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ZHIMING CHEN
C. M. ELLIOTT
TANG QI

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JUSTIFICATION OF A TWO DIMENSIONAL EVOLUTIONARY GINZBURG-LANDAU SUPERCONDUCTIVITY MODEL (*)

Zhiming CHEN (**) C. M. ELLIOTT (†) and Tang QI (†)

Abstract — *It is proved that a two dimensional evolutionary Ginzburg-Landau superconductivity model is an approximation of a corresponding thin plate three dimensional superconductivity model when the thickness of the plate uniformly approaches zero Some related topics such as existence of weak solutions to the three dimensional variable thickness model and the convergence when the variable thickness tends to zero are discussed A numerical experiment using the now model is reported* © *Elsevier, Paris*

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Résumé — *On introduit un modèle de superconductivité bi-dimensionnel instationnaire de type Ginzburg-Landau comme la limite du modèle de plaque mince tri-dimensionnel correspondant quand l'épaisseur de la plaque tend vers zéro On discute l'existence de solutions faibles du modèle et leur limite quand l'épaisseur tend vers zéro On présente une expérience numérique qui utilise le nouveau modèle* © *Elsevier, Paris*

1. INTRODUCTION

The Ginzburg-Landau Superconductivity model describes the phenomenon of vortex structure in the super-normal phase transition. From the mathematical point of view, the stationary two dimensional model allows a rigorous proof that for most of the physically relevant (gauge invariant) boundary conditions, the order parameter takes the value zero on isolated points (*cf.* [EMT 93]). This supports the theory of vortex structured phase transitions in the super-normal transition. In the three dimensional case, to the authors' best knowledge, there are no such results except for the study of Jaffe and Taubes in the self-dual case (*cf.* [JT 80]). We also observe that a two dimensional model is easier to be studied from the numerical point of view. It is therefore interesting to prove that the two dimensional model is a good approximation of the corresponding three dimensional model when the size of the sample is small in one particular dimension.

The particular model of evolutionary (or rather quasi-static) superconductivity dealt with in this paper was first studied in [GE 68]. The model involves three quantities, a magnetic potential, an electrical potential and an order parameter. The existence and uniqueness of solutions to such system subject to the homogeneous Neumann type boundary conditions are established in [CHL 93] and [Du 94] respectively. In this paper, we adopt the notation of [CHL 93] and study the convergence of the thin plate model. For the evolutionary equation with some other boundary conditions, existence and uniqueness of solutions have been established and properties of solutions have been analysed (*cf.* [T 95]). Recently, a more generalized result on existence (without the assumption that the initial data of the order parameter is bounded in L^∞) was proved in [TW 95]. It was also established in [TW 95] that the evolutionary system admits a global attractor. In this paper, we only give a brief sketch of the existence and uniqueness proof because the domain is not as smooth as in the previous papers and we want the paper to be self contained.

In [DG 93], the similar problem of showing that a two dimensional model is an approximation of the three dimensional model in the steady state case has been studied. Here, we allow the thin film to have different upper surface and lower surface and give the proofs in greater detail concerning certain regularity estimates. It is also worth noting that the geometry of the thin films with variable thickness is related to the pinning mechanism of the vortices in the superconducting material samples.

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(**) Lehrstuhl fur Angewandte Mathematik, Technische Universitat Munchen, Dachauerstr 9a, 80335 Munchen, Germany Present address Institute of Mathematics, Academia Sinica, Beijing 100080, PR China

(†) Centre for Mathematical Analysis and Its Applications, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom

2. PRELIMINARIES

Let Ω_0 be an open bounded subset of $\mathbb{R}^2$ with $\partial\Omega_0 \in C^2$. We consider a thin film of variable upper and lower surfaces defined by $\Omega_\varepsilon = \{x = (x_1, x_2, x_3) : x' = (x_1, x_2) \in \Omega_0,\ x_3 \in \varepsilon(-b(x'), a(x'))\}$ where $\varepsilon > 0$ is a small parameter, a, b are assumed to be functions in $C^2(\bar{\Omega}_0)$, and $a(x') \geqslant c_a > 0$, $b(x') \geqslant c_b > 0$ for all $x' \in \bar{\Omega}_0$. Throughout the paper we will assume that Ω_ε has a Lipschitz boundary $\partial\Omega_\varepsilon$.

Let $\mathscr{D}$ be an open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\mathscr{D}$, where $n = 2$ or 3. For $s \geqslant 1$, $p \geqslant 1$, $W^{s,p}(\mathscr{D})$ will denote the standard Sobolev space of real valued functions having all the derivatives of order up to s in the space $L^p(\mathscr{D})$. Let $H^s(\mathscr{D}) = W^{s,2}(\mathscr{D})$. We will also use the subspace

$$\mathbf{H}_n^1(\mathscr{D}) = \{\mathbf{A} \in \mathbf{H}^1(\mathscr{D}) : \mathbf{A}\,.\,\mathbf{n} = 0 \text{ on } \partial\mathscr{D}\}\,.$$

For any Banach space X and any integer $m \geqslant 0$, denote

$$W^{m,p}(0,T;X) = \left\{ u(t) \in X \text{ for a.e. } t \in (0,T), \int_0^T \{ \|u\|_X^p + \cdots + \|u^{(m)}\|_X^p\}\, dt < \infty \right\}.$$

Let $L^p(0,T;X) = W^{0,p}(0,T;X)$ and $H^1(0,T;X) = W^{1,2}(0,T;X)$. If X denotes some Banach space of real scalar functions, then the corresponding space of complex scalar functions will be denoted by its calligraphic form $\mathscr{X}$ and the corresponding space of real vector valued functions, each of its components belonging to X, will be denoted by its boldfaced form $\mathbf{X}$. However, we will use $\|\,.\,\|_X$ to denote the norm of the Banach space X, $\mathscr{X}$ or $\mathbf{X}$.

The standard gradient, divergence and curl operators in $\mathbb{R}^3$ will be denoted by **grad**, div and **curl**, respectively. Let $\mathbf{A}'$ denote the projection of a three dimensional vector $\mathbf{A} \in \mathbb{R}^3$ onto the (x_2, x_2)-plane. On the (x_1, x_2)-plane, it is convenient to introduce two curl operators

$$\operatorname{curl} \mathbf{B} = \frac{\partial \mathbf{B}_2}{\partial x_1} - \frac{\partial \mathbf{B}_1}{\partial x_2} \quad \text{and} \quad \mathbf{curl}'\,\psi = \left(\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)^T.$$

We also need the divergence and gradient operators

$$\operatorname{div}' \mathbf{B} = \frac{\partial \mathbf{B}_1}{\partial x_1} + \frac{\partial \mathbf{B}_2}{\partial x_2} \quad \text{and} \quad \mathbf{grad}'\,\psi = \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}\right)^T.$$

Here, we notice that the following inequality holds on $\mathbf{H}_n^1(\Omega_\varepsilon)$ (*cf.* [GR 86])

$$\|\mathbf{A}\|_{H^1(\Omega_\varepsilon)} \leqslant \hat{C}(\|\mathbf{A}\|_{L^2(\Omega_\varepsilon)} + \|\operatorname{div} \mathbf{A}\|_{L^2(\Omega_\varepsilon)} + \|\mathbf{curl}\,\mathbf{A}\|_{L^2(\Omega_\varepsilon)}), \quad \forall \mathbf{A} \in \mathbf{H}_n^1(\Omega_\varepsilon) \tag{2.1}$$

where the constant $\hat{C}$ depends on the domain Ω_ε. The fact that $\hat{C}$ in general depends on ε has been ignored in [DG 93] (*cf.* Lemma 3.1). In section 4.2 we will show that $\hat{C}$ is indeed independent of ε.

When the temperature is close to the critical temperature T_c where the transition from normal state to superconducting states starts taking place, the Ginzburg-Landau evolutionary superconductivity model is as follows

$$\eta \frac{\partial \psi}{\partial t} + i\eta\kappa\psi\phi + \left(\frac{i}{\kappa}\mathbf{grad} + \mathbf{A}\right)^2 \psi + \psi(|\psi|^2 - 1) = 0, \quad \text{in} \quad Q_\varepsilon \tag{2.2}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{grad}\,\phi + \mathbf{curl}^2\,\mathbf{A} + Re\left[\left(\frac{i}{\kappa}\mathbf{grad}\,\psi + \psi\mathbf{A}\right)\bar{\psi}\right] = \mathbf{curl}\,\mathbf{H}, \quad \text{in} \quad Q_\varepsilon \tag{2.3}$$

where $Q_\varepsilon = \Omega_\varepsilon \times (0,T)$. Here $\psi : \Omega_\varepsilon \to \mathscr{C}$ is a complex valued function and is usually referred to as the order parameter, $|\psi|^2$ represents the density of superconducting electron pairs; $\bar{\psi}$ is the complex conjugate of ψ;

$\mathbf{A} : \Omega_\varepsilon \to \mathbb{R}^3$ is a real vector potential for the total magnetic field; $\phi : \Omega_\varepsilon \to \mathbb{R}$ is a real scalar function called the electric potential which satisfies the constraint $\int_\Omega \phi \, dx = 0$ for almost every t; η, κ are physical constants; $\mathbf{H} : \Omega_\varepsilon \to \mathbb{R}^3$ is the applied magnetic field. The boundary and initial conditions are as follows:

$$\left(\frac{i}{\kappa}\,\mathbf{grad} + \mathbf{A}\right)\psi \,.\, \mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{A} \wedge \mathbf{n} = \mathbf{H} \wedge \mathbf{n}, \quad \text{on} \quad \Gamma_\varepsilon \tag{2.4}$$

$$\psi(x, 0) = \psi_0(x), \quad \mathbf{A}(x, 0) = \mathbf{A}_0(x), \quad \text{in} \quad \Omega_\varepsilon, \tag{2.5}$$

where $\Gamma_\varepsilon = \partial\Omega_\varepsilon \times (0, T)$. Here "." is used to denote the vector inner (scalar) product, "$\wedge$" is used to denote the vector exterior (vector) product and "$\mathbf{n}$" is the unit outward normal vector of $\partial\Omega_\varepsilon$.

The assumptions on the data are:

(A1) $\psi_0 \in \mathscr{H}^1(\Omega_\varepsilon)$, $|\psi_0| \leq 1$ a.e. in Ω_ε;

(A2) $\mathbf{A}_0 \in \mathbf{H}^1_n(\Omega_\varepsilon)$;

(A3) $\mathbf{H} \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega_\varepsilon))$.

It should be pointed out that this is a rescaled version of the original model. For the details of rescaling, we refer to [EMT 93], [Du 94] and [CHO 92].

It is straightforward to verify that for this model, if $(\psi, \phi, \mathbf{A})$ is a triple of solutions, then for any smooth function θ, $(\psi e^{i\kappa\theta}, \phi - \theta_t, \mathbf{A} + \mathbf{grad}\,\theta)$ is also a triple of solutions. This invariance property is called gauge invariance and a transformation of the type

$$(\psi, \phi, \mathbf{A}) \to (\psi e^{i\kappa\theta}, \phi - \theta_t, \mathbf{A} + \mathbf{grad}\,\theta) \tag{2.6}$$

is called a gauge transformation. It is therefore enough to discuss the properties of solutions for one particular gauge equivalent class of our choice.

PROPOSITION 2.1: *For any vector valued function* $\tilde{\mathbf{A}} \in \mathbf{H}^1(\Omega_\varepsilon \times (0, T))$ *and complex valued function* $\tilde{\phi} \in L^2(0, T; L^2(\Omega_\varepsilon))$ *with* $\int_\Omega \tilde{\phi}\, dx = 0$ *for almost every* t, *there exists a function* $\theta \in L^2(0, T; H^2(\Omega_\varepsilon)) \cap H^1(0, T; L^2(\Omega_\varepsilon))$ *such that* $\theta(x, 0) = 0$ *and* $\mathbf{A} = \tilde{\mathbf{A}} + \mathbf{grad}\,\theta$, $\phi = \tilde{\phi} - \theta_t$ *satisfies*

$$\begin{cases} \operatorname{div} \mathbf{A} + \phi = 0, & \text{in } Q_\varepsilon, \\ \mathbf{A} \,.\, \mathbf{n} = 0, & \text{on } \Gamma_\varepsilon. \end{cases} \tag{2.7}$$

The proof of this proposition will be given at the end of Section 4.2. In the following, we will only discuss solutions in the gauge equivalent class satisfying (2.7): under this gauge choice, we conclude that the system (2.2)-(2.5) can be rewritten as follows:

$$\begin{cases} \eta \dfrac{\partial \psi}{\partial t} - i\eta\kappa \operatorname{div} \mathbf{A}\psi + \left(\dfrac{i}{\kappa}\,\mathbf{grad} + \mathbf{A}\right)^2 \psi + \psi(|\psi|^2 - 1) = 0, & \text{in } Q_\varepsilon \\ \dfrac{\partial \mathbf{A}}{\partial t} - \Delta\mathbf{A} + Re\left[\left(\dfrac{i}{\kappa}\,\mathbf{grad}\,\psi + \psi\mathbf{A}\right)\bar{\psi}\right] = \mathbf{curl}\,\mathbf{H}, & \text{in } Q_\varepsilon \\ \mathbf{grad}\,\psi \,.\, \mathbf{n} = 0, \quad \mathbf{A} \,.\, \mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{A} \wedge \mathbf{n} = \mathbf{H} \wedge \mathbf{n}, & \text{on } \Gamma_\varepsilon \\ \psi(x, 0) = \psi_0(x), \quad \mathbf{A}(x, 0) = \mathbf{A}_0(x), & \text{in } \Omega_\varepsilon. \end{cases} \tag{2.8}$$

