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THE PERTURBED GENERALIZED PROXIMAL POINT ALGORITHM (*)

P. ALEXANDRE ⁽¹⁾, V. H. NGUYEN ⁽²⁾, P. TOSSINGS ⁽³⁾

Abstract — During the last years, different modifications were introduced in the proximal point algorithm developed by R T Rockafellar for searching a zero of a maximal monotone operator on a real Hilbert space. We combine these modifications to get a new version of this algorithm. We take simultaneously into account a variable metric, a perturbation and a kind of relaxation. Our work takes place in the context of the variational convergence theory. © Elsevier, Paris

Key words proximal regularization, perturbation, variable metric, relaxation, variational convergence
 AMS (MOS Mathematics Subject Classification): 65J15, 65K10

Résumé — Ces dernières années, plusieurs modifications ont été introduites dans l'algorithme du point proximal initialement développé par R T Rockafellar pour rechercher les zéros d'un opérateur maximal monotone sur un espace de Hilbert réel. Dans ce papier, nous combinons ces diverses modifications pour obtenir une nouvelle version de l'algorithme. Plus précisément, nous prenons simultanément en compte les notions de métrique variable, perturbation et relaxation. Nous plaçons notre étude dans le contexte de la convergence variationnelle. © Elsevier, Paris

Mots-Clés régularisation proximale, perturbation, métrique variable, relaxation, convergence variationnelle
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1. INTRODUCTION

Let \mathcal{H} be a real Hilbert space and T be a maximal monotone operator on \mathcal{H} . We consider the problem

$$(P) \quad \text{To find } \bar{x} \in \mathcal{H} \text{ such that } 0 \in T\bar{x}$$

The practical importance of this problem is well known, thanks to its applications (nondifferentiable convex optimization, minimax problems, variational inequalities, ...) and many authors work on its resolution.

R. T. Rockafellar [14] developed, in 1976, an algorithm for solving problem (P): *the proximal point algorithm*. Starting from an arbitrary point $x_0 \in \mathcal{H}$, this algorithm generates a sequence $\{x_n\} \subset \mathcal{H}$ by the recursive rule

$$(PR) \quad x_n = J_{\lambda_n}^T x_{n-1} + e_n, \quad \forall n \in \mathbb{N}^*,$$

$\{\lambda_n\}$ denoting a sequence of strictly positive real numbers, $\{e_n\} \subset \mathcal{H}$ a sequence approaching 0, introduced to take into account (in theory) the errors due (in applications) to numerical computation; $J_{\lambda_n}^T$ ($n \in \mathbb{N}^*$) denoting, with the notations of Section 2, the resolvent operator associated with T , with parameter λ_n .

Using the variational convergence theory, B. Lemaire [9] gave, in the eighties, a *perturbed version* of the proximal point algorithm for $T = \partial f$, subdifferential operator of a proper closed convex function f defined on \mathcal{H} . He replaced, in iteration $n \in \mathbb{N}^*$ of the algorithm, the function f by another proper closed convex function f_n , the sequence $\{f_n\}$ having to go to f in an appropriate manner.

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In 1990, P. Tossings [15] extended the notion of variational metric between proper closed convex functions to the context of maximal monotone operators and studied the perturbed version of the proximal point algorithm of R. T. Rockafellar. The operator T was replaced, in iteration $n \in \mathbb{N}^*$ of the algorithm, by another maximal monotone operator T_n , the sequence $\{T_n\}$ having to go to T in an appropriate manner.

In the same time, more and more authors began to modify the metric appearing in the proximal regularization. Two kinds of modifications arose. Some authors replaced the classical metric by a nonlinear fixed metric, based on an entropic method (G. Chen and M. Teboulle [4], J. Eckstein [7], S. Kabbadj [8], ...). Others let the metric change at each iteration (G. Cohen [5] and [6], M. Qian [11] and [12], J. F. Bonnans, J. C. Gilbert, C. Lemaréchal and C. Sagastizabal [3], A. Renaud [13], ...).

In the present paper, we mix these two approaches. We introduce simultaneously a perturbation and a variable metric in the classical proximal point algorithm. Moreover, we add to this combination a kind of relaxation. We place our study in the context of the variational convergence theory⁴.

In Section 2, we recall some definitions and results of the *generalized* variational convergence theory. In Section 3, we study the proximal point algorithm associated with a linear, continuous, self-adjoint positive definite transformation H , with linear continuous inverse H^{-1} . In Section 4, we present the *perturbed variable metric proximal point algorithm* or, more simply, *perturbed generalized proximal point algorithm* and its properties of convergence. Section 5 is devoted to the particularities of the *nonperturbed algorithm*.

The applications of our algorithm to convex optimization and variational inequalities will be studied in following papers.

Notations and conventions

In the following text, \mathcal{H} will always denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. T or $T_n (n \in \mathbb{N}^*)$ will denote a maximal monotone operator on \mathcal{H} and H or $H_n (n \in \mathbb{N}^*)$ a linear, continuous, self-adjoint positive definite transformation on \mathcal{H} , with linear continuous inverse H^{-1} or H_n^{-1} .

2. THE GENERALIZED VARIATIONAL CONVERGENCE THEORY

In this section are stated some extensions of the variational convergence theory introduced in P. Alexandre [1] and summarized in P. Alexandre and P. Tossings [2].

Let us first fix some notations.

★ We denote by $\langle \cdot, \cdot \rangle_H$ the inner product associated with H :

$$\langle x, y \rangle_H \stackrel{\text{def}}{=} \langle x, Hy \rangle, \quad \forall x, y \in \mathcal{H},$$

and by $\| \cdot \|_H$ the associated norm:

$$\|x\|_H \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle_H}, \quad \forall x \in \mathcal{H}.$$

This norm is connected with the initial one by the relations

$$\|x\|_H \leq \sqrt{\|H\|} \|x\|, \quad \forall x \in \mathcal{H}, \quad (2.1)$$

and

$$\|x\| \leq \sqrt{\|H^{-1}\|} \|x\|_H, \quad \forall x \in \mathcal{H}. \quad (2.2)$$

⁽⁴⁾ Let us mention that the notion of relaxation also appears in M. Qian [11] and [12], but *without perturbation*. As a consequence, the approach of this author does not take place in the context of the variational convergence theory – it is rather different from our own approach.

★ We denote by $J_\lambda^{H^{-1}T}$ the *generalized resolvent operator associated with T , with parameter λ* (i.e. the resolvent operator associated with $H^{-1}T$, with parameter λ),

$$J_\lambda^{H^{-1}T} \stackrel{def}{=} (I + \lambda H^{-1}T)^{-1} \tag{2.3}$$

which has sense because, under the assumptiuons on H , T is maximal monotone for the initial inner product on \mathcal{H} if and only if $H^{-1}T$ is maximal monotone for the inner product associated with H .

We denote by $A_\lambda^{H^{-1}T}$ the *generalized Yosida approximate of T , with parameter λ* ,

$$A_\lambda^{H^{-1}T} \stackrel{def}{=} \frac{I - J_\lambda^{H^{-1}T}}{\lambda}. \tag{2.4}$$

Definitions (2.3) and (2.4) imply

$$0 \in T\bar{x} \Leftrightarrow J_\lambda^{H^{-1}T}\bar{x} = \bar{x}, \quad \forall \lambda > 0 \Leftrightarrow A_\lambda^{H^{-1}T}\bar{x} = 0, \quad \forall \lambda > 0 \tag{2.5}$$

and

$$A_\lambda^{H^{-1}T}x \in H^{-1}T(J_\lambda^{H^{-1}T}x), \quad \forall \lambda > 0, \quad \forall x \in \mathcal{H}. \tag{2.6}$$

It is also possible to show that $J_\lambda^{H^{-1}T}(\lambda > 0)$ is a *strong contraction with respect to H* :

$$\|J_\lambda^{H^{-1}T}x_1 - J_\lambda^{H^{-1}T}x_2\|_H^2 + \lambda^2 \|A_\lambda^{H^{-1}T}x_1 - A_\lambda^{H^{-1}T}x_2\|_H^2 \leq \|x_1 - x_2\|_H^2, \quad \forall x_1, x_2 \in \mathcal{H}, \tag{2.7}$$

and, ensuring that $\|I - H\| < 1$, this operator is *Lipschitz continuous*:

$$\|J_\lambda^{H^{-1}T}x - J_\lambda^{H^{-1}T}y\| \leq \frac{\|H\|}{1 - \|I - H\|} \|x - y\|, \quad \forall \lambda > 0, \quad \forall x, y \in \mathcal{H}. \tag{2.8}$$

Let us finally note that the operators $J_\lambda^{H^{-1}T}$ and $J_\lambda^T(\lambda > 0)$ are connected by the following relation:

$$J_\lambda^{H^{-1}T}x = J_\lambda^T[J_\lambda^{H^{-1}T}x + Hx + HJ_\lambda^{H^{-1}T}x], \quad \forall \lambda > 0, \quad \forall x \in \mathcal{H}. \tag{2.9}$$

The previous notions are useful to define a *generalized variational metric* between operators on \mathcal{H} .

DEFINITION 2.1: Let $\lambda > 0$ and $\rho \geq 0$ be given. The *generalized variational metric between T_1 and T_2 , with parameters λ and ρ , is the metric*

$$\delta_{\lambda, \rho}(H_1^{-1}T_1, H_2^{-1}T_2) \stackrel{def}{=} \sup_{\|x\| \leq \rho} \|J_\lambda^{H_1^{-1}T_1}x - J_\lambda^{H_2^{-1}T_2}x\|. \tag{6}$$

It is possible to compare $\delta_{\lambda, \rho}(H_1^{-1}T_1, H_2^{-1}T_2)$ and $\delta_{\lambda, \rho}(T_1, T_2)$ ($\lambda > 0, \rho \geq 0$). We give here below a practical version of this comparison.

PROPOSITION 2.2: Assume that T has at least one zero \bar{x} and

$$(i) \quad 0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*,$$

(⁵) When H is the identity, these operators are nothing else but the classical resolvent operator associated with T , with parameter λ , and the corresponding Yosida approximate

(⁶) When H_1 and H_2 are the identity, this metric is nothing else but the classical variational metric between T_1 and T_2 , with parameters λ and ρ , introduced in P Tossings [15]

- (ii) $H_n \xrightarrow{u} H$,
- (iii) $\|I - H\| < 1$.

Then, for every $\rho \geq 0$, there are a range $N \in \mathbb{N}^*$ and strictly positive real numbers ρ^* , C and ε such that

$$\delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) \leq \frac{1}{\varepsilon} \left[C \|H_n - H\| + \frac{\lambda_n}{\underline{\lambda}} \delta_{\underline{\lambda}, \rho^*}(T_n, T) \right], \quad \forall n \geq N.$$

Let us yet mention, to conclude this section, the following technical result concerning the generalized variational metric.

