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MODELING THE VIBRATIONS OF A MULTI-ROD STRUCTURE (*)

by N. KERDID (1)

Abstract. — *In this article we show that the eigenvalues and eigenfunctions of three-dimensional linearized elasticity in a body consisting of two rods of thickness ε , converge toward the eigenvalues and eigenfunctions of a limit 1d-1d rod model, as the thickness of the rods goes to zero.*

Résumé. — *Dans cet article, on montre que les valeurs propres et les fonctions propres du problème d'élasticité linéarisée tridimensionnelle dans une multi-structure formée de deux poutres convergent, vers les valeurs propres et les fonctions propres d'un modèle 1d-1d bien posé quand l'épaisseur des poutres tend vers zéro.*

0. INTRODUCTION

In this article, we study the asymptotic behaviour of the eigenvalue problem for a multi-rod structure, when the thickness of the rods goes to zero.

Following the work of Ciarlet & Destuynder [6] and Ciarlet & Kesavan [7] on plates, a general method was developed and used successfully to study a large variety of problems for rods: Aganovič & Tutek [1], Bermudez & Viaño [2], Kerdid [14] and Trabucho & Viaño ([27]-[31]) for homogeneous isotropic linearly elastic rods, Cimetière, Geymonat, Le Dret, Raoult & Tutek [9] for nonlinear elastic rods, Sanchez-Hubert & Sanchez-Palencia [26] for nonhomogeneous anisotropic rods and Geymonat, Krasuki & Marigo [11] for rods formed of composite materials, among many others.

The problem of modeling the junction between different elastic structures (3d-bodies, plates and rods) has only recently been investigated from a mathematical viewpoint. The first works on this subject are due to Ciarlet, Le Dret & Nzengwa [8] for the problem of a plate inserted in a 3d body, Bourquin & Ciarlet [4] for the associated eigenvalue problem, Le Dret ([17], [18], [20]) for junctions between plates and Le Dret [19] for junctions between rods. The method used consists in scaling each structure independently of the other,

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passing then to the limit in the variational equations posed over domains independent of the thickness of different structures and finally identifying this limit. This idea was later used to study different problems of junctions in three-dimensional elasticity: Gruais ([12], [13]) for the static case, Kerdid [16] and Lods [23] for the eigenvalue problem and Le Dret [21], Raoult [24] for elastodynamics problems. For the numerical analysis of junctions between rods and plates see Bernadou Fayolle & L  n   [3] and Fayolle [10].

We consider here the same multi-structure as in Le Dret [19], i.e., an homogeneous isotropic linearly elastic body consisting of two rods of thickness ε perpendicular to each other. The multi-structure is assumed to be clamped at one of its ends only.

Our purpose is achieved by combining the techniques of Le Dret [19], who treat the problem of modeling the junction between two rods in the static case with the techniques of Ciarlet & Kesavan [7], who consider the limit eigenvalue problem for a single plate. Consequently, as parts of the proof are similar or identical to those of [19] and [7], we will repeatedly refer the reader to the aforementioned papers.

We show here that the eigenvalues η_ℓ^ε of the three-dimensional problem converge as $\varepsilon \rightarrow 0$ toward eigenvalues η_ℓ^0 of a well posed 1d-1d eigenvalue problem, which consists of two usual fourth-order rods equations coupled through a set of junction conditions.

The limit eigenfunctions are of Bernoulli-Navier type in each rod with no axial components. They are thus determined by a quadruple $(\zeta_2^{1,\ell}, \zeta_3^{1,\ell}, \zeta_1^{2,\ell}, \zeta_3^{2,\ell})$ of H^2 -functions of the longitudinal variable of each rod, that correspond to the flexural displacements of the rods. These displacements are solutions of the limit 1d-1d model and satisfy some junction conditions similar to those obtained in the static case [19]. These conditions reflect the translations and rotations of the free rod.

We note that the asymptotic behaviour of the eigenfunctions is similar to that of the displacements in the static case [19]. Indeed, the free rod undergoes a periodic motion of translation in its own direction of the order ε^{-1} and rotation whose axis is the clamped rod, which correspond to the translation and rotation of the free rod in the static case.

We see then that the motion of the clamped rod at the junction follows the translations of the free rod and that the rotation of the free rod creates a torsion in the clamped rod, whose angle is equal to the moment in the x_1 -direction of the inertia forces exerted by the free rod, divided by the scaled torsional rigidity coefficient of the clamped rod.

It is interesting to note that, contrarily to the static case, the contribution of the torsion of the clamped rod appears in the limit model through the torsional angles $\zeta_3^{2,\ell}(0)$. This is the main novel feature of the model. It was by no means obvious, judging solely from the static model of [19], that the limit eigenvalue problem would assume this form.

The results of this article were announced in Kerdid [15].

1. THE THREE-DIMENSIONAL PROBLEM

We use the following index convention: Latin indices take their values in $\{1, 2, 3\}$, Greek indices with exponent 1 in $\{2, 3\}$ and Greek indices with exponent 2 in $\{1, 3\}$.

We consider a family of open sets Ω^ϵ in \mathbb{R}^3 defined for $0 < \epsilon < 1$ as $\Omega^\epsilon = \Omega_1^\epsilon \cup \Omega_2^\epsilon$, where

$$\begin{cases} \Omega_1^\epsilon = \{x \in \mathbb{R}^3, 0 < x_1 < 1, 0 < x_2, x_3 < \epsilon\}, \\ \Omega_2^\epsilon = \{x \in \mathbb{R}^3, 0 < x_2 < 1, 0 < x_1, x_3 < \epsilon\}, \end{cases} \tag{1.1}$$

see *figure 1* below.

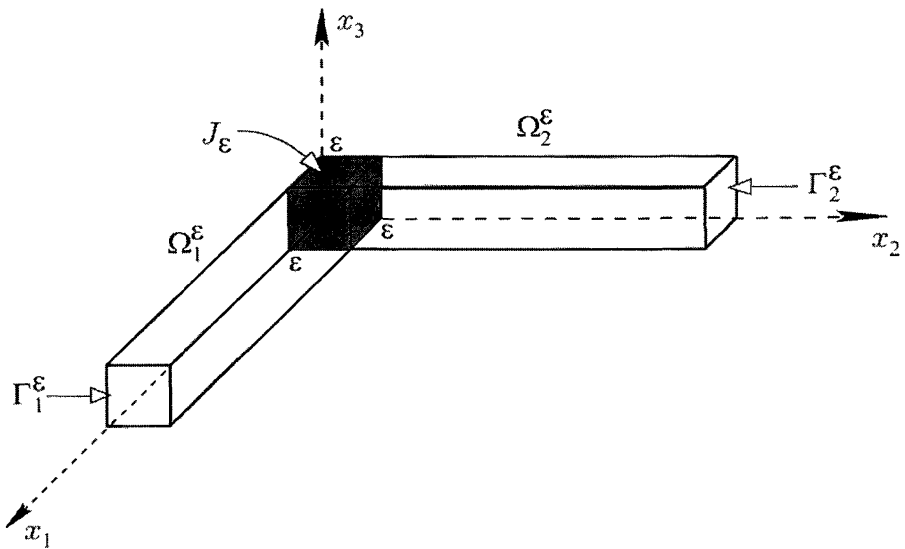


Figure 1.

We denote the ends of the rods by

$$\Gamma_1^\epsilon = \partial\Omega_1^\epsilon \cap \{x_1 = 1\} \quad \Gamma_2^\epsilon = \partial\Omega_2^\epsilon \cap \{x_2 = 1\}. \tag{1.2}$$

We suppose that Ω^ϵ is the reference configuration of an isotropic homogeneous linearly elastic body, with Lamé moduli $\mu > 0$ and $\lambda \geq 0$.

We assume that the bodies are clamped on Γ_1^e only. The associated eigenvalue problem then consists in finding displacements fields $u^e = (u_i^e)$ and real numbers η^e wich satisfy

$$\begin{cases} -\partial_j \sigma_{ij}^e = \eta^e u_i^e & \text{in } \Omega^e, \\ \sigma_{ij}^e = 2\mu e_{ij}(u^e) + \lambda e_{pp}(u^e) \delta_{ij} & \text{in } \Omega^e, \\ u^e = 0 & \text{on } \Gamma_1^e, \\ \sigma_{ij}^e \nu_j = 0 & \text{on } \partial\Omega^e \setminus \Gamma^e, \end{cases} \tag{1.3}$$

where $\Gamma^e = \Gamma_1^e \cup \Gamma_2^e$.

We introduce the spaces

$$\begin{aligned} V^e &= \{v = (v_i) \in H^1(\Omega^e; \mathbb{R}^3); \quad v = 0 \quad \text{on } \Gamma_1^e\}, \\ H^e &= L^2(\Omega^e; \mathbb{R}^3). \end{aligned} \tag{1.4}$$

Problem (1.3) admits the following variational formulation: Find (u^e, η^e) in $V^e \times \mathbb{R}$ such that

$$\forall v^e \in V^e, \quad \int_{\Omega^e} \sigma_{ij}(u^e) e_{ij}(v^e) dx = \eta^e \int_{\Omega^e} u_i^e v_i^e dx, \tag{1.5}$$

where

$$\sigma_{ij}(v^e) = 2\mu e_{ij}(v^e) + \lambda e_{pp}(v^e) \delta_{ij}. \tag{1.6}$$

By Korn’s inequality, the clamping condition and the positivity of the Lamé moduli, it is known that the symmetric bilinear form

$$(u^e, v^e) \in V^e \times V^e \rightarrow \int_{\Omega^e} \sigma_{ij}(u^e) e_{ij}(v^e) dx$$

is V^e -elliptic, in the sense that

$$\exists \alpha^e > 0, \quad \forall v^e \in V^e, \quad \int_{\Omega^e} \sigma_{ij}(v^e) e_{ij}(v^e) dx \geq \alpha^e \|v^e\|_{1, \Omega^e}.$$

This property and the compactness of the injection of V^e into H^e imply that problem (1.5), (1.6) possesses an increasing sequence of eigenvalues [25],

$$0 < \eta_1^e \leq \eta_2^e \leq \dots \leq \eta_\ell^e \leq \dots \quad \text{with} \quad \lim_{\ell \rightarrow \infty} \eta_\ell^e = +\infty, \tag{1.7}$$

associated with a sequence of eigenfunctions $u^{\varepsilon, \ell}$, which form a complete set in both V^ε and H^ε and which can be normalised in H^ε as follows

$$\int_{\Omega^\varepsilon} u^{\varepsilon, \ell} \cdot u^{\varepsilon, k} \, dx = \varepsilon^2 \delta_{k\ell}. \tag{1.8}$$

We introduce the Rayleigh quotient

$$R^\varepsilon(v^\varepsilon) = \frac{\int_{\Omega^\varepsilon} \sigma(v^\varepsilon) : e(v^\varepsilon) \, dx}{\int_{\Omega^\varepsilon} v^\varepsilon \cdot v^\varepsilon \, dx} \quad \forall v^\varepsilon \in V^\varepsilon \setminus \{0\}. \tag{1.9}$$

The eigenvalues η_ℓ^ε satisfy

$$\eta_\ell^\varepsilon = \min_{W \in E_\ell^\varepsilon} \max_{v \in W} R^\varepsilon(v^\varepsilon), \tag{1.10}$$

where E_ℓ^ε is the set of all subspaces of dimension ℓ of V^ε , see [25]. □

2. THE SCALED PROBLEM

The functional spaces V^ε to which the eigenfunctions $u^{\varepsilon, \ell}$ belong have the disadvantage of depending on ε . To remove this difficulty, we define a problem equivalent to problem (1.5)-(1.6) but posed over a domain which does not depend on ε , see [5].