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Since the domain is not as smooth as required in [CHL 93], we sketch the existence proof independently. As ε is considered to be a constant in this section, to simplify the notations, we use Ω for Ω_ε and drop the notation of dependence of solutions on ε. To begin with, we define

$$\begin{aligned} \mathscr{W} &= \mathscr{L}^2(0, T; \mathscr{H}^1(\Omega)) \cap \mathscr{H}^1(0, T; (\mathscr{H}^1(\Omega))') \\ \mathbf{W}_n &= \mathbf{L}^2(0, T; \mathbf{H}^1_n(\Omega)) \cap \mathbf{H}^1(0, T; (\mathbf{H}^1_n(\Omega))'). \end{aligned} \tag{3.1}$$

The weak formulation for the system (2 8) is then to find $(\psi, \mathbf{A}) \in \mathscr{W} \times \mathbf{W}_n$ such that

$$\psi(x,0) = \psi_0(x), \quad \mathbf{A}(x,0) = \mathbf{A}_0(x) \tag{3 2}$$

and

$$\eta \int_0^T \int_\Omega \frac{\partial \psi}{\partial t} \omega \, dx \, dt - i\eta\kappa \int_0^T \int_\Omega \operatorname{div} \mathbf{A} \psi \omega \, dx \, dt$$

$$+ \int_0^T \int_\Omega \left(\frac{i}{\kappa} \mathbf{grad}\, \psi + \mathbf{A}\psi \right) \left(-\frac{i}{\kappa} \mathbf{grad}\, \omega + \mathbf{A}\omega \right) dx \, dt$$

$$+ \int_0^T \int_\Omega (|\psi|^2 - 1) \psi \omega \, dx \, dt = 0, \quad \text{for any } \omega \in \mathscr{L}^2(0, T, \mathscr{H}^1(\Omega)) \tag{3 3}$$

$$\int_0^T \int_\Omega \frac{\partial \mathbf{A}}{\partial t} \mathbf{B} \, dx \, dt + \int_0^T \int_\Omega [\mathbf{curl}\, \mathbf{A}\, \mathbf{curl}\, \mathbf{B} + \operatorname{div} \mathbf{A} \operatorname{div} \mathbf{B}] \, dx \, dt$$

$$+ \int_0^T \int_\Omega Re \left[\left(\frac{i}{\kappa} \mathbf{grad}\, \psi + \mathbf{A}\psi \right) \bar{\psi} \right] \mathbf{B} \, dx \, dt$$

$$= \int_0^T \int_\Omega \mathbf{H}\, \mathbf{curl}\, \mathbf{B} \, dx \, dt, \quad \text{for any } \mathbf{B} \in \mathbf{L}^2(0, T, \mathbf{H}_n^1(\Omega)) \tag{3 4}$$

The purpose of this section is to prove existence and uniqueness of solutions to (3 2) (3 4)

THEOREM 3 1 *Let the assumptions (A1)-(A3) be satisfied Then (3 2)-(3 4) has a unique pair of solutions* $(\psi, \mathbf{A})$ *satisfying*

$$\psi \in \mathscr{L}^\infty(0, T, \mathscr{H}^1(\Omega)) \cap \mathscr{H}^1(0, T, \mathscr{L}^2(\Omega)), \tag{3 5}$$

$$\mathbf{A} \in \mathbf{L}^\infty(0, T, \mathbf{H}_n^1(\Omega)) \cap H^1(0, T, \mathbf{L}^2(\Omega)), \tag{3 6}$$

and

$$|\psi| \leqslant 1 \text{ almost everywhere on } \Omega \times (0, T) \tag{3 7}$$

The theorem is the consequence of the lemmas that follow

LEMMA 3 1 (Uniqueness) *The solution* $(\psi, \mathbf{A})$ *of (3 2)-(3 4) satisfying (3 5) (3 7) is unique*

Proof Let $(\psi_1, \mathbf{A}_1)$ and $(\psi_2, \mathbf{A}_2)$ be two solutions of (3 2)-(3 4) satisfying (3 5)-(3 7) and set $\tilde{\psi} = \psi_1 - \psi_2$, $\tilde{\mathbf{A}} = \mathbf{A}_1 - \mathbf{A}_2$ Then, subtracting the corresponding equations, we have

$$\eta \int_0^T \int_\Omega \frac{\partial \tilde{\psi}}{\partial t} \omega \, dx \, dt + \frac{1}{\kappa^2} \int_0^T \int_\Omega \mathbf{grad}\, \tilde{\psi}\, \mathbf{grad}\, \omega \, dx \, dt$$

$$= i\eta\kappa \int_0^T \int_\Omega (\operatorname{div} \tilde{\mathbf{A}} \psi_1 + \operatorname{div} \mathbf{A}_2 \, \tilde{\psi}) \, \omega \, dx \, dt$$

$$- \frac{i}{\kappa} \int_0^T \int_\Omega (\tilde{\mathbf{A}}\, \mathbf{grad}\, \psi_1 + \mathbf{A}_2\, \mathbf{grad}\, \tilde{\psi}) \, \omega \, dx \, dt$$

$$
\begin{aligned}
&+\frac{i}{\kappa}\int_0^T\int_\Omega (\tilde{\mathbf{A}}\psi_1 + \mathbf{A}_2\ \tilde{\psi})\,\mathbf{grad}\,\omega\,dx\,dt \\
&-\int_0^T\int_\Omega [\tilde{\mathbf{A}}(\mathbf{A}_1 + \mathbf{A}_2)\,\psi_1 + |\mathbf{A}_2|^2\,\tilde{\psi}]\,\omega\,dx\,dt \\
&+\int_0^T\int_\Omega [(|\psi_1|^2 + |\psi_2|^2 - 1)\,\tilde{\psi} + \psi_1\,\psi_2\,\bar{\tilde{\psi}}]\,\omega\,dx\,dt \\
&= (\mathrm{I})_1 + (\mathrm{I})_2 + (\mathrm{I})_3 + (\mathrm{I})_4 + (\mathrm{I})_5.
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial\tilde{\mathbf{A}}}{\partial t}\,\mathbf{B}\,dx\,dt + \int_0^T\int_\Omega [\mathbf{curl}\,\tilde{\mathbf{A}}\,\mathbf{curl}\,\mathbf{B} + \operatorname{div}\tilde{\mathbf{A}}\operatorname{div}\mathbf{B}]\,dx\,dt \\
&= -\int_0^T\int_\Omega Re\left[\left(\frac{i}{\kappa}\,\mathbf{grad}\,\tilde{\psi} + \tilde{\mathbf{A}}\psi_1 + \mathbf{A}_2\ \tilde{\psi}\right)\bar{\psi}_1\right]\mathbf{B}\,dx\,dt \\
&\quad -\int_0^T\int_\Omega Re\left[\left(\frac{i}{\kappa}\,\mathbf{grad}\,\psi_2 + \mathbf{A}_2\ \psi_2\right)\bar{\tilde{\psi}}\right]\mathbf{B}\,dx\,dt \\
&= (\mathrm{II})_1 + (\mathrm{II})_2.
\end{aligned}
\tag{3.9}
$$

We note that since $(\psi_1, \mathbf{A}_1)$, $(\psi_2, \mathbf{A}_2)$ satisfy (3.5)-(3.7), we have

$$
\|\psi_j\|_{L^\infty(\Omega)} \leqslant 1, \quad \|\psi_j\|_{H^1(\Omega)} \leqslant c, \quad \|\mathbf{A}_j\|_{L^4(\Omega)} \leqslant c, \quad \|\mathbf{A}_j\|_{H^1(\Omega)} \leqslant c
$$

for $j = 1, 2$ and for almost every $t \in (0, T)$. Let $\omega = \bar{\tilde{\psi}}\chi_{(0,t)}$ in (3.8) and take the real part of the equation, then

$$
|Re(\mathrm{I})_1| \leqslant \frac{1}{4}\int_0^t\int_\Omega |\operatorname{div}\tilde{\mathbf{A}}|^2\,dx\,dt + c\int_0^t\int_\Omega |\tilde{\psi}|^2\,dx\,dt\,;
$$

$$
\begin{aligned}
|Re(\mathrm{I})_2| &\leqslant \int_0^t [\,\|\tilde{\mathbf{A}}\|_{L^4(\Omega)}\|\mathbf{grad}\,\psi_1\|_{L^2(\Omega)}\|\tilde{\psi}\|_{L^4(\Omega)} + \|\mathbf{A}_2\|_{L^4(\Omega)}\|\mathbf{grad}\,\tilde{\psi}\|_{L^2(\Omega)}\|\tilde{\psi}\|_{L^4(\Omega)}]\,dt \\
&\leqslant \frac{1}{4\kappa^2}\int_0^t\int_\Omega |\mathbf{grad}\,\tilde{\psi}|^2\,dx\,dt + c\int_0^t [\,\|\tilde{\mathbf{A}}\|^2_{L^4(\Omega)} + \|\tilde{\psi}\|^2_{L^4(\Omega)}]\,dt\,;
\end{aligned}
$$

$$
\begin{aligned}
|Re(\mathrm{I})_3| &\leqslant \int_0^t [\,\|\tilde{\mathbf{A}}\|_{L^2(\Omega)}\|\psi_1\|_{L^\infty(\Omega)}\|\mathbf{grad}\,\tilde{\psi}\|_{L^2(\Omega)} + \|\mathbf{A}_2\|_{L^4(\Omega)}\|\tilde{\psi}\|_{L^4(\Omega)}\|\mathbf{grad}\,\tilde{\psi}\|_{L^2(\Omega)}]\,dt \\
&\leqslant \frac{1}{4\kappa^2}\int_0^t\int_\Omega |\mathbf{grad}\,\tilde{\psi}|^2\,dx\,dt + c\int_0^t\int_\Omega |\tilde{\mathbf{A}}|^2\,dx\,dt + c\int_0^t \|\tilde{\psi}\|^2_{L^4(\Omega)}\,dt\,;
\end{aligned}
$$

$$
\begin{aligned}
|Re(\mathrm{I})_4| &\leqslant \int_0^t [\,\|\tilde{\mathbf{A}}\|_{L^4(\Omega)}\|\mathbf{A}_1 + \mathbf{A}_2\|_{L^4(\Omega)}\|\psi_1\|_{L^\infty(\Omega)}\|\tilde{\psi}\|_{L^2(\Omega)} + \|\mathbf{A}_2\|^2_{L^4(\Omega)}\|\tilde{\psi}\|^2_{L^4(\Omega)}]\,dt \\
&\leqslant c\int_0^t [\,\|\tilde{A}\|^2_{L^4(\Omega)} + \|\tilde{\psi}\|^2_{L^4(\Omega)} + \|\tilde{\psi}\|^2_{L^2(\Omega)}]\,dt\,;
\end{aligned}
$$

$$
|Re(\mathrm{I})_5| \leqslant c\int_0^t\int_\Omega |\tilde{\psi}|^2\,dx\,dt\,.
$$

By Nirenberg's inequality, we have

$$\|\tilde\psi\|_{L^4(\Omega)} \leqslant c\|\tilde\psi\|^{3/4}_{H^1(\Omega)}\|\tilde\psi\|^{1/4}_{L^2(\Omega)} \leqslant \delta\|\mathbf{grad}\,\tilde\psi\|_{L^2(\Omega)} + c_\delta\|\tilde\psi\|_{L^2(\Omega)},$$

and, by (2.1),

$$\|\tilde{\mathbf{A}}\|_{L^4(\Omega)} \leqslant c\|\tilde{\mathbf{A}}\|^{3/4}_{H^1(\Omega)}\|\tilde{\mathbf{A}}\|^{1/4}_{L^4(\Omega)}$$
$$\leqslant \delta[\,\|\operatorname{div}\tilde{\mathbf{A}}\|_{L^2(\Omega)} + \|\operatorname{curl}\tilde{\mathbf{A}}\|_{L^2(\Omega)}] + c_\delta\|\tilde{\mathbf{A}}\|_{L^2(\Omega)}.$$

Consequently, by choosing δ appropriately, we have

$$\frac{\eta}{4}\int_\Omega|\tilde\psi|^2(x,t)\,dx + \frac{1}{\kappa^2}\int_0^t\int_\Omega|\mathbf{grad}\,\tilde\psi|^2\,dx\,dt$$
$$\leqslant \int_0^t\int_\Omega\left[\frac{3}{4\,\kappa^2}|\mathbf{grad}\,\tilde\psi|^2 + \frac{1}{2}|\mathbf{curl}\,\tilde{\mathbf{A}}|^2 + \frac{1}{2}|\operatorname{div}\tilde{\mathbf{A}}|^2 + c(|\tilde{\mathbf{A}}|^2 + |\tilde\psi|^2)\right]dx\,dt. \tag{3.10}$$

Similarly, taking $\mathbf{B} = \tilde{\mathbf{A}}$ in (3.9), we obtain

$$\frac{1}{2}\int_\Omega|\tilde{\mathbf{A}}|^2(x,t)\,dx + \int_0^t\int_\Omega[\,|\mathbf{curl}\,\tilde{\mathbf{A}}|^2 + |\operatorname{div}\tilde{\mathbf{A}}|^2]\,dx\,dt$$
$$\leqslant \int_0^t\int_\Omega\left[\frac{1}{8\,\kappa^2}|\mathbf{grad}\,\tilde\psi|^2 + \frac{1}{4}|\mathbf{curl}\,\tilde{\mathbf{A}}|^2 + \frac{1}{4}|\operatorname{div}\tilde{\mathbf{A}}|^2 + c(|\tilde{\mathbf{A}}|^2 + |\tilde\psi|^2)\right]dx\,dt. \tag{3.11}$$