PROPOSITION 2.3: Assume that $0 < \underline{\lambda} \leq \lambda_n$, $\forall n \in \mathbb{N}^*$, and

$$\lim_{n \rightarrow +\infty} \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) = 0, \quad \forall \rho \geq 0.$$

Then

$$\lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1} T_n} x - A_{\lambda_n}^{H^{-1} T} x\| = 0, \quad \forall x \in \mathcal{H}.$$

In particular,

$$\lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1} T_n} \bar{x}\| = 0, \quad \forall \bar{x} \in \mathcal{H} \text{ such that } 0 \in T\bar{x}.$$

3. THE PROXIMAL POINT ALGORITHM ASSOCIATED WITH H

The classical proximal point algorithm needs two transformations to become the proximal point algorithm associated with H .

First, using the norm associated with H and the related notions, we replace, in iteration $n \in \mathbb{N}^*$ of the classical algorithm, the resolvent operator $J_{\lambda_n}^T$ by the generalized resolvent operator $J_{\lambda_n}^{H^{-1} T}$, H denoting, with the previous conventions, a linear, continuous, symmetric positive definite transformation on \mathcal{H} , with linear continuous inverse H^{-1} .

Then, to introduce a relaxation, we replace the generalized resolvent operator $J_{\lambda_n}^{H^{-1} T}$ by the combination

$$I + \vartheta_n (J_{\lambda_n}^{H^{-1} T} - I),$$

ϑ_n being a strictly positive real number.

So, we are led to consider the algorithm that, starting from an arbitrary point $x_0 \in \mathcal{H}$, generates a sequence $\{x_n\} \subset \mathcal{H}$ defined by the recursive rule (called *basic generalized proximal rule*)

$$(BGPR) \quad x_n = S_{\lambda_n, \vartheta_n}^{H^{-1} T} x_{n-1} + e_n, \quad \forall n \in \mathbb{N}^*.$$

Before giving the fundamental result of convergence for the sequence generated by (BGPR), let us state two properties of the operator

$$S_{\lambda, \vartheta}^{H^{-1} T} \stackrel{\text{def}}{=} I + \vartheta (J_{\lambda}^{H^{-1} T} - I) \quad (\lambda, \vartheta > 0). \quad (3.1)$$

★ On the one hand, the previous definition and the properties of the generalized resolvent operator imply

$$0 \in T\bar{x} \Leftrightarrow S_{\lambda, \vartheta}^{H^{-1} T} \bar{x} = \bar{x}. \quad (3.2)$$

★ On the other hand, the following proposition shows that, when $\vartheta \leq 2$, the operator $S_{\lambda \vartheta}^{H^{-1}T}$ is a contraction for the norm generated by H

PROPOSITION 3.1 Let λ, ϑ be strictly positive real numbers and x_1, x_2 be given in \mathcal{H} . We have

$$\|S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2\|_H^2 + \vartheta(2 - \vartheta) \lambda^2 \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2 \leq \|x_1 - x_2\|_H^2$$

Proof It results from (2.4) and (3.1) that

$$x_1 = [S_{\lambda \vartheta}^{H^{-1}T} + \lambda \vartheta A_{\lambda}^{H^{-1}T}] x_1 \quad \text{and} \quad x_2 = [S_{\lambda \vartheta}^{H^{-1}T} + \lambda \vartheta A_{\lambda}^{H^{-1}T}] x_2$$

Therefore, we may write

$$\begin{aligned} \|x_1 - x_2\|_H^2 &= \|[S_{\lambda \vartheta}^{H^{-1}T} + \lambda \vartheta A_{\lambda}^{H^{-1}T}] x_1 - [S_{\lambda \vartheta}^{H^{-1}T} + \lambda \vartheta A_{\lambda}^{H^{-1}T}] x_2\|_H^2 \\ &= \|[S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2] + \lambda \vartheta [A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2]\|_H^2 \end{aligned}$$

and, thanks to a classical property of the norm,

$$\begin{aligned} \|x_1 - x_2\|_H^2 &= \|S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2\|_H^2 + \lambda^2 \vartheta^2 \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2 \\ &\quad + 2 \lambda \vartheta \langle S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2, A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H \end{aligned}$$

But the definitions of the operators $J_{\lambda}^{H^{-1}T}$, $A_{\lambda}^{H^{-1}T}$ and $S_{\lambda \vartheta}^{H^{-1}T}$ imply also

$$S_{\lambda \vartheta}^{H^{-1}T} x_1 = [J_{\lambda}^{H^{-1}T} + \lambda(1 - \vartheta) A_{\lambda}^{H^{-1}T}] x_1 \quad \text{and} \quad S_{\lambda \vartheta}^{H^{-1}T} x_2 = [J_{\lambda}^{H^{-1}T} + \lambda(1 - \vartheta) A_{\lambda}^{H^{-1}T}] x_2$$

and, therefore,

$$\begin{aligned} &\langle S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2, A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H \\ &= \langle [J_{\lambda}^{H^{-1}T} + \lambda(1 - \vartheta) A_{\lambda}^{H^{-1}T}] x_1 - [J_{\lambda}^{H^{-1}T} + \lambda(1 - \vartheta) A_{\lambda}^{H^{-1}T}] x_2, A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H \\ &= \langle [J_{\lambda}^{H^{-1}T} x_1 - J_{\lambda}^{H^{-1}T} x_2] + \lambda(1 - \vartheta) [A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2], A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H \\ &= \langle J_{\lambda}^{H^{-1}T} x_1 - J_{\lambda}^{H^{-1}T} x_2, A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H + \lambda(1 - \vartheta) \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2 \end{aligned}$$

That leads, by using inclusion (2.6) and the H -monotonicity of $H^{-1}T$, to

$$\langle S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2, A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2 \rangle_H \geq \lambda(1 - \vartheta) \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2$$

Finally, we may ensure that

$$\begin{aligned} \|x_1 - x_2\|_H^2 &\geq \|S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2\|_H^2 + \lambda^2 \vartheta^2 \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2 \\ &\quad + 2 \lambda^2 \vartheta(1 - \vartheta) \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2 \\ &\geq \|S_{\lambda \vartheta}^{H^{-1}T} x_1 - S_{\lambda \vartheta}^{H^{-1}T} x_2\|_H^2 + \lambda^2 \vartheta[\vartheta + 2(1 - \vartheta)] \|A_{\lambda}^{H^{-1}T} x_1 - A_{\lambda}^{H^{-1}T} x_2\|_H^2, \end{aligned}$$

what establishes the announced result

THEOREM 3.2: Assume that problem (P) has at least one solution \bar{x} and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$,
- (ii) $0 < \underline{\vartheta} \leq \vartheta_n \leq 2, \quad \forall n \in \mathbb{N}^*$,
- (iii) $\sum_{n=1}^{+\infty} \|e_n\| < +\infty$.

Then, the sequence $\{x_n\}$ generated by the rule (BGPR) weakly converges to a solution of (P) and is such that

$$\lim_{n \rightarrow +\infty} \|x_n - x_{n-1}\| = 0.$$

Proof: We establish this result in four steps.

1° The sequence $\{x_n\}$ is bounded.

The properties of the operator $S_{\lambda_n, \vartheta_n}^{H^{-1}T}$ ($n \in \mathbb{N}^*$) imply, by using relation (2.1),

$$\begin{aligned} \|x_n - \bar{x}\|_H &= \|S_{\lambda_n, \vartheta_n}^{H^{-1}T} x_{n-1} + e_n - \bar{x}\|_H \\ &\leq \|S_{\lambda_n, \vartheta_n}^{H^{-1}T} x_{n-1} - S_{\lambda_n, \vartheta_n}^{H^{-1}T} \bar{x}\|_H + \|e_n\|_H \\ &\leq \|x_{n-1} - \bar{x}\|_H + \sqrt{\|H\|} \|e_n\|. \end{aligned}$$

Therefore, from B. Martinet [10] (chapter V), the sequence $\{\|x_n - \bar{x}\|_H\}$ converges⁷. It is thus bounded and, from relation (2.2), so is the sequence $\{\|x_n - \bar{x}\|\}$.

2° The sequence $\{x_n\}$ is such that $\lim_{n \rightarrow +\infty} \|x_n - x_{n-1}\| = 0$.

The operator $J_{\lambda_n}^{H^{-1}T}$ ($n \in \mathbb{N}^*$) being a strong contraction for the norm generated by H , we have

$$\|J_{\lambda_n}^{H^{-1}T} x_{n-1} - J_{\lambda_n}^{H^{-1}T} \bar{x}\|_H^2 + \|(J_{\lambda_n}^{H^{-1}T} - I) x_{n-1} - (J_{\lambda_n}^{H^{-1}T} - I) \bar{x}\|_H^2 \leq \|x_{n-1} - \bar{x}\|_H^2.$$

This implies, from the definition of $\{x_n\}$ and the properties of the generalized resolvent operator,

$$\left\| \frac{x_n - x_{n-1}}{\vartheta_n} - \frac{e_n}{\vartheta_n} + x_{n-1} - \bar{x} \right\|_H^2 + \left\| \frac{x_n - x_{n-1}}{\vartheta_n} - \frac{e_n}{\vartheta_n} \right\|_H^2 \leq \|x_{n-1} - \bar{x}\|_H^2,$$

and, from the properties of the norm,

$$\left(\frac{\|x_n - \bar{x}\|_H - \|x_{n-1} - \bar{x}\|_H}{\vartheta_n} - \frac{\|e_n\|_H}{\vartheta_n} - \|x_{n-1} - \bar{x}\|_H \right)^2 + \left(\frac{\|x_n - x_{n-1}\|_H}{\vartheta_n} - \frac{\|e_n\|_H}{\vartheta_n} \right)^2 \leq \|x_{n-1} - \bar{x}\|_H^2,$$

for every $n \in \mathbb{N}^*$.

Let n go to infinity.