Let $\Omega^1 = \Omega_1^1$, $\Omega^2 = \Omega_2^1$ and $\Gamma^1 = \Gamma_1^1$. We set

$$\omega^1 = \partial\Omega^1 \cap \{x_1 = 0\}, \quad \omega^2 = \partial\Omega^2 \cap \{x_2 = 0\} \tag{2.1}$$

and

$$V = H_{\Gamma^1}^1(\Omega^1; \mathbb{R}^3) \times H^1(\Omega^2; \mathbb{R}^3). \tag{2.2}$$

As in [19], we define a scaling mapping

$$\begin{aligned} \varphi^\varepsilon : \bar{\Omega}^1 \cup \bar{\Omega}^2 &\rightarrow \Omega^\varepsilon \\ x &\rightarrow \begin{cases} (x_1, \varepsilon x_2, \varepsilon x_3) & \text{if } x \in \bar{\Omega}^1, \\ (\varepsilon x_1, x_2, \varepsilon x_3) & \text{if } x \in \bar{\Omega}^2 \end{cases} \end{aligned} \tag{2.3}$$

and we note

$$(u^1(\varepsilon), u^2(\varepsilon)) = ((\varepsilon^{-1} u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \circ \varphi^\varepsilon, (u_1^\varepsilon, \varepsilon^{-1} u_2^\varepsilon, u_3^\varepsilon) \circ \varphi^\varepsilon). \quad (2.4)$$

The junction region is the image by φ^ε of the open sets

$$J_\varepsilon^1 = \{x \in \Omega^1 : x_1 < \varepsilon\} \quad \text{and} \quad J_\varepsilon^2 = \{x \in \Omega^2 : x_2 < \varepsilon\}. \quad (2.5)$$

The fact that the displacements $u(\varepsilon)$ correspond on these two sets to the same displacement of the multi-structure gives the following relations called three-dimensional junction relations:

$$\begin{cases} \varepsilon u_1^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = u_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ u_2^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = \varepsilon u_2^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ u_3^1(\varepsilon)(\varepsilon x_1, x_2, x_3) = u_3^2(\varepsilon)(x_1, \varepsilon x_2, x_3). \end{cases} \quad (2.6)$$

Due to the scalings, we have

$$\begin{cases} e_{\alpha^1 \beta^1}(u^{1,\varepsilon}) = \varepsilon^{-1} e_{\alpha^1 \beta^1}(u^1(\varepsilon)), \\ e_{\alpha^1 1}(u^{1,\varepsilon}) = e_{\alpha^1 1}(u^1(\varepsilon)), \\ e_{11}(u^{1,\varepsilon}) = \varepsilon e_{11}(u^1(\varepsilon)) \end{cases} \quad (2.7)$$

and similar relations for $u^2(\varepsilon)$. Let us now introduce the scaled bilinear forms

$$\begin{aligned} b_\varepsilon^1(u, v) &= \varepsilon^{-4} [2\mu e_{\alpha^1 \beta^1}(u) e_{\alpha^1 \beta^1}(v) + \lambda e_{\alpha^1 \alpha^1}(u) e_{\beta^1 \beta^1}(v)] \\ &+ \varepsilon^{-2} [4\mu e_{\alpha^1 1}(u) e_{\alpha^1 1}(v) + \lambda(e_{\alpha^1 \alpha^1}(u) e_{11}(v) + e_{11}(u) e_{\alpha^1 \alpha^1}(v))] \\ &+ (\lambda + 2\mu) e_{11}(u) e_{11}(v) \end{aligned} \quad (2.8)$$

and the analogous formula for $b_\varepsilon^2(u, v)$.

Replacing formulas (2.4), (2.7) and (2.8) into equation (1.5) and performing the changes of variables in the integrals, we obtain the following set of variational equations for $u^\ell(\varepsilon)$:

for all $v(\varepsilon) \in V(\varepsilon)$,

$$\begin{aligned} & \int_{\Omega^1} b_\varepsilon^1(u^{1,\ell}(\varepsilon), v^1(\varepsilon)) \, dx + \int_{\Omega^2 \cup \mathcal{V}_\varepsilon^2} b_\varepsilon^2(u^{2,\ell}(\varepsilon), v^2(\varepsilon)) \, dx \\ &= \eta_\ell(\varepsilon) \int_{\Omega^1} [u_{\alpha^1}^{1,\ell}(\varepsilon) v_{\alpha^1}^1(\varepsilon) + \varepsilon^2 u_1^{1,\ell}(\varepsilon) v_1^1(\varepsilon)] \, dx \quad (2.9) \\ &+ \eta_\ell(\varepsilon) \int_{\Omega^2 \cup \mathcal{V}_\varepsilon^2} [u_{\alpha^2}^{2,\ell}(\varepsilon) v_{\alpha^2}^2(\varepsilon) + \varepsilon^2 u_2^{2,\ell}(\varepsilon) v_2^2(\varepsilon)] \, dx, \end{aligned}$$

where

$$\eta_\ell(\varepsilon) = \varepsilon^{-2} \eta_\ell^\varepsilon \quad (2.10)$$

and with the normalisation condition

$$\begin{aligned} & \int_{\Omega^1} [u_{\alpha^1}^{1,\ell}(\varepsilon) u_{\alpha^1}^{1,p}(\varepsilon) + \varepsilon^2 u_1^{1,\ell}(\varepsilon) u_1^{1,p}(\varepsilon)] \, dx \\ &+ \int_{\Omega^2 \cup \mathcal{V}_\varepsilon^2} [u_{\alpha^2}^{2,\ell}(\varepsilon) u_{\alpha^2}^{2,p}(\varepsilon) + \varepsilon^2 u_2^{2,\ell}(\varepsilon) u_2^{2,p}(\varepsilon)] \, dx = \delta_{\ell p}. \quad (2.11) \end{aligned}$$

We define the space of Bernoulli-Navier displacements on Ω^1 as

$$V_{BN}(\Omega^1) = \{v \in H^1(\Omega^1) : e_{\alpha^1,1}(v) = 0\}. \quad (2.12)$$

Elements of this space are of the form

$$\begin{cases} v_1(x_1, x_2, x_3) = \zeta_1(x_1) - (x_{\alpha^1} - 1/2) \dot{\zeta}_{\alpha^1}(x_1), \\ v_2(x_1, x_2, x_3) = \theta(x_3 - 1/2) + \zeta_2(x_1), \\ v_3(x_1, x_2, x_3) = -\theta(x_2 - 1/2) + \zeta_3(x_1), \end{cases} \quad (2.13)$$

where $\zeta_{\alpha^1} \in H^2(]0, 1[)$, $\zeta_1 \in H^1(]0, 1[)$ and $\theta \in \mathbb{R}$ and where we have denoted by a dot the derivative with respect to the longitudinal variable of Ω^1 . We define $V_{BN}(\Omega^2)$ by the analogous formulas.

We introduce the following limit spaces:

$$\mathcal{V} = \{ (\xi_{\alpha^1}^1, \xi_{\alpha^2}^2) \in H^2(]0, 1[; \mathbb{R}^4) ; \xi_{\alpha^1}^1(1) = \xi_{\alpha^1}^2(1) = 0, \xi_1^2(0) = 0, \xi_3^1(0) = \xi_3^2(0), \xi_2^1(0) = -\xi_1^2(0) \}, \tag{2.14}$$

$$\mathcal{V}_c = \left\{ (\xi_{\alpha^1}^1, \xi_{\alpha^2}^2) \in \mathcal{V} ; \xi_3^2(0) = \frac{c}{K^1} \int_0^1 x_2 \xi_3^2(x_2) dx_2 \right\}, \tag{2.15}$$

where

$$K^1 = 4 \mu \int_0^1 \|\nabla \chi^1\|^2 dx_2 dx_3 \tag{2.16}$$

is the scaled torsional rigidity coefficient of the rod Ω^1 and χ^1 the torsion function defined on ω^1 by

$$\begin{cases} \partial_{\alpha^1 \alpha^1} \chi^1 = 1 & \text{in } \omega^1, \\ \chi^1 = 0 & \text{on } \partial\omega^1. \end{cases} \tag{2.17}$$

3. THE LIMIT PROBLEM

THEOREM: *For all integers $\ell \geq 1$, we have*

$$\eta_\ell(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \eta_\ell(0), \tag{3.1}$$

where $\eta_\ell(0)$ is the ℓ th eigenvalue, counting multiplicity, of the following limit eigenvalue problem:

Find $(\zeta, \eta) \in \mathcal{V} \times \mathbb{R}$, such that for all $\xi \in \mathcal{V}$,

$$\begin{aligned} E \int_0^1 I_{\alpha^1 \beta^1}^1 \zeta_{\alpha^1}^1(x_1) \xi_{\beta^1}^1(x_1) dx_1 + \int_0^1 I_{\alpha^2 \beta^2}^2 \zeta_{\alpha^2}^2(x_2) \xi_{\beta^2}^2(x_2) dx_2 + K^1 \zeta_3^2(0) \xi_3^2(0) \\ = \eta \int_0^1 \zeta_{\alpha^1}^1(x_1) \xi_{\alpha^1}^1(x_1) dx_1 + \eta \int_0^1 \zeta_{\alpha^2}^2(x_2) \xi_{\alpha^2}^2(x_2) dx_2 + \eta \zeta_2^1(0) \xi_2^1(0), \end{aligned} \tag{3.2}$$

where $E = \frac{\mu(3\lambda + 2\mu)}{\mu + \lambda}$ is the scaled Young modulus and

$$\begin{cases} I_{\alpha^1 \beta^1}^1 = \int_{\omega^1} (x_{\alpha^1} - 1/2) (x_{\beta^1} - 1/2) dx_2 dx_3, \\ I_{\alpha^2 \beta^2}^2 = \int_{\omega^2} (x_{\alpha^2} - 1/2) (x_{\beta^2} - 1/2) dx_1 dx_3 \end{cases} \quad (3.3)$$

are the inertia tensors of each cross section.

