M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

PAUL DEURING

A stable mixed finite element method on truncated exterior domains

M2AN - Modélisation mathématique et analyse numérique, tome 32, nº 3 (1998), p. 283-305

<http://www.numdam.org/item?id=M2AN_1998__32_3_283_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHEMATIQUE ET ANALYSE NUMERIQUE

(Vol 32, n° 3, 1998, p 283 à 305)

A STABLE MIXED FINITE ELEMENT METHOD ON TRUNCATED EXTERIOR DOMAINS (*)

Paul DEURING (¹)

Abstract — *Let $\Omega \subset \mathbb{R}^3$ be a polyhedron Denote by Ω_R the intersection of the exterior domain $\mathbb{R}^3 \setminus \bar{\Omega}$ with a ball centered in the origin and of radius R Then we show that a mixed finite element method on Ω_R, based on the Mini element, satisfies the Babushka-Brezzi condition with a constant independent of R This result is exploited in [3] in order to approximate exterior Stokes flows © Elsevier, Paris*

Résumé — *On se donne un polyèdre Ω en $\mathbb{R}^3$ Dénotant par Ω_R l'intersection du domaine extérieur $\mathbb{R}^3 \setminus \bar{\Omega}$ avec une balle centrée à l'origine et de diamètre R, on considère une méthode d'éléments finis mixtes sur Ω_R Nous montrons que cette méthode vérifie la condition inf-sup avec une constante qui ne dépend pas du paramètre R Ce résultat est utilisé en [3] pour discrétiser des écoulements extérieurs de Stokes © Elsevier, Paris*

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^3$ with a connected Lipschitz boundary. Consider the Stokes system in the exterior domain $\mathbb{R}^3 \setminus \bar{\Omega}$, under Dirichlet boundary conditions:

$$- \Delta u + \nabla \pi = f, \quad \text{div } u = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}, \quad u_{|\partial\Omega} = 0, \quad \lim_{|x| \to \infty} u(x) = 0 . \tag{1.1}$$

In order to apply finite element methods to this problem, we proposed the following approach (see [3]): Let B_R denote the ball with radius R and centre in the origin, and set $\Omega_R := B_R \setminus \bar{\Omega}$ ("truncated exterior domain"). Consider the boundary value problem

$$- \Delta u_R + \nabla \pi_R = f_{|\Omega_R}, \quad \text{div } u_R = 0 \quad \text{in } \Omega_R , \tag{1.2}$$

$$u_{R|\partial\Omega} = 0, \quad \mathscr{L}_R(u_R, \pi_R) = 0 \quad \text{on } \partial B_R ,$$

where the symbol $\mathscr{L}_R$ denotes the operator defined by

$$\mathscr{L}_R(v, \rho)(x) := \left(\frac{3}{2R} v_j(x) + \frac{1}{R} \sum_{k=1}^{3} x_k \left(\frac{\partial v_j(x)}{\partial x_k} - \frac{1}{2} \frac{\partial v_k(x)}{\partial x_j} - \delta_{jk} \rho(x) \right) \right)_{1 \leq j \leq 3}$$

for $x \in \partial B_R$. The equation $\mathscr{L}_R(u_R, \pi_R) = 0$ in (1.2) is called an "artificial boundary condition". As we pointed out in [3], problem (1.2) may be written in a variational form with respect to which the artificial boundary condition $\mathscr{L}_R(u_R, \pi_R) = 0$ is natural. This variational problem has a uniquely determined solution (u_R, π_R) in a suitable function space. In [3], we estimated the difference between (u_R, π_R) and the solution (u, π) of (1.1) ("truncation error"). In addition, following Goldstein's study [6], [7] of the Laplace equation in exterior domains, we showed problem (1.2) may be discretized by means of finite element methods satisfying certain general

(*) Manuscript received Decembre 22, 1995 Revised September 16, 1996 and January 27, 1997.
(¹) Martin-Luther-Universitat Halle, Fachbereich Mathematik und Informatik, Institut fur Analysis, D-06099 Halle (Saale), Germany

assumptions; see [3, (6.7)-(6.17]). Under these assumptions, we estimated the corresponding discretization error, which turned out to be of the same order as in the case of the Laplace equation. However, we did not mention any concrete finite element spaces which would satisfy the assumptions required in [3]. It is the purpose of this paper to present such spaces. They should be of interest not only in the context of the Stokes problem (1.1). More generally, they may be used for discretizing other mixed variational problems in truncated exterior domains Ω_R, provided these problems involve a natural boundary condition on ∂B_R and their solutions are to be approximated by C^0 piecewise polynomials of degree 1.

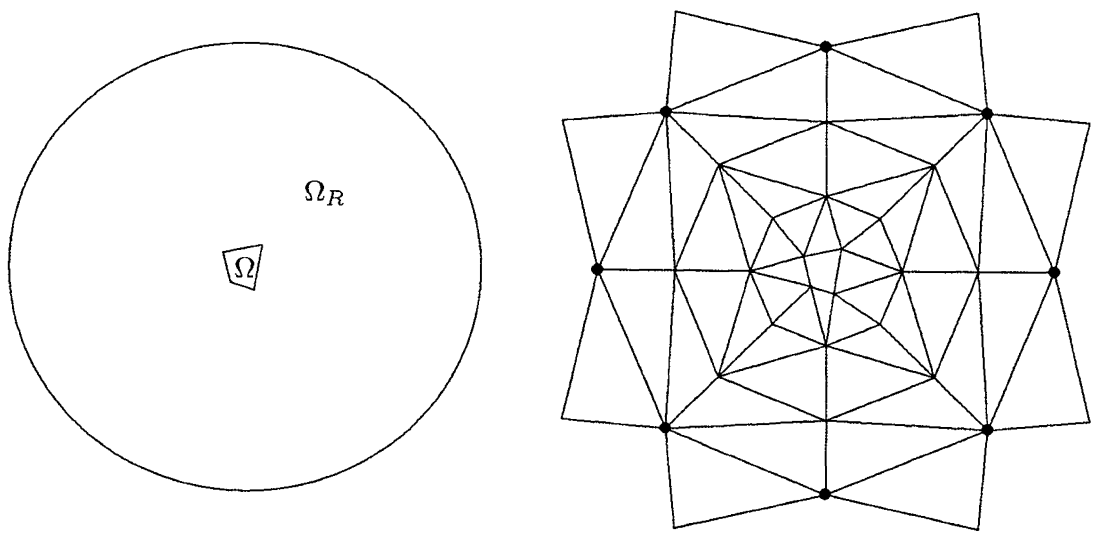

Figure 1. — Examples for domains Ω and Ω_R and for a triangulation $\mathcal{T}_h^R$ (2D representation). The large dots correspond to nodes located on ∂B_R

Let us explain what type of space we are looking for. Assume Ω is a polyhedral domain. Take $S \in (0, \infty)$ with $\bar{\Omega} \subset B_S$, $h_0 \in (0, S)$, $R_0 \in [8 \cdot S, \infty)$. Suppose that for $h \in (0, h_0]$ and $R \in [R_0, \infty)$, a triangulation

$$\mathcal{T}_h^R = (K_l(h, R))_{1 \leq l \leq k(h,R)}, \quad \text{for some} \quad k(h, R) \in \mathbb{N},$$

is given such that the ensuing conditions are satisfied: For $l \in \{1, ..., k(h, R)\}$, the set $K_l = K_l(h, R)$ is a tetrahedron with $K_l \subset \mathbb{R}^3 \backslash \bar{\Omega}$ and $K_l \cap B_R \neq \Phi$. The intersection of two different elements K_l and K_m is empty. Moreover,

$$\bar{\Omega}_R \subset \cup \{\bar{K}_m : 1 \leq m \leq k(h, R)\}, \tag{1.3}$$

and there is some number $\sigma_0 \in (0, 1)$ with

$$\sup \{r \in (0, \infty) : B_r(x) \subset K_l \text{ for some } x \in \mathbb{R}^3\} \geq \sigma_0 \cdot \text{diam } K_l$$

for $h \in (0, h_0]$, $R \in [R_0, \infty)$, $l \in \{1, ..., k(h, R)\}$, where $B_r(x)$ denotes a ball with radius $r > 0$ and center $x \in \mathbb{R}^3$.

Since all elements K_l of $\mathcal{T}_h^R$ are tetrahedrons, and because the sphere ∂B_R is part of the boundary of Ω_R, the mesh $\mathcal{T}_h^R$ cannot be a partitioning of Ω_R. In fact, it decomposes a region somewhat larger than Ω_R, as is implied by (1.3); see figure 1 for a two-dimensional representation of such a situation.

For any such triangulation $\mathcal{T}_h^R$, we look for a finite element space $W_h^R \times M_h^R$ with

$$W_h^R \subset \{u \in W^{1,2}(\Omega_R)^3 : u_{|\partial\Omega = 0}\}, \quad M_h^R \subset L^2(\Omega_R). \tag{1.4}$$

Furthermore, we require the dimension of $W_h^R \times M_h^R$ is bounded by the number $k(h, R)$ of elements of the mesh $\mathcal{T}_h^R$:

$$\dim W_h^R + \dim M_h^R \leqslant \mathcal{M} \cdot k(h, R) \quad \text{for} \quad h \in (0, h_0], R \in [R_0, \infty),$$

with a constant $\mathcal{M} > 0$ independent of h and R. In addition, the Babuska-Brezzi condition should be valid, with a constant $\beta \in (0, 1)$ independent of h and R:

$$\inf_{\rho \in M_h^R} \sup_{v \in W_h^R, v \neq 0} \frac{\int_{\Omega_R} \rho \operatorname{div} v \, dx}{|v|_{1,2}^{(R)} \|\rho\|_2} \geqslant \beta \tag{1.5}$$

for $h \in (0, h_0]$, $R \in [R_0, \infty)$. The norm $|\ |_{1,2}^{(R)}$ appearing in (1.5) is defined by

$$|v|_{1,2}^{(R)} := |v|_{1,2} + R^{-0.5} \cdot \|v_{|\partial B_R}\|_2,$$

with the seminorm $|\ \ |_{1,2}$ corresponding to the case $k = 1$ in the following definition:

$$|v|_{k,2} := \left(\int_{\Omega_R} \sum_{\alpha \in \mathbb{N}_0^3, |\alpha| = k} \sum_{j=1}^{3} \left| \frac{\partial^\alpha v_j(x)}{\partial x^\alpha} \right|^2 dx \right)^{1/2} \quad \text{for} \quad k \in \mathbb{N}_0, v \in W^{k,2}(\Omega_R)^3.$$

Finally the spaces W_h^R, M_h^R should fulfil some standard interpolation properties which we shall detail in Section 3.

For the construction of such spaces, some additional assumptions on the triangulations $\mathcal{T}_h^R$ will be necessary, but these conditions will be minor. It should be remarked that graded meshes as proposed by Goldstein [6], [7] and used in [3] are covered by our theory. We shall return to this point at the end of Section 3.

We call the functions from W_h^R "velocity functions" and those from M_h^R "pressure functions" because those spaces are constructed with an eye toward solving the Stokes problem by mixed finite element methods. In such a context, the velocity part of the solution is looked for in W_h^R, and the pressure part in M_h^R.

Our finite element spaces are related to the decompositions $(K_l \cap B_R)_{1 \leqslant l \leqslant k(h,R)}$ of Ω_R. Any element $K_l \cap B_R$ of these partitionings is tetrahedral if and only if $K_l \subset B_R$; otherwise some part of its boundary is curved. It is for technical reasons that we do not start out with the decompositions $(K_l \cap B_R)_{1 \leqslant l \leqslant k(h,R)}$, but introduce them via the triangulations $\mathcal{T}_h^R$.