On the one hand, hypothesis (iii) implies

$$\lim_{n \rightarrow +\infty} \|e_n\| = 0,$$

(⁷) In the following text, we set, for every weak cluster point \bar{x} of $\{x_n\}$,

$$L(\bar{x}) = \lim_{n \rightarrow +\infty} \|x_n - \bar{x}\|_H$$

thus, by using relation (2 1) and hypothesis (u),

$$\lim_{n \rightarrow +\infty} \frac{\|e_n\|_H}{\vartheta_n} = 0$$

On the other hand, the first part of the proof implies

$$\lim_{n \rightarrow +\infty} [\|x_n - \bar{x}\|_H - \|x_{n-1} - \bar{x}\|_H] = 0$$

or, by using once more hypothesis (u),

$$\lim_{n \rightarrow +\infty} \frac{\|x_n - \bar{x}\|_H - \|x_{n-1} - \bar{x}\|_H}{\vartheta_n} = 0$$

We deduce thence

$$\lim_{n \rightarrow +\infty} \frac{\|x_n - x_{n-1}\|_H}{\vartheta_n} = 0$$

and, from hypothesis (u) and relation (2 2), the announced result

3° Every weak cluster point x^* of $\{x_n\}$ (and, from 1°, there is at least one) is a solution of (P)
 Let x^* be a weak cluster point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$x_{n_k} \xrightarrow{w} x^*$$

We have (see (2 6))

$$H\left(\frac{x_{n_k-1} - J_{\lambda_{n_k}}^{H^{-1}T} x_{n_k-1}}{\lambda_{n_k}}\right) \in T(J_{\lambda_{n_k}}^{H^{-1}T} x_{n_k-1}), \quad \forall k \in \mathbb{N}^*$$

Let then $(x, w) \in \mathcal{H} \times \mathcal{H}$ be such that $w \in Tx$

The monotonicity of T implies

$$\left\langle x - J_{\lambda_{n_k}}^{H^{-1}T} x_{n_k-1}, w - H\left(\frac{x_{n_k-1} - J_{\lambda_{n_k}}^{H^{-1}T} x_{n_k-1}}{\lambda_{n_k}}\right) \right\rangle \geq 0, \quad \forall k \in \mathbb{N}^*,$$

or, by using the definition of $\{x_n\}$,

$$\left\langle x - \frac{x_{n_k} - x_{n_k-1}}{\vartheta_{n_k}} + \frac{e_{n_k}}{\vartheta_{n_k}} - x_{n_k-1}, w - H\left(\frac{e_{n_k}}{\vartheta_{n_k} \lambda_{n_k}} - \frac{x_{n_k} - x_{n_k-1}}{\vartheta_{n_k} \lambda_{n_k}}\right) \right\rangle \geq 0, \quad \forall k \in \mathbb{N}^*$$

Let k go to infinity

Since we know that

$$\begin{aligned} \lim_{k \rightarrow +\infty} \|e_{n_k}\| &= 0, \\ \lim_{k \rightarrow +\infty} \|x_{n_k} - x_{n_k-1}\| &= 0, \\ \vartheta_{n_k} &\geq \underline{\vartheta} > 0, \quad \forall k \in \mathbb{N}^*, \\ \lambda_{n_k} &\geq \underline{\lambda} > 0, \quad \forall k \in \mathbb{N}^*, \end{aligned}$$

we get, thanks to the linearity and the continuity of H ,

$$\langle x - x^*, w \rangle \geq 0.$$

This result being true for every $(x, w) \in \mathcal{H} \times \mathcal{H}$ such that $w \in Tx$, it derives from the maximality of T that $0 \in Tx^*$.

4° *The sequence $\{x_n\}$ weakly converges to \bar{x} .*

We know that the sequence $\{x_n\}$ is bounded, every weak cluster point x^* of $\{x_n\}$ being a solution of (P). Therefore, it suffices, to conclude, to show that $\{x_n\}$ possesses only one weak cluster point.

Assume that $\{x_n\}$ admits two weak cluster points x_1^* and x_2^* and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly convergent to x_2^* .

We have successively, with the notations introduced in the first part of the proof,

$$\begin{aligned} L^2(x_1^*) - L^2(x_2^*) &= \lim_{k \rightarrow +\infty} [\|x_{n_k} - x_1^*\|_H^2 - \|x_{n_k} - x_2^*\|_H^2] \\ &= \lim_{k \rightarrow +\infty} [\|x_1^* - x_2^*\|_H^2 + 2\langle x_{n_k} - x_2^*, x_2^* - x_1^* \rangle_H] \\ &= \lim_{k \rightarrow +\infty} \langle 2x_{n_k} - x_1^* - x_2^*, x_2^* - x_1^* \rangle_H \\ &= \|x_1^* - x_2^*\|_H^2. \end{aligned}$$

By another way, if $x_{n_k} \xrightarrow{w} x_1^*$, we obtain, with the same developments,

$$L^2(x_1^*) - L^2(x_2^*) = -\|x_1^* - x_2^*\|_H^2.$$

It follows that

$$\|x_1^* - x_2^*\|_H = 0$$

and $x_1^* = x_2^*$. ■

Remark 3.3: The proof here above brings out that, under hypothesis (i) to (iii) of theorem (3.2), problem (P) admits at least one solution if and only if the sequence $\{x_n\}$ generated by the rule (BGPR) is bounded.

4. THE PERTURBED GENERALIZED PROXIMAL POINT ALGORITHM

The *variable metric* or *generalized proximal point algorithm* is obtained from the proximal point algorithm associated with H by replacing, in iteration $n \in \mathbb{N}^*$, the transformation H by another transformation H_n depending on n . Starting from an arbitrary point $z_0 \in \mathcal{H}$, this algorithm generates a sequence $\{z_n\} \subset \mathcal{H}$ by the recursive rule (called *variable metric* or *generalized proximal rule*)

$$(GPR) \quad z_n = S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} z_{n-1} + e_n, \quad \forall n \in \mathbb{N}^*,$$

the sequence $\{H_n\}$ having to go to H in an appropriate manner.

To *perturbe* this algorithm, it suffices to replace, in (GPR), the operator T by another operator T_n . That leads to the *perturbed variable metric* or *perturbed generalized proximal rule*

$$(PGPR) \quad z_n = S_{\lambda_n, \vartheta_n}^{H_n^{-1}T_n} z_{n-1} + e_n, \quad \forall n \in \mathbb{N}^*,$$

the sequence $\{T_n\}$ having to go to T in an appropriate manner.

THEOREM 4.1: Assume that (P) has at least one solution \bar{z} and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$,
- (ii) $0 < \underline{\vartheta} \leq \vartheta_n \leq 2, \quad \forall n \in \mathbb{N}^*$,
- (iii) $\sum_{n=1}^{+\infty} \|e_n\| < +\infty,$
- (iv) $\sum_{n=1}^{+\infty} \lambda_n \delta_{\underline{\lambda}, \rho}(T_n, T) < +\infty, \quad \forall \rho \geq 0,$
- (v) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (vi) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty,$
- (vii) $\|I - H\| < 1.$

Then, the sequence $\{z_n\}$ generated by the rule (PGPR) weakly converges to a solution of (P) and is such that

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0.$$

Proof:

1° The sequence $\{z_n\}$ is bounded.

The definition of $z_n (n \in \mathbb{N}^*)$ and the equivalence (3.2) give

$$\|z_n - \bar{z}\| = \|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} z_{n-1} + e_n - S_{\lambda_n, \vartheta_n}^{H^{-1} T} \bar{z}\|. \tag{4.1}$$

Then, $S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n}$ being a contraction for the norm generated by H_n , relations (2.1) and (2.2), definition (2.1) and hypothesis (ii) lead to

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} z_{n-1} - S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} \bar{z}\| + \|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} \bar{z} - S_{\lambda_n, \vartheta_n}^{H^{-1} T} \bar{z}\| + \|e_n\| \\ &\leq \sqrt{\|H_n^{-1}\| \|H_n\|} \|z_{n-1} - \bar{z}\| + 2 \delta_{\lambda_n, \|\bar{z}\|}(H_n^{-1} T_n, H^{-1} T) + \|e_n\|. \end{aligned}$$

By repeating this operation for $\|z_{n-1} - \bar{z}\|$, we get

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \sqrt{\|H_n^{-1}\| \|H_n\|} \sqrt{\|H_{n-1}^{-1}\| \|H_{n-1}\|} \|z_{n-2} - \bar{z}\| \\ &\quad + 2 \sqrt{\|H_n^{-1}\| \|H_n\|} \delta_{\lambda_{n-1}, \|\bar{z}\|}(H_{n-1}^{-1} T_{n-1}, H^{-1} T) \\ &\quad + 2 \delta_{\lambda_n, \|\bar{z}\|}(H_n^{-1} T_n, H^{-1} T) \\ &\quad + \sqrt{\|H_n^{-1}\| \|H_n\|} \|e_{n-1}\| + \|e_n\| \end{aligned}$$

or, $\sqrt{\|H_n^{-1}\| \|H_n\|}$ being greater than 1,

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \sqrt{\|H_n^{-1}\| \|H_n\|} \sqrt{\|H_{n-1}^{-1}\| \|H_{n-1}\|} \|z_{n-2} - \bar{z}\| \\ &\quad + 2 \sqrt{\|H_n^{-1}\| \|H_n\|} [\delta_{\lambda_{n-1}, \|\bar{z}\|}(H_{n-1}^{-1} T_{n-1}, H^{-1} T) + \delta_{\lambda_n, \|\bar{z}\|}(H_n^{-1} T_n, H^{-1} T)] \\ &\quad + \sqrt{\|H_n^{-1}\| \|H_n\|} [\|e_{n-1}\| + \|e_n\|]. \end{aligned}$$

Step by step, we deduce thence

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \prod_{i=1}^n \sqrt{\|H_i^{-1}\| \|H_i\|} \|z_0 - \bar{z}\| \\ &\quad + \prod_{i=2}^n \sqrt{\|H_i^{-1}\| \|H_i\|} \left[\sum_{j=1}^n [2\delta_{\lambda_j, \|\bar{z}\|} (H_j^{-1} T_j, H^{-1} T) + \|e_j\|] \right]. \end{aligned}$$

Hypothesis (iii), (iv), (v) and (vi) and proposition (2.2) allow us to conclude.

In the second part of the proof, we will set

$$\rho = \sup_{n \in \mathbb{N}} \|z_n\|.$$

2° The sequence $\{z_n\}$ weakly converges to a solution of (P).

We have, by definition,

$$\begin{aligned} z_n &= (1 - \vartheta_n) z_{n-1} + \vartheta_n J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} + e_n \\ &= (1 - \vartheta_n) z_{n-1} + \vartheta_n J_{\lambda_n}^{H^{-1} T} z_{n-1} + \vartheta_n (J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H^{-1} T} z_{n-1}) + e_n, \end{aligned}$$

for every $n \in \mathbb{N}^*$.

Therefore, we can write

$$z_n = S_{\lambda_n, \vartheta_n}^{H^{-1} T} z_{n-1} + \tilde{e}_n$$

where

$$\tilde{e}_n = \vartheta_n (J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H^{-1} T} z_{n-1}) + e_n.$$

The sequence $\{z_n\}$ may thus be considered as generated by the rule (BGPR), what leads to the conclusion, ensuring that the sequence $\{\tilde{e}_n\}$ satisfies $\sum_{n=1}^{+\infty} \|\tilde{e}_n\| < +\infty$.

Let us establish this last assertion.

The first part of the proof, definition (2.1) and proposition (2.2) imply successively, ε and C denoting strictly positive real numbers,

$$\begin{aligned} \|\tilde{e}_n\| &\leq \vartheta_n \|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H^{-1} T} z_{n-1}\| + \|e_n\| \\ &\leq \vartheta_n \delta_{\lambda_n, \rho} (H_n^{-1} T_n, H^{-1} T) + \|e_n\| \\ &\leq \frac{\vartheta_n}{\varepsilon} \left[C \|H_n - H\| + \frac{\lambda_n}{\lambda} \delta_{\lambda, \rho} (T_n, T) \right] + \|e_n\|, \end{aligned}$$

for every $n \in \mathbb{N}^*$.

