{sgn}_j^{-1}(\sigma_j) \ni F_{xt}, \quad 1 \leq j \leq N. \tag{4.5}$$

It is convenient to write the system of equations (4.5) in a matrix form. Let Ψ denote the $N \times N$ diagonal matrix with diagonal entries $\{\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_N^{-1}\}$. Let A be the $N \times N$ matrix with all entries equal to 1. Let $\vec{\sigma} = (\sigma_1, \sigma_2, ... \sigma_N)$ and let

$$H(\vec{\sigma}) = \begin{bmatrix} \text{sgn}_1^{-1}(\sigma_1) & 0 & ... & 0 \\ 0 & \text{sgn}_2^{-1}(\sigma_2) & & \\ & & \ddots & \\ 0 & & & \text{sgn}_N^{-1}(\sigma_N) \end{bmatrix}.$$

We say that a matrix-valued function $h = (h_{ij})_{N \times N}$ is a selection of the graph $H(\vec{\sigma})$ if

$$h_{ij} = 0 \quad \text{for} \quad i \neq j,$$

$$h_{ii}(x, t) \in \text{sgn}_i^{-1}(\sigma_i(x, t)) \quad \text{a.e.,}$$

in which case we use the notation $h \in H(\vec{\sigma})$. We can now write (4.5) and the initial condition (3.4) in the form

$$\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_t - \frac{1}{\mu \ell} A \int_0^\ell \vec{\sigma}_t \, dy + \Psi H(\vec{\sigma}) \ni F_x \vec{e} \tag{4.6}$$

$$\vec{\sigma}(x, 0) = \vec{\varphi}(x) \tag{4.7}$$

where $\vec{\varphi} = (\varphi_1, \varphi_2, ... \varphi_N)$, $\vec{e} = (1, 1, ... 1)$.

We say that a vector-valued function $\vec{\sigma}$ is a strong solution to the integro-differential inclusions (4.6)-(4.7) if $\vec{\sigma} \in H^1(0, T ; [L^2(\Omega)]^N)$ such that (4.6)-(4.7) holds. As usual, the inclusion (4.6) is an abrieviation for the following statement: there exists a selection $h \in H(\vec{\sigma})$ such that

$$\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_t - \frac{1}{\mu\ell} A \int_0^\ell \vec{\sigma}_t \, dy + \Psi h = F_{xt} \vec{e} \, .$$

Careful observation reveals that the above calculations are reversible. We thus conclude the following lemma without a necessity for further justifications.

LEMMA 4.1: *Assume that the assumptions of Theorem 3.3 hold. Then the following statements hold. If $(\vec{\sigma}, u)$ is a strong solution to problem (3.1)-(3.5), then $(\vec{\sigma}, u)$ is a strong solution of the integro-differential inclusion (4.6)-(4.7). Conversely, if $\vec{\sigma}$ is a strong solution to (4.6)-(4.7) and let u be defined by (4.1), then $(\vec{\sigma}, u)$ is a strong solution to the problem (3.1)-(3.5).*

5. BASIC ESTIMATES

By virtue of Lemma 4.1, our task has thus reduced to the investigation of (4.6)-(4.7). To this end, we let $H_{j\epsilon}$, $\epsilon > 0$, be the Yosida approximation of sgn_j^{-1}, namely,

$$H_{j\epsilon}(\sigma_j) = \begin{cases} \frac{1}{\epsilon} (\sigma_j - \gamma_j) & \text{if } \sigma_j \geq \gamma_j, \\ 0 & \text{if } |\sigma_j| < \gamma_j, \\ \frac{1}{\epsilon} (\sigma_j + \gamma_j) & \text{if } \sigma_j \leq -\gamma_j \, . \end{cases}$$