Essentially our spaces consist of functions made up of P1-P1 elements, with the velocity fields enriched by bubble functions. Although this is a standard choice of shape functions, some effort will be necessary in order to prove the Babuska-Brezzi condition (1.5). In fact, our pressure functions $\pi \in M_h^R$ do not have mean value zero (see (1.4)), and the constant β in (1.5) must be independent of R. In this situation, the usual arguments for proving stability of piecewise polynomial mixed finite element spaces (see [5] and [2]) do not carry through. In fact, these arguments are based on estimating the H^1-seminorm of solutions to the divergence equation in bounded domains, under homogeneous Dirichlet boundary conditions. But the constant appearing in such an estimate depends on the respective domain. Thus, starting out with an estimate of this kind would lead to a constant β in (1.5) which depends on R. Moreover, existence of solutions to the problem just mentioned can be established only if the right-hand side in the divergence equation has mean value zero, a condition which translates into the requirement that the pressure has mean value zero. As mentioned above, our pressure functions do not satisfy this condition. Thus the standard theory on the divergence equation cannot be applied in our situation.

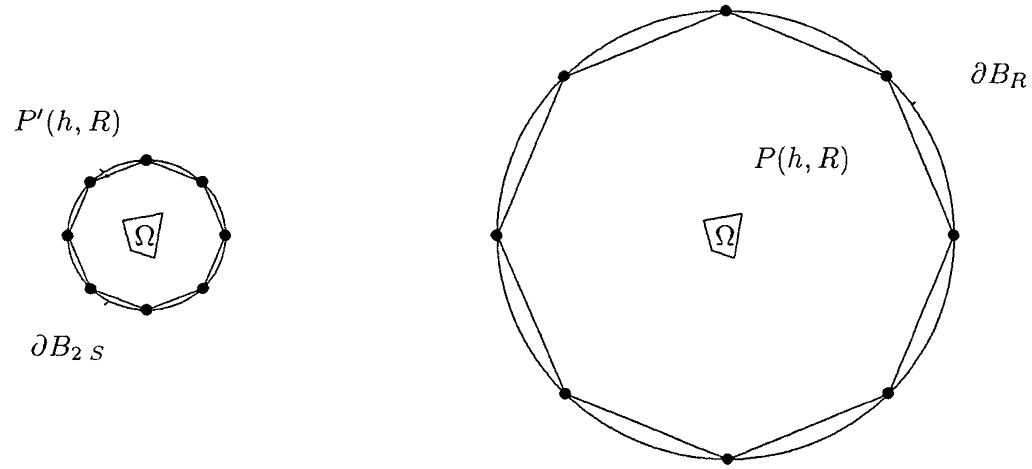

Figure 2 — Polyhedrons $P'(h\ R)$ and $P(h\ R)$ related to the triangulation $\mathcal{T}_h^R$ from fig 1 The large dots represent outer vertices of $P(h\ R)$ and $P'(h\ R)$

In order to overcome this difficulty, we shall point out that for a certain class of bounded domains, the divergence equation under homogeneous Dirichlet boundary conditions may be solved by functions which do not depend on the diameter of these domains These indications are made precise in Theorem 2 1 below

The domains in question are bounded by surfaces which, intuitively speaking, do not have folds We do not want to restrict our choice of Ω to such domains, so we cannot expect the domains Ω_R to belong to this class of sets This is the reason why for any $h \in (0, S)$, $R \in [8\ S, \infty)$, we shall introduce two polyhedrons, $P(h, R)$ and $P'(h, R)$ The first one — $P(h, R)$ — is to consist of those tetrahedrons of $\mathcal{T}_h^R$ which are contained in B_R As for the second one, namely $P'(h, R)$, it is defined as the union of the tetrahedrons in $\mathcal{T}_h^R$ which are a subset of $B_{2\ S}$ Thus $P(h, R)$ may be considered as large and $P'(h, R)$ as small, see figure 2 The decompositions $\mathcal{T}_h^R$ should be chosen in such a way that the surfaces of $P(h, R)$ and $P'(h, R)$ do not fold up Then the ring-shaped polyhedron $P(h, R)\backslash\overline{P'(h, R)}$ belongs to the class of sets considered in Theorem 2 1

It will turn out that our proof of (1 5) carries through if we solve the divergence equation twice, first on Ω_R and then on the ring shaped domain $P(h, R)\backslash\overline{P'(h, R)}$, see the proof of Theorem 4 1 In the first case, the solution has to vanish on $\partial\Omega$, but need not satisfy any boundary condition on ∂B_R This situation is covered by a theorem from [3] which we shall restate here as Theorem 2 2 As for the second case, we shall impose homogeneous Dirichlet conditions everywhere on the boundary and then refer to Theorem 2 1 In both cases we are able to estimate our solutions in such a way that the constants appearing in these estimates do not depend on the parameters h and R

We further remark that in the proof of (1 5), we shall use a perturbation argument in order to deal with the curved elements of the meshes $(K_l \cap B_R)_{1 \leqslant l \leqslant k(h\ R)}$ This argument only carries through if these curved elements are small To this end, we shall require the outer vertices of $P(h, R)$ are located on the sphere ∂B_R, see condition (A6) in Section 3 By "outer" vertices of $P(h, R)$, we mean those vertices which do not belong to $\partial\Omega$ (large dots in *fig* 2) Of course, this assumption on $P(h, R)$ implies the decompositions $(K_l \cap B_R)_{1 \leqslant l \leqslant k(h\ R)}$ degenerate near $\partial\Omega$ However, our reasoning only depends on the fact that the triangulations $\mathcal{T}_h^R$ are non degenerate

2 THE DIVERGENCE EQUATION

As indicated in the preceding section, we shall need some results on the divergence equation

$$\operatorname{div} v = f \quad \text{in a domain } \mathcal{A}, \tag{2 1}$$

under homogeneous Dirichlet boundary conditions

$$v_{|\partial \mathscr{A}} = 0\,, \tag{2.2}$$

with a given function $f \in L^2(\mathscr{A})$ satisfying the relation

$$\int_{\mathscr{A}} f\,dx = 0\,. \tag{2.3}$$

We are looking for solutions v of (2.1), (2.2) fulfilling the estimate

$$|v|_{1,2} \leqslant C \cdot \|f\|_2\,, \tag{2.4}$$

with the constant C independent of v, f and the diameter of $\mathscr{A}$. We begin our discussion by introducing some notations: If $\varphi \in (0, \pi/2)$, $z \in \mathbb{R}^3\backslash\{0\}$, we write

$$\mathbb{K}(\varphi, z) := \{x \in \mathbb{R}^3\backslash\{0\} : |x|^{-1} \cdot |z|^{-1} \cdot x \cdot z > \cos\varphi\}$$

for the infinite circular cone with vertex in the origin, semiaperture φ and axis in the direction of the vector z. Let $x \in \mathbb{R}^3$, $r > 0$. Following [1, p. 94], we call a domain $U \subset \mathbb{R}^3$ "star-shaped with respect to the ball $B_r(x)$", if for all $y \in U$, the closed convex hull of $\{y\} \cup B_r(x)$ is a subset of U.

Our main result on problem (2.1), (2.2) reads as follows:

THEOREM 2.1: *Let* $\varphi_0 \in (0, \pi/2)$. *Then there exists a constant* $\mathscr{C}_1(\varphi_0) > 0$ *with the ensuing properties: Let* $T \in (0, \infty)$, $R \in [4 \cdot T, \infty)$, *and let* $\mathscr{A}_1, \mathscr{A}_2 \subset \mathbb{R}^3$ *be domains with*

$$0 \in \mathscr{A}_1 \subset B_T \subset \overline{B_{R-T}} \subset \mathscr{A}_2 \subset B_R\,. \tag{2.5}$$

Assume

$$x + \mathbb{K}(\varphi_0, x) \subset \mathbb{R}^3\backslash\overline{\mathscr{A}_1} \quad \textit{for} \quad x \in \partial\mathscr{A}_1\,, \tag{2.6}$$

$$(x + \mathbb{K}(\varphi_0, -x)) \cap B_{R-T}(x) \subset \mathscr{A}_2 \quad \textit{for}\, x \in \partial\mathscr{A}_2\,. \tag{2.7}$$

Put $\mathscr{A} := \mathscr{A}_2\backslash\overline{\mathscr{A}_1}$. *Then, for any function* $f \in L^2(\mathscr{A})$ *satisfying (2.3), there is a function* $v := v(\mathscr{A}, f) \in W_0^{1,2}(\mathscr{A})^3$ *such that equations (2.1), (2.2) are valid and inequality (2.4) holds with* $C = \mathscr{C}_1(\varphi_0)$.

Figure 3 gives an example — in 2D representation — for domains $\mathscr{A}_1$, $\mathscr{A}_2$ satisfying (2.6), (2.7), respectively, and verifying the relations in (2.5). Note that assumptions (2.6) and (2.7) are stronger than the usual cone condition. To see this, consider the two-dimensional domain $\mathscr{A}_1$ shown in figure 4. This set fulfils the standard cone condition but not the two-dimensional analogue of (2.6). In fact, choosing the point x as indicated in figure 4, we cannot attach an infinite cone to x which has an empty intersection with $\mathscr{A}_1$. This example suggests an informal way — already indicated in Section 1 — for specifying the domains $\mathscr{A}$ admitted in Theorem 2.1: the surface of such domains should not be folded.

Proof of Theorem 2.1: Our aim is to apply [4, p. 124, Theorem III.3.1]. Let us verify the assumptions of that theorem.

A simple geometrical argument shows there is some $\gamma \in (0, 1/4)$, only depending on φ_0, such that

$$B_{\gamma R}((R/2) \cdot |x|^{-1} \cdot x) \subset x + \mathbb{K}(\varphi_0, x) \quad \text{for} \quad x \in \partial\mathscr{A}_1\,, \tag{2.8}$$

$$B_{\gamma R}((R/2) \cdot |x|^{-1}\, x) \subset x + \mathbb{K}(\varphi_0, -x) \quad \text{for} \quad x \in \partial\mathscr{A}_2\,;$$

see figure 5. Since $\gamma < 1/4$, $R \geqslant 4 \cdot T$ and $\overline{B_{R-T}}\backslash B_T \subset \mathscr{A}$, it further holds $\overline{B_{\gamma R}(z)} \subset \mathscr{A}$

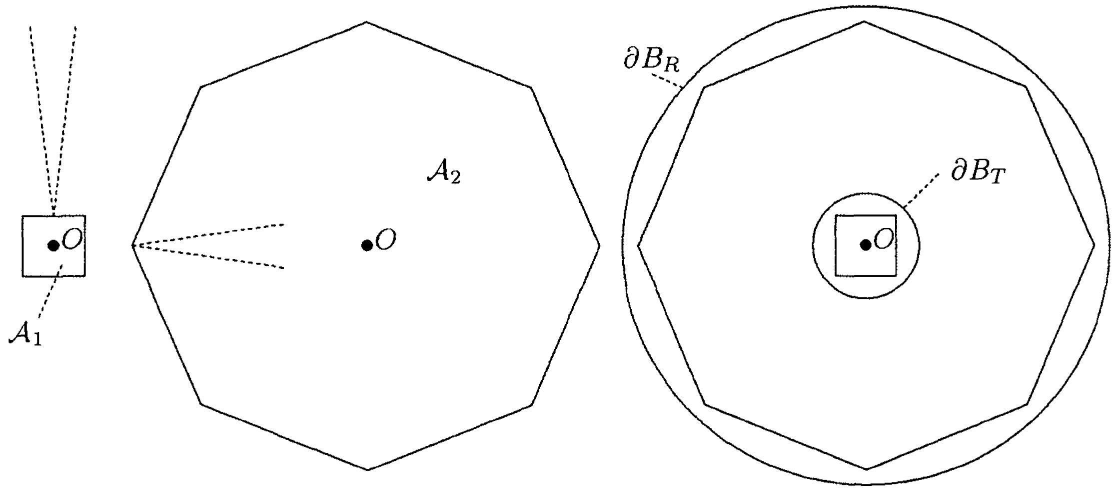

Figure 3. — Example for domains $\mathcal{A}_1$ (left) and $\mathcal{A}_2$ (center) satisfying (2.6) and (2.7) respectively (2D representation). Sample cones of the type appearing in (2.6) and (2.7) are also shown. The picture on the right represents the situation described in (2.5).