Now uniqueness follows from (3.10)-(3.11) by using Gronwall's inequality. □

In order to show the existence of the solutions of (3.2)-(3.4), we introduce the following semi-discretized approximation problem: let $N \geqslant 1$ be an integer, $\Delta t = T/N$ be the step size, $\mathbf{H}_j = \mathbf{H}(x, j\,\Delta t)$ for $j = 0, 1, 2, \dots, N$. The approximation problem is then to find $(\psi_j, \mathbf{A}_j) \in \mathscr{H}^1(\Omega) \times \mathbf{H}^1_n(\Omega)$, $j = 1, 2, \dots, N$ such that

$$\eta\int_\Omega\frac{\psi_j - \psi_{j-1}}{\Delta t}\,\omega\,dx - i\eta\kappa\int_\Omega\operatorname{div}\mathbf{A}_j\;\psi_j\,\omega\,dx$$
$$+\int_\Omega\left(\frac{i}{\kappa}\,\mathbf{grad}\,\psi_j + \mathbf{A}_j\,\psi_j\right)\left(-\frac{i}{\kappa}\,\mathbf{grad}\,\omega + \mathbf{A}_j\;\omega\right)dx$$
$$+\int_\Omega(\,|\psi_j|^2 - 1\,)\,\psi_j\,\omega\,dx = 0, \quad \text{for any } \omega \in \mathscr{H}^1(\Omega). \tag{3.12}$$

$$\int_\Omega\frac{\mathbf{A}_j - \mathbf{A}_{j-1}}{\Delta t}\,\mathbf{B}\,dx + \int_\Omega[\mathbf{curl}\,\mathrm{A}_j\;\mathbf{curl}\,\mathbf{B} + \operatorname{div}\mathbf{A}_j\;\operatorname{div}\mathbf{B}]\,dx$$
$$+\int_\Omega Re\left[\left(\frac{i}{\kappa}\,\mathbf{grad}\,\psi_{j-1} + \mathbf{A}_{j-1}\,\psi_{j-1}\right)\bar\psi_{j-1}\right]\mathbf{B}\,dx$$
$$= \int_\Omega\mathbf{H}_j\,\mathbf{curl}\,\mathbf{B}\,dx, \quad \text{for any } \mathbf{B} \in \mathbf{H}^1_n(\Omega), \tag{3.13}$$

where $(\psi_0, \mathbf{A}_0)$ is given by (3.2).

LEMMA 3.2 (Existence of solutions of (3.12)(3.13)): *Let* $\Delta t > 0$ *be sufficiently small. Then the problem (3.12)-(3.13) has a unique solution* $(\psi_j, \mathbf{A}_j) \in \mathscr{H}^1(\Omega) \times \mathbf{H}_n^1(\Omega)$ *for each* $j = 1, \dots, N$.

Proof: We first notice that (3.12)-(3.13) are independent of each other. (3.13) is a standard linear elliptic problem for $\mathbf{A}_j$ with ψ_{j-1} given by the previous step. Thus the existence and uniqueness of $\mathbf{A}_j \in \mathbf{H}_n^1(\Omega)$ follows from a standard argument. When $\mathbf{A}_j$ is determined, (3.12) is a semilinear alliptic problem with respect to ψ_j. The existence and uniqueness of ψ_j again follows by a standard argument. □

LEMMA 3.3: *For any* $j \geqslant 1$, $|\psi_j| \leqslant 1$ *for almost every* $x \in \Omega$.

Proof: This lemma can be proved by taking $\omega = (|\psi_j|^2 - 1)_+ \bar{\psi}_j$ in (3.12) and using the method in [CHL 93]. □

In the following, c is used to denote various constants independent of N, Δt and ε. We show a number of lemmas which will enable us to take the limit in (3.12)-(3.13) when $N \to \infty$ and consequently, prove that (3.2)-(3.4) admits solutions.

LEMMA 3.4: *We have*

$$\max_{1 \leqslant j \leqslant N} \int_\Omega |\psi_j|^2 \, dx + \sum_{j=1}^N \Delta t \int_\Omega \left| \frac{i}{\kappa} \mathbf{grad}\, \psi_j + \mathbf{A}_j \, \psi_j \right|^2 dx \leqslant c \int_\Omega |\psi_0|^2 \, dx \, .$$

Proof: Take $\omega = \bar{\psi}_j \, \Delta t$ in (3.12). □

LEMMA 3.5: *We have*

$$\max_{1 \leqslant j \leqslant N} \int_\Omega |A_j|^2 \, dx + \sum_{j=1}^N \Delta t \int_\Omega [\,|\mathrm{div}\, \mathbf{A}_j|^2 + |\mathbf{curl}\, \mathbf{A}_j|^2] \, dx$$

$$\leqslant c \int_\Omega [\,|\mathbf{A}_0|^2 + |\psi_0|^2] \, dx + c[\, \|\mathbf{H}\|^2_{H^1(0,T,L^2(\Omega))} + \|\mathbf{H}_0\|^2_{L^2(\Omega)}] \, .$$

Proof: Taking $\mathbf{B} = \mathbf{A}_j \, \Delta t$ in (3.13) and applying Lemma 3.4, we obtain

$$\sum_{j=1}^N \Delta t \int_\Omega |\mathbf{H}_j|^2 \, dx \leqslant c[\, \|\mathbf{H}_0\|^2_{L^2(\Omega)} + \|\mathbf{H}\|^2_{H^1(0,T,L^2(\Omega))}] \, .$$

This completes the proof. □

LEMMA 3.6: *We have*

$$\sum_{j=1}^N \Delta t \int_\Omega |\mathbf{grad}\, \psi_j|^2 \, dx \leqslant c \int_\Omega [\,|\mathbf{A}_0|^2 + |\psi_0|^2] \, dx + c[\, \|\mathbf{H}\|^2_{H^1(0,T,L^2(\Omega))} + \|\mathbf{H}_0\|^2_{L^2(\Omega)}] \, .$$

Proof: Use the results of Lemmas 3.4 and 3.5. □

LEMMA 3.7: *Let* $\partial \mathbf{A}_j = (\mathbf{A}_j - \mathbf{A}_{j-1})/\Delta t$. *We have*

$$\sum_{j=1}^N \Delta t \int_\Omega |\partial \mathbf{A}_j|^2 \, dx + \max_{1 \leqslant j \leqslant N} \int_\Omega [\,|\mathrm{div}\, \mathbf{A}_j|^2 + |\mathbf{curl}\, \mathbf{A}_j|^2] \, dx$$

$$\leqslant c[\, \|\mathbf{A}_0\|^2_{H^1(\Omega)} + \|\psi_0\|^2_{L^2(\Omega)} + \|\mathbf{H}\|^2_{H^1(0,T,L^2(\Omega))} + \|\mathbf{H}_0\|^2_{L^2(\Omega)}] \, .$$

Proof: Let $\mathbf{B} = \Delta t\, \partial \mathbf{A}_j$ in (3.13), then for any $m \leqslant N$,

$$\sum_{j=1}^{m} \Delta t \int_\Omega \left[|\partial \mathbf{A}_j|^2 + \frac{1}{2} |\operatorname{div} \mathbf{A}_m|^2 + \frac{1}{2} |\mathbf{curl}\, \mathbf{A}_m|^2 \right] dx$$

$$\leqslant \frac{1}{2} \int_\Omega [\, |\operatorname{div} \mathbf{A}_0|^2 + |\mathbf{curl}\, \mathbf{A}_0|^2]\, dx$$

$$- \sum_{j=1}^{m} \Delta t \int_\Omega Re \left[\left(\frac{i}{\kappa} \mathbf{grad}\, \psi_{j-1} + \mathbf{A}_{j-1}\, \psi_{j-1} \right) \bar{\psi}_{j-1} \right] \partial \mathbf{A}_j\, dx$$

$$+ \sum_{j=1}^{m} \int_\Omega \mathbf{H}_j\, \mathbf{curl}\, (A_j - \mathbf{A}_{j-1})\, dx$$

$$= (\mathrm{III})_1 + (\mathrm{III})_2 + (\mathrm{III})_3 .$$

Use the results of Lemma 3.4, we have

$$(\mathrm{III})_2 \leqslant \frac{1}{2} \sum_{j=1}^{m} \Delta t \left[\left\| \frac{i}{\kappa} \mathbf{grad}\, \psi_{j-1} + \mathbf{A}_{j-1}\, \psi_{j-1} \right\|^2_{L^2(\Omega)} + \| \partial \mathbf{A}_j \|^2_{L^2(\Omega)} \right]$$

$$\leqslant \frac{1}{2} \sum_{j=1}^{m} \Delta t \| \partial \mathbf{A}_j \|^2_{L^2(\Omega)} + c \| \psi_0 \|^2_{L^2(\Omega)} .$$

$$(\mathrm{III})_3 = \int_\Omega \mathbf{H}_m\; \mathbf{curl}\, \mathbf{A}_m\, dx - \int_\Omega \mathbf{H}_0\; \mathbf{curl}\, \mathbf{A}_0\, dx - \sum_{n=1}^{m} \int_\Omega (H_j - H_{j-1})\, \mathbf{curl}\, \mathbf{A}_{j-1}\, dx$$

$$\leqslant \frac{1}{4} \int_\Omega |\mathbf{curl}\, \mathbf{A}_m|^2\, dx + \| \mathbf{H}_m \|^2_{L^2(\Omega)} + \| \mathbf{H}_0 \|^2_{L^2(\Omega)} + \| \mathbf{curl}\, \mathbf{A}_0 \|^2_{L^2(\Omega)}$$

$$+ \sum_{n=1}^{m} \Delta t \left[\| \partial \mathbf{H}_j \|^2_{L^2(\Omega)} + \frac{1}{4} \| \mathbf{curl}\, \mathbf{A}_{j-1} \|^2_{L^2(\Omega)} \right]$$

$$\leqslant \frac{1}{4} \int_\Omega |\mathbf{curl}\, \mathbf{A}_m|^2\, dx + \frac{1}{4} \sum_{n=1}^{m} \Delta t \| \mathbf{curl}\, \mathbf{A}_{j-1} \|^2_{L^2(\Omega)}$$

$$+ c [\, \| \mathbf{H}_0 \|^2_{L^2(\Omega)} + \| \mathbf{A}_0 \|^2_{H^1(\Omega)} + \| \mathbf{H} \|^2_{H^1(0,T,L^2(\Omega))}] .$$

The Lemma then follows from the Gronwall's inequality. □

LEMMA 3.8: *Let* $\partial \psi_j = (\psi_j - \psi_{j-1})/\Delta t$. *We have*

$$\sum_{j=1}^{N} \Delta t \int_\Omega |\partial \psi_j|^2\, dx + \max_{1 \leqslant j \leqslant N} \int_\Omega |\mathbf{grad}\, \psi_j|^2\, dx$$

$$\leqslant c [\, \| \mathbf{A}_0 \|^2_{H^1(\Omega)} + \| \psi_0 \|^2_{H^1(\Omega)} + \| \mathbf{H} \|^2_{H^1(0,T;L^2(\Omega))} + \| \mathbf{H}_0 \|^2_{L^2(\Omega)}] .$$

Proof: Take $w = \partial \bar{\psi}_j \Delta t$ in (3.12) and apply Lemmas 3.3-3.7. □

Proof of Theorem 3.1: The uniqueness has been proved in Lemma 3.1. The existence of the solutions $(\psi, \mathbf{A})$ satisfying (3.5)-(3.7) can be proved from the estimates given in Lemmas 3.3-3.8 by employing standard convergence argument (*cf.* e.g. [T 95]). □

As a direct consequence of Theorem 3.1 and Lemmas 3.3-3.8 we obtain the following result.

COROLLARY 3.1: *The solution* $(\psi, \mathbf{A})$ *of (3.2)-(3.4) satisfies the following estimate*

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \int_\Omega [\,|\psi|^2 + |\mathbf{A}|^2 + |\mathbf{grad}\,\psi|^2 + |\operatorname{div}\mathbf{A}|^2 + |\mathbf{curl}\,\mathbf{A}|^2]\,dx$$

$$+ \int_0^T \int_\Omega \left[\left|\frac{\partial \psi}{\partial t}\right|^2 + \left|\frac{\partial \mathbf{A}}{\partial t}\right|^2\right] dx\,dt$$

$$\leqslant c[\,\|\psi_0\|^2_{H^1(\Omega)} + \|\mathbf{A}_0\|^2_{H^1(\Omega)} + \|\mathbf{H}_0\|^2_{L^2(\Omega)}] + c\int_0^T \int_\Omega \left|\frac{\partial \mathbf{H}}{\partial t}\right|^2 dx\,dt\,,$$

where the constant c is independent of ε.

4. THE LIMIT WHEN $\varepsilon \to 0$

From now on, $\Omega = \Omega_\varepsilon$ and we denote by $(\psi_\varepsilon, \mathbf{A}_\varepsilon)$ the solutions to the system (3.2)-(3.4). Let $\rho(x') = a(x') + b(x')$. For any (real, complex and/or vector valued) function $f(x, t) = f(x', x_3, t)$ defined on $\Omega_\varepsilon \times (0, T)$, we define the average

$$[\![f]\!]\,(x', t) = \frac{1}{\varepsilon \rho(x')} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} f(x', x_3, t)\,dx_3\,.$$

The purpose of this section is to prove that as $\varepsilon \to 0$, the average $([\![\psi_\varepsilon]\!], [\![\mathbf{A}_\varepsilon]\!])$ of the solutions $(\psi_\varepsilon, \mathbf{A}_\varepsilon)$ of the three dimensional problem (3.2)-(3.4) converge to solutions of a two dimensional problem.