Hence each $H_{j\epsilon}$ is a monotone, Lipschitz function. Let

$$H_\epsilon(\vec{\sigma}) = \begin{bmatrix} H_{1\epsilon}(\sigma_1) & & & 0 \\ & H_{2\epsilon}(\sigma_2) & & \\ & & \ddots & \\ 0 & & & H_{N\epsilon}(\sigma_N) \end{bmatrix} . \tag{5.1}$$

LEMMA 5.1: *The integral equation*

$$\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma} \, dy + \Psi \int_0^\ell \vec{\sigma} \, dy + \Psi \int_0^t H_\epsilon(\vec{\sigma}) \, d\tau = \vec{p} \tag{5.2}$$

has a unique solution $\vec{\sigma} \in H^2(0, T ; L^2(\Omega)]^N$ for each $\vec{p} \in H^2(0, T ; L^2(\Omega)]^N)$.

Proof: We first consider the integral equation

$$\left(\Psi + \frac{1}{\mu} A \right) \vec{\xi} - \frac{1}{\mu\ell} A \int_0^\ell \vec{\xi} \, dy = \vec{g} \tag{5.3}$$

on the space $[L^2(0,\ell)]^N$, where $= \vec{g} \in [L^2(0,\ell)]^N$. The matrix $\Psi + \frac{1}{\mu}A$ is nonsingular since $\Psi^{-1}\left(\Psi + \frac{1}{\mu}A\right) = I + \frac{1}{\mu}\check{A}$, where I is the identity matrix and $\check{A}$ is the matrix with all rows equal to $\{\alpha_1, \alpha_2, ..., \alpha_N\}$ whose eigenvalues are equal to 0 or $\alpha_1 + \cdots + \alpha_N$. Therefore the integral equation (5.3) is of Fredholm type for which the uniqueness implies the existence. Integrating (5.3) over the interval $(0,\ell)$ with $\vec{g} = 0$, we find that

$$\int_0^\ell \vec{\xi}\, dy = 0,$$

which in turns implies that the integral operator defined by

$$\mathcal{T}\vec{\xi} = \left(\Psi + \frac{1}{\mu}A\right)\vec{\xi} - \frac{1}{\mu\ell}A\int_0^\ell \vec{\xi}\, dy \tag{5.4}$$

is an isomorphism from $[L^2(0,\ell)]^N$ onto itself. We can now write (5.2) in an equivalent form

$$\sigma + \mathcal{T}^{-1}\Psi\int_0^t H_\epsilon(\vec{\sigma})\, d\tau = \mathcal{T}^{-1}\vec{p}. \tag{5.5}$$

Since (5.5) is a system of Voltera type integral equations over the space $L^2(0,T;[L^2(0,\ell)]^N)$, the kernel of which $H_\epsilon(\vec{\sigma})$ is Lipschitz continuous with respect to $\vec{\sigma}$, the existence of a solution $\vec{\sigma}$ to (5.5) follows from the standard theory. To obtain further differentiability of $\vec{\sigma}$, we integrate (5.2) over $(0,\ell)$. After cancellation of certain common terms, we obtain

$$\int_0^\ell \vec{\sigma}_\epsilon\, dy \in H^1(0,T)^N,$$

which in turn, together with (5.2) implies that $\vec{\sigma} \in H^1(0,T;[L^2(0,\ell)]^N)$. Thus we can differentiate (5.2) with respect to t and repeat the above argument to justify that $\vec{\sigma} \in H^2(0,T;[L^2(0,\ell)]^N)$.

A simple application of Lemma 5.1 implies that the initial boundary value problem

$$\left(\Psi + \frac{1}{\mu}A\right)\vec{\sigma}_{\epsilon t} - \frac{1}{\mu\ell}A\int_0^\ell \vec{\sigma}_{\epsilon t}\, dy + \Psi H_\epsilon(\vec{\sigma}) = F_{xt}\vec{e} \tag{5.6}$$

$$\vec{\sigma}_t(x,0) = \vec{\varphi}(x) \tag{5.7}$$

has a unique solution $\vec{\sigma}_\epsilon \in H^2(0,T;L^2(\Omega))$ for each $\epsilon > 0$. We next pass to the limit as $\epsilon \to 0$. Estimates on $\vec{\sigma}_\epsilon$ must be established to validate the limit. For this purpose, we need some additional lemmas.