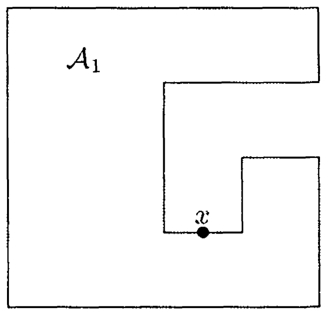

Figure 4. — 2D example for a domain $\mathcal{A}_1$ which does not fulfil (2.6).

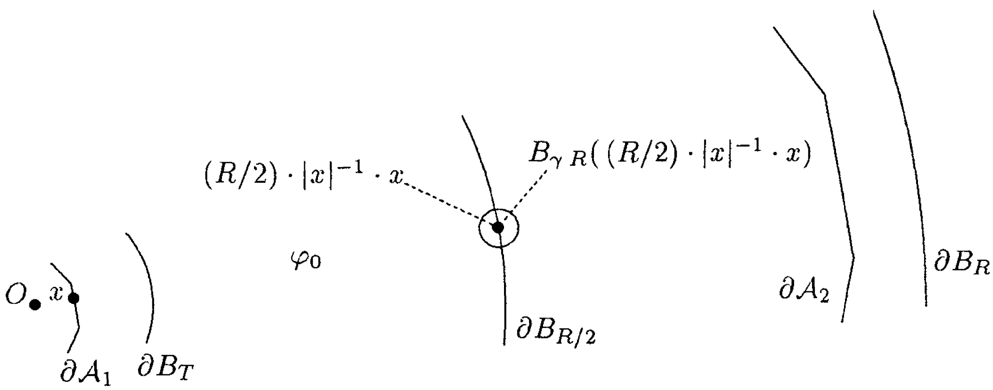

Figure 5. — 2D illustration of the relation in (2.8).

for $z \in \partial B_{R/2}$.

Next we choose $m \in \mathbb{N}$ and points $y_1, \ldots, y_m$ on the unit sphere ∂B_1 such that it holds for any $t \in (0, \infty)$:

$$\partial B_t \subset \bigcup_{j=1}^{m} B_{t\,\gamma/4}(t \cdot y_j)\,, \tag{2.9}$$

$$B_{t\,\gamma/4}(t \cdot y_i) \cap B_{t\,\gamma/4}(t \cdot y_{i+1}) \neq \Phi \quad \text{for} \quad 1 \leqslant i \leqslant m-1\,. \tag{2.10}$$

The relations in (2.9) and (2.10) amount to choosing m rays, each starting at the origin, with the following properties: if the intersection of each ray with the sphere ∂B_t is taken as the center of a ball with radius $\gamma \cdot t/4$, then the union of these balls contains the sphere ∂B_t. Moreover, their centers may be labeled in such a way that (2.10) holds. The important point is that the number m of balls does not depend on t. An easy way for proving (2.9), (2.10) consists in splitting the intervals $[0, 2 \cdot \pi]$ and $[0, \pi]$ into small enough parts, ordering them in a suitable way, and then making use of polar coordinates.

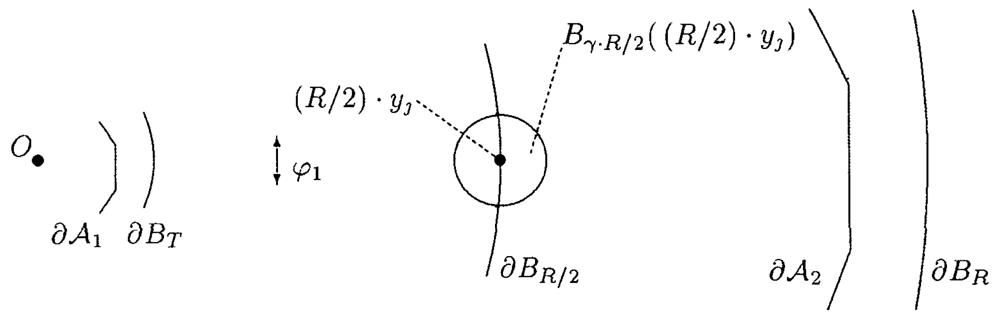

Figure 6. — The cone $\mathbb{K}(\varphi_1, y_j)$ envelops the ball $B_{R\,\gamma/2}((R/2)\ y_j)$.

By doubling the radius of the balls in (2.9), (2.10), we deduce from (2.10)

$$\text{Vol}\,(B_{t\,\gamma/2}(t \cdot y_j) \cap B_{t\,\gamma/2}(t \cdot y_{j+1})) \geqslant (4 \cdot \pi/3) \cdot (t \cdot \gamma/4)^3 \tag{2.11}$$

for $t \in (0, \infty)$, $1 \leqslant j \leqslant m-1$. Take $\varphi_1 \in (0, \pi/2)$ with $\sin \varphi_1 = \gamma$. Then for any $j \in \{1, \ldots, m\}$, the cone $\mathbb{K}(\varphi_1, y_j)$ envelops the ball $B_{R\,\gamma/2}((R/2) \cdot y_j)$ (*fig. 6*). Put

$$\Omega_j := \mathbb{K}(\varphi_1, y_j) \cap \mathscr{A} \quad \text{for} \quad 1 \leqslant j \leqslant m\,.$$

It follows from (2.9) that $\mathscr{A} \subset \bigcup_{j=1}^{m} \Omega_j$. Moreover, we have by (2.11), with $t = R/2$:

$$\text{Vol}\,(\Omega_j \cap \Omega_{j+1}) \geqslant (4 \cdot \pi/3) \cdot (R \cdot \gamma/8)^3 \quad \text{for} \quad 1 \leqslant j \leqslant m-1\,. \tag{2.12}$$

Let $j \in \{1, \ldots, m\}$. The set Ω_j is star-shaped with respect to the ball $\mathscr{B}_j := B_{R\,\gamma/4}((R/2) \cdot y_j)$. This may be shown by geometrical arguments involving assumptions (2.6) and (2.7). As an example, take $x_0 \in \mathscr{B}_j$ and $x_1 \in \partial\mathscr{A}_1 \cap \mathbb{K}(\varphi_1, y_j)$. Let L denote the line between x_0 and x_1 (*fig. 7*). We have to show that $L \backslash \{x_1\} \subset \Omega_j$. In fact, it holds on one hand $x_0 \in \mathscr{B}_j \subset \mathbb{K}(\varphi_1, y_j)$ and $x_1 \in \mathbb{K}(\varphi_1, y_j)$, hence $L \subset \mathbb{K}(\varphi_1, y_j)$. On the other hand, due to the relations $x_1 \in \mathbb{K}(\varphi_1, y_j)$, $\gamma = \sin \varphi_1$, we obtain by a simple calculation or by a geometrical argument

$$|\rho - (R/2) \cdot y_j| < (\sqrt{2}/2) \cdot R \cdot (1 - \cos \varphi_1)^{1/2} \leqslant (\sqrt{2}/2) \cdot \gamma \cdot R\,, \tag{2.13}$$

where we used the abbreviation $\rho := (R/2) \cdot |x_1|^{-1} \cdot x_1$; see figure 8. Now we may conclude $\mathscr{B}_j \subset B_{R\,\gamma}(\rho)$, hence by (2.8): $\mathscr{B}_j \subset x_1 + \mathbb{K}(\varphi_0, x_1)$. It follows $x_0 \in x_1 + \mathbb{K}(\varphi_0, x_1)$, hence $L\backslash\{x_1\} \subset x_1 + \mathbb{K}(\varphi_0, x_1)$. At this point we may refer to assumption (2.6) to obtain $L\backslash(x_1) \subset \mathscr{A}$. But we have already shown $L \subset \mathbb{K}(\varphi_1, y_j)$, so we finally obtain $L\backslash\{x_1\} \subset \Omega_j$.

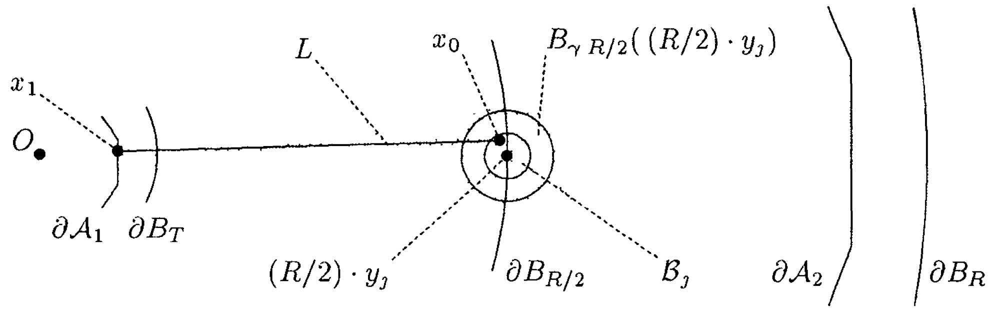

Figure 7. — This setting is considered in the proof that Ω_j is star-shaped. The shaded area represents the set Ω_j.

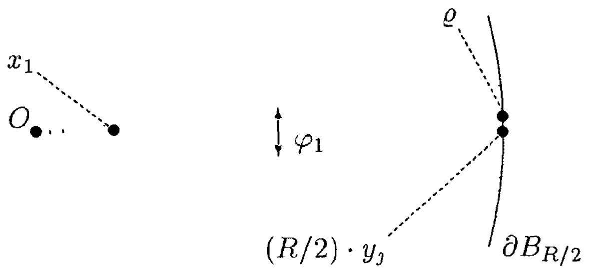

Figure 8. — The distance between ρ and $(R/2) . y_j$ (see (2.13)) may be estimated by a simple calculation or a geometrical argument.

Due to the properties of $\Omega_1, \ldots, \Omega_m$, we are in a position to apply [4, p. 124, Theorem II.3.1]. By this reference, problem (2.1), (2.2) may be solved by a function $v \in W_0^{1,2}(\mathscr{A})^3$, provided the function f is given in $L^2(\mathscr{A})$ and satisfies (2.3). According to [4, loc. cit.], this solution v fulfils the estimate $|v|_{1,2} \leqslant \alpha_1(\mathscr{A}) \cdot \|f\|_2$, with

$$\alpha_1(\mathscr{A}) := c \cdot \left[\sum_{k=1}^{m} \left(1 + \frac{\operatorname{Vol}(\Omega_k)}{\operatorname{Vol}(\Omega_k \cap \Omega_{k+1})} \right) \cdot \prod_{i=1}^{k-1} \left(1 + \frac{\operatorname{Vol}(\mathscr{A}\backslash\Omega_i)}{\operatorname{Vol}(\Omega_i \cap \Omega_{i+1})} \right)^{1/2} \right] \cdot \left(\frac{\operatorname{diam} \mathscr{A}}{R \cdot \gamma/4} \right)^3 \cdot \left(1 + \frac{\operatorname{diam} \mathscr{A}}{R \cdot \gamma/4} \right),$$

where we used the notation $\Omega_{m+1} := \Omega_m$. The letter c denotes a numerical constant. Since

$$\operatorname{diam} \mathscr{A} \leqslant 2 \cdot R, \quad \operatorname{Vol}(\Omega_k) \leqslant \operatorname{Vol} \mathscr{A} \leqslant (4 \cdot \pi/3) \cdot R^3,$$

and due to the relation in (2.12), the constant $\alpha_1(\mathscr{A})$ is bounded by a constant $\alpha_2(\gamma, m)$ which only depends on γ and m. But these parameters may be expressed in terms of φ_0, as follows by some tedious computations, which we omit here. Therefore the constant $\alpha_2(\gamma, m)$ depends only on φ_0, so Theorem 2.1 is valid with $\mathscr{C}_1(\varphi_0) := \alpha_2(\gamma, m)$. □