We first describe precisely the assumptions on the boundary $\partial\Omega_\varepsilon$. Let

$$\Gamma_1 = \{(x', x_3) \in \partial\Omega_\varepsilon : x_3 = \varepsilon a(x'), x' \in \Omega_0\}\,,$$

$$\Gamma_2 = \{(x', x_3) \in \partial\Omega_\varepsilon : x_3 = -\varepsilon b(x'), x' \in \Omega_0\}\,,$$

$$\Gamma_3 = \{(x', x_3) \in \partial\Omega_\varepsilon : x_3 \in (-\varepsilon b(x'), \varepsilon a(x')), x' \in \partial\Omega_0\}\,,$$

then it is clear that

$$\partial\Omega_\varepsilon = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup (\bar\Gamma_1 \cap \bar\Gamma_3) \cup (\bar\Gamma_2 \cap \bar\Gamma_3)\,.$$

From the assumptions made at the beginning of Section 2, we know that Γ_1, Γ_2, Γ_3 are of class C^2. We impose the following corner conditions for the boundary parts $\bar\Gamma_1 \cap \bar\Gamma_3$, $\bar\Gamma_2 \cap \bar\Gamma_3$. We point out that each Γ_j depends on ε, just for notational convenience, we drop the subscript ε.

HYPOTHESIS (H): *For every* $x_0 \in \bar{\Gamma}_1 \cap \bar{\Gamma}_3$ *(similar for* $x_0 \in \bar{\Gamma}_2 \cap \bar{\Gamma}_3$*) there exist a neighborhood* V *of* $x_0 \in \mathbb{R}^3$ *and a* C^2 *mapping* $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ *from* V *into* $\mathbb{R}^3$ *such that* $\boldsymbol{\eta}$ *is injective,* $\boldsymbol{\eta}$ *and* $\boldsymbol{\eta}^{-1}$ *(defined on* $(\boldsymbol{\eta}(V))$ *) are continuously differentiable, and*

$$\Omega_\varepsilon \cap V = \{x \in \Omega_\varepsilon : \eta_1(x) > 0, \eta_3(x) < 0\}$$

$$\Gamma_1 \cap V = \{x \in \partial\Omega_\varepsilon : \eta_1(x) > 0, \eta_3(x) = 0\}$$

$$\Gamma_3 \cap V = \{x \in \partial\Omega_\varepsilon : \eta_1(x) = 0, \eta_3(x) < 0\}$$

$$\bar{\Gamma}_1 \cap \bar{\Gamma}_3 \cap V = \{x \in \partial\Omega_\varepsilon : \eta_1(x) = 0, \eta_3(x) = 0\}.$$

Furthermore, we assume that the matrices $W = (\nabla_x \boldsymbol{\eta})^{-1}$ *and* $W^{-1} \in C^1(\bar{V})$, $\det W > 0$ *such that*

$$W|_{\Gamma_1 \cap V}(-\varepsilon a_x, -\varepsilon a_y, 1)^T / \sqrt{\varepsilon^2 |\mathbf{grad}\, a|^2 + 1} = (0, 0, 1)^T,$$

$$W|_{\Gamma_3 \cap V}\, \mathbf{n} = (-1, 0, 0)^T$$

where $\mathbf{n}$ *denotes the unit outward normal vector of* $\partial\Omega_\varepsilon$ *along* $\Gamma_3 \cap V$. □

The geometric meaning of this hypothesis is fairly obvious. We just point out here that if $a(x') \equiv$ constant, $b(x') \equiv$ constant (the case of a plate), it is straightforward to verify that Hypothesis (H) is satisfied provided that $\partial\Omega_0$ is sufficiently smooth. Another important remark is that the assumption $\det W > 0$ in (H) prevents that the edge of the plate forms a reentrant angle, this coincides with the assumption that a and b are C^1 functions up to the boundary and the requirement that $\boldsymbol{\eta}$ is a C^2 mapping which preserves the orientation of the domain. However, we choose not to verify rigorously these assumptions here and give the facts in the form of an hypothesis instead. For more detailed discussions about orientation preservation, image of a domain under a C^1 transformation, see [MTY 93] and references cited there.

We note that in general $(V, \boldsymbol{\eta})$ and the matrix W in Hypothesis (H) depend on ε, we drop the dependence on ε of the quantities and the quantities involved in the change of variables later in the proof of Lemma 4.2 for notational convenience. We made sure that the proofs will not be affected by this technical point.

In order to derive the two dimensional model, we make the following assumptions in this section in addition to (A1)-(A3):

(A1)′ $\psi_0 = \psi_0(x')$;

(A2)′ $\mathbf{A}_0 = (\mathbf{A}_0^1(x'), \mathbf{A}_0^2(x'), 0)^T$;

(A3)′ $\mathbf{H} = (0, 0, H(x', t))^T$.

4.1. The main result

The following theorem is the main result of this paper.

THEOREM 4.1: *Let the assumptions (A1)-(A3) and (A1)′-(A3)′ be satisfied. For any* $\varepsilon > 0$, *let* $(\psi_\varepsilon, \mathbf{A}_\varepsilon)$ *denote the solution of (3.2)-(3.4). Then, there exist two functions*

$$\psi \in \mathscr{H}^1(0, T; \mathscr{L}^2(\Omega_0)) \cap \mathscr{L}^\infty(0, T; \mathscr{H}^1(\Omega_0)) \text{ and } \mathbf{A}' \in \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega_0)) \cap \mathbf{L}^\infty(0, T; \mathbf{H}_n^1(\Omega_0))$$

such that as $\varepsilon \to 0$

$$[\![\psi_\varepsilon]\!] \to \psi \text{ weakly in } \mathscr{H}^1(0,T;\mathscr{L}^2(\Omega_0)) \text{ and weakly}^* \text{ in } \mathscr{L}^\infty(0,T;\mathscr{H}^1(\Omega_0)), \tag{4.1}$$

$$[\![\mathbf{A}_\varepsilon]\!] \to \mathbf{A} \text{ weakly in } \mathbf{H}^1(0,T;\mathbf{L}^2(\Omega_0)) \text{ and weakly}^* \text{ in } \mathbf{L}^\infty(0,T;\mathbf{H}^1_n(\Omega_0)), \tag{4.2}$$

where $\mathbf{A} = (\mathbf{A}', 0)$, *and* $(\psi, \mathbf{A}')$ *satisfies*

$$\psi(x', 0) = \psi_0(x'), \quad \mathbf{A}'(x', 0) = \mathbf{A}'_0(x') \tag{4.3}$$

and

$$\begin{aligned}
&\eta \int_0^T \int_{\Omega_0} \rho(x') \frac{\partial \psi}{\partial t} \omega \, dx' \, dt - i\eta\kappa \int_0^T \int_{\Omega_0} \operatorname{div}'(\rho(x')\mathbf{A}')\, \psi\omega \, dx' \, dt \\
&+ \int_0^T \int_{\Omega_0} \rho(x') \left(\frac{i}{\kappa} \mathbf{grad}' \, \psi + \mathbf{A}' \, \psi \right) \left(-\frac{i}{\kappa} \mathbf{grad}' \, \omega + \mathbf{A}' \, \omega \right) dx' \, dt \\
&+ \int_0^T \int_{\Omega_0} \rho(x') (|\psi|^2 - 1)\, \psi\omega \, dx' \, dt = 0, \quad \text{for any } \omega \in \mathscr{L}^2(0,T;\mathscr{H}^1(\Omega_0))
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} \rho(x') \frac{\partial \mathbf{A}'}{\partial t} \mathbf{B} \, dx' \, dt + \int_0^T \int_{\Omega_0} \rho(x') \operatorname{curl} \mathbf{A}' \operatorname{curl} \mathbf{B} \, dx' \, dt \\
&+ \int_0^T \int_{\Omega_0} \operatorname{div}'(\rho(x')\mathbf{A}') \operatorname{div}' \mathbf{B} \, dx' \, dt \\
&+ \int_0^T \int_{\Omega_0} \rho(x')\, Re\left[\left(\frac{i}{\kappa} \mathbf{grad}' \, \psi + \mathbf{A}' \, \psi \right) \bar{\psi} \right] \mathbf{B} \, dx' \, dt \\
&= \int_0^T \int_{\Omega_0} \rho(x')\, H \operatorname{curl} \mathbf{B} \, dx' \, dt, \quad \text{for any } \mathbf{B} \in \mathbf{L}^2(0,T;\mathbf{H}^1_n(\Omega_0)).
\end{aligned} \tag{4.5}$$

Moreover, the electric potential ϕ_ε *given in (2.7) satisfies, as* $\varepsilon \to 0$,

$$[\![\phi_\varepsilon]\!] \to -\frac{1}{\rho(x')} \operatorname{div}'(\rho(x')\mathbf{A}') \quad \text{weakly in } L^2(0,T;L^2(\Omega_0)). \tag{4.6}$$

Remark 4.1: (4.3)-(4.5) is the weak formulation of the following problem:

$$\begin{aligned}
&\eta\rho(x') \frac{\partial \psi}{\partial t} - i\eta\kappa \operatorname{div}'(\rho(x')\mathbf{A}')\, \psi + \left(\frac{i}{\kappa} \mathbf{grad}' + \mathbf{A}' \right) \left[\rho(x') \left(\frac{i}{\kappa} \mathbf{grad}' + \mathbf{A}' \right) \right] \psi \\
&\quad + \rho(x')(|\psi|^2 - 1)\, \psi = 0, \quad \text{in } Q_0 \\
&\rho(x') \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{curl}' \, [\rho(x') \, \mathbf{curl} \, \mathbf{A}'] - \mathbf{grad}' \, [\operatorname{div}'(\rho(x')\mathbf{A}')] \\
&\quad + \rho(x')\, Re\left[\left(\frac{i}{\kappa} \mathbf{grad}' \, \psi + \mathbf{A}' \, \psi \right) \bar{\psi} \right] = \mathbf{curl}' \, (\rho(x') H), \quad \text{in } Q_0 \\
&\quad \mathbf{grad}' \, \psi \,.\, \mathbf{n}' = 0, \quad \mathbf{A}' \,.\, \mathbf{n}' = 0, \quad \operatorname{curl} \mathbf{A}' = H, \quad \text{on } \Gamma_0 \\
&\quad \psi(x', 0) = \psi_0(x'), \quad \mathbf{A}'(x', 0) = \mathbf{A}'_0(x'), \quad \text{in } \Omega_0
\end{aligned}$$

where $Q_0 = \Omega_0 \times (0, T)$, $\Gamma_0 = \partial\Omega_0 \times (0, T)$, and $\mathbf{n}'$ is the unit outer normal of $\partial\Omega_0$.

Remark 4.2: It is clear from (4.3)-(4.5) that when $\rho(x')$ is a constant, we are able to derive the standard 2-dimensional Ginzburg-Landau equations.

Remark 4.3: We only have weak convergence in Theorem 4.1. Better convergence results require high regularity estimates independent of ε which we do not have.

The proof of Theorem 4.1 will be given in Section 4.3. We close this subsection by stating the following theorem concerning the solutions of (4.3)-(4.5) which can be proved by using the same methods used in the previous section and in [CHL 93].

THEOREM 4.2: *Let $\psi_0 \in \mathscr{H}^1(\Omega_0)$, $\mathbf{A}'_0 \in \mathbf{H}^1_n(\Omega_0)$ satisfying $|\psi_0| \leq 1$ a.e. in Ω_0. Assume that $H \in H^1(0, T; L^2(\Omega_0))$. Then (4.3)-(4.5) has a unique pair of solutions $(\psi, \mathbf{A}')$ satisfying*

$$\psi \in \mathscr{L}^\infty(0, T; \mathscr{H}^1(\Omega_0)) \cap \mathscr{H}^1(0, T; \mathscr{L}^2(\Omega_0));$$

$$\mathbf{A}' \in \mathbf{L}^\infty(0, T; \mathbf{H}^1_n(\Omega_0)) \cap \mathbf{H}^1(0, T; \mathbf{L}^2(\Omega_0));$$

and $|\psi| \leq 1$ a.e. in $\Omega_0 \times (0, T)$. Moreover, if $H \in H^1(0, T; H^1(\Omega_0))$, then we also have the regularity

$$\psi \in \mathscr{L}^2(0, T; \mathscr{H}^2(\Omega_0)), \quad \mathbf{A}' \in \mathbf{L}^2(0, T; \mathbf{H}^2(\Omega_0)).$$

4.2. Some estimates

In the following, c is used to denote the various constants independent of ε. We begin this subsection with some elementary results on the average operator $[\![\cdot]\!]$.

LEMMA 4.1: *Let $f \in L^2(\Omega_\varepsilon)$ be a given function, we have*

$$\| [\![f]\!] \|_{L^2(\Omega_0)} \leq \frac{c}{\sqrt{\varepsilon}} \|f\|_{L^2(\Omega_\varepsilon)}. \tag{4.7}$$

Let $f \in H^1(\Omega_\varepsilon)$ be a given function. For almost every $x' \in \Omega_0$, we have

$$|f(x', x_3) - [\![f]\!](x')| \leq c\sqrt{\varepsilon} \left\| \frac{\partial f}{\partial x_3} \right\|_{L^2(-\varepsilon b(x'), \varepsilon a(x'))} (x') \tag{4.8}$$

for all $x_3 \in [-\varepsilon b(x'), \varepsilon a(x')]$ where c is a constant independent of both ε and x'. Consequently,

$$\|f - [\![f]\!]\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon \left\| \frac{\partial f}{\partial x_3} \right\|_{L^2(\Omega_\varepsilon)}. \tag{4.9}$$

Proof: First, we note that by a regularity theorem due to Morrey and Necas (*cf.* [Mo 66], Chapter 3.1 or [Ne 67], Theorem 2.2.2, see also [MTY 93]): any $W^{1,1}(\Omega_\varepsilon)$ function $f(x', x_3)$ is absolutely continuous in x_3 for almost every $x' \in \Omega_0$ and the derivative in x_3 (which exists in an almost everywhere sense with respect to the Lebesgue measure of $\mathbb{R}^1$) coincides with the generalized derivative almost everywhere. Hence we have

$$f(x', x_3) - [\![f]\!](x') = \int_\xi^{x_3} \frac{\partial f}{\partial x_3} dx_3$$

for some $\xi \in (-\varepsilon b(x'), \varepsilon a(x'))$. Consequently, using the Cauchy-Schwartz inequality, we obtain

$$|f(x', x_3) - [\![f]\!](x')| \leq c\sqrt{\varepsilon}\left\|\frac{\partial f}{\partial x_3}\right\|_{L^2(-\varepsilon b(x'), \varepsilon a(x'))}(x') .$$

It is straightforward now to obtain (4.9). □

It is easy to see from (4.8) that for any $f \in H^1(\Omega_\varepsilon)$ and almost every $x' \in \Omega_0$ and $x_3 \in [-\varepsilon b(x'), \varepsilon a(x')]$, we have

$$|f(x', x_3)| \leq |[\![f]\!]|(x') + c\sqrt{\varepsilon}\left\|\frac{\partial f}{\partial x_3}\right\|_{L^2(-\varepsilon b(x'), \varepsilon a(x'))}(x') . \tag{4.10}$$

The next step is to show that the constant $\hat{C}$ in (2.1) is independent of ε. First, we prove a preliminary result.