LEMMA 5.2 ([3], Lemma G): *Let η, η^k, $k = 1, 2, ...$, be maximum monotone graphs in R^1 and*

$$\lim_{k \to \infty} (I + \eta^k)^{-1}(x) = (I + \eta)^{-1}(x), \quad \forall x \in R^1.$$

Let (S, μ) be a σ-finite measure space and let u_k, u, v_k and v be in $L^1(S;d\mu)$. Suppose $v_k \in \eta^k(u_k)$, $u_k \to u$ strongly in $L^1(S;d\mu)$ and $v_k \to v$ weakly in $L^1(S;d\mu)$ as $k \to \infty$. Then $v \in \eta(u)$.

The next lemma is a L^1 version of Grownall's inequality that is less used in practice, but it plays an important part in obtaining our desired estimates. The effect of this lemma will avoid the appearance of a large constant in the estimates that depends on T exponentially.

LEMMA 5.3: *Let* $y \in W^{1,1}(0, T)$ *and* $g \in L^1(0, T)$ *such that*

$$y \geq 0, \quad \frac{1}{2} y' \leq |g| \sqrt{y} \quad \text{a.e.} \quad \text{on } (0, T). \tag{5.8}$$

Then

$$\sqrt{y(t)} \leq \sqrt{y(0)} + \int_0^t |g| \, d\tau \quad \text{on } (0, T). \tag{5.9}$$

Proof: Let $E\{t ; t \in (0, T), y(t) > 0\}$. Since $y \in W^{1,1}(0, T)$, E is the union of at most countabally many disjoint open intervals, $E = \cup (a_i, b_i)$. We also have

$$\frac{1}{2} y' = \left(\sqrt{y(t)^2} \right)' = \sqrt{y(t)} \sqrt{y(t)}', \quad t \in E. \tag{5.10}$$

Fix $t \in (0, T)$. If $t \notin E$ no proof is necessary. If $t \in E$ we have $t \in (a_i, b_i)$ for some i.
Using (5.8) and (5.10) we obtain

$$\sqrt{y(\tau)}' \leq |g| \quad \forall \tau \in (a_i, b_i).$$

Integrating the inequality from a_i to t yields (5.9) since $a_i \geq 0$ and $y(0) \geq y(a_i)$.

In the rest of the estimates, we shall use the letter C to denote a constant that depends only on ℓ, N and $\{\alpha_j ; j = 1, ..., N\}$, whose value may vary from expression to expression, but whose dependence on these parameters will not change.

We now differentiate (5.6) in time and multiply the result by $\vec{\sigma}_{\epsilon t}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^\ell \left(\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t} \right) dy - \frac{1}{2} \frac{1}{\mu \ell} \frac{d}{dt} \left(A \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy, \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy \right)$$

$$+ \int_0^\ell (\Psi H'_\epsilon(\vec{\sigma}_\epsilon) \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t}) \, dy = \int_0^\ell (F_{xtt} \vec{e}, \vec{\sigma}_{\epsilon t}) \, dy. \tag{5.11}$$

Note that $(\Psi H'_\epsilon(\vec{\sigma}_\epsilon) \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t}) \geq 0$. The identity (5.11) implies

$$\frac{1}{2} \frac{d}{dt} \int_0^\ell \left(\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t} \right) dy - \frac{1}{2} \frac{1}{\mu \ell} \frac{d}{dt} \left(A \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy, \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy \right) \leq \int_0^\ell (F_{xtt} \vec{e}, \vec{\sigma}_{\epsilon t}) \, dy. \tag{5.12}$$