As indicated in Section 1, we shall need a second result on the divergence equation (2.1). This result was proved in [3]; see [3, Theorem 4.1]. We repeat it here in a form which is convenient for our purposes:

THEOREM 2.2: *Let $\mathscr{A} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a constant $\mathscr{C}_2(\mathscr{A})$ with the ensuing property:*

For any $R \in (0, \infty)$ with $\bar{\mathscr{A}} \subset B_R$, and for any $\pi \in L^2(B_R \backslash \bar{\mathscr{A}})$, there is a function $v := v(\mathscr{A}, R, \pi) \in W^{1,2}(B_R \backslash \bar{\mathscr{A}})^3$ such that

$$v_{|\partial \mathscr{A}} = 0, \quad \operatorname{div} v = \pi, \quad |v|_{1,2}^{(R)} \leqslant \mathscr{C}_2(\mathscr{A}) \cdot \|\pi\|_2 .$$

3. FINITE ELEMENT SPACES ON TRUNCATED EXTERIOR DOMAINS

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain with Lipschitz boundary. Suppose $0 \in \Omega$, and fix $S \in (0, \infty)$ with $\bar{\Omega} \subset B_S$. For any $R \in [8 \cdot S, \infty)$, $h \in (0, S)$, let $\mathscr{T}_h^R$ be a partitioning with $k(h, R)$ elements ($k(h, R) \in \mathbb{N}$). These elements are denoted by $K_1(h, R), \ldots, K_{k(h,R)}(h, R)$, so that

$$\mathscr{T}_h^R = (K_l(h, R))_{1 \leqslant l \leqslant k(h,R)} .$$

The properties of the decompositions $\mathscr{T}_h^R$ will be specified by assumptions (A1)-(A9) below. For brevity, we shall only write K_l instead of $K_l(h, R)$, except at some rare occasions when the choice of the parameters h and R may not be clear from context.

Our first assumptions state that the meshes $\mathscr{T}_h^R$ are tetrahedral and decompose a region around Ω which is larger than Ω_R (see *fig.* 1):

(A1) For $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, $l \in \{1, \ldots, k(h, R)\}$, the set K_l is an open tetrahedron with $K_l \subset \mathbb{R}^3 \backslash \bar{\Omega}$.

(A2) $\bar{\Omega}_R \subset \cup \{\bar{K}_l : 1 \leqslant l \leqslant k(h, R)\}$ for h, R as in (A1).

We do not admit hanging nodes:

(A3) If $h \in (0, S)$, $R \geqslant 8 \cdot S$, $l, m \in \{1, \ldots, k(h, R)\}$ with $l \neq m$ and $F := \bar{K}_l \cap \bar{K}_m \neq \theta$, then F is either a common face or side or vertex of K_l and K_m.

Moreover, the decompositions $\mathscr{T}_h^R$ are supposed to be non-degenerate:

(A4) There is a constant $\sigma_0 \in (0, 1)$ with

$$\sup \{r \in (0, \infty) : B_r(x) \subset K_l \text{ for some } x \in K_l\} \geqslant \sigma_0 \cdot \operatorname{diam} K_l$$

for h, R, l as in (A1).

The mesh size of elements is allowed to become larger with increasing distance from Ω, but this growth should not be too strong:

(A5) If $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, it holds

$$\operatorname{diam} K_l \leqslant 2^{j-2} \cdot S \quad \text{for } l \in \{1, \ldots, k(h, R)\}, 1 \leqslant j \leqslant 3 \quad \text{with } K_l \cap \bar{B}_{2^j \cdot S} \neq \Phi ,$$

$$\operatorname{diam} K_l \leqslant R \cdot h/(4 \cdot S) \quad \text{for } l \in \{1, \ldots, k(h, R)\} .$$

Thus the element mesh size of $\mathscr{T}_h^R$ may be as large as $R \cdot h/(4 \cdot S)$. As we shall see at the end of this section, the graded meshes used in [3] satisfy assumption (A5).

For $h \in (0, S)$, $R \geqslant 8 \cdot S$, we define the polyhedron $P(h, R)$ as the interior of

$$\cup \{\bar{K}_l : l \in \{1, \ldots, k(h, R)\}, K_l \subset \Omega_R\} ,$$

and the polyhedron $P'(h, R)$ as the interior of

$$\cup \{\bar{K}_l : l \in \{1, \ldots, k(h, R)\}, K_l \subset B_{2 \cdot S}\} ;$$

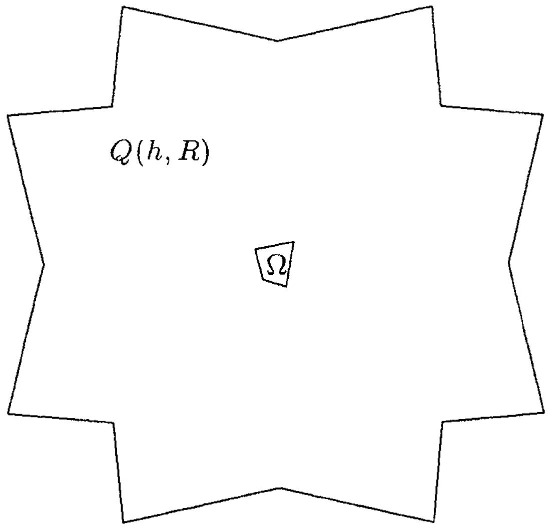

Figure 9. — Domain $Q(h, R)$ (2D representation) related to the mesh $\mathcal{T}_h^R$ from figure 1.

see figure 2. Moreover, we denote the interior of the set $\cup\{\bar{K}_l : 1 \leqslant l \leqslant k(h, R)\}$ by $Q(h, R)$. This means $Q(h, R)$ is the region decomposed by $\mathcal{T}_h^R$. Thus, if $\mathcal{T}_h^R$ is given as in figure 1, the corresponding set $Q(h, R)$ has the form shown in figure 9. We further point out that $P(h, R) \subset B_R$ and $P'(h, R) \subset B_{2\ S}$.

For reasons already indicated in Section 1, we require the outer vertices of $P(h, R)$ belong to ∂B_R :

(A6) If $h \in (0, S)$, $R \geqslant 8 \cdot S$, and if $x \in \partial P(h, R)$ is a vertex of $P(h, R)$, then it holds $x \in \partial B_R \cup \partial \Omega$.

This condition may be stated equivalently by requiring that for h, R, l as in (A1), the tetrahedron K_l is either a subset of Ω_R, or all its vertices belong to $\mathbb{R}^3 \backslash B_R$. Thus a tetrahedron with a face as shown in figure 10 is not admitted because such a tetrahedron has vertices inside Ω_R as well as outside $\bar{B}_R$. On the other hand, a setting as in figure 11 does not imply a contradiction to (A6) since the three vertices of the tetrahedral face shown there are located in $\mathbb{R}^3 \backslash B_R$.

Without loss of generality, we may assume that for any $h \in (0, S)$, $R \geqslant 8 \cdot S$, the sets K_l are labeled in such a way that there are indices $\kappa(h, R)$, $\tau(h, R) \in \{1, \ldots, k(h, R)\}$ with $1 < \kappa(h, R) < \tau(h, R) < k(h, R)$ and

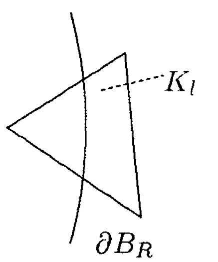

Figure 10. — Tetrahedrons with such faces are not admitted.

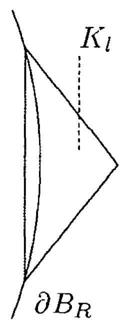

Figure 11. — Example for the face of a tetrahedron which conforms to assumption (A6).

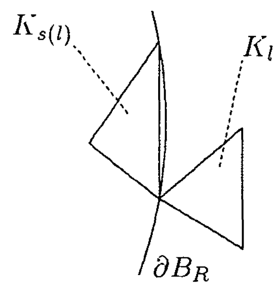

Figure 12. — Any tetrahedron K_l not contained in Ω_R should touch at least one element $K_{s(l)}$ with $K_{s(l)} \subset \Omega_R$.

$K_l \subset \Omega_{2\,S}$ for $1 \leqslant l \leqslant \kappa(h,R)$, $K_l \subset \Omega_R$ for $1 \leqslant l \leqslant \tau(h,R)$, $K_l \backslash \bar{B}_R \neq \Phi$ for $\tau(h,R)+1 \leqslant l \leqslant k(h,R)$.

This means the polyhedrons $P(h,R)$, $P'(h,R)$ and $P(h,R)\backslash\overline{P'(h,R)}$ coincide with the interior of the sets

$$\cup\{\bar{K}_l : 1 \leqslant l \leqslant \tau(h,R)\}, \quad \cup\{\bar{K}_l : 1 \leqslant l \leqslant \tau(h,R)\}$$

and $\cup\{\bar{K}_l : \tau(h,R)+1 \leqslant l \leqslant \tau(h,R)\}$, respectively. Next we want to exclude the case that our triangulations $\mathcal{T}_h^R$ extend too far beyond B_R. Therefore we require

(A7) For $h \in (0,S)$, $R \geqslant 8 \cdot S$, $l \in \{\tau(h,R)+1, \ldots, k(h,R)\}$, there is an index $s(l) = s(l,h,R) \in \{1, \ldots, \tau(h,R)\}$ such that $\bar{K}_{s(l)} \cap \bar{K}_l \neq \Phi$.

Thus any tetrahedron K_l not contained in Ω_R should touch at least one element $K_{s(l)}$ with $K_{s(l)} \subset \Omega_R$ (*fig.* 12). This implies K_l touches several such elements, but we pick just one, namely $K_{s(l)}$.

As explained in Section 1, we have to require the outer surfaces of $P'(h,R)$ $P(h,R)$ do not fold up:

(A8) There is an angle $\varphi_0 \in (0, \pi/2)$ such that for $h \in (0,S)$, $R \geqslant 8 \cdot S$, it holds

$$x + \mathbb{K}(\varphi_0, x) \subset \mathbb{R}^3 \backslash \overline{P'(h,R) \cup \Omega} \quad \text{for } x \in \partial P'(h,R)\backslash\partial\Omega\,,$$

$$(x + \mathbb{K}(\varphi_0, -x)) \cap B_R(x) \subset P(h,R) \cup \overline{\Omega} \quad \text{for } x \in \partial P(h,R)\backslash\partial\Omega\,.$$

This assumption is not much of a restriction in practice. In fact, it suffices to take care that the outer faces of $P'(h,R)$ and $P(h,R)$ have a normal roughly pointing in radial direction.

For h, R, l as in (A1), we introduce the usual macroelement $(K_l)_\Delta$ made of the elements in $\mathcal{T}_h^R$ neighbouring K_l. More precisely, the set $(K_l)_\Delta$ is defined as the interior of

$$\cup\{\bar{K}_m : 1 \leqslant m \leqslant k(h,R), \bar{K}_m \cap \bar{K}_l \neq \Phi\}\,.$$

These macroelements should have a shape which will allow us to apply the interpolation result in [1, p. 100/101, Lemma 4.3.8]. Therefore we require

(A9) There is a constant $\sigma_1 \in (0,1)$ such that for h, R, l as in (A1), the set $(K_l)_\Delta$ is star-shaped with respect to the ball $B_{\sigma_1 \operatorname{diam} K_l}(x)$, for some $x \in (K_l)_\Delta$.

As in the case of assumption (A8), it should not be too difficult to satisfy assumption (A9) in practice. For example, one may think of starting with a non-degenerate decomposition into 3-rectangles and then split up each rectangle into tetrahedrons.