In addition, there exists a subsequence (still denoted ε) such that for all integers $\ell \geq 1$, there exists $b_2^{\varepsilon, \ell} \in \mathbb{R}$ such that setting

$$u^{2, \ell}(\varepsilon) = \bar{u}^{2, \ell}(\varepsilon) + \begin{pmatrix} 0 \\ b_2^{\varepsilon, \ell} \\ 0 \end{pmatrix}, \quad (3.4)$$

then there exists $(u^{1, \ell}(0), \bar{u}^{2, \ell}(0)) \in V$ and $\bar{b}^{0, \ell} \in \mathbb{R}$ such that

$$(u^{1, \ell}(\varepsilon), \bar{u}^{2, \ell}(\varepsilon))_{\varepsilon \rightarrow 0} (u^{1, \ell}(0), \bar{u}^{2, \ell}(0)) \quad \text{strongly in } V \quad (3.5)$$

$$\text{and } \varepsilon b_2^{\varepsilon, \ell} \xrightarrow{\varepsilon \rightarrow 0} \bar{b}^{0, \ell}. \quad (3.6)$$

The pair $(u^{1, \ell}(0), \bar{u}^{2, \ell}(0))$ is such that

$$\begin{aligned} u^{1, \ell}(0) &= (- (x_{\alpha^1} - 1/2) \zeta_{\alpha^1}^{1, \ell}(x_1), \zeta_2^{1, \ell}(x_1), \zeta_3^{1, \ell}(x_1)), \\ \bar{u}^{2, \ell}(0) &= (\zeta_1^{2, \ell}(x_2), - (x_{\alpha^2} - 1/2) \zeta_{\alpha^2}^{2, \ell}(x_2), \zeta_3^{2, \ell}(x_2)), \end{aligned} \quad (3.7)$$

where $(\zeta_{\alpha^1}^{1, \ell}, \zeta_{\alpha^2}^{2, \ell})$ is the ℓ th eigenfunction of the limit problem (3.2) and satisfy

$$\forall \ell \geq 1 \quad (\zeta_{\alpha^1}^{1, \ell}, \zeta_{\alpha^2}^{2, \ell}) \in \mathcal{V}_{n_\ell(0)}. \quad (3.8)$$

These eigenfunctions satisfy the following normalisation condition

$$\forall k, \ell \geq 1 \quad \int_0^1 \zeta_{\alpha^1}^{1, \ell} \zeta_{\alpha^1}^{1, k} dx_1 + \int_0^1 \zeta_{\alpha^2}^{2, \ell} \zeta_{\alpha^2}^{2, k} dx_2 + \zeta_2^{1, \ell}(0) \zeta_2^{1, k}(0) = \delta_{k\ell}. \quad (3.9)$$

As last, if $\eta_\varepsilon(0)$ is a simple eigenvalue then the whole family of flexural displacements converges to $\pm (\zeta_{\alpha^1}^{1, \ell}, \zeta_{\alpha^2}^{2, \ell})$ as $\varepsilon \rightarrow 0$.

Remark: In our case, both inertia tensors are equal to $\frac{1}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

To prove this theorem, we combine the techniques of [7], [14] and [19]. First we show that the eigenvalues $\eta_\varepsilon(\varepsilon)$ are bounded independently of ε .

LEMMA 1: For each integer $\ell \geq 1$, there exists $\delta_\ell > 0$ such that for all $\varepsilon \leq 1$,

$$\eta_\ell(\varepsilon) \leq \delta_\ell. \tag{3.10}$$

Proof: Let us define

$$R(\varepsilon)(v(\varepsilon)) = \frac{N(\varepsilon)(v(\varepsilon))}{D(\varepsilon)(v(\varepsilon))} \quad \forall v(\varepsilon) \in V(\varepsilon) \setminus \{0\}$$

where

$$N(\varepsilon)(v(\varepsilon)) = \int_{\Omega^1} b_\varepsilon^1(v^1(\varepsilon), v^1(\varepsilon)) dx + \int_{\Omega^2} b_\varepsilon^2(v^2(\varepsilon), v^2(\varepsilon)) dx$$

and

$$D(\varepsilon)(v(\varepsilon)) = \int_{\Omega^1} [v_{\alpha^1}^1(\varepsilon) v_{\alpha^1}^1(\varepsilon) + \varepsilon^2 v_1^1(\varepsilon) v_1^1(\varepsilon)] dx + \int_{\Omega^2} [v_{\alpha^2}^2(\varepsilon) v_{\alpha^2}^2(\varepsilon) + \varepsilon^2 v_2^2(\varepsilon) v_2^2(\varepsilon)] dx.$$

Then it is easily checked that

$$R(\varepsilon)(v(\varepsilon)) = \varepsilon^{-2} R^\varepsilon(v^\varepsilon). \tag{3.11}$$

Combining (1.9), (1.10), (2.10) and (3.11), we obtain

$$\eta_\ell(\varepsilon) = \min_{W^\ell(\varepsilon) \in E_\ell(\varepsilon)} \max_{v(\varepsilon) \in W^\ell(\varepsilon)} R(\varepsilon)(v(\varepsilon)) \tag{3.12}$$

where $E_\ell(\varepsilon)$ is the family of all subspaces of $V(\varepsilon)$ which are of dimension ℓ .

Let $W \subset V$ the sapce of pairs (v^1, v^2) such that there exists $\xi_{\alpha^1}^1 \in H^2([0, 1])$ satisfying $\xi_{\alpha^1}^1(x_1) = 0$ for $x_1 \leq 1/2$ with

$$\begin{cases} v^1(x) = -(x_{\alpha^1} - 1/2) \xi_{\alpha^1}^1(x_1), \xi_2^1(x_1), \xi_3^1(x_1) & \text{in } \Omega^1 \\ v^2(x) = 0 & \text{in } \Omega^2. \end{cases}$$

It is clear that $W \subset V(\varepsilon)$ for $\varepsilon < 1/2$ and consequently,

$$\eta_\ell(\varepsilon) \leq \min_{U^\ell \in E_\ell} \max_{v \in U^\ell} R(\varepsilon)(v) \tag{3.13}$$

where E_ℓ is the set of all subspaces of dimension ℓ of W .

If $v \in W$ then $e_{\alpha^1, i}(v^1) = 0$ and we obtain

$$\begin{aligned} N(\varepsilon)(v) &= (\lambda + 2\mu) \int_{\Omega^i} e_{11}(v^1) e_{11}(v^1) dx \\ &= (\lambda + 2\mu) \int_{1/2}^1 I_{\alpha^1 \beta^1}^1 \ddot{\xi}_{\alpha^1}^1(x_1) \ddot{\xi}_{\beta^1}^1(x_1) dx_1. \end{aligned}$$

Moreover,

$$D(\varepsilon)(v) \geq \int_{1/2}^1 \xi_{\alpha^1}^1(x_1) \xi_{\alpha^1}^1(x_1) dx_1,$$

then

$$R(\varepsilon)(v) \leq (\lambda + 2\mu) \frac{\int_{1/2}^1 I_{\alpha^1 \beta^1}^1 \ddot{\xi}_{\alpha^1}^1(x_1) \ddot{\xi}_{\beta^1}^1(x_1) dx_1}{\int_{1/2}^1 \xi_{\alpha^1}^1(x_1) \xi_{\alpha^1}^1(x_1) dx_1} = r(v) \quad \forall v \in W$$

where r is the Rayleigh quotient associated with the operator

$$(H_0^2([1/2, 1]); \mathbb{R}^2) \rightarrow L^2([-1, 1]; \mathbb{R}^2)$$

$$(\xi_1, \xi_2) \mapsto (\mu + 2\lambda) I_{\alpha^1 \beta^1}^1 \frac{\partial^4 \xi_{\alpha^1}^1}{\partial x_1^4}$$

which is self-adjoint with compact inverse. Consequently,

$$\eta_\ell(\varepsilon) \leq \min_{U \in E_\ell^*} \max_{(\xi_2^1, \xi_3^1) \in U} r(v) = \delta_\ell$$

where E_ℓ^* is the set of all subspaces of dimension ℓ of $(H_0^2([1/2, 1]); \mathbb{R}^2)$. □

As in the static case [19], the eigenfunctions $u^\ell(\varepsilon)$ are not bounded in V . To remove this difficulty, we will use the following result:

LEMMA 2: *Let $y^\varepsilon = (1/2, \varepsilon/2, 1/2)$. There exists a constant $C > 0$ such that for all $v(\varepsilon) \in V(\varepsilon)$, there exists two vectors $a^\varepsilon(v^2(\varepsilon))$ and $b^\varepsilon(v^2(\varepsilon))$ in \mathbb{R}^3 such that if we set*

$$v^2(\varepsilon) = \bar{v}^2(\varepsilon) + a^\varepsilon(v^2(\varepsilon)) \wedge (x - y^\varepsilon) + b^\varepsilon(v^2(\varepsilon)) \tag{3.14}$$

then we have the following estimates:

$$\| \bar{v}^2(\varepsilon) \|_{H^1(\Omega^2, \mathbb{R}^3)} \leq C \| e(\bar{v}^2(\varepsilon)) \|_{L^2(\Omega^2, M^3)}, \tag{3.15}$$

$$\begin{cases} |a_\tau^\varepsilon(v^2(\varepsilon))| \leq C \|v(\varepsilon)\|_\varepsilon, \\ |b_{\alpha^2}^\varepsilon(v^2(\varepsilon))| \leq C \|v(\varepsilon)\|_\varepsilon, \\ \varepsilon |b_2^\varepsilon(v^2(\varepsilon))| \leq C \|v(\varepsilon)\|_\varepsilon, \end{cases} \tag{3.16}$$

where

$$\|v(\varepsilon)\|_\varepsilon = \int_{\Omega^1} b_\varepsilon^1(v^1(\varepsilon), v^1(\varepsilon)) dx + \int_{\Omega^2} b_\varepsilon^2(v^2(\varepsilon), v^2(\varepsilon)) dx. \tag{3.17}$$

Proof: See [19]. □

We can now prove an appropriate bound for the eigenfunctions.