LEMMA 4.2: *Under Hypothesis (H),* $\mathbf{H}_n^1(\Omega_\varepsilon) \cap \mathbf{H}^2(\Omega_\varepsilon)$ *is dense in* $\mathbf{H}_n^1(\Omega_\varepsilon)$.

Proof: For any $\mathbf{Q} \in \mathbf{H}_n^1(\Omega_\varepsilon)$, with the help of a family of local charts and of the partition of unity, we need to consider only $\phi\mathbf{Q}$ with ϕ being a nonnegative $C_0^\infty(\mathbb{R}^3)$ function such that supp $\phi \cap \bar{\Gamma}_1 \cap \bar{\Gamma}_3 \neq \Phi$ (same for supp $\phi \cap \bar{\Gamma}_2 \cap \bar{\Gamma}_3 \neq \Phi$ and simpler for supp $\phi \cap \Gamma_j \neq \Phi$, $j = 1, 2, 3$). In this case, let $(V, \boldsymbol{\eta})$ be the corresponding local chart in Hypothesis (H) and $\mathbf{P}(\,.\,) = (W^{-T}\phi\mathbf{Q})(\boldsymbol{\eta}^{-1}(\,.\,))$, the condition that $\mathbf{Q}\,.\,\mathbf{n} = \mathbf{Q}^T\,\mathbf{n} = 0$ on $\partial\Omega_\varepsilon$ is transformed to (note that $\mathbf{n}$ is transformed to $W\mathbf{n}$) $P_1 = 0$ on $y_1 = 0$, $P_3 = 0$ on $y_3 = 0$. Our problem is then to approximate such H^1 functions while keeping these boundary conditions.

We just sketch how to approximate P_1 and the rest of the proof can be carried out in a similar way. We define a new function P_1' as follows:

$$P_1' = \begin{cases} P_1(y_1, y_2, y_3) & \text{when } y_1 > 0, y_3 < 0, \\ P_1(y_1, y_2, -y_3) & \text{when } y_1 > 0, y_3 > 0, \\ -P_1(-y_1, y_2, -y_3) & \text{when } y_1 < 0, y_3 > 0, \\ -P_1(-y_1, y_2, y_3) & \text{when } y_1 < 0, y_3 < 0. \end{cases}$$

P_1' can be regarded as a function defined in $\mathbb{R}^3$ with compact support. Use a radially symmetric mollifier $\sigma_\mu(\,.\,)$ where μ is the standard mollifier parameter which goes to zero, we have obviously $P_1' * \sigma_\mu|_{y_1 = 0} = 0$ ($*$ denotes the standard convolution) and the sequence $\{P_1' * \sigma_\mu\}_\mu$ is the desired approximation of P_1 when restricted to the region $\{y \in \mathbb{R}^3 : y_1 > 0,\ y_3 < 0\}$. This completes the proof. □

LEMMA 4.3: *For any* $\mathbf{Q} \in \mathbf{H}_n^1(\Omega_\varepsilon)$, *we have*

$$\|\mathbf{Q}\|_{H^1(\Omega_\varepsilon)} \leq \hat{C}(1+\varepsilon)\left(\|\mathbf{Q}\|_{L^2(\Omega_\varepsilon)} + \|\operatorname{div}\mathbf{Q}\|_{L^2(\Omega_\varepsilon)} + \|\mathbf{curl}\,\mathbf{Q}\|_{L^2(\Omega)}\right), \quad \forall \mathbf{Q} \in \mathbf{H}_n^1(\Omega_\varepsilon)$$

where the constant $\hat{C}$ *is independent of* ε.

Proof: By Lemma 4.2, it is obvious that we only need to prove the lemma for $\mathbf{Q} \in \mathbf{H}^2(\Omega_\varepsilon) \cap \mathbf{H}_n^1(\Omega_\varepsilon)$. For any $\mathbf{Q} \in \mathbf{H}^2(\Omega_\varepsilon) \cap \mathbf{H}_n^1(\Omega_\varepsilon)$, we have (*cf.* [G 85], Theorem 3.1.1.2),

$$\int_{\Omega_\varepsilon} |\operatorname{div}\mathbf{Q}|^2\,dx + \int_{\Omega_\varepsilon} |\mathbf{curl}\,\mathbf{Q}|^2\,dx - \sum_{i,j=1}^{3}\int_{\Omega_\varepsilon}\left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx = \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \mathscr{B}(\mathbf{Q}\,;\mathbf{Q})\,d\sigma\,, \tag{4.11}$$

where $\mathscr{B}$ is the second fundamental quadratic form of Γ_j, $j = 1, 2, 3$, and $d\sigma$ is the surface element of $\partial\Omega_\varepsilon$. An elementary definition of $\mathscr{B}$ is recalled in [G 85], p. 133. If x_0 is a point of Γ_j, we consider a relatednew orthogonal coordinates system $\{y_1, y_2, y_3\}$ with origin at x_0 defined as follows: there exist a cube $V = \{(y_1, y_2, y_3) : \ -a_j < y_j < a_j,\ \ j = 1, 2, 3\}$ and a function φ of class C^2 in

$V' = \{(y_1, y_2) : -a_j < y_j < a_j, \quad j = 1, 2\}$ such that $|\varphi(y')| \leqslant a_3/2$ for every $y' = (y_1, y_2) \in V'$, $\Omega_\varepsilon \cap V = \{y = (y', y_3) \in V : y_3 < \varphi(y')\}$, $\Gamma_j \cap V = \{y = (y', y_3) \in V : \{y_3 = \varphi(y')\}$. We can even choose the new coordinates so that the plane $y_3 = 0$ is tangent to Γ_j at x_0, which implies $\nabla\varphi(0) = 0$. Then, if ξ, η are tangent vectors to Γ_j at x_0 with components (ξ_1, ξ_2) and (η_1, η_2) in the direction of $\{y_1, y_2\}$, we have

$$\mathscr{B}_{x_0}(\xi, \eta) = \sum_{j,k=1}^{2} \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(0)\, \xi_k \xi_j .$$

It is obvious that

$$|\mathscr{B}_{x_0}(\xi, \eta)| \leqslant c|\xi|\,|\eta|, \quad \text{for all } x_0 \in \Gamma_3 ,$$

for any tangent vectors ξ, η to Γ at x_0, and the constant c is independent of ε. Thus

$$\left|\int_{\Gamma_3} \mathscr{B}(\mathbf{Q} ; \mathbf{Q})\, d\sigma\right| \leqslant c \int_{\partial\Omega_0} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\mathbf{Q}|^2\, dx_3\, d\sigma' ,$$

where $d\sigma'$ is the integral element on $\partial\Omega_0$. By applying the trace theorem

$$\int_{\partial\Omega_0} |u|\, d\sigma' \leqslant C(\Omega_0) \left[\int_{\Omega_0} |\mathbf{grad}'\, u|\, dx' + \int_{\Omega_0} |u|\, dx'\right]$$

for all $u \in W^{1,1}(\Omega_0)$, we get

$$\left|\int_{\Gamma_3} \mathscr{B}(\mathbf{Q} ; \mathbf{Q})\, d\sigma\right| \leqslant c \int_{\Omega_0} \left|\mathbf{grad}'\left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\mathbf{Q}|^2\, dx_3\right]\right| dx' + c \int_{\Omega_0} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\mathbf{Q}|^2\, dx_3\, dx' .$$

It is easy to see that

$$\mathbf{grad}'\left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\mathbf{Q}|^2\, dx_3\right] = \int_{-\varepsilon b(x')}^{\varepsilon a(x')} \mathbf{grad}'\, |\mathbf{Q}|^2\, dx_3$$

$$+ \varepsilon|\mathbf{Q}(x', \varepsilon a(x'))|^2\, \mathbf{grad}'\, a(x') - \varepsilon|\mathbf{Q}(x', -\varepsilon b(x'))|^2\, \mathbf{grad}'\, b(x') .$$

But as in the proof of Lemma 4.1, we have

$$\begin{aligned}
&\int_{\Omega_0} |\varepsilon|\mathbf{Q}(x', \varepsilon a(x'))|^2\, \mathbf{grad}'\, a(x')|\, dx' \\
&\leqslant c\varepsilon \int_{\Omega_0} |\mathbf{Q}(x', \varepsilon a(x'))|^2\, dx' \\
&\leqslant c\varepsilon \int_{\Omega_0} [\![|\mathbf{Q}|^2]\!]\, dx' + c\varepsilon \int_{\Omega_0} ||\mathbf{Q}(x', \varepsilon a(x'))|^2 - [\![|\mathbf{Q}|^2]\!](x')|\, dx' \\
&\leqslant c \int_{\Omega_\varepsilon} |\mathbf{Q}|^2\, dx' + c\varepsilon \int_{\Omega_0} \left[\int_{\xi}^{\varepsilon a(x')} \frac{\partial|\mathbf{Q}|^2}{\partial x_3}\, dx_3\right] dx'
\end{aligned}$$

for some $\xi \in (-\varepsilon b(x'), \varepsilon a(x'))$. Thus we get

$$\int_{\Omega_0} |\varepsilon|\mathbf{Q}(x', \varepsilon a(x'))|^2 \,\mathbf{grad}'\, a(x')| \, dx' \leqslant c \int_{\Omega_\varepsilon} |\mathbf{Q}|^2 \, dx' + c\varepsilon \int_{\Omega_\varepsilon} \left|\mathbf{Q} \frac{\partial \mathbf{Q}}{\partial x_3}\right| dx$$

$$\leqslant \varepsilon^2 \delta \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c\left(1 + \frac{1}{\delta}\right) \|Q\|^2_{L^2(\Omega_\varepsilon)},$$

for any $\delta > 0$, where the constant c is independent of ε and δ.

Similarly, we get

$$\int_{\Omega_0} |\varepsilon|\mathbf{Q}(x', -\varepsilon b(x'))|^2 \,\mathbf{grad}'\, b(x')| \, dx' \leqslant \varepsilon^2 \delta \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c\left(1 + \frac{1}{\delta}\right) \|\mathbf{Q}\|^2_{L^2(\Omega_\varepsilon)},$$

for any $\delta > 0$, where the constant c is independent of ε and δ. By applying Cauchy's inequality and Young's inequality, we have

$$\int_{\Omega_0} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} \mathbf{grad}'\, |\mathbf{Q}|^2 \, dx_3 \, dx' \leqslant c \|\mathbf{Q}\|_{L^2(\Omega_\varepsilon)} \left[\sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx \right]^{\frac{1}{2}}$$

$$\leqslant \frac{1}{2} \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c \|\mathbf{Q}\|^2_{L^2(\Omega_\varepsilon)},$$

so that

$$\left|\int_{\Gamma_3} \mathscr{B}(\mathbf{Q} ; \mathbf{Q}) \, d\sigma\right| \leqslant \left(\frac{1}{2} + 2\varepsilon^2 \delta\right) \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c\left(1 + \frac{1}{\delta}\right) \|\mathbf{Q}\|^2_{L^2(\Omega_\varepsilon)}, \tag{4.12}$$

for any $\delta > 0$, where the constant c is independent of ε and δ. On Γ_1, it is easy to obtain that

$$|\mathscr{B}_{x_0}(\xi, \eta)| \leqslant c\varepsilon |\xi| \, |\eta|, \quad \text{for all } x_0 \in \Gamma_3,$$

for any tangent vectors ξ, η to Γ at x_0, and the constant c is independent of ε. Therefore

$$\left|\int_{\Gamma_1} \mathscr{B}(Q ; Q) \, d\sigma\right| \leqslant c\varepsilon \int_{\Omega_0} |Q(x', \varepsilon a(x'))|^2 \, dx'$$

$$\leqslant \varepsilon^2 \delta \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c\left(1 + \frac{1}{\delta}\right) \|\mathbf{Q}\|^2_{L^2(\Omega_\varepsilon)}, \tag{4.13}$$

for any $\delta > 0$ with the constant c independent of ε and δ. In deriving (4.13), we have used the argument leadint to (4.12). Similarly, we have

$$\left|\int_{\Gamma_2} \mathscr{B}(\mathbf{Q} ; \mathbf{Q}) \, d\sigma\right| \leqslant \varepsilon^2 \delta \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} \left|\frac{\partial Q_i}{\partial x_j}\right|^2 dx + c\left(1 + \frac{1}{\delta}\right) \|\mathbf{Q}\|^2_{L^2(\Omega_\varepsilon)}, \tag{4.14}$$

for any $\delta > 0$ with the constant c independent of ε and δ. Now the lemma follows from (4.11)-(4.14) by choosing $\delta = 1/16$ for $\varepsilon \leqslant 1$ and $\delta = 1/(16\varepsilon^2)$ for $\varepsilon \geqslant 1$. □

In the following we will always assume that $\varepsilon \leqslant 1$. From Lemma 4.3 and Corollary 3.1 we obtain the following lemma.