Then, assumptions (ii), (iii), (iv) and (v) allow us to conclude. ■

Remark 4.2: The proof here above and remark (3.3) bring out that, under the hypothesis (i) to (vii) of theorem (4.1), problem (P) has at least one solution if and only if the sequence $\{z_n\}$ generated by the rule (PGPR) is bounded.

Remark 4.3: It derives from the properties of infinite products that hypothesis (vi) of theorem (4.1) is satisfied if and only if

$$\prod_{n=1}^{+\infty} [\|H_n^{-1}\| \|H_n\|] < +\infty .$$

LEMMA 4.4: Under the hypothesis of theorem (4.1), the sequence $\{z_n\}$ generated by the rule (PGPR) satisfies

$$\lim_{n \rightarrow +\infty} \|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - z_{n-1}\| = 0 .$$

Proof: The definition of $\{z_n\}$ implies

$$\begin{aligned} \|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - z_{n-1}\| &= \frac{\|z_n - z_{n-1} - e_n\|}{\vartheta_n} \\ &\leq \frac{\|z_n - z_{n-1}\| + \|e_n\|}{\vartheta} , \quad \forall n \in \mathbb{N}^* , \end{aligned}$$

and, under the hypothesis of theorem (4.1),

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|e_n\| = 0 .$$

Remark 4.5: The assumption $0 < \underline{\lambda} \leq \lambda_n, \forall n \in \mathbb{N}^*$, allows us to write the result of lemma (4.4) under the equivalent form ■

$$\lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1}\| = 0 .$$

THEOREM 4.6: Assume that the operators $T_n^{-1} (n \in \mathbb{N}^*)$ are uniformly locally Lipschitz continuous at 0, i.e. there are two strictly positive constants a, τ such that

$$\|w_1\|, \|w_2\| < \tau \Rightarrow \|z_1 - z_2\| \leq a \|w_1 - w_2\|, \quad \forall z_1 \in T_n^{-1} w_1, \quad \forall z_2 \in T_n^{-1} w_2,$$

for every $n \in \mathbb{N}^*$, and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \forall n \in \mathbb{N}^*$, with $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$,
- (ii) $1 \leq \vartheta_n \leq \bar{\vartheta} < 2, \forall n \in \mathbb{N}^*$,
- (iii) the sequence $\{z_n\}$ generated by the rule (PGPR) is bounded,
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \forall n \in \mathbb{N}^*$, with $\sum_{n=1}^{+\infty} \delta_n < +\infty$,
- (v) $\sum_{n=1}^{+\infty} \lambda_n \delta_{\underline{\lambda}, \rho}(T_n, T) < +\infty, \forall \rho \geq 0$,
- (vi) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty$,
- (vii) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty$,
- (viii) $\|I - H\| < 1$.

Then problem (P) admits a unique solution \bar{z} and the sequence $\{z_n\}$ strongly converges to this solution. Moreover, there are real numbers $\eta \in]0, 1[$, $C > 0$ and a range $N \in \mathbb{N}^*$ such that

$$\|z_n - \bar{z}\| \leq \eta \|z_{n-1} - \bar{z}\| + C \delta_{\lambda_n, \|\bar{z}\|}(H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq N . \tag{4.2}$$

Proof: The existence of a unique solution to problem (P) is an immediate consequence of theorem (4.1) and P. Tossings [15] (proof of theorem (II.3.4)).

Let us denote this solution by \bar{z} and try to establish (4.2).

Proposition (2.3) and remark (4.5) imply

$$\lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1}\| = 0.$$

Since $H_n \xrightarrow{u} H$, from hypothesis (vi), we deduce thence

$$\lim_{n \rightarrow +\infty} \|H_n A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|H_n A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1}\| = 0.$$

Therefore, there is a range $N_1 \in \mathbb{N}^*$ from which

$$\|H_n A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| < \tau \quad \text{and} \quad \|H_n A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1}\| < \tau.$$

Now, formula (2.6) gives

$$J_{\lambda_n}^{H_n^{-1} T_n} \bar{z} \in T_n^{-1}(H_n A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}) \quad \text{and} \quad J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} \in T_n^{-1}(H_n A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1}),$$

for every $n \in \mathbb{N}^*$.

Therefore, the Lipschitz condition imposed on the operators $T_n^{-1} (n \in \mathbb{N}^*)$ leads to

$$\|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| \leq a \|H_n A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - H_n A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|,$$

for every $n \geq N_1$ or, from the linearity and the continuity of the operators $H_n (n \in \mathbb{N}^*)$,

$$\|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| \leq a \|H_n\| \|A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|, \quad \forall n \geq N_1.$$

Since $a \neq 0$, by hypothesis, and $\|H_n\| \neq 0, \forall n \in \mathbb{N}^*$, from the inversibility of H_n , we deduce thence

$$\frac{1}{a \|H_n\|} \|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\| \leq \|A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|, \quad \forall n \geq N_1. \quad (4.3)$$

By another way, the operator $J_{\lambda_n}^{H_n^{-1} T_n} (n \in \mathbb{N}^*)$ being a strong contraction for the norm generated by H_n , relations (2.1) and (2.2) imply

$$\|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|^2 + \lambda_n^2 \|A_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - A_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|^2 \leq \|H_n\| \|H_n^{-1}\| \|z_{n-1} - \bar{z}\|^2, \quad (4.4)$$

for every $n \in \mathbb{N}^*$.

Relations (4.3) and (4.4) allow us to write

$$\left(1 + \frac{\lambda_n^2}{a^2 \|H_n\|^2}\right) \|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|^2 \leq \|H_n\| \|H_n^{-1}\| \|z_{n-1} - \bar{z}\|^2, \quad \forall n \geq N_1,$$

or, equivalently,

$$\|J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1} T_n} \bar{z}\|^2 \leq \|H_n\| \|H_n^{-1}\| \frac{a^2 \|H_n\|^2}{a^2 \|H_n\|^2 + \lambda_n^2} \|z_{n-1} - \bar{z}\|^2, \quad \forall n \geq N_1.$$

It follows that

$$\|J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1}T_n} \bar{z}\| \leq \mu_n \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N_1,$$

where

$$\mu_n = \sqrt{\frac{a \|H_n\| \|H_n^{-1}\|}{\sqrt{a^2 \|H_n\|^2 + \lambda_n^2}}}, \quad \forall n \geq N_1.$$

Before trying to estimate $\|z_n - \bar{z}\|$ ($n \in \mathbb{N}^*$), let us just note that we have, from the definition of $S_{\lambda_n, \vartheta_n}^{H_n^{-1}T_n} z_{n-1}$ ($n \in \mathbb{N}^*$),

$$\begin{aligned} z_n - J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} &= [z_n - S_{\lambda_n, \vartheta_n}^{H_n^{-1}T_n} z_{n-1}] + z_{n-1} + \vartheta_n [J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - z_{n-1}] - J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} \\ &= e_n + (\vartheta_n - 1) (J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - z_{n-1}) \\ &= \frac{1}{\vartheta_n} e_n + \frac{\vartheta_n - 1}{\vartheta_n} (z_n - z_{n-1}), \end{aligned}$$

for every $n \in \mathbb{N}^*$.

Therefore, we get, by using hypothesis (iv),

$$\|z_n - J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1}\| \leq \frac{\delta_n + \vartheta_n - 1}{\vartheta_n} \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*.$$

We are now able to study the sequence $\{\|z_n - \bar{z}\|\}$. The point \bar{z} being a solution of problem (P) (what implies $J_{\lambda_n}^{H_n^{-1}T_n} \bar{z} = \bar{z}$, $\forall n \in \mathbb{N}^*$), the properties of the norm and the results established here above lead to the overestimations

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \|z_n - J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1}\| + \|J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - J_{\lambda_n}^{H_n^{-1}T_n} \bar{z}\| + \|J_{\lambda_n}^{H_n^{-1}T_n} \bar{z} - \bar{z}\| \\ &\leq \frac{\delta_n + \vartheta_n - 1}{\vartheta_n} \|z_n - z_{n-1}\| + \mu_n \|z_{n-1} - \bar{z}\| + \delta_{\lambda_n, \|\bar{z}\|} (H_n^{-1} T_n, H^{-1} T), \end{aligned}$$

for every $n \geq N_1$.

It follows that

$$\frac{1 - \delta_n}{\vartheta_n} \|z_n - \bar{z}\| \leq \left(\frac{\delta_n + \vartheta_n - 1}{\vartheta_n} + \mu_n \right) \|z_{n-1} - \bar{z}\| + \delta_{\lambda_n, \|\bar{z}\|} (H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq N_1. \quad (4.5)$$

Now, the conditions imposed on the sequences $\{\vartheta_n\}$ and $\{\delta_n\}$ ensure the existence of a range $N_2 \in \mathbb{N}^*$ such that

$$\frac{1 - \delta_n}{\vartheta_n} > 0, \quad \forall n \geq N_2. \quad (4.6)$$

Therefore, relation (4.5) leads to

$$\|z_n - \bar{z}\| \leq \eta_n \|z_{n-1} - \bar{z}\| + C_n \delta_{\lambda_n, \|\bar{z}\|} (H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq \max\{N_1, N_2\}, \quad (4.7)$$

where

$$\eta_n = \frac{\delta_n + \vartheta_n - 1 + \vartheta_n \mu_n}{1 - \delta_n} \quad \text{and} \quad C_n = \frac{\vartheta_n}{1 - \delta_n}.$$

η_n being strictly positive, for every $n \geq \max\{N_1, N_2\}$.

Moreover, hypothesis (ii), (i) and (iv) ensure successively the existence of real numbers ν and ε such that

$$0 < \nu < 2 - \bar{\vartheta} \quad \text{and} \quad 0 < \varepsilon < \frac{2 - \nu - \bar{\vartheta}}{2 - \nu}$$

and of a range $N_3 \in \mathbb{N}^*$ from which

$$0 < \mu_n < \frac{(1 - \varepsilon)(2 - \nu)}{\bar{\vartheta}} - 1 \quad \text{and} \quad 0 \leq \delta_n < \varepsilon.$$

We deduce thence, for $n \geq N_3$,

$$\vartheta_n(1 + \mu_n) \leq \bar{\vartheta}(1 + \mu_n) < (1 - \varepsilon)(2 - \nu)$$

and

$$1 - \varepsilon < 1 - \delta_n. \quad (4.8)$$

It follows that

$$\vartheta_n(1 + \mu_n) < (1 - \delta_n)(2 - \nu)$$

or

$$\vartheta_n(1 + \mu_n) - (1 - \delta_n) < (1 - \delta_n)(1 - \nu)$$

and, for $n \geq N = \max\{N_1, N_2, N_3\}$,

$$\eta_n < 1 - \nu. \quad (4.9)$$

Concerning C_n , relations (4.6) and (4.8) and hypothesis (ii) imply

$$0 < C_n < \frac{\bar{\vartheta}}{1 - \varepsilon}, \quad \forall n \geq N. \quad (4.10)$$

Finally, relations (4.7), (4.9) and (4.10) lead to

$$\|z_n - \bar{z}\| \leq \eta \|z_{n-1} - \bar{z}\| + C \delta_{\lambda_n, \|\bar{z}\|} (H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq N, \quad (4.11)$$

with

$$\eta = \sup_{n \geq N} \eta_n \quad (0 < \eta < 1) \quad \text{and} \quad C = \frac{\bar{\vartheta}}{1 - \varepsilon}, \quad (4.12)$$

that is to say to relation (4.2).