By the Cauchy-Schwartz inequality,

$$\left(A \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy, \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy \right) = \left| \int_0^\ell A^{1/2} \vec{\sigma}_{\epsilon t} \, dy \right|^2$$

$$\leq \ell \int_0^\ell |A^{1/2} \vec{\sigma}_{\epsilon t}|^2 \, dy$$

$$= \ell \int_0^\ell (A \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t}) \, dy.$$

Therefore, the quantity

$$D(t) \equiv \int_0^\ell \left(\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_{\epsilon t}, \vec{\sigma}_{\epsilon t} \right) dy - \frac{1}{\mu\ell} \left(A \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy, \int_0^\ell \vec{\sigma}_{\epsilon t} \, dy \right)$$

is bounded below by

$$D(t) \geq \int_0^\ell |\vec{\sigma}_{\epsilon t}|^2 \, dy \,. \tag{5.13}$$

Using (5.13), we derive from (5.12) that

$$\frac{1}{2} \frac{d}{dt} D \leq C \left(\int_0^\ell |F_{xtt}\vec{e}|^2 \, dx \right)^{1/2} \left(\int_0^\ell |\vec{\sigma}_{\epsilon t}|^2 \, dx \right)^{1/2}$$

$$\leq C \left(\int_0^\ell |F_{xtt}\vec{e}|^2 \, dx \right)^{1/2} D^{1/2} \,. \tag{5.14}$$

Using and Lemma 5.3, we obtain from (5.14) that

$$\sqrt{D(t)} \leq \sqrt{D(0)} + C \int_0^t \left(\int_0^\ell |F_{xtt}\vec{e}|^2 \, dx \right)^{1/2} d\tau \,. \tag{5.15}$$

We now set $t = 0$ in (5.6) to get

$$\left(\Psi + \frac{1}{\mu} A \right) \vec{\sigma}_{\epsilon t}(x, 0) - \frac{1}{\mu\ell} A \int_0^\ell \vec{\sigma}_{\epsilon t}(x, 0) \, dx = F_{xt}(x, 0) \,. \tag{5.16}$$

Multiplying both sides of (5.16) by $\vec{\sigma}_{\epsilon t}(x, 0)$, integrating the result over $(0, \ell)$ and using (5.13) with $t = 0$ we further get

$$\int_0^\ell |\vec{\sigma}_{\epsilon t}(x, 0)|^2 \, dx \leq D(0) \leq C \left(\int_0^\ell |F_{xt}(x, 0)|^2 \, dx \right)^{1/2} \left(\int_0^\ell |\vec{\sigma}_{\epsilon t}(x, t)|^2 \, dx \right)^{1/2},$$

$$\left(\int_0^\ell |\vec{\sigma}_{\epsilon t}(x, 0)|^2 \, dx \right)^{1/2} \leq C \left(\int_0^\ell |F_{xt}(x, 0)|^2 \, dx \right)^{1/2}$$

therefore,

$$\sqrt{D(0)} \leq C \left(\int_0^\ell |F_{xt}(x, 0)|^2 \, dx \right)^{1/2} \,. \tag{5.17}$$

We obtain from (5.15), (5.16) and (5.17) that

$$\int_0^\ell |\vec{\sigma}_{\epsilon t}|^2 \, dx \leq C \|F_x\|_{H^2(0, T; L^2(\Omega))} \,. \tag{5.18}$$

Next, we estimate $\vec{\sigma}_{ex}$. To this end, we choose $x_1 > x_2$ in $(0, \ell)$ and evaluate (5.6) at x_1 and x_2. We take the difference between these expressions and multiply it by $\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)$. By virtue of the monotonicity of H_ϵ, the above calculations lead to

$$\left(\left(\Psi + \frac{1}{\mu} A \right) (\vec{\sigma}_{\epsilon t}(x_1, t) - \vec{\sigma}_{\epsilon t}(x_2, t)), \vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t) \right)$$

$$\leq \left(F_{xt}(x_1, t) \, \vec{e} - F_{xt}(x_2, t) \, \vec{e}, \vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t) \right) .$$