LEMMA 4.4: *We have*

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \left[\| \psi_\varepsilon \|^2_{H^1(\Omega_\varepsilon)} + \| \mathbf{A}_\varepsilon \|^2_{H^1(\Omega_\varepsilon)} \right] + \int_0^T \int_{\Omega_\varepsilon} \left[\left| \frac{\partial \psi_\varepsilon}{\partial t} \right|^2 + \left| \frac{\partial \mathbf{A}_\varepsilon}{\partial t} \right|^2 \right] dx\, dt \leqslant c\varepsilon .$$

To proceed further, note that

$$\begin{aligned} \mathbf{grad}' [\![f]\!] = {} & \mathbf{grad}' \left[\frac{1}{\varepsilon \rho(x')} \right] \int_{-\varepsilon b(x')}^{\varepsilon a(x')} f(x', x_3)\, dx_3 \\ & + \frac{1}{\varepsilon \rho(x')} \left[\varepsilon f(x', \varepsilon a(x'))\, \mathbf{grad}'\, a(x') + \varepsilon f(x', -\varepsilon b(x'))\, \mathbf{grad}'\, b(x') \right] \\ & + \frac{1}{\varepsilon \rho(x')} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} \mathbf{grad}' f(x', x_3)\, dx_3 \\ = {} & \frac{1}{\rho(x')} \mathbf{grad}'\, a(x') \left[f(x', \varepsilon a(x')) - [\![f]\!](x') \right] \\ & + \frac{1}{\rho(x')} \mathbf{grad}'\, b(x') \left[f(x', -\varepsilon b(x')) - [\![f]\!](x') \right] \\ & + [\![\mathbf{grad}'\, f]\!] . \end{aligned} \tag{4.15}$$

Therefore, by (4.7) and (4.8), we get

$$\| \mathbf{grad}' [\![f]\!] \|_{L^2(\Omega_0)} \leqslant \frac{c}{\sqrt{\varepsilon}} \| \mathbf{grad}'\, f \|_{L^2(\Omega_\varepsilon)} + c\sqrt{\varepsilon} \left\| \frac{\partial f}{\partial x_3} \right\|_{L^2(\Omega_\varepsilon)} . \tag{4.16}$$

LEMMA 4.5: *We have*

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \| [\![\psi_\varepsilon]\!] \|^2_{H^1(\Omega_0)} + \int_0^T \left\| \frac{\partial}{\partial t} [\![\psi_\varepsilon]\!] \right\|^2_{L^2(\Omega_0)} dt \leqslant c ,$$

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \| [\![\mathbf{A}_\varepsilon]\!] \|^2_{H^1(\Omega_0)} + \int_0^T \left\| \frac{\partial}{\partial t} [\![\mathbf{A}_\varepsilon]\!] \right\|^2_{L^2(\Omega_0)} dt \leqslant c .$$

Proof: It is obvious from Lemma 4.4 and (4.16). □

LEMMA 4.6: *Let* $\mathbf{A}_\varepsilon = (A^1_\varepsilon, A^2_\varepsilon, A^3_\varepsilon)$. *We have*

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \left[\| A^3_\varepsilon(\,\cdot\,, \varepsilon a(\,\cdot\,), t) \|_{L^2(\Omega_0)} + \| A^3_\varepsilon(\,\cdot\,, -\varepsilon b(\,\cdot\,), t) \|_{L^2(\Omega_0)} + \| [\![A^3_\varepsilon]\!](\,\cdot\,, t) \|_{L^2(\Omega_0)} \right] \leqslant c\varepsilon .$$

Proof: From the boundary condition $\mathbf{A}_\varepsilon \cdot \mathbf{n} = 0$, we obtain

$$A^3_\varepsilon(x', \varepsilon a(x'), t) = \varepsilon\, \mathbf{grad}'\, a(x') \cdot \mathbf{A}'_\varepsilon(x', \varepsilon a(x'), t) , \tag{4.17}$$

$$A^3_\varepsilon(x', -\varepsilon b(x'), t) = -\varepsilon\, \mathbf{grad}'\, b(x') \cdot \mathbf{A}'_\varepsilon(x', -\varepsilon b(x'), t) . \tag{4.18}$$

By (4.10) and Lemma 4.4, we get

$$\| A^3_\varepsilon(\,\cdot\,, \varepsilon a(\,\cdot\,), t) \|_{L^2(\Omega_0)} \leqslant c\varepsilon \| \mathbf{A}'_\varepsilon(\,\cdot\,, \varepsilon a(\,\cdot\,), t) \|_{L^2(\Omega_0)} \leqslant c\varepsilon . \tag{4.19}$$

Similarly, we have

$$\|A_\varepsilon^3(\,.\,, -\varepsilon b(\,.\,), t)\|_{L^2(\Omega_0)} \leqslant c\varepsilon \|\mathbf{A}'_\varepsilon(\,.\,, -\varepsilon b(\,.\,), t)\|_{L^2(\Omega_0)} \leqslant c\varepsilon\,. \tag{4.20}$$

Finally, by (4.8), Lemma 4.4 and (4.19), we have

$$\begin{aligned}\|[\![A_\varepsilon^3]\!](\,.\,, t)\|_{L^2(\Omega_0)} &\leqslant \|[\![A_\varepsilon^3]\!](\,.\,, t) - A_\varepsilon^3(\,.\,, \varepsilon a(\,.\,), t)\|_{L^2(\Omega_0)} + \|A_\varepsilon^3(\,.\,, \varepsilon a(\,.\,), t)\|_{L^2(\Omega_0)} \\ &\leqslant c\sqrt{\varepsilon}\left\|\frac{\partial A_\varepsilon^3}{\partial x_3}\right\|_{L^2(\Omega_\varepsilon)} + c\varepsilon \leqslant c\varepsilon\,,\end{aligned}$$

which completes the proof. □

Now we can give a sketch of proof for Proposition 2.1.

Sketch of proof of Proposition 2.1: In the following, to simplify notation, we denote Ω_ε by Ω. The function θ satisfies

$$\begin{cases}\theta_t - \Delta\theta = \operatorname{div}\tilde{\mathbf{A}} + \tilde{\phi}, & \text{in } \Omega \times (0, T)\,,\\ \partial\theta/\partial n = -\tilde{\mathbf{A}}\,.\,\mathbf{n}, & \text{on } \partial\Omega\,,\\ \theta(x, 0) = 0, & \text{in } \Omega\,.\end{cases}$$

Using a standard energy estimate, it is easy to show that $\theta_t \in L^2((0, T) \times \Omega)$ by, for example, the Faedo-Galerkin method. The remaining problem is then to show that, for a solution of

$$\begin{cases}-\Delta\varphi = g, & \text{in} \quad \Omega\,,\\ \partial\varphi/\partial\mathbf{n} = -\mathbf{B}\,.\,\mathbf{n} & \text{on} \quad \partial\Omega\end{cases}$$

with $\mathbf{B} \in \mathbf{H}^1(\Omega)$ and $g \in L^2(\Omega)$ such that $\int_\Omega g = \int_{\partial\Omega} \mathbf{B}\,.\,\mathbf{n}$, we have

$$\varphi \in H^2(\Omega)\,.$$

Had the boundary of the domain been C^2, the regularity would be obvious. But our domain is only Lipschitz. Now let's look at the problem from a different perspective. Let $\sigma = \mathbf{grad}\,\varphi$, we need to show that

$$\sigma \in \mathbf{H}^1(\Omega)$$

knowing that

$$\operatorname{div}\sigma = \Delta\varphi \in L^2(\Omega), \quad \mathbf{curl}\,\sigma = \mathbf{curl}\,(\mathbf{grad}\,\phi) = \mathbf{0}\,.$$

Introducing $w = \sigma + \mathbf{B} \in \mathbf{L}^2(\Omega)$, we wish to show that

$$w \in \mathbf{H}^1(\Omega)$$

while knowing

$$\begin{cases}\operatorname{div} w = \operatorname{div}\sigma + \operatorname{div}\mathbf{B} \in \mathrm{L}^2(\Omega)\,,\\ \mathbf{curl}\, w = \mathbf{curl}\,\mathbf{B} \in \mathbf{L}^2(\Omega)\,,\\ w\,.\,\mathbf{n} = \sigma\,.\,\mathbf{n} + \mathbf{B}\,.\,\mathbf{n} = 0\,.\end{cases}$$

Or equivalently, we want to show that if $w \in X$, where

$$X = \{ v \in \mathbf{L}^2(\Omega), \operatorname{div} v \in L^2(\Omega), \mathbf{curl}\, v \in \mathbf{L}^2(\Omega), v \,.\, \mathbf{n}|_{\partial\Omega} = 0 \},$$

then $w \in \mathbf{H}^1(\Omega)$ But we know from Lemma 4 3 that under Hypothesis (H) for the domain Ω, we have

$$\| w \|_{\mathbf{H}^1(\Omega)} \leqslant C(\| \operatorname{div} w \|_{L^2} + \| \mathbf{curl}\, w \|_{L^2} + \| w \|_{L^2}), \quad \forall w \in \mathbf{H}^1_n(\Omega)$$

If we can show that $\mathbf{H}^1_n(\Omega)$ is dense in X, then Proposition 2 1 is proved

LEMMA 4 7 *Let Ω be a bounded domain in R^3 with piecewise C^2 boundary satisfying (H) and having the following property*

for any point $x_0 \in \partial\Omega$, there exists a neighborhood of x_0, say $B(x_0) \subset R^3$ such that there is a one to one mapping

$$V \quad B(x_0) \to R^3$$

with **grad** *V and* **grad** *V^{-1} bounded in L^∞ and det* **grad** *$V > 0$ such that $W(B(x_0) \cap \partial\Omega)$ is a bounded smooth domain in the plane $x_3 = 0$*

Then $\mathbf{H}^1_n(\Omega)$ is dense in X

The proof of Lemma 2 1 then follows almost identically from that of [DL 76], pp 362-364

Our domain obviously satisfies the conditions in the above lemma, Proposition 2 1 now follows □

4.3. The convergence

In the weak formulation (3 3-(3 4), choosing the test functions ω, $\mathbf{B}$ as

$$\omega = \omega(x', t), \quad \mathbf{B} = (\mathbf{B}^1(x', t), \mathbf{B}^2(x', t), 0)$$

and noting that

$$\operatorname{div} \mathbf{B} = \operatorname{div}' \mathbf{B}', \quad \mathbf{curl}\, \mathbf{B} = (0, 0, \operatorname{curl} \mathbf{B}'),$$

we have

$$\begin{aligned} &\eta \int_0^T \int_{\Omega_0} \rho(x') \left[\!\left[\frac{\partial \psi_\varepsilon}{\partial t} \right]\!\right] \omega \, dx' \, dt - i\eta\kappa \int_0^T \int_{\Omega_0} \rho(x') [\![\operatorname{div} A_\varepsilon \, \psi_\varepsilon]\!] \, \omega \, dx' \, dt \\ &+ \int_0^T \int_{\Omega_0} \rho(x') \left[\!\left[\left(\frac{i}{\kappa} \mathbf{grad}' \, \psi_\varepsilon + \mathbf{A}'_\varepsilon \, \psi_\varepsilon \right) \left(- \frac{i}{\kappa} \mathbf{grad}' \, \omega + \mathbf{A}'_\varepsilon \, \omega \right) \right]\!\right] dx' \, dt \\ &+ \int_0^T \int_{\Omega_0} \rho(x') \left[\!\left[\left(\frac{i}{\kappa} \frac{\partial \psi_\varepsilon}{\partial x_3} + A^3_\varepsilon \, \psi_\varepsilon \right) A^3_\varepsilon \right]\!\right] \omega \, dx' \, dt \\ &+ \int_0^T \int_{\Omega_0} \rho(x') [\![(|\psi_\varepsilon|^2 - 1) \, \psi_\varepsilon]\!] \, \omega \, dx' \, dt = 0, \quad \text{for any } \omega \in \mathscr{L}^2(0, T, \mathscr{H}^1(\Omega_0)) \end{aligned} \tag{4 21}$$

$$\int_0^T \int_{\Omega_0} \rho(x') \left[\!\left[\frac{\partial \mathbf{A}'_\varepsilon}{\partial t}\right]\!\right] \mathbf{B}' \, dx' \, dt + \int_0^T \int_{\Omega_0} \rho(x') \, [\![\operatorname{curl} \mathbf{A}'_\varepsilon]\!] \operatorname{curl} \mathbf{B}' \, dx' \, dt$$

$$+ \int_0^T \int_{\Omega_0} \rho(x') \, [\![\operatorname{div} \mathbf{A}_\varepsilon]\!] \operatorname{div}' \mathbf{B}' \, dx' \, dt$$

$$+ \int_0^T \int_{\Omega_0} \rho(x') \left[\!\left[Re\left[\left(\frac{i}{\kappa} \mathbf{grad}' \psi_\varepsilon + \mathbf{A}'_\varepsilon \, \psi_\varepsilon \right) \bar{\psi}_\varepsilon \right] \right]\!\right] \mathbf{B}' \, dx' \, dt$$

$$= \int_0^T \int_{\Omega_0} \rho(x') \, H \operatorname{curl} \mathbf{B}' \, dx' \, dt, \quad \text{for any } \mathbf{B}' \in \mathbf{L}^2(0, T; \mathbf{H}^1_n(\Omega_0)) . \tag{4.22}$$

In order to let $\varepsilon \to 0$ in (4.21)-(4.22), we prove several lemmas.