It remains to prove that $\{z_n\}$ strongly converges to \bar{z} .

In fact, since hypothesis (i), (v) and (vi) and proposition (2.2) imply

$$\sum_{n=1}^{+\infty} \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) < +\infty, \quad \forall \rho \geq 0,$$

relations (4.11) and (4.12) ensure the convergence of the sequence $\{\|z_n - \bar{z}\|\}$ (see B. Martinet [10], chapter V) with

$$\lim_{n \rightarrow +\infty} \|z_n - \bar{z}\| = 0.$$

Remark 4.7: The Lipschitz condition in theorem (4.6) holds when the operators T_n^{-1} ($n \in \mathbb{N}^*$) are *uniformly globally Lipschitz continuous*, i.e. T_n^{-1} ($n \in \mathbb{N}^*$) is a one to one mapping and there is a constant $a \geq 0$ such that

$$\|T_n^{-1} w_1 - T_n^{-1} w_2\| \leq a \|w_1 - w_2\|, \quad \forall w_1, w_2 \in R(T_n),$$

for every $n \in \mathbb{N}^*$.

This last condition is itself satisfied when the operators T_n ($n \in \mathbb{N}^*$) are *uniformly strongly monotone*, i.e. there is a constant $\alpha > 0$ ($a = \alpha^{-1}$) such that

$$\langle w_1 - w_2, z_1 - z_2 \rangle \geq \alpha \|z_1 - z_2\|^2, \quad \forall z_1, z_2 \in D(T_n), \quad \forall w_1 \in T_n z_1, \quad \forall w_2 \in T_n z_2,$$

for every $n \in \mathbb{N}^*$.

Remark 4.8: Let us replace, in theorem (4.6), hypothesis (i) and (ii) by

(i)' $1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$

(ii)' $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*.$

Then, the overestimation (4.2) remains true with η_n in place of η and C_n in place of C , the sequences $\{\eta_n\}$ and $\{C_n\}$ satisfying

$$\lim_{n \rightarrow +\infty} \eta_n = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} C_n = 1.$$

Effectively, these new hypothesis allow us to write relation (4.7) with

$$\begin{aligned} \eta_n &= \frac{\delta_n + \frac{1 + \lambda_n}{\lambda_n} - 1 + \frac{1 + \lambda_n}{\lambda_n} \mu_n}{1 - \delta_n} \\ &= \frac{\lambda_n \delta_n + (1 + \lambda_n) - \lambda_n + (1 + \lambda_n) \mu_n}{\lambda_n(1 - \delta_n)} \\ &= \frac{\delta_n + \mu_n}{1 - \delta_n} + \frac{1 + \mu_n}{\lambda_n(1 - \delta_n)} \end{aligned}$$

and

$$C_n = \frac{1 + \lambda_n}{\lambda_n(1 - \delta_n)} = \frac{1 + \frac{1}{\lambda_n}}{1 - \delta_n},$$

where, following the proof of theorem (4.6),

$$\lim_{n \rightarrow +\infty} \delta_n = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mu_n = 0.$$

THEOREM 4.9: Assume that

$$\exists \bar{z} \in \mathcal{H} \text{ such that } 0 \in \text{int } T\bar{z}, \quad (4.13)$$

and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$,
- (ii) $0 < \underline{\vartheta} \leq \vartheta_n \leq \bar{\vartheta} < 2, \quad \forall n \in \mathbb{N}^*$,
- (iii) the sequence $\{z_n\}$ generated by the rule (PGPR) is bounded,
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*$, with $\sum_{n=1}^{+\infty} \delta_n < +\infty$,
- (v) $\sum_{n=1}^{+\infty} \lambda_n \delta_{\underline{\lambda}, \rho}(T_n, T) < +\infty, \quad \forall \rho \geq 0$,
- (vi) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty$,
- (vii) $\prod_{n=1}^{+\infty} \sqrt{\|H_n\| \|H_n^{-1}\|} < +\infty$,
- (viii) $\|I - H\| < 1$.

Then \bar{z} is the unique solution of problem (P) and the sequence $\{z_n\}$ strongly converges to \bar{z} . Moreover, there are real numbers $\eta \in]0, 1[$, $C > 0$ and a range $N \in \mathbb{N}^*$ such that

$$\|z_n - \bar{z}\| \leq \eta \|z_{n-1} - \bar{z}\| + C \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq N. \quad (4.14)$$

Proof: Let us define an auxiliary sequence $\{\tilde{z}_n\}$ by

$$\begin{cases} \tilde{z}_0 = z_0, \\ \tilde{z}_n = S_{\lambda_n, \vartheta_n}^{H^{-1} T} z_{n-1}, \quad \forall n \in \mathbb{N}^*, \end{cases}$$

and proceed by steps.

1° The distance between two corresponding iterates of the sequences $\{z_n\}$ and $\{\tilde{z}_n\}$ goes to 0 when n goes to infinity:

$$\lim_{n \rightarrow +\infty} \|z_n - \tilde{z}_n\| = 0.$$

Let $n \in \mathbb{N}^*$ be given.

We have, by definition of z_n and \tilde{z}_n ,

$$z_n - \tilde{z}_n = [z_{n-1} + \vartheta_n (J_{\lambda_n}^{H^{-1} T_n} z_{n-1} - z_{n-1}) + e_n] - [z_{n-1} + \vartheta_n (J_{\lambda_n}^{H^{-1} T} z_{n-1} - z_{n-1})].$$

It follows that

$$\|z_n - \tilde{z}_n\| \leq \vartheta_n \|J_{\lambda_n}^{H^{-1} T_n} z_{n-1} - J_{\lambda_n}^{H^{-1} T} z_{n-1}\| + \|e_n\|.$$

The boundedness of the sequence $\{z_n\}$ leads to

$$\|z_n - \tilde{z}_n\| \leq \vartheta_n \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) + \|e_n\|,$$

where

$$\rho = \sup_{n \in \mathbb{N}} \|z_n\|. \tag{4.15}$$

The hypothesis of this theorem and proposition (2.2) allow us to conclude.

2° *The distance between two consecutive iterates of the sequence $\{\tilde{z}_n\}$ goes to 0 when n goes to infinity:*

$$\lim_{n \rightarrow +\infty} \|\tilde{z}_n - \tilde{z}_{n-1}\| = 0.$$

Let, once more, $n \in \mathbb{N}^*$ be given.

The definition of \tilde{z}_n gives

$$\tilde{z}_n - \tilde{z}_{n-1} = [z_{n-1} + \vartheta_n(J_{\lambda_n}^{H^{-1}T} z_{n-1} - z_{n-1})] - [z_{n-2} + \vartheta_{n-1}(J_{\lambda_{n-1}}^{H^{-1}T} z_{n-2} - z_{n-2})].$$

It follows, from the properties of the norm, hypothesis (ii) and (4.15), that

$$\begin{aligned} \|\tilde{z}_n - \tilde{z}_{n-1}\| &\leq \|z_{n-1} - z_{n-2}\| + \vartheta_n \|J_{\lambda_n}^{H^{-1}T} z_{n-1} - z_{n-1}\| + \vartheta_{n-1} \|J_{\lambda_{n-1}}^{H^{-1}T} z_{n-2} - z_{n-2}\| \\ &\leq \|z_{n-1} - z_{n-2}\| + \bar{\vartheta} [\|J_{\lambda_n}^{H^{-1}T} z_{n-1} - J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1}\| + \|J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - z_{n-1}\| \\ &\quad + \|J_{\lambda_{n-1}}^{H^{-1}T} z_{n-2} - J_{\lambda_{n-1}}^{H_{n-1}^{-1}T_{n-1}} z_{n-2}\| + \|J_{\lambda_{n-1}}^{H_{n-1}^{-1}T_{n-1}} z_{n-2} - z_{n-2}\|] \\ &\leq \|z_{n-1} - z_{n-2}\| + \bar{\vartheta} [\delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) + \|J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - z_{n-1}\| \\ &\quad + \delta_{\lambda_{n-1}, \rho}(H_{n-1}^{-1} T_{n-1}, H^{-1} T) + \|J_{\lambda_{n-1}}^{H_{n-1}^{-1}T_{n-1}} z_{n-2} - z_{n-2}\|]. \end{aligned}$$

But the hypothesis of the present theorem recover those of theorem (4.1).

Therefore, we know that

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0$$

and (see lemma (4.4))

$$\lim_{n \rightarrow +\infty} \|J_{\lambda_n}^{H_n^{-1}T_n} z_{n-1} - z_{n-1}\| = 0.$$

It follows that result 2° is an immediate consequence of the hypothesis of this theorem and proposition (2.2).

3° *There is a range $N_1 \in \mathbb{N}^*$ from which $J_{\lambda_n}^{H^{-1}T} z_{n-1} = \tilde{z}_n$.*

It derives from the definition of $\{\tilde{z}_n\}$ that

$$\frac{\tilde{z}_n - z_{n-1} + \vartheta_n z_{n-1}}{\vartheta_n} = (I + \lambda_n H^{-1} T)^{-1} z_{n-1}, \quad \forall n \in \mathbb{N}^*. \tag{4.16}$$

That implies

$$z_{n-1} \in (I + \lambda_n H^{-1} T) \frac{\tilde{z}_n - z_{n-1} + \vartheta_n z_{n-1}}{\vartheta_n}, \quad \forall n \in \mathbb{N}^*,$$

thus

$$\frac{z_{n-1} - \bar{z}_n}{\lambda_n \vartheta_n} \in H^{-1} T \left(\frac{\bar{z}_n - z_{n-1} + \vartheta_n z_{n-1}}{\vartheta_n} \right), \quad \forall n \in \mathbb{N}^*,$$

and, finally,

$$\frac{\bar{z}_n - z_{n-1} + \vartheta_n z_{n-1}}{\vartheta_n} \in T^{-1} H \frac{z_{n-1} - \bar{z}_n}{\lambda_n \vartheta_n}. \quad (4.17)$$

On the one hand, we have

$$\|z_{n-1} - \bar{z}_n\| \leq \|z_{n-1} - \bar{z}_{n-1}\| + \|\bar{z}_{n-1} - \bar{z}_n\|, \quad \forall n \in \mathbb{N}^*,$$

where, by steps 1° and 2°,

$$\lim_{n \rightarrow +\infty} \|z_{n-1} - \bar{z}_{n-1}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|\bar{z}_{n-1} - \bar{z}_n\| = 0.$$

That implies

$$\lim_{n \rightarrow +\infty} \|z_{n-1} - \bar{z}_n\| = 0,$$

and, by using hypothesis (i) and (ii) and the continuity of H ,

$$\lim_{n \rightarrow +\infty} \left\| H \frac{z_{n-1} - \bar{z}_n}{\lambda_n \vartheta_n} \right\| = 0. \quad (4.18)$$

On the other hand, hypothesis (4.13) and R. T. Rockafellar [14], theorem 3, imply the existence of a strictly positive real number ε such that

$$\|z\| \leq \varepsilon \Rightarrow T^{-1} z = \{\bar{z}\}.$$

Therefore, from relations (4.17) and (4.18), there is a range $N_1 \in \mathbb{N}^*$ from which

$$\frac{\bar{z}_n - z_{n-1} + \vartheta_n z_{n-1}}{\vartheta_n} = \bar{z},$$

what, thanks to (4.16), establishes 3°.