LEMMA 3: For each integer $\ell \geq 1$, there exists $b_2^{\varepsilon, \ell} \in \mathbb{R}$ such that setting

$$u^{2, \ell}(\varepsilon) = \bar{u}^{2, \ell}(\varepsilon) + \begin{pmatrix} 0 \\ b_2^{\varepsilon, \ell} \\ 0 \end{pmatrix},$$

then there exists a constant $C_\ell > 0$ independent of ε such that

$$\|u^{1, \ell}(\varepsilon)\|_{H^1(\Omega^1)} + \|\bar{u}^{2, \ell}(\varepsilon)\|_{H^1(\Omega^2)} \leq C_\ell \tag{3.18}$$

and

$$\varepsilon |b_2^{\varepsilon, \ell}| \leq C_\ell. \tag{3.19}$$

Proof: Since the Lamé constants are positive we have

$$\begin{aligned} & 2(\|e(v^1(\varepsilon))\|_{L^2(\Omega^1, M^3)} + \|e(v^2(\varepsilon))\|_{L^2(\Omega^1, M^3)}) \\ & \leq \int_{\Omega^1} b_\varepsilon^1(v^1(\varepsilon), v^1(\varepsilon)) dx + \int_{\Omega^2} b_\varepsilon^2(v^2(\varepsilon), v^2(\varepsilon)) dx. \end{aligned} \tag{3.20}$$

Letting $v(\varepsilon) = (u^{1,\ell}(\varepsilon), u^{2,\ell}(\varepsilon))$ in the variational equation (2.9), we obtain

$$\begin{aligned} & \int_{\Omega^1} b_\varepsilon^1(u^{1,\ell}(\varepsilon), u^{1,\ell}(\varepsilon)) \, dx + \int_{\Omega^2 \cup \mathcal{V}_\varepsilon^2} b_\varepsilon^2(u^{2,\ell}(\varepsilon), u^{2,\ell}(\varepsilon)) \, dx \\ &= \eta_\ell(\varepsilon) \int_{\Omega^1} [u_{\alpha^1}^{1,\ell}(\varepsilon) u_{\alpha^1}^{1,\ell}(\varepsilon) + \varepsilon^2 u_1^{1,\ell}(\varepsilon) u_1^{1,\ell}(\varepsilon)] \, dx \\ & \quad + \eta_\ell(\varepsilon) \int_{\Omega^2 \cup \mathcal{V}_\varepsilon^2} [u_{\alpha^2}^{2,\ell}(\varepsilon) u_{\alpha^2}^{2,\ell}(\varepsilon) + \varepsilon^2 u_2^{2,\ell}(\varepsilon) u_2^{2,\ell}(\varepsilon)] \, dx \quad (3.21) \\ &= \eta_\ell(\varepsilon) \\ &\leq \delta_\ell, \end{aligned}$$

by the normalisation condition (2.11) and estimate (3.10). Then, we deduce that $e(u^{1,\ell}(\varepsilon))$ and $e(u^{2,\ell}(\varepsilon))$ are bounded in $L^2(\Omega^1; M^3)$ and $L^2(\Omega^2; M^3)$ respectively. Consequently, by Korn's inequality, the clamping condition on Γ^1 and estimate (3.10), we obtain (3.18).

It follows from estimate (3.20) that $|a_i^\varepsilon(u^{2,\ell}(\varepsilon))|$, $|b_1^\varepsilon(u^{2,\ell}(\varepsilon))|$ and $|b_3^\varepsilon(u^{2,\ell}(\varepsilon))|$ are bounded in \mathbb{R} . Then, if we set

$$\begin{aligned} \bar{u}^{2,\ell}(\varepsilon) &= \bar{u}^{2,\ell}(\varepsilon) + \\ & \left(\begin{array}{l} a_2^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_3 - \frac{1}{2}\right) - a_3^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_2 - \frac{\varepsilon}{2}\right) + b_1^\varepsilon(u^{2,\ell}(\varepsilon)) \\ a_3^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_1 - \frac{1}{2}\right) - a_1^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_3 - \frac{1}{2}\right) \\ a_1^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_3 - \frac{\varepsilon}{2}\right) - a_2^\varepsilon(u^{2,\ell}(\varepsilon)) \left(x_2 - \frac{1}{2}\right) + b_3^\varepsilon(u^{2,\ell}(\varepsilon)) \end{array} \right) \end{aligned}$$

and $b_2^{\varepsilon,\ell} = b_2^\varepsilon(u^{2,\ell}(\varepsilon))$, we see that $(u^{1,\ell}(\varepsilon), \bar{u}^{2,\ell}(\varepsilon))$ is bounded in $V(\varepsilon)$ and $\varepsilon b_2^{\varepsilon,\ell}$ is bounded in \mathbb{R} .

Consequence: For each integer $\ell \geq 1$, we can extract from the family ε a subsequence (still denoted ε), such that

$$\begin{aligned} \eta_\ell(\varepsilon) &\xrightarrow{\varepsilon \rightarrow 0} \eta_\ell(0) \quad \text{in } \mathbb{R}, \\ \varepsilon b_2^{\varepsilon,\ell} &\xrightarrow{\varepsilon \rightarrow 0} \bar{b}^{0,\ell} \quad \text{in } \mathbb{R} \end{aligned}$$

and

$$(u^{1,\ell}(\varepsilon), \bar{u}^{2,\ell}(\varepsilon)) \rightharpoonup (u^{1,\ell}(0), \bar{u}^{2,\ell}(0)) \quad \text{weakly in } V. \quad (3.22)$$

Remark: Using the diagonal procedure we can choose the same subsequence for all ℓ .

LEMMA 4: For each integer $\ell \geq 1$,

$$(u^{1,\ell}(0), \bar{u}^{2,\ell}(0)) \in V_{BN}(\Omega^1) \times V_{BN}(\Omega^2). \tag{3.23}$$

Proof: for all $\ell \geq 1$ we define the tensors $\kappa^{1,\ell}(\varepsilon)$ by

$$\begin{cases} \kappa_{\alpha^1 \beta^1}^{1,\ell}(\varepsilon) = \varepsilon^{-2} e_{\alpha^1 \beta^1}(u^{1,\ell}(\varepsilon)), \\ \kappa_{\alpha^1 1}^{1,\ell}(\varepsilon) = \varepsilon^{-1} e_{\alpha^1 1}(u^{1,\ell}(\varepsilon)), \\ \kappa_{11}^{1,\ell}(\varepsilon) = e_{11}(u^{1,\ell}(\varepsilon)) \end{cases} \tag{3.24}$$

and similarly for $\kappa^{2,\ell}(\varepsilon)$.

Using (3.21) and the positivity of the Lamé constants, we obtain

$$2 \mu (\|\kappa^{1,\ell}(\varepsilon)\|_{0,\Omega^1}^2 + \|\kappa^{2,\ell}(\varepsilon)\|_{0,\Omega^2}^2) \leq C_\ell. \tag{3.25}$$

Therefore,

$$\begin{aligned} \|e_{\alpha^1 \beta^1}(u^{1,\ell}(\varepsilon))\|_{0,\Omega^1}^2 &\leq \varepsilon^2 C_\ell, \\ \|e_{\alpha^1 1}(u^{1,\ell}(\varepsilon))\|_{0,\Omega^1}^2 &\leq \varepsilon C_\ell \end{aligned}$$

and consequently,

$$e_{\alpha^1 i}(u^{1,\ell}(\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^2(\Omega^1). \tag{3.26}$$

Since $u^{1,\ell}(\varepsilon) \rightharpoonup u^{1,\ell}(0)$ weakly in $H^1(\Omega^1)$, it follows that $e_{ij}(u^{1,\ell}(\varepsilon)) \rightharpoonup e_{ij}(u^{1,\ell}(0))$ in $L^2(\Omega^1)$ and we deduce that $e_{\alpha^1 i}(u^{1,\ell}(0)) = 0$. Thus, $u^{1,\ell}(0) \in V_{BN}(\Omega^1)$.

Similarly, we show that $\bar{u}^{2,\ell}(0) \in V_{BN}(\Omega^2)$. □

Consequently there exists $\theta^{1,\ell}, \theta^{2,\ell} \in \mathbb{R}$ and $\zeta_i^{1,\ell}, \zeta_i^{2,\ell}$, such that $\zeta_{\alpha^1}^{1,\ell}, \zeta_{\alpha^2}^{2,\ell} \in H^2([0, 1])$, $\zeta_1^{1,\ell}, \zeta_2^{2,\ell} \in H^1([0, 1])$ and

$$\begin{cases} u_1^{1,\ell}(0)(x) = \zeta_1^{1,\ell}(x_1) - (x_{\alpha^1} - 1/2) \zeta_{\alpha^1}^{1,\ell}(x_1), \\ u_2^{1,\ell}(0)(x) = \zeta_2^{1,\ell}(x_1) + \theta^{1,\ell}(x_3 - 1/2), \\ u_3^{1,\ell}(0)(x) = \zeta_3^{1,\ell}(x_1) - \theta^{1,\ell}(x_2 - 1/2) \end{cases} \tag{3.27}$$

and

$$\begin{cases} \bar{u}_1^{2,\ell}(0)(x) = \zeta_1^{2,\ell}(x_2) + \theta^{2,\ell}(x_3 - 1/2), \\ \bar{u}_2^{2,\ell}(0)(x) = \zeta_2^{2,\ell}(x_2) - (x_{\alpha^2} - 1/2) \zeta_{\alpha^2}^{2,\ell}(x_2), \\ \bar{u}_3^{2,\ell}(0)(x) = \zeta_3^{2,\ell}(x_1) - \theta^{2,\ell}(x_1 - 1/2). \end{cases} \tag{3.28}$$

Because of the clamping condition on Γ^1 we have

$$\theta^{1,\ell} = 0 \quad \text{and} \quad \zeta_i^{1,\ell}(1) = \dot{\zeta}_i^{1,\ell}(1) = 0. \tag{3.29}$$

Let us identify the limit junction relations.