LEMMA 4.8: *We have*

$$[\![\operatorname{div} \mathbf{A}_\varepsilon]\!] = \frac{1}{\rho(x')} \operatorname{div}'(\rho(x') \, [\![\mathbf{A}'_\varepsilon]\!])$$

Proof: By (4.15) and (4.17)-(4.18), we have

$$[\![\operatorname{div} \mathbf{A}'_\varepsilon]\!]$$

$$= [\![\operatorname{div}' \mathbf{A}'_\varepsilon]\!] + \frac{1}{\varepsilon \rho(x')} \int_{-\varepsilon b(x')}^{\varepsilon a(x')} \frac{\partial A^3_\varepsilon}{\partial x_3} \, dx_3$$

$$= [\![\operatorname{div}' \mathbf{A}'_\varepsilon]\!] + \frac{1}{\varepsilon \rho(x')} \left[A^3_\varepsilon(x', \varepsilon a(x'), t) - A^3_\varepsilon(x', -\varepsilon b(x'), t) \right]$$

$$= [\![\operatorname{div}' \mathbf{A}'_\varepsilon]\!] + \frac{1}{\rho(x')} \left[\mathbf{grad}' \, a(x') \, . \, \mathbf{A}'_\varepsilon(x', \varepsilon a(x'), t) + \mathbf{grad}' \, b(x') \, . \, \mathbf{A}'_\varepsilon(x', -\varepsilon b(x'), t) \right]$$

$$= [\![\operatorname{div}' \mathbf{A}'_\varepsilon]\!] + \frac{1}{\rho(x')} \left[\mathbf{grad}' \, a(x') + \mathbf{grad}' \, b(x') \right] [\![\mathbf{A}'_\varepsilon]\!]$$

$$= \frac{1}{\rho(x')} \operatorname{div}'(\rho(x') \, [\![\mathbf{A}'_\varepsilon]\!]) .$$

This completes the proof. □

LEMMA 4.9: *We have*

$$\operatorname{esssup}_{0 \leq t \leq T} \| [\![\operatorname{curl} \mathbf{A}'_\varepsilon]\!] - \operatorname{curl} [\![\mathbf{A}'_\varepsilon]\!] \|_{L^2(\Omega_0)} \leq c\varepsilon ; \tag{4.23}$$

$$\operatorname{esssup}_{0 \leq t \leq T} \| [\![\mathbf{grad}' \, \psi_\varepsilon]\!] - \mathbf{grad}' [\![\psi_\varepsilon]\!] \|_{L^2(\Omega_0)} \leq c\varepsilon . \tag{4.24}$$

Proof: We only prove (4.23). The estimate (4.24) can be proved similarly. By (4.15), we have

$$[\![\operatorname{curl} \mathbf{A}'_\varepsilon]\!] - \operatorname{curl} [\![\mathbf{A}'_\varepsilon]\!]$$

$$= \frac{1}{\rho(x')} \mathbf{curl}' \, a(x') \, [\mathbf{A}'_\varepsilon(x', \varepsilon a(x'), t) - [\![\mathbf{A}'_\varepsilon]\!] (x', t)]$$

$$+ \frac{1}{\rho(x')} \mathbf{curl}' b(x') \, [\mathbf{A}'_\varepsilon(x', -\varepsilon b(x'), t) - [\![\mathbf{A}'_\varepsilon]\!] (x', t)] \, .$$

Now (4.23) follows from (4.8) and Lemma 4.4. □

LEMMA 4.10: *We have*

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \| \rho [\![\operatorname{div} \mathbf{A}_\varepsilon \; \psi_\varepsilon]\!] - \operatorname{div}'(\rho [\![\mathbf{A}'_\varepsilon]\!]) \, [\![\psi_\varepsilon]\!] \|_{L^1(\Omega_0)} \leqslant c\varepsilon \, ; \tag{4.25}$$

$$\operatorname{esssup}_{0 \leqslant t \leqslant T} \| [\![\mathbf{grad}' \; \psi_\varepsilon \, . \, \mathbf{A}'_\varepsilon]\!] - \mathbf{grad}' [\![\psi_\varepsilon]\!] \, . \, [\![\mathbf{A}'_\varepsilon]\!] \|_{L^1(\Omega_0)} \leqslant c\varepsilon \, . \tag{4.26}$$

Proof: We only prove (4.25). The estimate (4.26) can be proved similarly. By Lemma 4.8 we have

$$\rho [\![\operatorname{div} \mathbf{A}_\varepsilon \; \psi_\varepsilon]\!] - \operatorname{div}'(\rho [\![\mathbf{A}'_\varepsilon]\!]) \, [\![\psi_\varepsilon]\!] = \rho [\![\operatorname{div} \mathbf{A}_\varepsilon \, (\psi_\varepsilon - [\![\psi_\varepsilon]\!])]\!] \, .$$

Note that by (4.9) and Lemma 4.4, we have

$$\| \rho [\![\operatorname{div} \mathbf{A}_\varepsilon \, (\psi_\varepsilon - [\![\psi_\varepsilon]\!])]\!] \|_{L^1(\Omega_0)} \leqslant \frac{c}{\varepsilon} \, \| \operatorname{div} \mathbf{A}_\varepsilon \, \|_{L^2(\Omega_\varepsilon)} \, \| \psi_\varepsilon - [\![\psi_\varepsilon]\!] \|_{L^2(\Omega_\varepsilon)}$$

$$\leqslant \frac{c}{\varepsilon} \cdot c \sqrt{\varepsilon} \cdot c\varepsilon \left\| \frac{\partial \psi_\varepsilon}{\partial x_3} \right\|_{L^2(\Omega_\varepsilon)} \leqslant c\varepsilon \, .$$

This completes the proof. □

LEMMA 4.11: *Let* $Q_0 = \Omega_0 \times (0, T)$. *We have*

$$\| [\![\mathbf{A}'_\varepsilon \, \psi_\varepsilon]\!] - [\![\mathbf{A}'_\varepsilon]\!] \, [\![\psi_\varepsilon]\!] \|_{L^2(Q_0)} \leqslant c\varepsilon \, ; \tag{4.27}$$

$$\| [\![|A'_\varepsilon|^2 \, \psi_\varepsilon]\!] - | [\![A'_\varepsilon]\!] |^2 \, [\![\psi_\varepsilon]\!] \|_{L^1(Q_0)} \leqslant c\varepsilon \, ; \tag{4.28}$$

$$\| [\![(|\psi_\varepsilon|^2 - 1) \, \psi_\varepsilon]\!] - (| [\![\psi]\!] |^2 - 1) \, [\![\psi_\varepsilon]\!] \|_{L^1(Q_0)} \leqslant c\varepsilon \, . \tag{4.29}$$

Proof: We only prove (4.27) and (4.28). The estimate (4.29) can be proved similarly. Since $|\psi_\varepsilon| \leqslant 1$ a.e. in Q_ε, we have

$$\int_0^T \int_{\Omega_0} | [\![\mathbf{A}'_\varepsilon \, \psi_\varepsilon]\!] - [\![\mathbf{A}'_\varepsilon]\!] \, [\![\psi_\varepsilon]\!] |^2 \, dx' \, dt$$

$$= \int_0^T \int_{\Omega_0} | [\![(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!]) \, \psi_\varepsilon]\!] |^2 \, dx' \, dt$$

$$\leqslant \int_0^T \int_{\Omega_0} \frac{1}{\varepsilon^2 \, \rho(x')^2} \left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} | \mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!] | \, . \, |\psi_\varepsilon| \, dx_3 \right]^2 dx' \, dt$$

$$\leqslant \int_0^T \int_{\Omega_0} \frac{1}{\varepsilon^2 \, \rho(x')^2} \left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} \left| \frac{\partial \mathbf{A}'_\varepsilon}{\partial x_3} \right| dx_3 \right]^2 \left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\psi_\varepsilon| \, dx_3 \right]^2 dx' \, dt$$

$$\leqslant \int_0^T \int_{\Omega_0} \left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')} \left| \frac{\partial \mathbf{A}'_\varepsilon}{\partial x_3} \right| dx_3 \right]^2 dx' \, dt$$

$$\leqslant c\varepsilon \int_0^T \int_{\Omega_\varepsilon} \left| \frac{\partial \mathbf{A}'_\varepsilon}{\partial x_3} \right|^2 dx \, dt \leqslant c\varepsilon^2 \, ,$$

where in the last inequality we have used Lemma 4.4. This proves (4.27). In order to show (4.28) we observe first that

$$\begin{aligned}&[\![|\mathbf{A}'_\varepsilon|^2\,\psi_\varepsilon]\!] - |[\![\mathbf{A}'_\varepsilon]\!]|^2\,[\![\psi_\varepsilon]\!]\\ &= [\![|\mathbf{A}'_\varepsilon|^2(\psi_\varepsilon - [\![\psi_\varepsilon]\!])]\!] + [\![\mathbf{A}'_\varepsilon(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!])]\!]\,[\![\psi_\varepsilon]\!] \end{aligned}\tag{4.30}$$

But

$$\begin{aligned}&\int_{\Omega_0} |[\![|A'_\varepsilon|^2(\psi_\varepsilon - [\![\psi_\varepsilon]\!])]\!]|\,dx'\\ &\leqslant \int_{\Omega_0}\frac{1}{\varepsilon p(x')}\int_{-\varepsilon b(x')}^{\varepsilon a(x')} |\mathbf{A}'_\varepsilon|^2\,|\psi_\varepsilon - [\![\psi_\varepsilon]\!]|\,dx_3\,dx'\\ &\leqslant \int_{\Omega_0}\left[\frac{1}{\varepsilon p(x')}\int_{-\varepsilon b(x')}^{\varepsilon a(x')}|A'_\varepsilon|^2\,dx_3\right]\cdot\left[\int_{-\varepsilon b(x')}^{\varepsilon a(x')}\left|\frac{\partial\psi_\varepsilon}{\partial x_3}\right|dx_3\right]dx'\\ &\leqslant \|[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^2(\Omega_0)}\left[\int_{\Omega_0}\left|\int_{-\varepsilon b(x')}^{\varepsilon a(x')}\left|\frac{\partial\psi_\varepsilon}{\partial x_3}\right|dx_3\right|^2 dx'\right]^{1/2}\\ &\leqslant c\varepsilon\,\|[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^2(\Omega_0)}\,,\end{aligned}\tag{4.31}$$

where we have used Lemma 4.4. By the embedding theorem $W^{1,1}(\Omega_0) \hookrightarrow L^4(\Omega_0)$, we get

$$\|[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^2(\Omega_0)} \leqslant C(\Omega_0)\,[\,\|\mathbf{grad}'[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^1(\Omega_0)} + \|[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^1(\Omega_0)}]\,.$$

It is easy to see that

$$\|[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^1(\Omega_0)} \leqslant \frac{c}{\varepsilon}\,\|\mathbf{A}'_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leqslant c\,,\tag{4.32}$$

where we have used Lemma 4.4. By (4.15), we have

$$\begin{aligned}&\|\mathbf{grad}'[\![|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^1(\Omega_0)}\\ &\qquad\leqslant \|[\![\mathbf{grad}'\,|\mathbf{A}'_\varepsilon|^2]\!]\|_{L^1(\Omega_0)} + c\,\|\,|\mathbf{A}'_\varepsilon|^2(\,.\,,\varepsilon a(\,.\,),t) - [\![|\mathbf{A}'_\varepsilon|^2]\!](\,.\,,t)\|_{L^1(\Omega_0)}\\ &\qquad\qquad + c\,\|\,|\mathbf{A}'_\varepsilon|^2(\,.\,,-\varepsilon b(\,.\,),t) - [\![|\mathbf{A}'_\varepsilon|^2]\!](\,.\,,t)\|_{L^1(\Omega_0)}\\ &\qquad\leqslant \frac{c}{\varepsilon}\,\|\mathbf{A}'_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,\|\mathbf{A}'_\varepsilon\|_{H^1(\Omega_\varepsilon)} + c\,\|\mathbf{A}'_\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|\frac{\partial\mathbf{A}'_\varepsilon}{\partial x_3}\right\|_{L^2(\Omega_\varepsilon)}\\ &\qquad\leqslant \frac{c}{\varepsilon}\cdot c\sqrt{\varepsilon}\cdot c\sqrt{\varepsilon} + c\sqrt{\varepsilon}\cdot c\sqrt{\varepsilon} \leqslant c\,,\end{aligned}\tag{4.33}$$

where we have used Lemma 4.4. Thus, by (4.31)-(4.33), we have

$$\|[\![|\mathbf{A}'_\varepsilon|^2(\psi_\varepsilon - [\![\psi_\varepsilon]\!])]\!]\|_{L^1(\Omega_0)} \leqslant c\varepsilon\,.\tag{4.34}$$

On the other hand, since $|\psi_\varepsilon| \leq 1$ a.e. in $\Omega_\varepsilon \times (0,T)$, we have $\|[\![\psi_\varepsilon]\!]\|_{L^\infty(Q_0)} \leq 1$. Thus

$$\begin{aligned}
\|[\![\mathbf{A}'_\varepsilon(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!])]\!]\,[\![\psi_\varepsilon]\!]\|_{L^1(Q_0)} &\leq \int_0^T \|[\![\mathbf{A}'_\varepsilon(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!])]\!]\|_{L^1(\Omega_0)}\,dt \\
&\leq \frac{c}{\varepsilon}\int_0^T \|\mathbf{A}'_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,\|\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!]\|_{L^2(\Omega_\varepsilon)}\,dt \\
&\leq \frac{c}{\varepsilon}\int_0^T c\sqrt{\varepsilon}\,.\,c\varepsilon\left\|\frac{\partial \mathbf{A}'_\varepsilon}{\partial x_3}\right\|_{L^2(\Omega_\varepsilon)} dt \\
&\leq c\varepsilon\,, \qquad (4.35)
\end{aligned}$$

where we have used Lemma 4.4 and (4.9). Now (4.28) follows from (4.30) and (4.34)-(4.35). This completes the proof. □