4° *Let us now try to conclude.*

We have successively, for every fixed $n \in \mathbb{N}^*$,

$$\begin{aligned} \|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} z_{n-1} - \bar{z}\| &\leq \|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} z_{n-1} - S_{\lambda_n, \vartheta_n}^{H^{-1} T} z_{n-1}\| + \|S_{\lambda_n, \vartheta_n}^{H^{-1} T} z_{n-1} - \bar{z}\| \\ &\leq \|[z_{n-1} + \vartheta_n (J_{\lambda_n}^{H_n^{-1} T_n} z_{n-1} - z_{n-1})] - [z_{n-1} + \vartheta_n (J_{\lambda_n}^{H^{-1} T} z_{n-1} - z_{n-1})]\| \\ &\quad + \|[z_{n-1} + \vartheta_n (J_{\lambda_n}^{H^{-1} T} z_{n-1} - z_{n-1})] - \bar{z}\| \\ &\leq \vartheta_n \delta_{\lambda_n, \rho} (H_n^{-1} T_n, H^{-1} T) + \|[z_{n-1} + \vartheta_n (J_{\lambda_n}^{H^{-1} T} z_{n-1} - z_{n-1})] - \bar{z}\|. \end{aligned}$$

For $n \geq N_1$, we deduce thence, by using step 3°,

$$\|S_{\lambda_n, \vartheta_n}^{H_n^{-1} T_n} z_{n-1} - \bar{z}\| \leq \vartheta_n \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) + |1 - \vartheta_n| \|z_{n-1} - \bar{z}\|$$

thus

$$\|z_n - \bar{z}\| \leq \vartheta_n \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T) + |1 - \vartheta_n| \|z_{n-1} - \bar{z}\| + \|e_n\|$$

where, from hypothesis (iv) and the properties of the norm,

$$\|e_n\| \leq \delta_n [\|z_n - \bar{z}\| + \|z_{n-1} - \bar{z}\|].$$

So, we obtain

$$(1 - \delta_n) \|z_n - \bar{z}\| \leq [|1 - \vartheta_n| + \delta_n] \|z_{n-1} - \bar{z}\| + \vartheta_n \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq N_1,$$

or, hypothesis (ii) ensuring the existence of a range $N_2 \in \mathbb{N}^*$ from which $1 - \delta_n$ is strictly positive,

$$\|z_n - \bar{z}\| \leq \eta_n \|z_{n-1} - \bar{z}\| + C_n \delta_{\lambda_n, \rho}(H_n^{-1} T_n, H^{-1} T), \quad \forall n \geq \max\{N_1, N_2\}, \tag{4.19}$$

where, for $n \geq \max\{N_1, N_2\}$,

$$\eta_n = \frac{|1 - \vartheta_n| + \delta_n}{1 - \delta_n} \quad \text{and} \quad C_n = \frac{\vartheta_n}{1 - \delta_n}.$$

We have to estimate η_n and C_n for $n \geq \max\{N_1, N_2\}$.

Let us define Θ by

$$\Theta = \max\{\bar{\vartheta} - 1, 1 - \underline{\vartheta}\}.$$

Hypothesis (ii) implies

$$0 < \Theta < 1$$

and

$$|1 - \vartheta_n| \leq \Theta, \quad \forall n \in \mathbb{N}^*. \tag{4.20}$$

Let then ν be a real number in $]0, 1 - \Theta[$.

Hypothesis (iv) ensures the existence of a range $N_3 \in \mathbb{N}^*$ such that

$$\delta_n \leq \frac{1 - \Theta - \nu}{2}, \quad \forall n \geq N_3. \tag{4.21}$$

It derives from relations (4.20) and (4.21) that

$$|1 - \vartheta_n| \leq 1 - \nu - 2\delta_n, \quad \forall n \geq N_3,$$

or, equivalently,

$$|1 - \vartheta_n| + \delta_n \leq (1 - \delta_n)(1 - \nu), \quad \forall n \geq N_3.$$

Since $0 < 1 - \delta_n < 1$, for every $n \geq N_2$, it follows that

$$\eta_n \leq 1 - \nu, \quad \forall n \geq \max \{N_2, N_3\}.$$

Concerning C_n , hypothesis (ii) and relation (4.21) imply directly

$$C_n \leq \frac{2\bar{\vartheta}}{1 + \Theta + \nu}, \quad \forall n \geq \max \{N_2, N_3\}.$$

It suffices, to conclude, to set

$$N = \max \{N_1, N_2, N_3\}, \quad \eta = \sup_{n \geq N} \eta_n \quad \text{and} \quad C = \frac{2\bar{\vartheta}}{1 + \Theta + \nu}$$

and to work as in the proof of theorem (4.6).

Remark 4.10: Let us replace, in theorem (4.9), hypothesis (i) and (ii) by

(i)' $1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$, with $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$,

(ii)' $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*$.

Then, the overestimation (4.14) remains true with η_n in place of η and C_n in place of C , the sequences $\{\eta_n\}$ and $\{C_n\}$ satisfying

$$\lim_{n \rightarrow +\infty} \eta_n = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} C_n = 1.$$

Effectively, these new hypothesis allow us to write relation (4.19) with

$$\eta_n = \frac{\frac{1 + \lambda_n}{\lambda_n} - 1 + \delta_n}{1 - \delta_n} = \frac{\frac{1}{\lambda_n} + \delta_n}{1 - \delta_n}$$

and

$$C_n = \frac{1 + \lambda_n}{\lambda_n(1 - \delta_n)} = \frac{1 + \frac{1}{\lambda_n}}{1 - \delta_n}$$

where

$$\lim_{n \rightarrow +\infty} \delta_n = 0.$$

We close this section with a special result in which we obtain a *super-linear rate of convergence* for the sequence generated by (PGPR).

THEOREM 4.11: *Assume that the operators T_n^{-1} ($n \in \mathbb{N}^*$) are uniformly differentiable at 0, i.e. there are a point $\bar{z} \in \mathcal{H}$, a real number $\tau > 0$ and a sequence of linear transformations $\{A_n\}$ such that*

$$\begin{cases} T_n^{-1} 0 = \{\bar{z}\}, \quad \forall n \in \mathbb{N}^*, \\ \|\omega\| \leq \tau \Rightarrow [T_n^{-1} \omega - \bar{z} - A_n \omega] \subset o(\|\omega\|) B, \quad \forall n \in \mathbb{N}^*, \\ \sup_{n \in \mathbb{N}^*} \|A_n\| < +\infty, \end{cases}$$

(⁸) We denote by B the unit ball in \mathcal{H} :

$$B = \{x \in \mathcal{H} : \|x\| \leq 1\}.$$

and

- (i) $0 < 1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$
- (ii) $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*,$
- (iii) *the sequence $\{z_n\}$ generated by the rule (PGPR) is bounded,*
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \sum_{n=1}^{+\infty} \delta_n < +\infty,$
- (v) $\sum_{n=1}^{+\infty} \lambda_n \delta_{\lambda_n, \rho}(T_n, T) < +\infty, \quad \forall \rho \geq 0,$
- (vi) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (vii) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty,$
- (viii) $\|I - H\| < 1.$

Then \bar{z} is the unique solution of problem (P) and the sequence $\{z_n\}$ strongly converges to this solution. Moreover, there are a sequence $\{\eta_n\}$ convergent to 0 and a range $N \in \mathbb{N}^*$ such that

$$\|z_n - \bar{z}\| \leq \eta_n \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N. \tag{4.22}$$

Proof: Let us first note that the assumption

$$T_n^{-1} 0 = \bar{z}, \quad \forall n \in \mathbb{N}^*,$$

is equivalent to

$$0 \in T_n \bar{z}, \quad \forall n \in \mathbb{N}^*.$$

Since hypothesis (v) implies that

$$T_n \xrightarrow{G} T,$$

we deduce thence

$$0 \in T \bar{z}.$$

Following theorem (4.1), the sequence $\{z_n\}$ is therefore such that

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0. \tag{4.23}$$

Let us set

$$\tilde{z}_n = S_{\lambda_n \vartheta_n}^{H_n^{-1} T_n} z_{n-1}, \quad \forall n \in \mathbb{N}^*.$$

The rule (PGPR) and hypothesis (iv) give us the overestimation

$$\|z_n - \tilde{z}_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*. \tag{4.24}$$

Now, it is easy to prove that

$$I - J_{\lambda}^{H^{-1} T} = \left(I + T^{-1} H \frac{1}{\lambda} \right)^{-1}.$$

Therefore, we have, thanks to hypothesis (ii),

$$\begin{aligned}
S_{\lambda_n \vartheta_n}^{H_n^{-1} T_n} &= I + \vartheta_n (J_{\lambda_n}^{H_n^{-1} T_n} - I) \\
&= I - \vartheta_n \left(I + T_n^{-1} H_n \frac{1}{\lambda_n} \right)^{-1} \\
&= I - \left(\left[I + T_n^{-1} H_n \frac{1}{\lambda_n} \right] \frac{1}{\vartheta_n} \right)^{-1} \\
&= I - \left(\left[I + T_n^{-1} H_n \frac{1}{\lambda_n} \right] \frac{\lambda_n}{1 + \lambda_n} \right)^{-1} \\
&= I - \left(\frac{\lambda_n}{1 + \lambda_n} I + T_n^{-1} H_n \frac{1}{1 + \lambda_n} \right)^{-1} \\
&= I - \left(I + [T_n^{-1} H_n - I] \frac{1}{1 + \lambda_n} \right)^{-1}, \quad \forall n \in \mathbb{N}^*,
\end{aligned}$$

or, from the definition of \tilde{z}_n ($n \in \mathbb{N}^*$),

$$z_{n-1} - \tilde{z}_n = \left(I + [T_n^{-1} H_n - I] \frac{1}{1 + \lambda_n} \right)^{-1} z_{n-1}, \quad \forall n \in \mathbb{N}^*.$$

This allows us to write

$$\tilde{z}_n \in [T_n^{-1} H_n - I] \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n), \quad \forall n \in \mathbb{N}^*,$$

and, by the way,

$$\begin{aligned}
(\tilde{z}_n - \bar{z}) &\in T_n^{-1} H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) - \bar{z} - A_n H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) \\
&\quad + \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_{n-1} - z_n) \\
&\quad + \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_n - \tilde{z}_n), \quad \forall n \in \mathbb{N}^*.
\end{aligned}$$

We will first study the expression

$$T_n^{-1} H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) - \bar{z} - A_n H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) \quad (n \in \mathbb{N}^*).$$

The uniform convergence of the sequence $\{H_n\}$ ensures the existence of a strictly positive real number μ_1 such that

$$\|H_n\| \leq \frac{1}{\mu_1}, \quad \forall n \in \mathbb{N}^*.$$

Let us set $\mu_2 = \mu_1 \tau$.