LEMMA 5: For each integer $\ell \geq 1$, we have

$$\begin{cases} \zeta_2^{1,\ell}(0) = \bar{b}^{0,\ell}, \\ \zeta_1^{2,\ell}(0) = 0, \\ \zeta_3^{1,\ell}(0) = \zeta_3^{2,\ell}(0), \\ \dot{\zeta}_2^{1,\ell}(0) = -\dot{\zeta}_1^{2,\ell}(0), \\ \theta^{2,\ell} = 0. \end{cases} \tag{3.30}$$

Proof: See [19].

LEMMA 6: We have for each integer $\ell \geq 1$

$$\begin{aligned} \kappa_{i1}^{1,\ell}(\varepsilon) &\rightharpoonup \kappa_{i1}^{1,\ell}(0) \quad \text{weakly in } L^2(\Omega^1), \\ \kappa_{i2}^{2,\ell}(\varepsilon) &\rightharpoonup \kappa_{i2}^{2,\ell}(0) \quad \text{weakly in } L^2(\Omega^2), \end{aligned} \tag{3.31}$$

where the limits are given by

$$\begin{cases} \kappa_{11}^{1,\ell}(0) = \dot{\zeta}_1^{1,\ell}(x_1) - (x_{\alpha^1} - 1/2) \ddot{\zeta}_{\alpha^1}^{1,\ell}(x_1), \\ \kappa_{22}^{2,\ell}(0) = \dot{\zeta}_2^{2,\ell}(x_2) - (x_{\alpha^2} - 1/2) \ddot{\zeta}_{\alpha^2}^{2,\ell}(x_2), \end{cases} \tag{3.32}$$

$$\begin{cases} \kappa_{21}^{1,\ell}(0) = M(\zeta^\ell) \partial_3 \chi^1, \\ \kappa_{31}^{1,\ell}(0) = -M(\zeta^\ell) \partial_2 \chi^1 \end{cases} \tag{3.33}$$

and

$$\kappa_{\alpha^2 2}^{2,\ell}(0) = 0, \tag{3.34}$$

where

$$M(\zeta^\ell) = \frac{\eta_\ell(0)}{K^1} \int_0^1 x_2 \zeta_3^{2,\ell}(x_2) dx_2. \tag{3.35}$$

Proof: It follows from formula (3.25) that the tensors $\kappa_y^\ell(\varepsilon)$ are bounded independently of ε in L^2 . Then, using the diagonal procedure, we can extract a subsequence, which will be the same for all ℓ , such that

$$\begin{cases} \kappa_y^{1,\ell}(\varepsilon) \rightharpoonup \kappa_y^{1,\ell}(0) & \text{weakly in } L^2(\Omega^1) \\ \kappa_y^{2,\ell}(\varepsilon) \rightharpoonup \kappa_y^{2,\ell}(0) & \text{weakly in } L^2(\Omega^2) \end{cases} \tag{3.36}$$

and we obtain (3.31) immediately.

We introduce now, the scaled stress tensor in Ω^1 by

$$\begin{cases} \sigma_{\alpha^i \beta^i}^{1,\ell} = \varepsilon^{-2} [2 \mu \kappa_{\alpha^i \beta^i}^{1,\ell}(\varepsilon) + \lambda \kappa_u^{1,\ell}(\varepsilon) \delta_{\alpha^i \beta^i}], \\ \sigma_{\alpha^1 1}^{1,\ell} = \varepsilon^{-1} [2 \mu \kappa_{\alpha^1 1}^{1,\ell}(\varepsilon)], \\ \sigma_{11}^{1,\ell} = 2 \mu \kappa_{11}^{1,\ell}(\varepsilon) + \lambda \kappa_u^{1,\ell}(\varepsilon) \end{cases} \tag{3.37}$$

and by analogous formulas in Ω^2 .

This tensor is related to the actual stress tensor $\sigma(u^{\varepsilon,\ell})$ in Ω^ε via the scaling formulas

$$\begin{cases} \sigma_{\alpha^i \beta^i}^{1,\ell}(\varepsilon) = \varepsilon^{-3} \sigma_{\alpha^i \beta^i}(u^{\varepsilon,\ell}) \circ \varphi_\varepsilon, \\ \sigma_{\alpha^1 1}^{1,\ell}(\varepsilon) = \varepsilon^{-2} \sigma_{\alpha^1 1}(u^{\varepsilon,\ell}) \circ \varphi_\varepsilon, \\ \sigma_{11}^{1,\ell}(\varepsilon) = \varepsilon^{-1} \sigma_{11}(u^{\varepsilon,\ell}) \circ \varphi_\varepsilon. \end{cases} \tag{3.38}$$

The scaled stresses satisfy the variational equation

$$\begin{aligned} & \int_{\Omega^1} \sigma_y^{1,\ell}(\varepsilon) e_y(v^1(\varepsilon)) dx + \int_{\Omega^2 \cup \Omega^2_\varepsilon} \sigma_y^{2,\ell}(\varepsilon) e_y(v^2(\varepsilon)) dx \\ &= \eta_\ell(\varepsilon) \int_{\Omega^1} [u_{\alpha^1}^{1,\ell}(\varepsilon) v_{\alpha^1}^1(\varepsilon) + \varepsilon^2 u_1^{1,\ell}(\varepsilon) v_1^1(\varepsilon)] dx \\ &+ \eta_\ell(\varepsilon) \int_{\Omega^2 \cup \Omega^2_\varepsilon} [u_{\alpha^2}^{2,\ell}(\varepsilon) v_{\alpha^2}^2(\varepsilon) + \varepsilon^2 u_2^{2,\ell}(\varepsilon) v_2^2(\varepsilon)] dx. \end{aligned} \tag{3.39}$$

We proceed as in [19] to prove that there exists two scalars $\tau^{1,\ell}, \tau^{2,\ell}$ such that

$$\begin{aligned} \text{rot}(\kappa^{1,\ell}(\varepsilon)) &\rightharpoonup \tau^{1,\ell} \quad \text{weakly in } L^2(0, 1; H^{-1}(\omega^1)), \\ \text{rot}(\kappa^{2,\ell}(\varepsilon)) &\rightharpoonup \tau^{2,\ell} \quad \text{weakly in } L^2(0, 1; H^{-1}(\omega^2)), \end{aligned}$$

so that

$$\begin{cases} \kappa_{21}^{1,\ell}(0)(x) = \tau^{1,\ell} \partial_3 \chi^1(x_{\alpha^1}), \\ \kappa_{31}^{1,\ell}(0)(x) = -\tau^{1,\ell} \partial_2 \chi^1(x_{\alpha^1}), \end{cases} \quad \text{and} \quad \begin{cases} \kappa_{12}^{2,\ell}(0)(x) = \tau^{2,\ell} \partial_3 \chi^2(x_{\alpha^1}), \\ \kappa_{32}^{2,\ell}(0)(x) = -\tau^{2,\ell} \partial_1 \chi^2(x_{\alpha^2}). \end{cases} \tag{3.40}$$

To determine $\tau^{1,\ell}$, we use a test-function of the form

$$v^1(\varepsilon)(x) = \begin{pmatrix} 0 \\ x_3 - 1/2 \\ 1/2 - x_2 \end{pmatrix} \xi^\varepsilon(x_1); \quad v^2(\varepsilon)(x) = \begin{pmatrix} 0 \\ \varepsilon^{-1}(x_3 - 1/2) \\ 1/2 - \varepsilon^{-1}x_2 \end{pmatrix},$$

with

$$\xi^\varepsilon(x_1) = \begin{cases} 1 & 0 < x_1 < \varepsilon, \\ \frac{1-x_1}{1-\varepsilon} & \varepsilon < x_1 < 1. \end{cases}$$

This function satisfy the three-dimensional junction relations (2.6) and the clamping condition on Γ^1 , so that multiplying equation (2.9) by ε and passing to the limit, we then obtain

$$\begin{aligned} 2 \mu \tau^{1,\ell} \int_{\omega^1} [(x_3 - 1/2) \partial_3 \chi^1(x_{\alpha^1}) + (x_2 - 1/2) \partial_2 \chi^1(x_{\alpha^1})] dx_2 dx_3 \\ = \eta_\ell \int_{\Omega^2} x_2 u_3^{2,\ell}(0)(x) dx. \end{aligned}$$

Hence

$$4 \mu \tau^{1,\ell} \int_{\omega^1} \|\nabla \chi^1\|^2 dx_2 dx_3 = \eta_\ell(0) \int_0^1 x_2 \zeta_3^{2,\ell}(x_2) dx_2$$

which gives

$$\tau^{1,\ell} = M(\zeta^\ell). \tag{3.41}$$

Replacing (3.41) in formulas (3.40) we obtain (3.33).

We show that $\tau^{2,\ell} = 0$ by using a test-function of the form

$$v^1(\varepsilon) = 0 \quad v^2(\varepsilon) = \begin{pmatrix} 1/2 - x_3 \\ 0 \\ x_1 - 1/2 \end{pmatrix} \zeta^\varepsilon(x_2),$$

where

$$\zeta^\varepsilon(x_2) = \begin{cases} 0 & 0 < x_2 < \varepsilon, \\ x_2 - \varepsilon & \varepsilon < x_2 < 1. \end{cases}$$

□

Remark: $M(\zeta^\ell)$ is the moment in the x_1 -direction of the vertical forces exerted by the free rod divided by the scaled torsional rigidity of the clamped rod.

The rotation of the free rod in the (x_2, x_3) -plane create moments which tend to twist the clamped rod. Let us identify these torsional angles.