Hence

$$\frac{1}{2} \frac{d}{dt} \left(\left(\Psi + \frac{1}{\mu} A \right) (\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)), \vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t) \right)$$

$$\leq C |F_{xt}(x_1, t) - F_{xt}(x_2, t)| \, |\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)| . \tag{5.19}$$

Arguing as before we introduce

$$\mathscr{D}(t) \equiv \left(\left(\Psi + \frac{1}{\mu} A \right) (\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)), \vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t) \right)$$

$$\geq C |\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)|^2 . \tag{5.20}$$

Then (5.19) and (5.20) lead to

$$\frac{d}{dt} \mathscr{D}(t) \leq C |F_{xt}(x_1, t) - F_{xt}(x_2, t)| \, \mathscr{D}^{1/2}(t) ,$$

and Lemma 5.3 implies that

$$\sqrt{\mathscr{D}(t)} \leq \sqrt{\mathscr{D}(0)} + C \int_0^t |F_{xt}(x_1, \tau) - F_{xt}(x_2, \tau)| \, d\tau . \tag{5.21}$$

Note that

$$\mathscr{D}(0) \leq C |\vec{\varphi}(x_1) - \vec{\varphi}(x_2)|^2 . \tag{5.22}$$

Hence (5.20)-(5.22) imply

$$|\vec{\sigma}_\epsilon(x_1, t) - \vec{\sigma}_\epsilon(x_2, t)| \leq C \left\{ |\vec{\varphi}(x_1) - \vec{\varphi}(x_2)| + \int_0^t |F_{xt}(x_1, t) - F_{xt}(x_2, t)| \, d\tau \right\}$$

from which we obtain

$$|\vec{\sigma}_{ex}(x, t)| \leq C \left\{ |\vec{\varphi}_x(x)| + \int_0^t |F_{xxt}(x, \tau)| \, d\tau \right\} . \tag{5.23}$$

In light of the estimates (5.18) and (5.23), we can extract a subsequence of $\{\vec{\sigma}_\epsilon\}$, which is still denoted by $\{\vec{\sigma}_\epsilon\}$, such that

$$\vec{\sigma}_\epsilon \to \vec{\sigma} \quad \text{weak}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{5.24}$$

$$\vec{\sigma}_{\epsilon t} \to \vec{\sigma}_t \quad \text{weak}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{5.25}$$

$$\vec{\sigma}_{\epsilon x} \to \vec{\sigma}_x \quad \text{weak}^* \text{ in } L^\infty(0, T; L^2(\Omega)). \tag{5.26}$$

Moreover,

$$|\vec{\sigma}_x(x, t)| \leq C\left\{ |\vec{\varphi}_x(x)| + \int_0^t |F_{xxt}(x, \tau)| \, d\tau \right\}. \tag{5.27}$$

Indeed, denoting the right hand side of (5.27) by $\tilde{f}(x, t)$, (5.23) implies that

$$\|\vec{\sigma}_{\epsilon x}\|_{L^p(E)} \leq \|\tilde{f}\|_{L^p(E)}, \quad 1 < p < +\infty \tag{5.28}$$

for all measurable subset $E \subset Q_T$. Letting $\epsilon \to 0$ in (5.28) we get, using the lower semi-continuity of the L^p norm

$$\|\vec{\sigma}_x\|_{L^p(E)} \leq \|\tilde{f}\|_{L^p(E)},$$

from which as $p \to +\infty$ we obtain

$$\|\vec{\sigma}_x\|_{L^\infty(E)} \leq \|\tilde{f}\|_{L^\infty(E)}. \tag{5.29}$$

Suppose now (5.27) does not hold. Then there exist a set $S \subset Q_T$ of positive measure such that $|\vec{\sigma}_x(x, t)| > \tilde{f}(x, t)$ on S. By Lusin's theorem, there exists a compact set $E \subset S$ of positive measure such that

$$\vec{\sigma}_x|_E \quad \text{and} \quad \tilde{f}|_E \quad \text{are continuous.}$$

Hence

$$\delta = \inf_{(x, t) \in E} (|\vec{\sigma}_x| - \tilde{f})(x, t) > 0,$$

which contradicts to (5.29). Thus, (5.27) is true.