The uniform differentiability of $\{T_n\}$ leads to

$$\|\omega\| \leq \mu_2 \Rightarrow [T_n^{-1} H_n \omega - \bar{z} - A_n H_n \omega] \subset o(\|H_n \omega\|) B \subset o(\|\omega\|) B, \quad \forall n \in \mathbb{N}^*. \quad (4.25)$$

Furthermore, thanks to hypothesis (iv) and relation (4.23), there is a range $N_0 \in \mathbb{N}^*$ from which

$$\delta_n \leq 1 \quad \text{and} \quad \|z_n - z_{n-1}\| \leq \mu_2.$$

From the overestimation (4.24), we deduce thence

$$\begin{aligned} \|z_{n-1} - \tilde{z}_n\| &\leq \|z_n - z_{n-1}\| + \|z_n - \tilde{z}_n\| \\ &\leq (1 + \delta_n) \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \end{aligned}$$

and

$$\|z_{n-1} - \tilde{z}_n\| \leq 2\mu_2, \quad \forall n \geq N_0.$$

Since $\lambda_n \geq 1, \forall n \in \mathbb{N}^*$, we get finally

$$\left\| \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) \right\| \leq \mu_2, \quad \forall n \geq N_0.$$

Therefore, it is possible to write inclusion (4.25) at the point $\frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n)$. That leads, for $n \geq N_0$, to

$$\begin{aligned} &\left[T_n^{-1} H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) - \bar{z} - A_n H_n \frac{1}{1 + \lambda_n} (z_{n-1} - \tilde{z}_n) \right] \\ &\subset o\left(\frac{\|z_{n-1} - \tilde{z}_n\|}{1 + \lambda_n} \right) B \\ &\subset o\left(\frac{1 + \delta_n}{1 + \lambda_n} \|z_n - z_{n-1}\| \right) B \\ &\subset o(\|z_n - z_{n-1}\|) B. \end{aligned}$$

Let us now work on the expressions

$$\frac{1}{1 + \lambda_n} [A_n H_n - I] (z_{n-1} - z_n) \quad \text{and} \quad \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_n - \tilde{z}_n) \quad (n \in \mathbb{N}^*).$$

Set

$$\nu = \sup_{n \in \mathbb{N}^*} \|A_n\|$$

and

$$\varepsilon_n = \frac{\nu + 1}{1 + \lambda_n}, \quad \forall n \in \mathbb{N}^*.$$

Assumption (i) implies

$$\lim_{n \rightarrow +\infty} \varepsilon_n = 0.$$

and the previous developments ensure that

$$\begin{aligned} \left\| \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_{n-1} - z_n) \right\| &\leq \frac{\|A_n\| \|H_n\| + 1}{1 + \lambda_n} \|z_n - z_{n-1}\| \\ &\leq \varepsilon_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_n - \tilde{z}_n) \right\| &\leq \frac{\|A_n\| \|H_n\| + 1}{1 + \lambda_n} \|z_n - \tilde{z}_n\| \\ &\leq \varepsilon_n \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \end{aligned}$$

or, taking into account the properties of δ_n ($n \in \mathbb{N}^*$),

$$\left\| \frac{1}{1 + \lambda_n} [A_n H_n - I] (z_n - \tilde{z}_n) \right\| \leq \varepsilon_n \|z_n - z_{n-1}\|, \quad \forall n \geq N_0.$$

All these relations allow us to write

$$(\tilde{z}_n - \bar{z}) \in [o(\|z_n - z_{n-1}\|) + 2\varepsilon_n \|z_n - z_{n-1}\|] B, \quad \forall n \geq N_0,$$

what ensures the existence of a sequence $\{\alpha_n\}$, going to 0, such that

$$\|\tilde{z}_n - \bar{z}\| \leq \alpha_n \|z_n - z_{n-1}\|, \quad \forall n \geq N_0.$$

On the one hand, we have

$$\|z_n - \bar{z}\| \leq \|z_n - \tilde{z}_n\| + \|\tilde{z}_n - \bar{z}\| \leq (\delta_n + \alpha_n) \|z_n - z_{n-1}\|, \quad \forall n \geq N_0,$$

and, on the other hand,

$$\|z_n - z_{n-1}\| \leq \|z_n - \bar{z}\| + \|z_{n-1} - \bar{z}\|, \quad \forall n \in \mathbb{N}^*.$$

We deduce thence

$$[1 - (\delta_n + \alpha_n)] \|z_n - \bar{z}\| \leq (\delta_n + \alpha_n) \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N_0.$$

It suffices, to achieve the proof, to recall that

$$\lim_{n \rightarrow +\infty} (\delta_n + \alpha_n) = 0,$$

what implies the existence of a range $N_1 \in \mathbb{N}^*$ from which

$$1 - (\delta_n + \alpha_n) > 0.$$

The conclusion arises immediately by setting $N = \max\{N_0, N_1\}$ and

$$\eta_n = \frac{\delta_n + \alpha_n}{1 - (\delta_n + \alpha_n)}, \quad \forall n \geq N.$$

■

5. THE NONPERTURBED GENERALIZED PROXIMAL POINT ALGORITHM

In the present section, we adapt the results of the previous one to the *nonperturbed* generalized proximal point algorithm. We point out the simplifications appearing in the proof of these results and give, when possible, their concrete interpretation. We also establish an additional criteria for getting the strong convergence of the sequence generated by the rule (GPR).

COROLLARY 5.1: Assume that (P) has at least one solution \bar{z} and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$,
- (ii) $0 < \underline{\vartheta} \leq \vartheta_n \leq 2, \quad \forall n \in \mathbb{N}^*$,
- (iii) $\sum_{n=1}^{+\infty} \|e_n\| < +\infty,$
- (iv) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (v) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty,$
- (vi) $\|I - H\| < 1.$

Then the sequence $\{z_n\}$ generated by the rule (GPR) weakly converges to a solution of (P) and is such that

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0.$$

Proof: It suffices to replace, in the proof of theorem (4.1), T_n ($n \in \mathbb{N}^*$) by T .

• In the first part of the proof, thanks to equivalence (3.2), we may write directly equality (4.1) with the same operator $S_{\lambda_n, \vartheta_n}^{H_n^{-1}T}$ working on z_{n-1} and \bar{z} respectively: for every $n \in \mathbb{N}^*$, we get

$$\|z_n - \bar{z}\| = \|S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} z_{n-1} + e_n - S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} \bar{z}\|,$$

what leads to

$$\begin{aligned} \|z_n - \bar{z}\| &\leq \|S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} z_{n-1} - S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} \bar{z}\| + \|e_n\| \\ &\leq \sqrt{\|H_n^{-1}\| \|H_n\|} \|z_{n-1} - \bar{z}\| + \|e_n\| \end{aligned}$$

and, step by step, to

$$\|z_n - \bar{z}\| \leq \prod_{i=1}^n \sqrt{\|H_i^{-1}\| \|H_i\|} \|z_0 - \bar{z}\| + \prod_{i=2}^n \sqrt{\|H_i^{-1}\| \|H_i\|} \left[\sum_{j=1}^n \|e_j\| \right].$$

• In the second part of the proof, the term

$$\frac{\lambda_n}{\underline{\lambda}} \delta_{\underline{\lambda}, \rho}(T, T),$$

appearing in the overestimation of

$$\delta_{\lambda_n, \rho}(H_n^{-1}T, H^{-1}T),$$

is evidently equal to 0. ■

Remark 5.2: Under the hypothesis (i) to (vi) of corollary (5.1), problem (P) has at least one solution if and only if the consequence $\{z_n\}$ generated by the rule (GPR) is bounded.

It suffices, to be convinced, to review the proof of this corollary.

COROLLARY 5.3: *Under the hypothesis of corollary (5.1), the sequence $\{z_n\}$ generated by the rule (GPR) satisfies*

$$\lim_{n \rightarrow +\infty} \|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} - z_{n-1}\| = 0.$$

Proof: It is an immediate consequence of lemma (4.4). ■

Remark 5.4: Thanks to the hypothesis $0 < \underline{\lambda} \leq \lambda_n$, $\forall n \in \mathbb{N}^*$, the result of corollary (5.3) is equivalent to

$$\lim_{n \rightarrow +\infty} \|A_{\lambda_n}^{H_n^{-1}T} z_{n-1}\| = 0.$$

Corollary (5.3) and remark (5.4) lead, by introducing a Lipschitz condition on T^{-1} , to a result of *strong convergence* for the sequence generated by the rule (GPR).

COROLLARY 5.5: *Assume that the hypothesis of corollary (5.1) hold and the operator T^{-1} is locally Lipschitz continuous at 0, i.e. problem (P) admits a unique solution \bar{z} and there are two strictly positive real numbers a , τ such that*

$$\|w\| \leq \tau \Rightarrow \|z - \bar{z}\| \leq a\|w\|, \quad \forall z \in T^{-1}w.$$

Then, the sequence $\{z_n\}$ generated by the rule (GPR) satisfies

$$\lim_{n \rightarrow +\infty} \|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} - \bar{z}\| = 0.$$

Proof: Set, by using hypothesis (iv),

$$C = \sup_{n \in \mathbb{N}^*} \|H_n\|.$$

Corollary (5.3) implies the existence of a range $N \in \mathbb{N}^*$ such that

$$\|(J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1}\| \leq \frac{T\underline{\lambda}}{C}, \quad \forall n \geq N.$$

It derives thence, by using the linearity of H_n ($n \in \mathbb{N}^*$),

$$\left\| \frac{1}{\lambda_n} H_n (J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1} \right\| \leq \frac{\|H_n\|}{\underline{\lambda}} \|(J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1}\| \leq \tau, \quad \forall n \geq N.$$

Therefore, the Lipschitz condition introduced in this corollary leads to

$$\|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} - \bar{z}\| \leq a \left\| \frac{H_n (J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1}}{\lambda_n} \right\|.$$

It follows that

$$\|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} - \bar{z}\| \leq \frac{aC}{\underline{\lambda}} \|(J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1}\|$$

and corollary (5.3) gives the conclusion ■

THEOREM 5.6 *Assume that the hypothesis of corollary (5.1) hold and the operator T^{-1} is locally Lipschitz continuous at 0 (see corollary (5.5))*

Then, the sequence $\{z_n\}$ generated by the rule (GPR) strongly converges to the solution \bar{z} of problem (P)