Let us note two consequences of assumptions (A3) and (A4): Firstly, there is some integer $Z \in \mathbb{N}$ with

$$\operatorname{card}\{m \in \{1, \ldots, k(h,R)\} : \bar{K}_m \cap \bar{K}_l \neq \Phi\} \leqslant Z \quad \text{for } h, R, l \text{ as in (A1)}\,. \tag{3.1}$$

Secondly, there is a constant $\mathcal{D}_1 > 0$ such that

$$\operatorname{diam}(K_l)_\Delta \leqslant \mathcal{D}_1 \cdot \operatorname{diam} K_l \quad \text{for } h, R, l \text{ as in (A1)}\,. \tag{3.2}$$

Observe that inequality (3.1) implies for $f \in L^1(Q(h,R))$, h, R as in (A1):

$$\sum_{l=1}^{k(h,R)} \int_{(K_l)_\Delta} f\,dx \leqslant Z \cdot \sum_{l=1}^{k(h,R)} \int_{K_l} f\,dx\,. \tag{3.3}$$

Next we introduce our finite element spaces. To this end, we use the following notations: If $\mathscr{A} \subset \mathbb{R}^3$, we write $Pol_1(\mathscr{A})$ for the set of all polynomials on $\mathscr{A}$ with degree less than or equal to 1. If $T \subset \mathbb{R}^3$ is a tetrahedron with vertices $a_1, ..., a_4$, and if $j \in \{1, ..., 4\}$, then let $\lambda_j = \lambda(a_j, T)$ denote the polynomial from $Pol_1(\mathbb{R}^3)$ satisfying the equation $\lambda_j(a_i) = \delta_{ij}$ for $1 \leqslant i \leqslant 4$. Define the bubble function b_T by $b_T := \lambda_1 \cdot ... \cdot \lambda_4$.

Take $h \in (0, S)$, $R \in [8 \cdot S, \infty)$. Then let the space W_h^R contain all the functions $v \in C^0(\bar{\Omega}_R)^3$ with $v_{|\partial\Omega} = 0$,

$$v_{j|K_l} \in \operatorname{span}(Pol_1(K_l) \cup \{b_{K_l}\}) \quad \text{for } 1 \leqslant l \leqslant \tau(h, R), 1 \leqslant j \leqslant 3,$$

$$v_{j|K_l \cap \Omega_R} \in Pol_1(K_l \cap \Omega_R) \quad \text{for } \tau(h, R) + 1 \leqslant l \leqslant k(h, R), 1 \leqslant j \leqslant 3.$$

Moreover, let M_h^R denote the space of all functions $\rho : \overline{P(h, R)} \cup \Omega_R \mapsto \mathbb{R}$ with

$$\rho_{|\overline{P(h,R)}} \in C^0(\overline{P(h, R)}), \quad \rho_{|K_l} \in Pol_1(K_l) \quad \text{for } 1 \leqslant l \leqslant \tau(h, R),$$

$$\rho_{|(K_l \cap \Omega_R) \cup \overline{K_{s(l)}}} \in Pol_1((K_l \cap \Omega_R) \cup \overline{K_{s(l)}}) \quad \text{for } \tau(h, R) + 1 \leqslant l \leqslant k(h, R).$$

Note that $W_h^R \subset W^{1,2}(\Omega)^3$ and $M_h^R \subset L^2(\Omega_R)$. We further remark that on tetrahedrons K_l contained in Ω_R, our spaces reduce to the shape functions of the Mini element ([2, p. 213]). On curved domains $K_l \cap \Omega_R$, however, any function from the velocity space W_h^R equals a polynomial of order 1, whereas any function from the pressure space M_h^R coincides with one and the same polynomial of first order both on the curved element $K_l \cap \Omega_R$ and on its related tetrahedron $K_{s(l)} \subset \Omega_R$. This means our pressure functions may be estimated against their restrictions to $P(h, R)$. For a proof, we first point out a consequence of (3.2):

LEMMA 3.1: *There is a constant* $\mathscr{D}_2 > 0$ *such that*

$$\|p_{|(K_l)_\Delta}\|_2 \leqslant \mathscr{D}_2 \cdot \|p_{|K_l}\|_2 \quad \textit{for } h, R, l \textit{ as in } (A1), p \in Pol_1(\mathbb{R}^3).$$

In particular, since $K_l \subset (K_{s(l)})_\Delta$ *for* $\tau(h, R) + 1 \leqslant l \leqslant k(h, R)$, *it holds*

$$\|p_{|(K_l)_\Delta}\|_2 \leqslant \mathscr{D}_2 \cdot \|p_{|K_{s(l)}}\|_2 \quad \textit{for } h \in (0, S), R \geqslant 8 \cdot S, \tau(h, R) + 1 \leqslant l \leqslant k(h, R).$$

Now it follows by (3.3):

LEMMA 3.2: *For* $h \in (0, S)$, $R \geqslant 8 \cdot S$, $\rho \in M_h^R$, *it holds*

$$\|\rho\|_2 \leqslant \mathscr{D}_3 \cdot \|\rho_{|P(h,R)}\|_2, \tag{3.4}$$

$$\|\rho_{|\Omega_R \setminus \overline{P'(h,R)}}\|_2 \leqslant \mathscr{D}_3 \cdot \|\rho_{|P(h,R) \setminus \overline{P'(h,R)}}\|_2, \tag{3.5}$$

with a constant $\mathscr{D}_3$ *independent of* h, R *and* ρ.

For technical reasons, we shall need certain spaces of C^0 piecewise polynomials of degree 1 over the sets $Q(h, R)$. These spaces are defined by

$$V_h^R := \{u \in C^0(\overline{Q(h, R)})^3 : u_{|K_l} \in Pol_1(K_l)^3 \quad \text{for } 1 \leqslant l \leqslant k(h, R); u_{|\partial\Omega} = 0\},$$

for $h \in (0, S), R \geqslant 8 \cdot S$.

Due to (A9), we may use the result in [1, p. 100/101, Lemma 4.3.8] in order to approximate functions on $(K_l)_\Delta$ by polynomials. It follows by a reasoning as in [1, p. 118/119]:

THEOREM 3.1: *There is a constant $\mathscr{D}_4 > 0$ and for any $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, a bounded linear operator*

$$\mathcal{I}_h^R : \{w \in W^{1,2}(Q(h,R))^3 : w_{|\partial\Omega} = 0\} \mapsto W_h^R$$

such that

$$|(w - \mathcal{I}_h^R(w))_{|K_l \cap \Omega_R}|_{v,2} \cdot (diam\, K_l)^{-m+v} \leqslant \mathscr{D}_4 \cdot |w_{|(K_l)_\Delta}|_{m,2}$$

for h, R, l as in (A1), $m \in \{0, 1, 2\}$, $v \in \{0, ..., m\}$, $w \in W^{1,2}(Q(h,R))^3$ with $w_{|\partial\Omega} = 0$ and $w_{|(K_l)_\Delta} \in W^{m,2}((K_l)_\Delta)^3$.

THEOREM 3.2: *There is a constant $\mathscr{D}_5 > 0$ and for any $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, a projection*

$$\Pi_h^R : \{w \in W^{1,2}(Q(h,R))^3 : w_{|\partial\Omega} = 0\} \mapsto V_h^R$$

such that it holds

$$|(w - \Pi_h^R(w))_{|K_l}|_{v,2} \cdot (diam\, K_l)^{-1+v} \leqslant \mathscr{D}_5 \cdot |w_{|(K_l)_\Delta}|_{1,2}$$

for h, R, l as in (A1), $v \in \{0, 1\}$, $w \in W^{1,2}(Q(h,R))^3$ with $w_{|\partial\Omega} = 0$.

THEOREM 3.3: *There is a constant $\mathscr{D}_6 > 0$ and for any $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, a projection*

$$\tilde{\Pi}_h^R : W_0^{1,2}(P(h,R)\backslash\overline{P'(h,R)})^3 \mapsto \{u \in C^0(\overline{P(h,R)}\backslash P'(h,R))^3 :$$

$$u_{|K_l} \in Pol_1(K_l)^3 \quad for\ \kappa(h,R) + 1 \leqslant l \leqslant \tau(h,R), \quad u_{|\partial P(h,R) \cup \partial P'(h,R)} = 0\}$$

such that it holds

$$|(w - \tilde{\Pi}_h^R(w))_{|K_l}|_{v,2} \cdot (diam\, K_l)^{-1+v} \leqslant \mathscr{D}_6 \cdot |w_{|(K_l)_\Delta \cap P(h,R)\backslash\overline{P'(h,R)}}|_{1,2}$$

for h, R as before, $w \in W_0^{1,2}(P(h,R)\backslash\overline{P'(h,R)})^3$, $l \in \{\kappa(h,R) + 1, ..., \tau(h,R)\}$, $v \in \{0, 1\}$.

By slightly modifying the arguments in [1, p. 108-110] and using Lemma 3.1, one may show

THEOREM 3.4: *There is a constant $\mathscr{D}_7 > 0$ and for any $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, a bounded linear operator*

$$\tilde{\mathcal{I}}_h^R : L^2(Q(h,R)) \mapsto M_h^R$$

such that

$$\|(\pi - \tilde{\mathcal{I}}_h^R(\pi))_{|K_l \cap \Omega_R}\|_2 \leqslant \mathscr{D}_7 \cdot diam\, K_l \cdot |\pi_{|(K_{\rho(l)})_\Delta}|_{1,2}$$

for h, R, l as in (A1), $\pi \in L^2(Q(h,R))$ with $\pi_{|(K_{\rho(l)})_\Delta} \in W^{1,2}((K_{\rho(l)})_\Delta)$, where $\rho(l) := l$ if $l \leqslant \tau(h,R)$, and $\rho(l) := s(l)$ if $l > \tau(h,R)$.

In Theorem 3.1 and 3.4, we did not use the notion of "projection" because the range of the operators $\mathcal{I}_h^R$ and $\tilde{\mathcal{I}}_h^R$ may not be considered a subset of their respective domain.

We note a consequence of Theorem 3.2 and (A5):

COROLLARY 3.1: *There is a constant $\mathscr{D}_8 > 0$ with*

$$R^{-1/2} \cdot \|(w - \Pi_h^R(w))_{|\partial B_R}\|_2 \leqslant \mathscr{D}_8 \cdot |w|_{1,2}$$

for $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, $w \in W^{1,2}(Q(h,R))^3$ *with* $w_{|\partial\Omega} = 0$.

Proof: Let v be the trivial extension of $w - \Pi_h^R(w)$ to $A := Q(h,R) \cup \overline{\Omega}$. Since $(w - \Pi_h^R(w))_{|\partial\Omega} = 0$, it follows $v \in W^{1,2}(A)$. A scaling argument together with a standard trace theorem yields

$$\| v_{|\partial B_R} \|_2^2 \leqslant (R^{-1} \cdot \| v \|_2^2 + R \cdot \| \nabla v \|_2^2)$$
$$= \mathscr{K} \cdot \sum_{l=1}^{k(h,R)} (R^{-1} \cdot \| (w - \Pi_h^R(w))_{|K_l} \|_2^2 + R \cdot \| \nabla (w - \Pi_h^R(w))_{|K_l} \|_2^2),$$

with a constant $\mathscr{K}$ independent of h and R. Now the corollary follows by (3.3), Theorem 3.2 and the second inequality in (A5).

Let us compare our assumptions on the triangulations $\mathscr{T}_h^R$ and spaces W_h^R, M_h^R with the corresponding assumptions in [3]. To this end, we set

$$U_2 := B_{4\ S} \backslash \bar{\Omega}, \quad U_j := B_{2^j\ S} \backslash \overline{B_{2^{j-1}\ S}} \quad \text{for} \quad j \in \mathbb{N},\ j \geqslant 3 .$$

As in [3], we only consider meshes $\mathscr{T}_h^R$ with $R = 2^J \cdot S$, for $J \in \mathbb{N}$, $J \geqslant 3$. We replace (A4), (A5) with the following stronger assumptions:

(A4') There is some constant $\sigma_2 \in (0, 1)$ with

$$\sup \{ r \in (0, \infty) : B_r(x) \subset K_l \text{ for some } x \in K_l \} \geqslant \sigma_2 \cdot 2^{j-2} \cdot h$$

for $h \in (0, S)$, $J \in \mathbb{N}$, $J \geqslant 3$, $j \in \{2, ..., J\}, 1 \leqslant l \leqslant k(h, 2^J \cdot S)$ with $K_l \cap U_j \neq \Phi$, where $K_l = K_l(j, 2^J \cdot S)$.