LEMMA 4.12: *We have*

$$\operatorname{esssup}_{0 \leq t \leq T}\left\|\left[\!\!\left[\left(\frac{i}{\kappa}\frac{\partial \psi_\varepsilon}{\partial x_3} + A_\varepsilon^3\,\psi_\varepsilon\right)A_\varepsilon^3\right]\!\!\right]\right\|_{L^1(\Omega_0)} \leq c\sqrt{\varepsilon}\,.$$

Proof: At first, by using (4.9), Lemma 4.4 and Lemma 4.6, we have

$$\begin{aligned}
\|A_\varepsilon^3\|_{L^2(\Omega_\varepsilon)} &\leq \|A_\varepsilon^3 - [\![A_\varepsilon^3]\!]\|_{L^2(\Omega_\varepsilon)} + \|[\![A_\varepsilon^3]\!]\|_{L^2(\Omega_\varepsilon)} \\
&\leq c\varepsilon\left\|\frac{\partial A_\varepsilon^3}{\partial x_3}\right\|_{L^2(\Omega_\varepsilon)} + c\varepsilon \\
&\leq c\varepsilon\,.\,c\sqrt{\varepsilon} + c\varepsilon \leq c\varepsilon\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|\left[\!\!\left[\left(\frac{i}{\kappa}\frac{\partial \psi_\varepsilon}{\partial x_3} + A_\varepsilon^3\,\psi_\varepsilon\right)A_\varepsilon^3\right]\!\!\right]\right\|_{L^1(\Omega_0)} \\
&\leq \frac{c}{\varepsilon}\left\|\frac{i}{\kappa}\frac{\partial \psi_\varepsilon}{\partial x_3} + A_\varepsilon^3\,\psi_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} \|A_\varepsilon^3\|_{L^2(\Omega_\varepsilon)} \\
&\leq c\sqrt{\varepsilon}\,,
\end{aligned}$$

where we have used Lemma 4.4. □

LEMMA 4.13: *We have*

$$\|[\![\mathbf{grad}'\,\psi_\varepsilon\,.\,\bar{\psi}_\varepsilon]\!] - \mathbf{grad}'[\![\psi_\varepsilon]\!]\,.\,[\![\bar{\psi}_\varepsilon]\!]\|_{L^2(Q_0)} \leq c\varepsilon\,; \qquad (4.36)$$

$$\|[\![\mathbf{A}'_\varepsilon|\psi_\varepsilon|^2]\!] - [\![\mathbf{A}'_\varepsilon]\!]\,|[\![\psi_\varepsilon]\!]|^2\|_{L^2(0,T,L^{\frac{4}{3}}(\Omega_0))} \leq c\varepsilon\,. \qquad (4.37)$$

Proof: The estimate (4.36) can be proved by using (4.15), (4.8), Lemma 4.4 and the fact that $|\psi_\varepsilon| \leqslant 1$ a.e. in Q_ε. To prove (4.37) we observe first that

$$[\![\mathbf{A}'_\varepsilon |\psi_\varepsilon|^2]\!] - [\![\mathbf{A}'_\varepsilon]\!]\, |[\![\psi_\varepsilon]\!]|^2$$

$$= [\![(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!])\, |\psi_\varepsilon|^2]\!] + [\![\mathbf{A}'_\varepsilon]\!]\, [\![(\psi_\varepsilon - [\![\psi_\varepsilon]\!])\, \bar{\psi}_\varepsilon]\!] \,. \tag{4.38}$$

On the one hand, similar to the proof of (4.27) in Lemma 4.11, we can obtain

$$\| [\![(\mathbf{A}'_\varepsilon - [\![\mathbf{A}'_\varepsilon]\!])]\!]\, |\psi_\varepsilon|^2 \|_{L^2(\Omega_0)} \leqslant c\varepsilon \,. \tag{4.39}$$

On the other hand, by Hölder's inequality, the embedding theorem $L^4(\Omega_0) \hookrightarrow H^1(\Omega_0)$ and Lemma 4.5, we have

$$\| [\![\mathbf{A}'_\varepsilon]\!]\, [\![(\psi_\varepsilon - [\![\psi_\varepsilon]\!])\, \bar{\psi}_\varepsilon]\!] \|_{L^{\frac{4}{3}}(\Omega_0)} \leqslant \| [\![\mathbf{A}'_\varepsilon]\!] \|_{L^4(\Omega_0)}\, \| [\![(\psi_\varepsilon - [\![\psi_\varepsilon]\!])\, \bar{\psi}_\varepsilon]\!] \|_{L^2(\Omega_0)}$$

$$\leqslant c \| [\![(\psi_\varepsilon - [\![\psi_\varepsilon]\!])\, \bar{\psi}_\varepsilon]\!] \|_{L^2(\Omega_0)} \,.$$

Again similar to the proof of (4.27) in Lemma 4.11, we can prove

$$\| [\![(\psi_\varepsilon - [\![\psi_\varepsilon]\!]) \bar{\psi}_\varepsilon]\!] \|_{L^2(\Omega_0)} \leqslant c\varepsilon \,. \tag{4.40}$$

Now the estimate (4.37) follows from (4.38)-(4.40). □

Proof of Theorem 4.1: From Lemma 4.5 we know that there exist two functions ψ, $\mathbf{A}$ such that after possibly extracting a subsequence, (4.1)-(4.2) are satisfied. From Lemma 4.6 we have that $A^3 = 0$, that is, $\mathbf{A} = (\mathbf{A}', 0)$. By letting $\varepsilon \to 0$ in (4.21) and using Lemma 4.7-4.10 we know that $(\psi, \mathbf{A}')$ satisfies (4.4) for any $\omega \in \mathscr{L}^2(0, T; \mathscr{H}^1(\Omega_0)) \cap \mathscr{L}^\infty(Q_0)$ thus also for any $\omega \in \mathscr{L}^2(0, T; \mathscr{H}^1(\Omega_0))$ by the standard density argument. That $(\psi, \mathbf{A}')$ satisfies (4.5) can be obtained by letting $\varepsilon \to 0$ in (4.22) by using Lemma 4.7-4.12, the embedding theorem $\mathbf{H}^1_n(\Omega_0) \hookrightarrow \mathbf{L}^4(\Omega_0)$ and employing standard convergence argument. □

5. A NUMERICAL EXAMPLE

We now present some numerical computations obtained by using the model derived in Section 4. The numerical method used to solve the system (4.4)-(4.5) is based on a semi-implicit finite element scheme using piecewise continuous biquadratic polynomials based on a subdivision of Ω_0 into a quadrilateral grid. This scheme was proposed and analyzed in [CH 95] for the case $\rho =$ constant. One of the purposes of the computations is to show that the model derived in this paper is effective in simulating the pinning mechanism of the vortices in superconducting thin films with variable thickness. A more detailed study of the numerical aspects of the model and more numerical simulations will be reported elsewhere.

In the numerical example here, we take the domain $\Omega_0 = (0, 1) \times (0, 1)$, the length of the time interval $T = 50$, $\kappa = 10$, the time step size $\Delta t = 0.1$. The grid over Ω_0 is obtained by subdividing Ω_0 into a uniform grid having 40 intervals in each direction. The vertical shape of the thin film is created by setting $\rho(x_1, x_2) = 0.9$ for (x_1, x_2) in the circle centered at (0.25, 0.25) with radius 0.1, $\rho(x_1, x_2) = 1$ for (x_1, x_2) outside the circle centered at (0.25, 0.25) with radius 0.2, and in between, ρ is smooth. The applied magnetic field H depends on t with $H(t) = 0, 1, 2, 3, 4$ for $t \in [0, 10), [10, 20), [20, 30), [30, 40), [40, 50)$, respectively. The contour plots of the magnitude of the density $|\psi|^2$ are given in Figure 1. We observe that in the magnetization process, one vortex first forms in the region near the lower left corner of Ω_0 (where the film is thinner) when the applied magnetic field H is increased to 2, and later this region is kept in the normal phase as the applied magnetic field is increased. Moreover, this region absorbs the new vortex coming from the left side of the domain as the pictures in Figure 1 indicated. This simple numerical example shows that the model derived in Section 4 can indeed be used to simulate the "pinning" mechanism of the vortices in variable thickness thin films.

6. CONCLUDING REMARKS

In this paper, a new two-dimensional evolutionary Ginzburg-Landau model for thin super-conducting films has been derived by letting the thickness of the film uniformly approach zero in the corresponding three-dimensional Ginzburg-Landau model. When the thickness function is constant, the standard two-dimensional Ginzburg-Landau

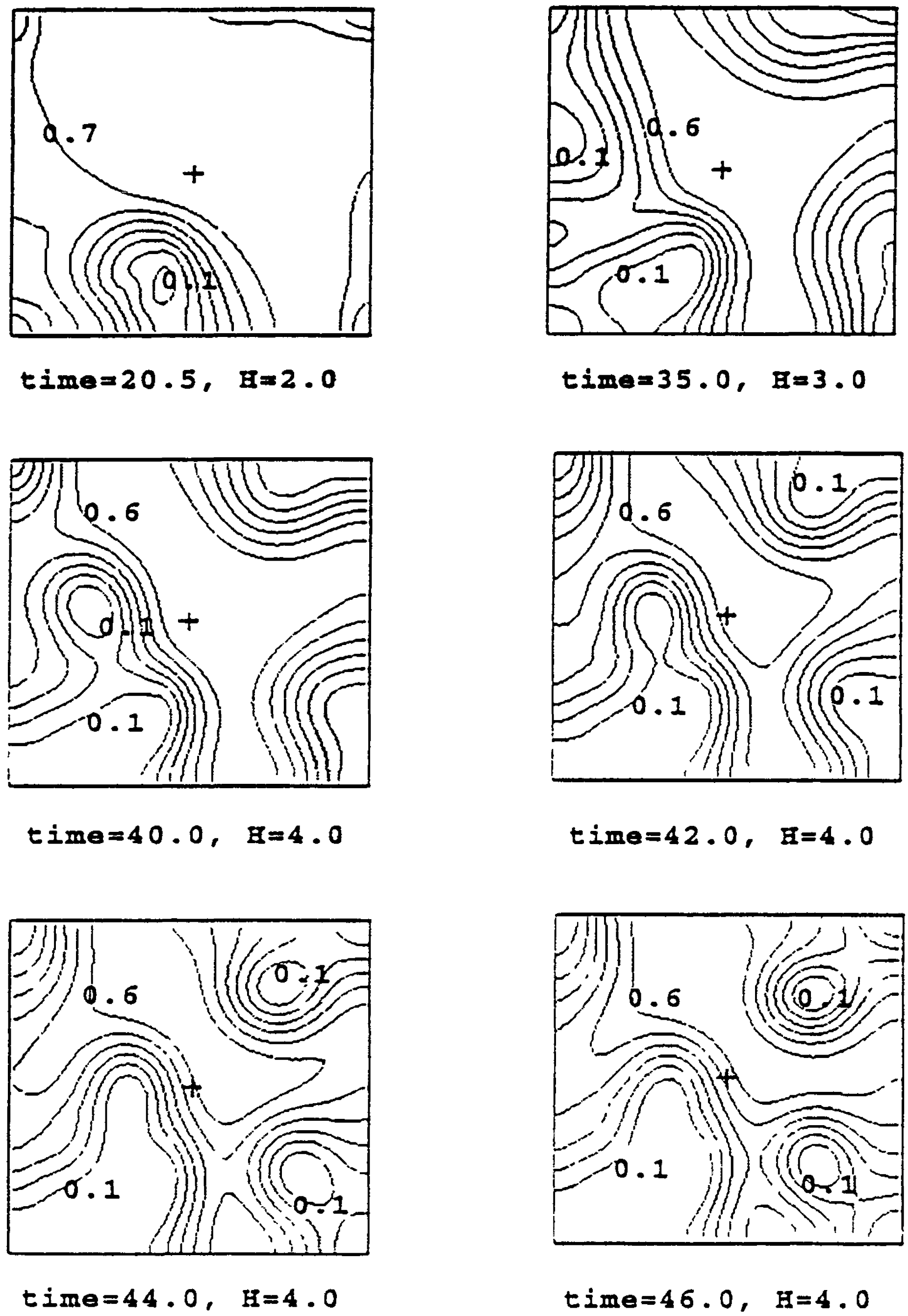

Figure 1. — Contour plots of the magnitude of the density.

model is recovered. The derivation was carried out under the Lorentz gauge (2.7). However, the method in this paper can also be used to prove the convergence in the other gauges, for example, Coulomb's gauge. Here we describe briefly this case.

By Coulomb's gauge, we refer to the gauge equivalent class where

$$\operatorname{div} \mathbf{A}_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon \times (0, T),$$

$$\mathbf{A}_\varepsilon \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \times (0, T),$$

$$\int_{\Omega_\varepsilon} \phi_\varepsilon \, dx = 0 \quad \text{a.e. in} \quad (0, T).$$

This is different from our previous case because the electrical potential will then be involved in the formulation of the problem explicitly and the condition $\operatorname{div} \mathbf{A}_\varepsilon = 0$ becomes an explicit constraint. However, since all the estimates in this paper holds, we have no difficulty in proving that the variational formulation converges as $\varepsilon \to 0$. For the explicit constraint $\operatorname{div} \mathbf{A}_\varepsilon = 0$, let $\mathbf{A}$ be the weak limit of $[\![\mathbf{A}_\varepsilon]\!]$, it is then straightforward to show that $\operatorname{div} \mathbf{A}_\varepsilon$ tends to

$$\frac{1}{\rho(x')} \operatorname{div}'(\rho(x') \, \mathbf{A}')$$

weakly in $L^2(\Omega_0 \times (0, T))$ (*cf.* Lemma 4.7).

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