Proof We have successively

$$\begin{aligned} \|z_n - \bar{z}\| &= \|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} + (\vartheta_n - 1) (J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1} + e_n - \bar{z}\| \\ &\leq \|J_{\lambda_n}^{H_n^{-1}T} z_{n-1} - \bar{z}\| + |1 - \vartheta_n| \| (J_{\lambda_n}^{H_n^{-1}T} - I) z_{n-1} \| + \|e_n\|, \end{aligned}$$

and, by using hypothesis (ii) and (iii) and corollaries (5.3) and (5.5),

$$\lim_{n \rightarrow +\infty} \|z_n - \bar{z}\| = 0$$
■

Theorems (4.6) and (4.9), on the one hand, remarks (4.8) and (4.10) and theorem (4.11), on the other hand, adapted to the nonperturbed context, lead to criteria for getting *linear* and *super-linear* rates of convergence for the sequence generated by the rule (GPR)

COROLLARY 5.7 *Assume that the operator T^{-1} is locally Lipschitz continuous at 0 (see corollary (5.5)) and*

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$
- (ii) $1 \leq \vartheta_n \leq \bar{\vartheta} < 2, \quad \forall n \in \mathbb{N}^*,$
- (iii) *the sequence $\{z_n\}$ generated by the rule (GPR) is bounded,*
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \sum_{n=1}^{+\infty} \delta_n < +\infty,$
- (v) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (vi) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty,$
- (vii) $\|I - H\| < 1$

Then, the sequence $\{z_n\}$ strongly converges to the unique solution \bar{z} of problem (P)

Moreover, there are a real number $\eta \in]0, 1[$ and a range $N \in \mathbb{N}^$ such that*

$$\|z_n - \bar{z}\| \leq \eta \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N \tag{5.1}$$

Proof Replace, in the proof of theorem (4.6), $T_n (n \in \mathbb{N}^*)$ by T and recall that

$$0 \in T\bar{z} \Rightarrow J_{\lambda_n}^{H_n^{-1}T}\bar{z} = \bar{z} \quad \text{and} \quad A_{\lambda_n}^{H_n^{-1}T}\bar{z} = 0, \quad \forall n \in \mathbb{N}^*$$
■

Remark 5.8 Let us replace, in corollary (5.7), hypothesis (i) and (ii) by

- (i)' $1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$
- (ii)' $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*$

Then, the overestimation (5.1) remains true with η_n in place of η , the sequence $\{\eta_n\}$ being such that

$$\lim_{n \rightarrow +\infty} \eta_n = 0$$

It suffices, to be convinced, to join the ideas contained in remark (4.8) and in the proof of theorem (5.7)

COROLLARY 5.9: Assume that

$$\exists \bar{z} \in \mathcal{H} \text{ such that } 0 \in \text{int } T\bar{z}, \quad (5.2)$$

and

- (i) $0 < \underline{\lambda} \leq \lambda_n, \quad \forall n \in \mathbb{N}^*$,
- (ii) $0 < \underline{\vartheta} \leq \vartheta_n \leq \bar{\vartheta} < 2, \quad \forall n \in \mathbb{N}^*$,
- (iii) the sequence $\{z_n\}$ generated by the rule (GPR) is bounded,
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \quad \text{with } \sum_{n=1}^{+\infty} \delta_n < +\infty,$
- (v) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (vi) $\prod_{n=1}^{+\infty} \sqrt{\|H_n\| \|H_n^{-1}\|} < +\infty,$
- (vii) $\|I - H\| < 1.$

Then \bar{z} is the unique solution of problem (P) and the sequence $\{z_n\}$ strongly converges to \bar{z} . Moreover, there are a real number $\eta \in]0, 1[$ and a range $N \in \mathbb{N}^*$ such that

$$\|z_n - \bar{z}\| \leq \eta \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N. \quad (5.3)$$

Proof: In this context, the introduction of an auxiliary sequence is no more necessary. It is possible to establish directly the existence of a range $N \in \mathbb{N}^*$ from which

$$J_{\lambda_n}^{H_n^{-1}T} z_{n-1} = \bar{z}.$$

Let us do that.

On the one hand, the definition of the sequence $\{z_n\}$ implies

$$\frac{z_n - z_{n-1} + \vartheta_n z_{n-1} - e_n}{\vartheta_n} = J_{\lambda_n}^{H_n^{-1}T} z_{n-1}, \quad \forall n \in \mathbb{N}^*, \quad (5.4)$$

or, equivalently,

$$z_{n-1} \in (I + \lambda_n H_n^{-1}T) \frac{z_n - z_{n-1} + \vartheta_n z_{n-1} - e_n}{\vartheta_n}, \quad \forall n \in \mathbb{N}^*,$$

and, therefore,

$$\frac{z_{n-1} - z_n + e_n}{\lambda_n \vartheta_n} \in H_n^{-1}T \left(\frac{z_n - z_{n-1} + \vartheta_n z_{n-1} - e_n}{\vartheta_n} \right), \quad \forall n \in \mathbb{N}^*,$$

or yet

$$\frac{z_n - z_{n-1} + \vartheta_n z_{n-1} - e_n}{\vartheta_n} \in T^{-1} H_n \frac{z_{n-1} - z_n + e_n}{\lambda_n \vartheta_n}. \quad (5.5)$$

On the other hand, the hypothesis of corollary (5.9), lead to

$$\lim_{n \rightarrow +\infty} \|z_n - z_{n-1}\| = 0$$

(see corollary (5.1)) and, by using hypothesis (i), (ii), (iii) and (v), to

$$\lim_{n \rightarrow +\infty} \left\| H_n \frac{z_{n-1} - z_n + e_n}{\lambda_n \vartheta_n} \right\| = 0. \tag{5.6}$$

Assumption (5.2) implying the existence of a neighbourhood of 0 on which T^{-1} takes the unique value \bar{z} (see R. T. Rockafellar [14], theorem 3), assertions (5.5) and (5.6) ensure the existence of a range $N \in \mathbb{N}^*$ from which

$$\frac{z_n - z_{n-1} + \vartheta_n z_{n-1} - e_n}{\vartheta_n} = \bar{z},$$

that is to say, thanks to (5.4), the announced result.

This result leads to

$$S_{\lambda_n, \vartheta_n}^{H_n^{-1}T} z_{n-1} - \bar{z} = (1 - \vartheta_n) (z_{n-1} - \bar{z}), \quad \forall n \geq N.$$

The conclusion may then be obtained by proceeding as in the proof of theorem (4.9). ■

Remark 5.10: Let us replace, in corollary (5.9), hypothesis (i) and (ii) by

- (i)' $1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$
- (ii)' $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*.$

Then, the overestimation (5.3) remains true with η_n in place of η , the sequence $\{\eta_n\}$ being such that

$$\lim_{n \rightarrow +\infty} \eta_n = 0.$$

It suffices, to be convinced, to join the ideas contained in remark (4.10) and in the proof of theorem (5.9).

COROLLARY 5.11: *Assume that the operator T^{-1} is Fréchet-differentiable at 0 and*

- (i) $0 < 1 \leq \lambda_n, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty,$
- (ii) $\vartheta_n = \frac{1 + \lambda_n}{\lambda_n}, \quad \forall n \in \mathbb{N}^*,$
- (iii) *the sequence $\{z_n\}$ generated by the rule (GPR) is bounded,*
- (iv) $\|e_n\| \leq \delta_n \|z_n - z_{n-1}\|, \quad \forall n \in \mathbb{N}^*, \quad \text{with} \quad \sum_{n=1}^{+\infty} \delta_n < +\infty,$
- (v) $\sum_{n=1}^{+\infty} \|H_n - H\| < +\infty,$
- (vi) $\prod_{n=1}^{+\infty} \sqrt{\|H_n^{-1}\| \|H_n\|} < +\infty,$
- (vii) $\|I - H\| < 1.$

Then \bar{z} is the unique solution of problem (P) and the sequence $\{z_n\}$ strongly converges to this solution.

Moreover, there are a sequence $\{\eta_n\}$ convergent to 0 and a range $N \in \mathbb{N}^$ such that*

$$\|z_n - \bar{z}\| \leq \eta_n \|z_{n-1} - \bar{z}\|, \quad \forall n \geq N.$$

Proof: Replace $T_n (n \in \mathbb{N}^*)$ by T everywhere in the proof of theorem (4.11) and conclude by considering theorem (5.1) in place of theorem (4.1). ■

We close this section with the presentation of a very special case for which the nonperturbed generalized proximal point algorithm needs an unique iteration to work.

THEOREM 5.12: *Assume that the operator T is defined by*

$$Tz = H(z - \bar{z}), \quad \forall z \in \mathcal{H},$$

\bar{z} denoting an arbitrary element of \mathcal{H}

Then the generalized proximal point algorithm, working with

$$\lambda_1 \geq 1, \quad \vartheta_1 = \frac{1 + \lambda_1}{\lambda_1}, \quad H_1 = H \text{ and } e_1 = 0,$$

stops at \bar{z} after an unique iteration, for every z_0 choosen in \mathcal{H}

Proof Let $z_0 \in \mathcal{H}$ be given

The first iteration of the generalized proximal point algorithm leads to

$$\begin{aligned} z_1 &= z_0 + \frac{1 + \lambda_1}{\lambda_1} (J_{\lambda_1}^{H^{-1}T} - I) z_0 \\ &= z_0 - (1 + \lambda_1) \frac{I - J_{\lambda_1}^{H^{-1}T}}{\lambda_1} z_0 \end{aligned}$$

where, from the definition of $A_{\lambda_1}^{H^{-1}T}$, inclusion (2.6) and the particular definition of T ,

$$\frac{I - J_{\lambda_1}^{H^{-1}T}}{\lambda_1} z_0 = A_{\lambda_1}^{H^{-1}T} z_0,$$

$$A_{\lambda_1}^{H^{-1}T} z_0 \in H^{-1}T(J_{\lambda_1}^{H^{-1}T} z_0)$$

and

$$H^{-1}T(J_{\lambda_1}^{H^{-1}T} z_0) = J_{\lambda_1}^{H^{-1}T} z_0 - \bar{z}$$

It follows that

$$\frac{I - J_{\lambda_1}^{H^{-1}T}}{\lambda_1} z_0 = A_{\lambda_1}^{H^{-1}T} z_0 = J_{\lambda_1}^{H^{-1}T} z_0 - \bar{z},$$

or, equivalently,

$$J_{\lambda_1}^{H^{-1}T} z_0 = \frac{1}{1 + \lambda_1} (z_0 + \lambda_1 \bar{z})$$

That leads to

$$\begin{aligned} z_1 &= z_0 + \frac{1 + \lambda_1}{\lambda_1} \left(\frac{1}{1 + \lambda_1} (z_0 + \lambda_1 \bar{z}) - z_0 \right) \\ &= z_0 + \frac{1 + \lambda_1}{\lambda_1} \left(\frac{\lambda_1}{1 + \lambda_1} (\bar{z} - z_0) \right) \\ &= \bar{z} \end{aligned}$$

Since the definition of T implies

$$T\bar{z} = 0,$$

the conclusion arises immediately.

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