(A5') diam $K_l \leqslant 2^{j-2} \cdot h$ for h, J, j, l as in (A4').

These conditions mean that any element $K_l = K_l(h, 2^J \cdot S)$ intersecting the annular region U_j has a diameter of order $2^{j-2} \cdot h \cdot S$. In particular, the element mesh size doubles from one annular region to the next in outward direction.

With these stronger assumptions, which will not be needed for the proof of (1.5) in Section 4, the meshes $\mathscr{T}_h^{2^J\ S}$ belong to the type of decompositions considered in [3]. Moreover, the results stated in Theorem 3.1 and 3.4 imply the spaces $W_h^{2^J\ S}, M_h^{2^J\ S}$ fulfil the interpolation properties listed in [3, (6.10)-(6.17)]. Obviously, it holds

$$N := N(h, J) := \dim W_h^{2^J\ S} + \dim W_h^{2^J\ S} \leqslant 20 \cdot k(h, 2^J \cdot S) \tag{3.6}$$

$(h \in (0, S), J \in \mathbb{N}, J > 2)$. Thus, if they satisfies the Babuska-Brezzi condition (1.5), the spaces $W_h^{2^J\ S}, M_h^{2^J\ S}$ exhibit all the features required in [3, Section 6]. But of course, the proof of (1.5) represents the main difficulty of our theory.

Let us draw some conclusions from (3.6). To this end, we remark that the left-hand side N in (3.6) corresponds to the number of unknowns which arise when a mixed finite element problem is solved in the space $W_h^{2^J\ S} \times M_h^{2^J\ S}$. To give an example, consider a situation as in [3], where an exterior Stokes flow (u, π) in $\mathbb{R} \backslash \bar{\Omega}$ is approximated in $\Omega_{2^J\ S}$ by the solution

$$(v(h,J), \rho(h,J)) \in W_h^{2^J\ S} \times M_h^{2^J\ S}$$

of a certain mixed finite element problem. According to [3, Corollary 6.1], applied with $k = 2$, $\sigma > 4$, $\delta = 1$, $h < \min\{1, S\}$, $J = \left[-\frac{2}{3 \cdot \ln 2} \cdot \ln h \right] + 1$, it holds

$$\| \nabla u_{|\Omega_{2^J\ S}} - \nabla v(h,J) \|_2 + \| \pi_{|\Omega_{2^J\ S}} - \rho(h,J) \|_2 \leqslant \mathscr{K} \cdot h . \tag{3.7}$$

Here and in the rest of this section, the letter $\mathcal{K}$ denotes constants which are independent of h and N. As may be seen by [3, Theorem 6.2], the assumption $J = \left[-\frac{2}{3 \cdot \ln 2} \cdot \ln h \right] + 1$ amounts to balancing the truncation and discretization errors mentioned in Section 1.

The order of the system of equations arising in a computation of $v(h, J)$ and $\rho(h, J)$ equals the number N on the left-hand side of (3.6). This number N may be estimated by combining (3.6) with the upper bound for $k(h, 2^J \cdot S)$ given by [3, Lemma 6.1]. It follows

$$N \leqslant \mathcal{K} \cdot h^{-3} \cdot |\ln h| \,. \tag{3.8}$$

Thus we see that in spite of our graded mesh, the number N of unknowns may grow with the radius $R = 2^J \cdot S$ of the truncating sphere ∂B_R. However, this growth is much slower than in the case of uniform mesh size of elements. As a consequence, the mesh grading process which is described by (A4'), (A5') and which goes back to Goldstein [6], [7] leads to a considerable saving of computational effort. To be more precise, we transform (3.7) into an effective error estimate. For any $\epsilon \in (0, 1)$, inequality (3.8) yields $N \leqslant \mathcal{K} \cdot \epsilon^{-1} \cdot h^{-3-\epsilon}$, hence by (3.7):

$$\| \nabla u_{|\Omega_{2^J \cdot S}} - \nabla v(h, J) \|_2 + \| \pi_{|\Omega_{2^J \cdot S}} - \rho(h, J) \|_2 \leqslant \mathcal{K} \cdot \epsilon^{-1} \cdot N^{-1/(3+\epsilon)} \,. \tag{3.9}$$

Now consider a triangulation with a uniform mesh size of elements, that is, replace (A4'), (A5') by the ensuing assumption:

(A4") There exists some number $\sigma_3 \in (0, 1)$ with $h \geqslant \operatorname{diam} K_l \geqslant \sigma_3 \cdot h$, for h, R, l as in (A1).
This conditions implies $k(h, 2^J \cdot S) \leqslant \mathcal{K} \cdot h^{-3} \cdot (2^J \cdot S)^3$, so it follows from (3.6): $N \leqslant \mathcal{K} \cdot h^{-6}$, and we may deduce from (3.7)

$$\| \nabla u_{|\Omega_{2^J \cdot S}} - \nabla v(h, J) \|_2 + \| \pi_{|\Omega_{2^J \cdot S}} - \rho(h, J) \|_2 \leqslant \mathcal{K} \cdot N^{-1/6} \,,$$

which is a result much worse than the estimate in (3.9).

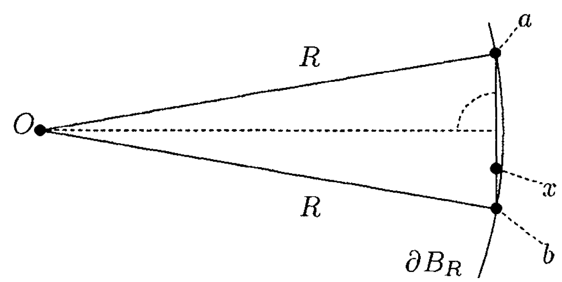

Figure 13. — Geometrical interpretation of inequality (4.1).

4. PROOF OF THE STABILITY CONDITION (1.5)

We begin by some technical lemmas. First we note a consequence of the fact that the shape functions related to the standard Lagrangian $P1$ finite element are linearly independent:

LEMMA 4.1: *There is some number $\mathcal{D}_9 \in (0, \infty)$ such that for $h \in (0, S)$, $R \in [8 \cdot S, \infty)$, $l \in \{1, ..., k(h, R)\}$, $p, q \in Pol_1(K_l)$ with $q(x) = p(x)$ or $q(x) = 0$ for any vertex x of K_l, the following inequality is valid:*

$$\| q \|_2 \leqslant \mathcal{D}_9 \cdot \| p \|_2 \,.$$

The next lemma ındicates ın what sense the set $\Omega_R \backslash P(h, R)$ ıs small

LEMMA 4 2 *Let* $h \in (0, S)$, $R \geqslant 8\ S$ *Then*

$$B_{R\ \sqrt{1 - h^2/(16\ S^2)}} \backslash B_S \subset P(h, R)$$

Proof The polyhedron $P(h, R)$ has two kind of faces "ınner" faces, which are part of $\partial\Omega$, and "outer" faces, which are triangles with vertices on ∂B_R, see (A6) Let F be an outer face of $P(h, R)$ By the definition of $P(h, R)$, there ıs some $l \in \{1, \ldots, k(h, R)\}$ such that F ıs a face of K_l Let a, b be two of the vertices of F Then we have $|a| = |b| = R$, $|a - b| \leqslant \operatorname{diam} K_l$ It follows by a simple geometrical argument (*fig* 13)

$$|x|^2 \geqslant R^2 - |a - b|^2/2 \geqslant R^2 - (\operatorname{diam} K_l)^2/2 \tag{4 1}$$

for any point x on that side of F which has endpoints a and b

Now take a point $y \in F$ We may choose points c, d, each located on a side of F, such that $y = t\ c + (1 - t)\ d$ for some $t \in [0, 1]$ A simple calculation yields

$$|y|^2 \geqslant \min\{|c|^2, |d|^2\} - (1/2)\ |c - d|^2$$

But $|c - d| \leqslant \operatorname{diam} K_l$, so we obtain by (4 1) and (A5)

$$|y|^2 \geqslant R^2 - (\operatorname{diam} K_l)^2 \geqslant R^2 - h^2\ R^2/(16\ S^2) \tag{4 2}$$

Since F was an arbitrarily chosen outer face of $P(h, R)$, inequality (4 2) holds for any $y \in \partial P(h, R)\backslash\partial\Omega$ On the other hand, we have $\bar{\Omega} \subset B_S$, so the lemma ıs proved □

When a fixed bounded domain ıs considered, a standard method for validating the Babuska-Brezzı condition ıs based on the next two results (see [2, pp 220/221])

LEMMA 4 3 *There ıs a constant* $\mathscr{D}_{10} > 0$ *such that ıt holds for* $h \in (0, S)$, $R \in [8\ S, \infty)$, $l \in \{1, \ldots, k(h, R)\}$

$$|b_{K_l}|_{1\ 2} \leqslant \mathscr{D}_{10}\ (\operatorname{diam} K_l)^{-5/2} \int_{K_l} b_{K_l}\, dx\,,$$

where b_{K_l} *ıs the bubble function associated to* K_l *(see Section 3)*

The proof of this lemma ıs easy and may be omitted

LEMMA 4 4 *There exists a constant* $\mathscr{D}_{11} > 0$ *with properties as follows*

Let $h \in (0, S)$, $R \in [8\ S, \infty)$, $w \in W^{1\ 2}(Q(h, R))^3$ *with* $w_{|\partial\Omega} = 0$ *Furthermore, take* $v \in W_0^{1\ 2}(P(h, R)\backslash\overline{P'(h, R)})^3$, $l \in \{1, \ldots, k(h, R)\}$, $m \in \{\kappa(h, R) + 1, \ldots, \tau(h, R)\}$ *and set*

$$c_l = \left(\int_{K_l} b_{K_l}\, dx\right)^{-1} \int_{K_l} (w - \Pi_h^R(w))\, dx\,,$$

$$d_m = \left(\int_{K_m} b_{K_m}\, dx\right)^{-1} \int_{K_m} (v - \tilde{\Pi}_h^R(v))\, dx$$

Then it holds

$$|c_l \cdot b_{K_l}|_{1,2} \leqslant \mathscr{D}_{11} \cdot |w_{|(K_l)_\Delta}|_{1,2},$$

$$|d_m \cdot b_{K_m}|_{1,2} \leqslant \mathscr{D}_{11} \cdot |v_{|(K_m)_\Delta}|_{1,2}.$$

This lemma is an immediate consequence of Lemma 4.3, Theorem 3.2 and 3.3.

Now we are in a position to prove our main result,

THEOREM 4.1: *There are constants $\beta > 0$, $\mathscr{D}_{17} \geqslant 8 \cdot S$ such that it holds for $R \in [\mathscr{D}_{17}, \infty)$, $h \in (0, S)$ with $h \leqslant S/\mathscr{D}_3$:*

$$\inf_{\rho \in M_h^R} \sup_{v \in W_h^R, v \neq 0} \frac{\int_{\Omega_R} \rho \operatorname{div} v \, dx}{|v|_{1,2}^{(R)} \, \|\rho\|_2} \geqslant \beta .$$

We remark that the constant $\mathscr{D}_{17}$ will be defined explicitly via equations (4.19), (4.28) and (4.29) below.