LEMMA 7: For each $\ell \geq 1$ we have

$$\dot{\zeta}_3^{2,\ell}(0) = M(\zeta^\ell). \tag{3.42}$$

Proof: We have seen that

$$\text{rot}(\kappa^{1,\ell}(\varepsilon)) \rightharpoonup M(\zeta^\ell) \quad \text{weakly in } L^2(0, 1; H^{-1}(\omega^1)).$$

Therefore, for almost all x_1 ,

$$\text{rot}(\kappa^{1,\ell}(\varepsilon))(x_1) \rightharpoonup M(\zeta^\ell) \quad \text{weakly in } H^{-1}(\omega^1).$$

Since $\text{rot}(\kappa^{1,\ell}(\varepsilon)) = \partial_1(\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon)))/2$ and $\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon))|_{x_1=1} = 0$, we have

$$\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon))(x_1, x_2, x_3) = 2 \int_1^{x_1} \text{rot}(\kappa^{1,\ell}(\varepsilon))(s, x_2, x_3) ds. \tag{3.43}$$

In addition, by definition

$$\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon)) = \partial_3(\varepsilon^{-1} u_2^{1,\ell}(\varepsilon)) - \partial_2(\varepsilon^{-1} u_3^{1,\ell}(\varepsilon)).$$

Let us differentiate the last equation in the three-dimensional junction relations (2.6) with respect to x_2 . We obtain

$$\partial_2(\varepsilon^{-1} u_3^{1,\ell}(\varepsilon))(\varepsilon x_1, x_2, x_3) = \partial_2 u_3^{2,\ell}(\varepsilon)(x_1, \varepsilon x_2, x_3),$$

so that

$$\begin{aligned} \text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon))(\varepsilon x_1, x_2, x_3) &= [\partial_3 \bar{u}_2^{2,\ell}(\varepsilon) - \partial_2 \bar{u}_3^{2,\ell}(\varepsilon)(\bar{x}_1, \bar{\varepsilon} x_2, x_3)], \\ &= 2[\partial_3 \bar{u}_2^{2,\ell}(\varepsilon) - e_{23}(\bar{u}^{2,\ell}(\varepsilon))](x_1, \varepsilon x_2, x_3). \end{aligned} \tag{3.44}$$

Let us multiply equation (3.44) by $\varphi(x) = \prod_{i=1}^3 \varphi_i(x_i)$ with $\varphi_i \in \mathcal{D}(]0, 1[)$ and $\int_0^1 \varphi_i(s) ds = 1$ and integrating on $\Omega = [0, 1]^3$, we obtain

$$\begin{aligned} &2 \int_{\Omega} [\partial_3 \bar{u}_2^{2,\ell}(\varepsilon) - e_{23}(\bar{u}^{2,\ell}(\varepsilon))](x_1, \varepsilon x_2, x_3) \varphi(x) dx \\ &= \int_0^1 \varphi(x_1) \left(\int_{\omega^1} (\text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon))(\varepsilon x_1, x_2, x_3)) \varphi_2(x_2) \varphi_3(x_3) dx_2 dx_3 \right) dx_1. \end{aligned} \tag{3.45}$$

As $\text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon))$ belongs to $H^1(0, 1; H^{-1}(\omega^1))$, we have for almost all x_1 , see [20],

$$\begin{aligned} &\| \text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon))(\varepsilon x_1) - \text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon))(0) \|_{H^{-1}(\omega^1)} \\ &\leq C\varepsilon^{1/2} \| \text{rot}(\varepsilon^{-1} u_{\alpha}^{1,\ell}(\varepsilon)) \|_{H^1(0, 1; H^{-1}(\omega^1))} \\ &\leq C\varepsilon^{1/2} \| \text{rot}(\kappa^{1,\ell}(\varepsilon)) \|_{L^2(0, 1; H^{-1}(\omega^1))} \\ &\leq C\varepsilon^{1/2}, \end{aligned} \tag{3.46}$$

by the clamping condition on Γ^1 . Thus, using (3.43) and (3.46), we obtain

$$\begin{aligned} & \int_0^1 \varphi(x_1) \left(\int_{\omega^1} (\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon))(\varepsilon x_1, x_2, x_3)) \varphi_2(x_2) \varphi_3(x_3) dx_2 dx_3 \right) dx_1 \\ &= \int_0^1 \varphi(x_1) \left(\int_{\omega^1} (\text{rot}(\varepsilon^{-1} u_{\alpha^1}^{1,\ell}(\varepsilon))(0, x_2, x_3)) \varphi_2(x_2) \varphi_3(x_3) dx_2 dx_3 \right) dx_1 \\ & \quad + o(\varepsilon^{1/2}) \\ &= 2 \int_{\omega^1} \left(\int_0^1 \text{rot}(\kappa^{1,\ell}(\varepsilon))(s, x_2, x_3) ds \right) \varphi_2(x_2) \varphi_3(x_3) dx_2 dx_3 + o(\varepsilon^{1/2}) \\ &= -2 \int_0^1 \left(\int_{\omega^1} \text{rot}(\kappa^{1,\ell}(\varepsilon))(s, x_2, x_3) \varphi_2(x_2) \varphi_3(x_3) dx_2 dx_3 \right) ds + o(\varepsilon^{1/2}). \end{aligned}$$

Consequently, the right-hand side of (3.45) tends to $-2M(\zeta^\ell)$ when $\varepsilon \rightarrow 0$.

For the left-hand side of (3.45), we have the estimate

$$\| \partial_3 \bar{u}_2^{2,\ell}(\varepsilon)(x_1, \varepsilon x_2, x_3) - \partial_3 \bar{u}_2^{2,\ell}(\varepsilon)(x_1, 0, x_3) \|_{H^{-1}(\omega^2)} \leq C\varepsilon^{1/2},$$

by the continuity of the derivative mapping from $H^1(\Omega^2)$ into $H^1(0, 1; H^{-1}(\omega^2))$ and the estimate (3.18). In addition, because of the compactness of the injection of $H^1(0, 1; H^{-1}(\omega^2))$ into $\mathcal{C}^0([0, 1]; H^{-1}(\omega^2))$ we have

$$\begin{aligned} & \int_{\Omega} \partial_3 \bar{u}_2^{2,\ell}(\varepsilon)(x_1, \varepsilon x_2, x_3) \varphi(x) dx \xrightarrow{\varepsilon \rightarrow 0} 0 \\ & \int_{\omega^2} \partial_3 \bar{u}_2^{2,\ell}(0)(x_1, 0, x_3) \varphi(x_1) \varphi(x_3) dx_1 dx_3 = -\dot{\zeta}_3^2(0). \end{aligned}$$

Finally, we show that

$$\int_{\Omega} e_{23}(\bar{u}^{2,\ell}(\varepsilon))(x_1, \varepsilon x_2, x_3) \varphi(x) dx \xrightarrow{\varepsilon \rightarrow 0} 0 \tag{3.47}$$

by using the Cauchy-Schwartz inequality and the fact that $e_{23}(\bar{u}^{2,\ell}(\varepsilon))$ tends to 0 strongly in $L^2(\Omega^2)$.

Thus, we have shown that the left-hand side of (3.45) tends to $-2 \dot{\zeta}_3^2(0)$ while the right-hand side tends to $-2 M(\zeta^\ell)$, which gives (3.42). \square

LEMMA 8: For each integer $\ell \geq 1$, we have

$$\begin{aligned} \kappa_{\alpha^1 \beta^1}^{1, \ell}(\varepsilon) &\rightarrow \kappa_{\alpha^1 \beta^1}^{1, \ell}(0) \quad \text{weakly in } L^2(\Omega^1), \\ \kappa_{\alpha^2 \beta^2}^{2, \ell}(\varepsilon) &\rightarrow \kappa_{\alpha^2 \beta^2}^{2, \ell}(0) \quad \text{weakly in } L^2(\Omega^2), \end{aligned} \tag{3.48}$$

with

$$\begin{cases} \kappa_{22}^{1, \ell}(0) = \kappa_{33}^{1, \ell}(0) = \frac{-\lambda}{2(\mu + \lambda)} \dot{\zeta}_1^{1, \ell}(x_1) + \frac{\lambda}{2(\mu + \lambda)} (x_{\alpha^1} - 1/2) \ddot{\zeta}_{\alpha^1}^{1, \ell}(x_1), \\ \kappa_{23}^{1, \ell}(0) = 0 \end{cases} \tag{3.49}$$

and

$$\begin{cases} \kappa_{11}^{2, \ell}(0) = \kappa_{33}^{2, \ell}(0) = \frac{-\lambda}{2(\mu + \lambda)} \dot{\zeta}_2^{2, \ell}(x_2) + \frac{\lambda}{2(\mu + \lambda)} (x_{\alpha^2} - 1/2) \ddot{\zeta}_{\alpha^2}^{2, \ell}(x_2), \\ \kappa_{13}^{2, \ell}(0) = 0. \end{cases} \tag{3.50}$$

Proof: See [19]. \square

We are now able to determine the variational equations which the flexural components satisfy.

LEMMA 9: The flexural displacements $(\zeta_{\alpha^1}^{1, \ell}, \zeta_{\alpha^2}^{2, \ell})$ belong to \mathcal{V} for each integer $\ell \geq 1$ and satisfy, $\forall (\zeta_{\alpha^1}^1, \zeta_{\alpha^2}^2) \in \mathcal{V}$,

$$\begin{aligned} &E \int_0^1 I_{\alpha^1 \beta^1}^1 \ddot{\zeta}_{\alpha^1}^{1, \ell}(x_1) \zeta_{\beta^1}^1(x_1) dx_1 \\ &+ E \int_0^1 I_{\alpha^2 \beta^2}^2 \ddot{\zeta}_{\alpha^2}^{2, \ell}(x_2) \zeta_{\beta^2}^2(x_2) dx_2 + K^1 \zeta_3^{2, \ell}(0) \zeta_3^2(0) \\ &= \eta_\ell(0) \int_0^1 \zeta_{\alpha^1}^{1, \ell}(x_1) \zeta_{\alpha^1}^1(x_1) dx_1 \\ &+ \eta_\ell(0) \int_0^1 \zeta_{\alpha^2}^{2, \ell}(x_2) \zeta_{\alpha^2}^2(x_2) dx_2 + \eta_\ell(0) \zeta_2^{1, \ell}(0) \zeta_2^1(0). \end{aligned} \tag{3.51}$$