LEMMA 5.4: *Assume that the conditions on Theorem 3.3 hold. Then the system of the integro-differential inclusions (4.6)-(4.7) has a unique strong solution* $\vec{\sigma} = (\sigma_1, \sigma_2, \dots \sigma_N)$. *In addition, the following estimates hold.*

$$\int_0^\ell |\vec{\sigma}_t(x, t)|^2 \, dx \leq C\|f\|_{H^2(0, T, L^2(\Omega))}, \tag{5.30}$$

$$|\vec{\sigma}_x(x, t)| \leq C\left\{ |\vec{\varphi}_x(x)| + \int_0^t f_t(x, \tau)| \, d\tau \right\}. \tag{5.31}$$

Proof: In light of the integro-differential equations (5.6) and the estimate (5.18), it is clear that the $[L^2(Q_T)]^{N \times N}$ norm of $H_\epsilon(\vec{\sigma}_\epsilon)$ is independent of ϵ. Therefore,

$$H_\epsilon(\vec{\sigma}_\epsilon) \to \vec{\lambda} \in H(\vec{\sigma}) \quad \text{weakly in } [L^2(Q_T)]^{N \times N}. \tag{5.32}$$

The existence then follows from Lemma 5.2 and the estimates (5.24)-(5.26) as $\epsilon \to 0$. Recall the definition of $F(x, t)$ given by (4.2). The 1 estimates (5.30)-(5.31) then follow from (5.18) and (5.27). The uniqueness part can be carried out in a straightforward manner. Suppose

$$\vec{\sigma}^i = (\sigma_1^i, \sigma_2^i, \dots \sigma_N^i), \quad i = 1, 2,$$

$$\vec{\lambda}^i = (\lambda_1^i, \lambda_2^i, \dots \lambda_N^i), \quad i = 1, 2,$$

are two pair of solutions to (4.6)-(4.7). Then

$$\left(\Psi + \frac{1}{\mu} A \right) (\vec{\sigma}^1 - \vec{\sigma}^2) - \frac{1}{\mu \ell} A \int_0^\ell (\vec{\sigma}^1 - \vec{\sigma}^2) \, dx + \Psi(\vec{\lambda}^1 - \vec{\lambda}^2) = 0. \tag{5.33}$$

Note that

$$(\Psi(\vec{\lambda}^1 - \vec{\lambda}^2), \vec{\sigma}^1 - \vec{\sigma}^2) \geqslant 0. \tag{5.34}$$

We first multiply both sides of (5.33) by $\vec{\sigma}^1 - \vec{\sigma}^2$, integrating over $(0, \ell)$, and dropping out the nonnegative term using (5.34), to obtain

$$\int_0^\ell |\vec{\sigma}^1 - \vec{\sigma}^2|^2 \, dx \leqslant 0.$$

This implies $\vec{\sigma}^1 = \vec{\sigma}^2$. Hence (5.33) becomes $\Psi(\vec{\lambda}^1 - \vec{\lambda}^2) = 0$ and $\vec{\lambda}^1 = \vec{\lambda}^2$.

Lemma 5.4 asserts that integro-differential inclusions (4.6)-(4.7) has a unique strong solution $\vec{\sigma} = (\sigma_1, \sigma_2, \dots \sigma_N)$. Lemma 4.1 implies that $\vec{\sigma} = (\sigma_1, \sigma_2, \dots \sigma_N, u)$ is also the unique strong solution to problem (3.1)-(3.5). All estimates in Theorem 3.3 follow from Lemma 5.4 and the representation of the displacement (4.1) by straightforward calculations.

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