We shall prove (1.5) in the following way: for any pressure function $\pi \in M_h^R$, we shall distinguish three cases. In the first one, it will be assumed that the L^2-norm of π is concentrated on $P'(h, R)$. Then we shall start out with Theorem 2.2 which deals with the divergence equation in truncated exterior domains. The second case arises if the L^2-norm of π is concentrated on $\Omega_R \backslash P'(h, R)$ and the mean value of π is small in a certain sense. Under these assumptions, our arguments will be based on the solution theory for the divergence equation given by Theorem 2.1. Finally, if the L^2-norm of the pressure function π is again concentrated on $\Omega_R \backslash P'(h, R)$, but if the mean value of π is large, we shall consider this function π as a perturbation of its mean value. It is for this last step that we assumed the parameter h is not too large, the radius R is not too small, and the element mesh size does not grow too strongly with increasing distance from Ω. The latter restriction is formalized by the second equation in assumption (A5).

Proof of Theorem 4.1: In the following, the symbol $\mathscr{K}$ will denote constants which may depend on Ω, σ_0, S and Z. We shall use this symbol whenever there is no need to explicitly define the respective constant.

Take $h \in (0, S)$ with $h \leqslant S/\mathscr{D}_3$, $R \in [8 \cdot S, \infty)$, $\rho \in M_h^R$.

Set $\epsilon := \min\{1/2, (4 \cdot \mathscr{D}_{14})^{-2}\}$, where the constant $\mathscr{D}_{14}$ will be defined in (4.6) below. As indicated above, we shall distinguish three cases. First, we assume

$$\|\rho_{|P'(h,R)}\|_2^2 \geqslant (1 - \epsilon) \cdot \|\rho_{|P(h,R)}\|_2^2 . \tag{4.3}$$

Then we define the function $\rho_1 \in M_h^R$ in the following way: For

$$x \in \{y \in \bar{\Omega}_R : y \text{ is a vertex of } K_l \text{ for some } l \in \{1, \ldots, \tau(h, R)\}\},$$

we set

$$\rho_1(x) := \rho(x) \quad \text{if} \quad x \in \overline{P'(h, R)}, \quad \rho_1(x) := 0 \text{ if } x \in \bar{\Omega}_R \backslash \overline{P'(h, R)} .$$

We further put

$$w := v(\Omega, 2 \cdot R, \bar{\rho}_1)_{|Q(h,R)},$$

where $\tilde{\rho}_1$ denotes the zero extension of ρ_1 to $\Omega_{2\,R}$ The function $v(\Omega, 2\ R, \tilde{\rho}_1)$ is to be chosen according to Theorem 2 2, with $\mathscr{A}, R, \pi$ replaced by $\Omega, 2\ R, \tilde{\rho}_1$ For $l \in \{1, \ , \tau(h, R)\}$, set

$$c_l = \left(\int_{K_l} b_{K_l}\, dx\right)^{-1} \int_{K_l} (w - \Pi_h^R(w))\, dx$$

Define $u\ \ \Omega_R \mapsto \mathbb{R}^3$ by $u(x) = 0$ for $x \in \Omega_R \backslash P(h, R)$,

$$u(x) = c_l\ \ b_{K_l}(x) \quad \text{for} \quad x \in K_l \quad \text{with} \quad l \in \{1, \ , \tau(h, R)\}$$

Then we have $u \in W_h^R$ Combining (3 3), Theorem 2 2 and Lemma 4 4, we obtain

$$|u|_{1\,2} \leqslant \mathscr{D}_{11}\ \ Z^{1/2}\ \ |w|_{1\,2} \leqslant \mathscr{D}_{12}\ \ \|\rho_1\|_2 \tag{4 4}$$

with $\mathscr{D}_{12} = \mathscr{D}_{11}\ \ Z^{1/2}\ \ \mathscr{C}_2(\Omega)$ Moreover, observing that

$$\rho_{1|\Omega_R \backslash B_{3\,S}} = 0, \quad \rho_{1|\overline{P(h\ R)}} \in C^0(\overline{P(h, R)}), \quad \rho_{1|P(h\ R)} \in W^{1\ 2}(P(h, R)),$$

$\nabla \rho_{1|K_l}$ constant for $1 \leqslant l \leqslant \tau(h, R)$, $u_{|\partial\Omega} = 0$, $(w - \Pi_h^R(w))_{|\partial\Omega} = 0$,
we conclude as in [2, pp 220/221]

$$\int_{\Omega_R} \rho_1\ \ \operatorname{div} u\, dx = -\int_{P(h\ R)} \nabla\rho_1\ \ u\, dx$$

$$= -\int_{P(h\ R)} \nabla\rho_1\ \ (w - \Pi_h^R(w))\, dx = \int_{\Omega_R} \rho_1\ \ \operatorname{div}(w - \Pi_h^R(w))\, dx,$$

hence

$$\int_{\Omega_R} \rho_1\ \ \operatorname{div}(u + \Pi_h^R(w))\, dx = \|\rho_1\|_2^2 \tag{4 5}$$

On the other hand, it holds by (3 3), (4 4), Theorem 3 2 and 2 2

$$\|\operatorname{div}(u + \Pi_h^R(w))\|_2 \leqslant |u + \Pi_h^R(w)|_{1\,2}$$

$$\leqslant |u|_{1\,2} + |\Pi_h^R(w) - w|_{1\,2} + |w|_{1\,2}$$

$$\leqslant \mathscr{D}_{12}\ \ \|\rho_1\|_2 + (\mathscr{D}_5\ \ Z^{1/2} + 1)\ \ |w|_{1\,2} \leqslant \mathscr{D}_{13}\ \ \|\rho_1\|_2,$$

with

$$\mathscr{D}_{13} = \mathscr{D}_{12} + (\mathscr{D}_5\ \ Z^{1/2} + 1)\ \ \mathscr{C}_2(\Omega)$$

Setting $\rho_2 = \rho - \rho_1$ and

$$\mathscr{D}_{14} = 2^{1/2}\ \ \mathscr{D}_{13}\ \ \mathscr{D}_9\ \ \mathscr{D}_3, \tag{4 6}$$

we now find by recurring to (4 3), (3 5) and Lemma 4 1

$$\left|\int_{\Omega_R} \rho_2 \ \operatorname{div}\,(u + \Pi_h^R(w))\,dx\right| \leqslant \|\rho_2\|_2 \ \ \mathscr{D}_{13} \ \ \|\rho_1\|_2 \tag{4 7}$$

$$\leqslant \mathscr{D}_{13} \ \ \mathscr{D}_9 \ \ \|\rho_{|\Omega_R \setminus P\,(h\ R)}\|_2 \ \ \|\rho_1\|_2$$

$$\leqslant 2^{-1/2} \ \ \mathscr{D}_{14} \ \ \|\rho_{|P(h\ R) \setminus P\,(h\ R)}\|_2 \ \ \|\rho_1\|_2$$

$$\leqslant 2^{-1/2} \ \ \mathscr{D}_{14} \ \epsilon^{1/2} \ \ (1-\epsilon)^{-1/2} \ \ \|\rho_1\|_2^2$$

$$\leqslant \mathscr{D}_{14} \ \epsilon^{1/2} \ \ \|\rho_1\|_2^2 \leqslant (1/4) \ \ \|\rho_1\|_2^2 ,$$

where the last inequality follows by the choice of ϵ Combining (4 5) and (4 7) yields

$$\int_{\Omega_R} \rho \ \ \operatorname{div}\,(u + \Pi_h^R(w))\,dx \geqslant (3/4) \ \ \|\rho_1\|_2^3 \tag{4 8}$$

Finally, observe that

$$|\Pi_h^R(w)_{|\Omega_R}|_{1\ 2}^{(R)} \leqslant |\Pi_h^R(w)_{|\Omega_R} - w|_{1\ 2}^{(R)} + |w|_{1\ 2}^{(R)} \tag{4 9}$$

$$\leqslant \mathscr{K} \ \ |w|_{1\ 2}^{(R)} \leqslant \mathscr{K} \ \ \|\rho_1\| ,$$

where we used Corollary 3 1, Theorem 2 2 and 3 2 We deduce from (4 4), (4 8), (4 9) and (4 3)

$$(|u + \Pi_h^R(w)|_{1\ 2}^{(R)})^{-1} \int_{\Omega_R} \rho \ \ \operatorname{div}\,(u + \Pi_h^R(w))\,dx \geqslant \mathscr{K} \ \ \|\rho_1\|_2$$

$$\geqslant \mathscr{K} \ \ \|\rho_{|P\,(h\ R)}\|_2 \geqslant \mathscr{K} \ \ \|\rho_{|P(h\ R)}\|_2$$

Next consider the case

$$\|\rho_{|P\,(h\ R)}\|_2^2 \leqslant (1-\epsilon) \ \ \|\rho_{|P(h\ R)}\|_2^2 \tag{4 10}$$

For brevity we set

$$\mathscr{P} = P(h,R)\backslash\overline{P'(h,R)}, \quad p = \operatorname{Vol}(\mathscr{P})^{-1} \int_{\mathscr{P}} \rho\,dx$$

Assume $\operatorname{Vol}(\mathscr{P}) \ \ p^2 \leqslant (9/16) \ \ \|\rho_{|\mathscr{P}}\|_2^2$
Then we get

$$\|(\rho - p)_{|\mathscr{P}}\|_2 = (\|\rho_{|\mathscr{P}}\|_2^2 - \operatorname{Vol}(\mathscr{P}) \ \ p^2)^{1/2} \geqslant (7^{1/2}/4) \ \ \|\rho_{|\mathscr{P}}\|_2 \tag{4 11}$$

The function $(\rho - p)_{|\mathscr{P}}$ satisfies (2.3) with $\mathscr{A} = \mathscr{P}$. Thus, recalling (A8), we see Theorem 2.1 may be applied with $T = 2 \cdot S$, $\mathscr{A}_1 = P'(h, R) \cup \overline{\Omega}$, $\mathscr{A}_2 = P(h, R) \cup \overline{\Omega}$. Therefore the corresponding function $v := v(\mathscr{P}, (\rho - p)_{|\mathscr{P}})$ from Theorem 2.1 is well defined. For $m \in \{\kappa(h, R) + 1, \ldots, \tau(h, R)\}$, put

$$d_m := \left(\int_{K_m} b_{K_m}\, dx\right)^{-1} \cdot \int_{K_m} (v - \tilde{\Pi}_h^R(v))\, dx .$$

Define $u : \Omega_R \mapsto \mathbb{R}^3$ by

$$u(x) := d_m \cdot b_{K_m}(x), \text{ if } x \in \bar{K}_m \text{ for some } m \in \{\kappa(h, R) + 1, \ldots, \tau(h, R)\},$$

$u(x) := 0$ for $x \in \Omega_R \backslash \bar{\mathscr{P}}$. Then $u \in W_h^R$ and

$$|u|_{1,2}^{(R)} = |u|_{1,2} \leqslant \mathscr{K} \cdot |v|_{1,2} \leqslant \mathscr{K} \cdot \|(\rho - p)_{|\mathscr{P}}\|_2 , \tag{4.12}$$

where we used (3.3), Lemma 4.4 and Theorem 2.1. Denote by g the zero extension of $\tilde{\Pi}_h^R(v)$ to Ω_R. Then $g \in W_h^R$, and we find by referring to (3.3), Theorem 3.3 and 2.1:

$$\begin{aligned} |g|_{1,2}^{(R)} = |\tilde{\Pi}_h^R(v)|_{1,2} &\leqslant |\tilde{\Pi}_h^R(v) - v|_{1,2} + |v|_{1,2} \\ &\leqslant \mathscr{K} \cdot \|(\rho - p)_{|\mathscr{P}}\|_2 . \end{aligned} \tag{4.13}$$

We further compute

$$\begin{aligned} \int_{\Omega_R} \rho \cdot \operatorname{div}(u + g)\, dx &= \int_{\mathscr{P}} \rho \cdot \operatorname{div}(u + \tilde{\Pi}_h^R(v))\, dx \\ &= \int_{\mathscr{P}} -\nabla\rho \cdot (u + \tilde{\Pi}_h^R(v))\, dx \\ &= \int_{\mathscr{P}} -\nabla\rho \cdot (v - \tilde{\Pi}_h^R(v) + \tilde{\Pi}_h^R(v))\, dx \\ &= \int_{\mathscr{P}} (\rho - p) \cdot \operatorname{div} v\, dx = \|(\rho - p)_{|\mathscr{P}}\|_2^2 . \end{aligned} \tag{4.14}$$