Proof: Let $(\xi_{\alpha^1}^1, \xi_{\alpha^2}^2)$ belong to \mathcal{V} , and be \mathcal{C}^∞ in $[0,1]$. We denote by (v^1, \bar{v}^2) the corresponding Bernoulli-Navier displacements

$$\begin{cases} v^1(x) = (-(x_2 - 1/2) \xi_2^1(x_1) - (x_3 - 1/2) \xi_3^1(x_1), \xi_2^1(x_1), \xi_3^1(x_1)), \\ \bar{v}^2(x) = (\xi_1^2(x_2), -(x_1 - 1/2) \xi_1^2(x_2) - (x_3 - 1/2) \xi_3^2(x_2), \xi_3^2(x_2)). \end{cases} \tag{3.52}$$

As in [19] we construct an approximation $(v^1(\varepsilon), \bar{v}^2(\varepsilon))$ of (v^1, \bar{v}^2) such that the functions $v(\varepsilon) = (v^1(\varepsilon), v^2(\varepsilon))$, with

$$v^2(\varepsilon) = \bar{v}^2(\varepsilon) + \begin{pmatrix} 0 \\ \varepsilon^{-1} \xi_2^1(0) \\ 0 \end{pmatrix}, \tag{3.53}$$

belong to $V(\varepsilon)$ and satisfy the following convergence properties:

$$\begin{cases} (v^1(\varepsilon), \bar{v}^2(\varepsilon)) \rightarrow (v^1, \bar{v}^2) & \text{strongly in } L^2(\Omega^1) \times L^2(\Omega^2), \\ e_{11}(v^1(\varepsilon)) \rightarrow e_{11}(v^1) & \text{strongly in } L^2(\Omega^1), \\ e_{22}(\bar{v}^2(\varepsilon)) \rightarrow e_{22}(\bar{v}^2) & \text{strongly in } L^2(\Omega^2), \\ \varepsilon^{-1} e_{12}(v^1(\varepsilon)) \rightarrow \frac{1}{2} [\xi_2^1(0)/2 + (x_3 - 1/2) \xi_3^2(0)] & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-1} e_{13}(v^1(\varepsilon)) \rightarrow \frac{1}{2} x_2 \xi_3^2(0) & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-1} e_{2\alpha^2}(v^2(\varepsilon)) \rightarrow 0 & \text{strongly in } L^2(\Omega^2), \\ \varepsilon^{-2} e_{\alpha^1\alpha^1}(v^1(\varepsilon)) = \varepsilon^{-2} e_{\alpha^2\alpha^2}(v^2(\varepsilon)) = 0, \\ \varepsilon^{-2} e_{23}(v^1(\varepsilon)) \rightarrow g^1(x) & \text{strongly in } L^2(\Omega^1), \\ \varepsilon^{-2} e_{13}(v^2(\varepsilon)) \rightarrow g^2(x) & \text{strongly in } L^2(\Omega^2). \end{cases} \tag{3.54}$$

Using relations (3.31)-(3.34), (3.48)-(3.50) and (3.54) we obtain

$$\left\{ \begin{aligned}
 \int_{\Omega^1} \sigma_{11}^{1,\ell}(\varepsilon) e_{11}(v^1(\varepsilon)) dx &\rightarrow \int_{\Omega^1} \left[\lambda + 2\mu - \frac{\lambda^2}{\mu + \lambda} \right] e_{11}(u^{1,\ell}(0)) e_{11}(v^1), \\
 \int_{\Omega^1} \sigma_{12}^{1,\ell}(\varepsilon) e_{12}(v^1(\varepsilon)) dx &\rightarrow \mu \check{\zeta}_3^2(0) \tau^{1,\ell} \int_{\Omega^1} x_3 \partial_3 \chi^1 dx, \\
 \int_{\Omega^1} \sigma_{13}^{1,\ell}(\varepsilon) e_{13}(v^1(\varepsilon)) dx &\rightarrow \mu \check{\zeta}_3^2(0) \tau^{1,\ell} \int_{\Omega^1} x_2 \partial_2 \chi^1 dx, \\
 \int_{\Omega^1} \sigma_{23}^{1,\ell}(\varepsilon) e_{23}(v^1(\varepsilon)) dx &\rightarrow 0, \\
 \int_{\Omega^1} \sigma_{\alpha^1 \alpha^1}^{1,\ell}(\varepsilon) e_{\alpha^1 \alpha^1}(v^1(\varepsilon)) dx &\rightarrow 0, \\
 \int_{\Omega^2} \sigma_{22}^{2,\ell}(\varepsilon) e_{11}(v^2(\varepsilon)) dx &\rightarrow \int_{\Omega^2} \left[\lambda + 2\mu - \frac{\lambda^2}{\mu + \lambda} \right] e_{22}(u^{2,\ell}(0)) e_{22}(v^2) dx, \\
 \int_{\Omega^2} \sigma_{\alpha^2 i}^{2,\ell}(\varepsilon) e_{\alpha^2 i}(v^2(\varepsilon)) dx &\rightarrow 0.
 \end{aligned} \right. \tag{3.55}$$

Passing to the limit in equation (2.9) and using convergences (3.55), we obtain

$$\begin{aligned}
 &\int_{\Omega^1} \frac{\mu(3\lambda + 2\mu)}{\mu + \lambda} e_{11}(u^{1,\ell}(0)) e_{11}(v^1) dx \\
 &\quad + \int_{\Omega^2 \cup \Omega^2_\varepsilon} \frac{\mu(3\lambda + 2\mu)}{\mu + \lambda} e_{22}(\bar{u}^{2,\ell}(0)) e_{22}(\bar{v}^2) dx \\
 &\quad + 2\mu M(\check{\zeta}^\ell) \check{\zeta}_3^2(0) \int_{\omega^1} (x_3 \partial_3 \chi^1 + x_2 \partial_2 \chi^1) dx_2 dx_3 \\
 &= \eta_\ell(0) \int_{\Omega^1} u_{\alpha^1}^{1,\ell}(0) v_{\alpha^1}^1 dx \\
 &\quad + \eta_\ell(0) \int_{\Omega^2 \cup \Omega^2_\varepsilon} \bar{u}_{\alpha^2}^{2,\ell}(0) \bar{v}_{\alpha^2}^2 dx + \eta_\ell(0) \bar{b}^{0,\ell} \check{\zeta}_2^1(0). \tag{3.56}
 \end{aligned}$$

Replacing formulas (3.27) and (3.28) in (3.56), we obtain

$$E \int_{\Omega^1} [\check{\zeta}_1^{1,\ell}(x_1) - (x_{\alpha^1} - 1/2) \check{\zeta}_{\alpha^1}^{1,\ell}(x_1)] [- (x_{\beta^1} - 1/2) \check{\zeta}_{\beta^1}^1(x_1)] dx$$

$$\begin{aligned}
 &+ E \int_{\Omega^2 \cup \mathbb{V}_\varepsilon^2} [\zeta_2^2{}^\ell(x_2) - (x_{\alpha^2} - 1/2) \zeta_{\alpha^2}^2{}^\ell(x_2)] [-(x_{\beta^2} - 1/2) \zeta_{\beta^2}^2{}^\ell(x_2)] dx \\
 &\quad + 4 \mu \zeta_3^2{}^\ell(0) \zeta_3^2(0) \int_{\omega^1} \|\nabla_{\mathcal{L}^1}\|^2 dx_2 dx_3 \\
 = &\eta_\ell(0) \int_0^1 \zeta_{\alpha^1}^1{}^\ell(x_1) \zeta_{\alpha^1}^1(x_1) dx_1 \\
 &\quad + \eta_\ell(0) \int_0^1 \zeta_{\alpha^2}^2{}^\ell(x_2) \zeta_{\alpha^2}^2(x_2) dx_2 + \eta_\ell(0) \zeta_2^1{}^\ell(0) \zeta_2^1(0),
 \end{aligned}$$

which gives (3 51) □

Remark The last term in the left-hand side of formula (3 51) reflects the contribution of the torsion of the clamped rod in the limit model. This result was not foreseeable because usually, in rod problems, torsion effects do not appear in the limit models. The last term in the right hand side of (3 51) reflects the translation of the free rod in the x_2 -direction.

LEMMA 10 *The limit normalisation condition is given by*

$$\forall \ell, k \geq 1 \quad \int_0^1 \zeta_{\alpha^1}^{1,k} \zeta_{\alpha^1}^1{}^\ell dx_1 + \int_0^1 \zeta_{\alpha^2}^{2,k} \zeta_{\alpha^2}^2{}^\ell dx_2 + \zeta_2^k(0) \zeta_2^1{}^\ell(0) = \delta_{\ell k} \quad (3 57)$$

Proof As $u^\ell(\varepsilon)$ converge to $u^\ell(0)$ weakly in V and the injection of V into $L^2(\Omega^1, \mathbb{R}^3) \times L^2(\Omega^2, \mathbb{R}^3)$ is compact, it follows that

$$u^\ell(\varepsilon) \rightarrow u^\ell(0) \quad \text{strongly in } L^2(\Omega^1, \mathbb{R}^3) \times L^2(\Omega^2, \mathbb{R}^3) \quad (3 58)$$

Passing to the limit in (2 9), we obtain the result □

LEMMA 11 *The axial components of the limit displacements vanish and we have*

$$(u^1{}^\ell(\varepsilon), \bar{u}^2{}^\ell(\varepsilon)) \rightarrow (u^1{}^\ell(0), \bar{u}^2{}^\ell(0)) \quad \text{strongly in } V$$

Proof Let A be the elasticity tensor defined by

$$\forall \tau \in \mathcal{M}^3, \quad A\tau = 2\mu\tau + \lambda \operatorname{tr}(\tau) Id$$

Because the Lamé constants are positive, there exists $C > 0$ such that

$$\begin{aligned}
 & C(\|\kappa^{1,\ell}(\varepsilon) - \kappa^{1,\ell}(0)\|_{L^2(\Omega^1, M^3)} + \|\kappa^{2,\ell}(\varepsilon) - \kappa^{2,\ell}(0)\|_{L^2(\Omega^2, M^3)}) \\
 & \leq \int_{\Omega^1} A(\kappa^{1,\ell}(\varepsilon) - \kappa^{1,\ell}(0)) : (\kappa^{1,\ell}(\varepsilon) - \kappa^{1,\ell}(0)) \, dx \quad (3.59) \\
 & + \int_{\Omega^2 \mathcal{V}_\varepsilon^2} A(\kappa^{2,\ell}(\varepsilon) - \kappa^{2,\ell}(0)) : (\kappa^{2,\ell}(\varepsilon) - \kappa^{2,\ell}(0)) \, dx .
 \end{aligned}$$