Here we exploited the fact that $u, v, \tilde{\Pi}_h^R(v) \in W_0^{1,2}(\mathscr{P})^3$. Combining (4.12), (4.13) and (4.14) yields

$$(|u + g|_{1,2}^{(R)})^{-1} \cdot \int_{\Omega_R} \rho \cdot \operatorname{div}(u + g)\, dx \geqslant \mathscr{K} \cdot \|(\rho - p)_{|\mathscr{P}}\|_2 .$$

Now inequalities (4.10) and (4.11) imply

$$(|u + g|_{1,2}^{(R)})^{-1} \cdot \int_{\Omega_R} \rho \cdot \operatorname{div}(u + g)\, dx \geqslant \mathscr{K} \cdot \|\rho_{|\mathscr{P}}\|_2 \geqslant \mathscr{K} \cdot \|\rho_{|P(h,R)}\|_2 .$$

This leaves us to consider the case (4.10) under the additional assumption

$$\operatorname{Vol}(\mathscr{P}) \cdot p^2 \geqslant (9/16) \cdot \|\rho_{|\mathscr{P}}\|_2^2 . \tag{4.15}$$

Define $\Phi : \mathbb{R} \mapsto \mathbb{R}$ by $\Phi(r) := 1$ for $r \in [7 \cdot S/2, \infty)$, $\Phi(r) := (r - 5 \cdot S/2)/S$ for $r \in (5 \cdot S/2, 7 \cdot S/2)$, $\Phi(r) := 0$ else. Furthermore, set

$$v_1(x) := \Phi(|x|) \cdot x_1 \cdot p, \quad v_2(x) := v_3(x) := 0 \quad \text{for } x \in Q(h, R) .$$

Observe that

$$\operatorname{div} v(x) = p \quad \text{for} \quad x \in Q(h, R) \backslash B_{7 \cdot S/2}, \quad v_{|B_{2 \cdot S} \backslash \bar{\Omega}} = 0 ,$$

$$|\nabla v(x)| \leqslant 12 \cdot |p| \quad \text{for} \quad x \in Q(h, R) \text{ with } |x| \notin \{5 \cdot S/2, 7 \cdot S/2\} .$$

We may conclude by (A5)

$$\Pi_h^R(v)_{|P'(h, R)} = 0 , \tag{4.16}$$

$$\nabla \Pi_h^R(v)_{|K_l} = (p, 0, 0) \quad \text{for} \quad 1 \leqslant l \leqslant k(h, R) \text{ with } \bar{K}_l \cap (\Omega_R \backslash B_{7 \cdot S}) \neq \Phi , \tag{4.17}$$

$$|\Pi_h^R(v)_{|K_l}|_{1,2} \leqslant (1 + \mathscr{D}_5) \cdot 12 \cdot |p| \cdot \sqrt{\operatorname{Vol}((K_l)_\Delta)} \tag{4.18}$$

for any index $l \in \{1, ..., k(h, R)\}$. The last inequality follows by Theorem 3.2. Setting

$$\mathscr{D}_{15} := (1 + \mathscr{D}_5) \cdot 12 \cdot ((4 \cdot \pi/3) \cdot (9 \cdot S \cdot \mathscr{D}_1 / \sigma_0)^3)^{1/2} , \tag{4.19}$$

we get by (4.18), (A4), (A5) and (3.2):

$$|\Pi_h^R(v)_{|B_{7 \cdot S} \backslash \overline{P'(h, R)}}|_{1,2} \leqslant \mathscr{D}_{15} \cdot |p| , \tag{4.20}$$

$$|\Pi_h^R(v)_{|\Omega_R}|_{1,2}^{(R)} \leqslant \mathscr{K} \cdot R^{3/2} \cdot |p| . \tag{4.21}$$

Furthermore, recurring to (4.16) and (4.17), we obtain

$$\int_{\Omega_R} \rho \cdot \operatorname{div} \Pi_h^R(v)\, dx = A + B + C , \tag{4.22}$$

with

$$A: = \int_{\Omega_R \backslash P(h, R)} \rho \cdot p\, dx, \quad B := \int_{B_{7 \cdot S} \backslash \overline{P'(h, R)}} \rho \cdot (\operatorname{div} \Pi_h^R(v) - p)\, dx ,$$

$$C: = \operatorname{Vol}(\mathscr{P}) \cdot |p|^2 ,$$

where we used the fact that

$$\int_{\mathscr{P}} (\rho - p) \cdot p\, dx = 0 .$$

By Lemma 4.2 and the assumption $R \geqslant 8 \cdot S$, we may conclude

$$B_{R \sqrt{1 - h^2/(16 \cdot S^2)}} \backslash \bar{B}_{R/4} \subset P(h, R) \backslash \overline{B_{2 \cdot S}} \subset \mathscr{P}\,, \tag{4.23}$$

$$\Omega_R \backslash \overline{P(h, R)} \subset B_R \backslash B_{R \sqrt{1 - h^2/(16 \cdot S^2)}}\,. \tag{4.24}$$

The relation in (4.23) yields a lower bound for $\mathrm{Vol}\,(\mathscr{P})$. In fact, applying Bernoulli's inequality, we get

$$\begin{aligned}\mathrm{Vol}\,(\mathscr{P}) &\geqslant (4 \cdot \pi/3) \cdot R^3 \cdot ((1 - h^2/(16 \cdot S^2))^{3/2} - 1/64) \\ &\geqslant (4 \cdot \pi/3) \cdot R^3 \cdot (1 - 3 \cdot h^2/(32 \cdot S^2) - 1/64) \geqslant (19 \cdot \pi/16) \cdot R^3\,.\end{aligned} \tag{4.25}$$

We make use of (4.24) in order to find an upper bound for $\mathrm{Vol}\,(\Omega_R \backslash \overline{P(h, R)})$.
We further find

$$\begin{aligned}&\mathrm{Vol}\,(\Omega_R \backslash \overline{P(h, R)}) \leqslant \mathrm{Vol}\left(B_R \backslash B_{R \sqrt{1 - h^2/(16 \cdot S^2)}}\right) \\ &\leqslant (4 \cdot \pi/3) \cdot R^3 \cdot (1 - (1 - h^2/(16 \cdot S^2))^{3/2}) \leqslant \pi \cdot R^3 \cdot h^2/(8 \cdot S^2)\,.\end{aligned} \tag{4.26}$$

As a consequence, we get

$$\mathrm{Vol}\,(\Omega_R \backslash \overline{P(h, R)}) \cdot \mathrm{Vol}\,(\mathscr{P})^{-1} \leqslant 2 \cdot h^2/(19 \cdot S^2)\,. \tag{4.27}$$

Now we obtain by referring to (4.20) and (4.25):

$$\begin{aligned}|B| &\leqslant (\mathscr{D}_{15} + (4 \cdot \pi/3)^{1/2} \cdot (7 \cdot S)^{3/2}) \cdot |p| \cdot \|\rho_{|\mathscr{P}}\|_2 \\ &\leqslant \mathscr{D}_{16} \cdot R^{-3/2} \cdot \mathrm{Vol}\,(\mathscr{P})^{1/2} \cdot |p| \cdot \|\rho_{|\mathscr{P}}\|_2\,,\end{aligned}$$

with

$$\mathscr{D}_{16} := (\mathscr{D}_{15} + (4 \cdot \pi/3)^{1/2} \cdot (7 \cdot S)^{3/2}) \cdot 4/\sqrt{19 \cdot \pi}\,. \tag{4.28}$$

We further obtain by referring to (4.27):

$$\begin{aligned}|A| &\leqslant \mathrm{Vol}\,(\Omega_R \backslash P(h, R))^{1/2} \cdot |p| \cdot \|\rho_{|\Omega_R \backslash P(h, R)}\|_2 \\ &\leqslant \sqrt{2} \cdot h/(\sqrt{19} \cdot S) \cdot \mathrm{Vol}\,(\mathscr{P})^{1/2} \cdot |p| \cdot \|\rho_{|\Omega_R \backslash P(h, R)}\|_2\,.\end{aligned}$$

It follows by (3.5) and our assumption $h \leqslant S/\mathscr{D}_3$:

$$\begin{aligned}|A| &\leqslant \sqrt{2} \cdot h/(\sqrt{19} \cdot S) \cdot \mathscr{D}_3 \cdot \mathrm{Vol}\,(\mathscr{P})^{1/2} \cdot |p| \cdot \|\rho_{|\mathscr{P}}\|_2 \\ &\leqslant 1/2 \cdot \mathrm{Vol}\,(\mathscr{P})^{1/2} \cdot |p| \cdot \|\rho_{|\mathscr{P}}\|_2\,.\end{aligned}$$

Now returning to (4.22) and using (4.15), we deduce

$$\int_{\Omega_R} \rho \cdot \operatorname{div} \Pi_h^R(v)\, dx$$

$$\geqslant (3/4) \cdot \|\rho_{|\mathcal{P}}\|_2 \cdot \operatorname{Vol}(\mathcal{P})^{1/2} \cdot |p| - |A| - |B|$$

$$\geqslant (3/4 - 1/2 - \mathcal{D}_{16} \cdot R^{-3/2}) \cdot \operatorname{Vol}(\mathcal{P})^{1/2} \cdot |p| \cdot \|\rho_{|\mathcal{P}}\|_2 .$$

Thus, setting

$$\mathcal{D}_{17} := \max \{8 \cdot S, (8 \cdot \mathcal{D}_{16})^{2/3}\}, \tag{4.29}$$

we get for $R \in [\mathcal{D}_{17}, \infty)$:

$$\int_{\Omega_R} \rho \cdot \operatorname{div} \Pi_h^R(v)\, dx \geqslant (1/8) \cdot \operatorname{Vol}(\mathcal{P})^{1/2} \cdot |p| \cdot \|\rho_{|\mathcal{P}}\|_2 . \tag{4.30}$$

Inequalities (4.21), (4.25), (4.30) and (4.10) yield

$$(|\Pi_h^R(v)_{|\Omega_R}|_{1,2}^{(R)})^{-1} \cdot \int_{\Omega_R} \rho \cdot \operatorname{div} \Pi_h^R(v)\, dx \geqslant \mathcal{K} \cdot \|\rho_{|P(h,R)}\|_2 .$$

Thus we have shown for any case there is a function $w \in W_h^R$ with

$$(|w|_{1,2}^{(R)})^{-1} \cdot \int_{\Omega_R} \rho \cdot \operatorname{div} w\, dx \geqslant \mathcal{K} \cdot \|\rho_{|P(h,R)}\|_2 .$$

Our proof is completed by referring to inequality (3.4).

REFERENCES

[1] S. C. BRENNER, L. R. SCOTT, *The mathematical theory of finite element methods,* Springer, New York e.a., 1994.

[2] F. BREZZI, M. FORTIN, *Mixed and hybrid finite element methods,* Springer, New York e.a., 1991.

[3] P. DEURING, *Finite element methods for the Stokes system in exterior domains,* Meth. Meth. Appl. Sci., 20 (1997), pp. 245-269.

[4] G. P. GALDI, *An introduction to the mathematical theory of the Navier-Stokes equations. Volume I. Linearized steady problems,* Springer, New York e.a., 1994.

[5] V. GIRAULT, P.-A. RAVIART, *Finite element methods for Navier-Stokes equations,* Springer, Berlin e.a., 1986.

[6] C. I. GOLDSTEIN, *The finite element method with nonuniform mesh sizes for unbounded domains,* Math. Comp., 36 (1981), pp. 387-404.

[7] C. I. Goldstein, *Multigrid methods for elliptic problems in unbounded domains,* SIAM J. Numer. Anal., 30 (1993), pp. 159-183.