Let $I(\varepsilon)$ the right-hand side of (3.59). Because of (2.11) we have

$$\begin{aligned}
 I(\varepsilon) = \eta_\ell(\varepsilon) + \int_{\Omega^1} A(\kappa^{1,\ell}(0) - 2\kappa^{1,\ell}(\varepsilon)) : \kappa^{1,\ell}(0) \, dx \\
 + \int_{\Omega^2 \mathcal{V}_\varepsilon^2} A(\kappa^{2,\ell}(0) - 2\kappa^{2,\ell}(\varepsilon)) : \kappa^{2,\ell}(0) \, dx . \quad (3.60)
 \end{aligned}$$

Then

$$I(\varepsilon) \rightarrow I(0) \geq 0 , \quad (3.61)$$

with

$$I(0) = \eta_\ell(0) - \int_{\Omega^1} A\kappa^{1,\ell}(0) : \kappa^{1,\ell}(0) \, dx - \int_{\Omega^2 \mathcal{V}_\varepsilon^2} A\kappa^{2,\ell}(0) : \kappa^{2,\ell}(0) \, dx . \quad (3.62)$$

Using formulas (3.32)-(3.34), (3.49) and (3.50) we obtain

$$\begin{aligned}
 I(0) &= \eta_\ell(0) - E \int_{\Omega^1} [\kappa_{11}^{1,\ell}(0)]^2 \, dx - E \int_{\Omega^2} [\kappa_{22}^{2,\ell}(0)]^2 \, dx \\
 & - 4\mu \int_{\Omega^1} [\kappa_{\alpha^1 1}^{1,\ell}(0)]^2 \, dx \\
 &= \eta_\ell(0) - E \int_0^1 I_{\alpha^1 \beta^1}^1 \zeta_{\alpha^1}^{\dot{\gamma}^1, \ell} \zeta_{\beta^1}^{\dot{\gamma}^1, \ell} \, dx_1 - E \int_0^1 I_{\alpha^2 \beta^2}^2 \zeta_{\alpha^2}^{\dot{\gamma}^2, \ell} \zeta_{\beta^2}^{\dot{\gamma}^2, \ell} \, dx_2 \\
 & - E \int_0^1 (\dot{\zeta}_1^{1,\ell})^2 \, dx_1 - E \int_0^1 (\dot{\zeta}_2^{2,\ell})^2 \, dx_2 - 4\mu \int_{\omega^1} \|\nabla \chi^1\|^2 \, dx_2 \, dx_3 M(\zeta^\ell)^2 \\
 &= - E \int_0^1 (\dot{\zeta}_1^{1,\ell})^2 \, dx_1 - E \int_0^1 (\dot{\zeta}_2^{2,\ell})^2 \, dx_2 \leq 0 \quad (3.63)
 \end{aligned}$$

by (3.51) and the normalisation condition (3.57).

We deduce then that $I(0) = 0$ and consequently,

$$(i) \quad (\kappa^{1,\ell}(\varepsilon), \kappa^{2,\ell}(\varepsilon)) \rightarrow (\kappa^{1,\ell}(0), \kappa^{2,\ell}(0))$$

$$\text{strongly in } L^2(\Omega^1; M^3) \times L^2(\Omega^2; M^3)$$

which gives

$$(e(u^{1,\ell}(\varepsilon)), e(u^{2,\ell}(\varepsilon))) \rightarrow (e(u^{1,\ell}(0)), e(u^{2,\ell}(0)))$$

$$\text{strongly in } L^2(\Omega^1; M^3) \times L^2(\Omega^2; M^3). \tag{3.64}$$

Therefore, by Korn’s inequality, the clamping condition on I^1 and estimate (3.10) we have

$$(\nabla u^{1,\ell}(\varepsilon), \nabla \bar{u}^{2,\ell}(\varepsilon)) \rightarrow (\nabla u^{1,\ell}(0), \nabla \bar{u}^{2,\ell}(0))$$

$$\text{strongly in } L^2(\Omega^1; \mathbb{R}^3) \times L^2(\Omega^2; \mathbb{R}^3). \tag{3.65}$$

We thus obtain the strong convergence of displacements in V from (3.58).

(ii) $\zeta_1^{1,\ell}$ and $\zeta_2^{2,\ell}$ are constants, wich gives $\zeta_1^{1,\ell} = 0$ since $\zeta_1^{1,\ell}(1) = 0$ and $\zeta_2^{2,\ell} = 0$ by the junction relations at $x_2 = 0$. □

We consider now the following eigenvalue problem:

Find $(\zeta, \eta) \in \mathcal{V} \times \mathbb{R}$ such that $\forall (\zeta_{\alpha^1}^1, \zeta_{\alpha^2}^2) \in \mathcal{V}$,

$$\int_0^1 I_{\alpha^1 \beta^1}^1 \check{\zeta}_{\alpha^1}^1(x_1) \check{\zeta}_{\beta^1}^1(x_1) dx_1 + \int_0^1 I_{\alpha^2 \beta^2}^2 \check{\zeta}_{\alpha^2}^2(x_2) \check{\zeta}_{\beta^2}^2(x_2) dx_2 + K^1 \check{\zeta}_3^2(0) \check{\zeta}_3^2(0)$$

$$= \eta \int_0^1 \zeta_{\alpha^1}^1(x_1) \xi_{\alpha^1}^1(x_1) dx_1 + \eta \int_0^1 \zeta_{\alpha^2}^2(x_2) \xi_{\alpha^2}^2(x_2) dx_2 + \eta \zeta_2^1(0) \xi_2^1(0).$$

$$\tag{3.66}$$

LEMMA 12: *Problem (3.66) possesses an infinite sequence of eigenvalues $(\eta_\ell)_{\ell \geq 1}$ which satisfy*

$$0 < \eta_1 \leq \eta_2 \leq \dots \leq \eta_\ell \leq \dots \quad \text{with} \quad \lim_{\ell \rightarrow +\infty} \eta_\ell = +\infty \tag{3.67}$$

and the associated eigenfunctions form an orthonormal set in \mathcal{H} and an orthogonal set in \mathcal{V} equipped with the scalar product defined by the left-hand side of (3.66).

Proof: We consider the bilinear form a defined by

$$a(\zeta, \xi) = \int_0^1 I_{\alpha^1 \beta^1}^1 \check{\zeta}_{\alpha^1}^1(x_1) \check{\xi}_{\beta^1}^1(x_1) dx_1 + \int_0^1 I_{\alpha^2 \beta^2}^2 \check{\zeta}_{\alpha^2}^2(x_2) \check{\xi}_{\beta^2}^2(x_2) dx_2 + K^1 \check{\zeta}_3^2(0) \check{\xi}_3^2(0). \quad (3.68)$$

By definition, there exists $c > 0$ such that

$$\forall \xi \in \mathcal{V}, \quad a(\xi, \xi) \geq c(\|\check{\xi}_{\alpha^1}^1\|_{L^2(0,1)}^2 + \|\check{\xi}_{\alpha^2}^2\|_{L^2(0,1)}^2) + K^1(\check{\xi}_3^2(0))^2, \quad (3.69)$$

thus if $a(\xi, \xi) = 0$, then the functions $\xi_{\alpha^1}^1$ and $\xi_{\alpha^2}^2$ are affine and

$$\check{\xi}_3^2(0) = 0. \quad (3.70)$$

The clamping conditions imply directly that $\xi_{\alpha^1}^1 = 0$ and the junction conditions give

$$\xi_1^2(0) = 0, \quad \xi_3^2(0) = \xi_3^1(0) = 0, \quad \xi_1^1(0) = \xi_2^1(0) = 0. \quad (3.71)$$

Relations (3.70) and (3.71) imply that $\xi_{\alpha^2}^2 = 0$.

The symmetric bilinear form a is coercive, if and only if, there exists a constant $\alpha > 0$ such that

$$a(\xi, \xi) \geq \alpha \|\xi\|_{H^2(0,1;\mathbb{R}^4)}. \quad (3.72)$$

To prove this, we argue by contradiction.

We suppose that there exists a sequence $\xi_n \in \mathcal{V}$ such that

$$a(\xi_n, \xi_n) \rightarrow 0 \quad \text{and} \quad \|\xi_n\|_{H^2(0,1;\mathbb{R}^4)} = 1.$$

Then, there exists a subsequence (still denoted ξ_n) such that $\xi_n \rightharpoonup_{n \rightarrow +\infty} \check{\xi}$ weakly in $H^2(0,1;\mathbb{R}^4)$, thus weakly in \mathcal{V} . As the injection from \mathcal{V} into $H^1(0,1;\mathbb{R}^4)$ is compact, we have

$$\xi_n \xrightarrow{n \rightarrow +\infty} \check{\xi} \quad \text{strongly in } H^1(0,1;\mathbb{R}^4).$$

From the definition of the bilinear form a , it is easy to see that

$$\begin{aligned} \|\xi_n - \bar{\xi}\|_{H^2(0,1;\mathbb{R}^4)}^2 &= \|\xi_n - \bar{\xi}\|_{H^1(0,1;\mathbb{R}^4)}^2 + \|\ddot{\xi}_n - \ddot{\bar{\xi}}\|_{L^2(0,1;\mathbb{R}^4)}^2 \\ &\leq \|\xi_n - \bar{\xi}\|_{H^1(0,1;\mathbb{R}^4)}^2 + \frac{1}{c} a(\xi_n - \bar{\xi}, \xi_n - \bar{\xi}) \tag{3.74} \\ &= \|\xi_n - \bar{\xi}\|_{H^1(0,1;\mathbb{R}^4)}^2 + \frac{1}{c} [a(\xi_n, \xi_n) \\ &\quad - 2 a(\xi_n, \bar{\xi}) + a(\bar{\xi}, \bar{\xi})] . \end{aligned}$$

Then, we deduce from the first part of assumption (3.73) that the right-hand side of (3.74) goes to $-\frac{1}{c} a(\bar{\xi}, \bar{\xi}) \leq 0$, as $n \rightarrow 0$, which shows that $\xi_n \rightarrow \bar{\xi}$ strongly in $H^2(0,1;\mathbb{R}^4)$ then strongly in \mathcal{V} .

Consequently, the first part of (3.73) implies that $a(\bar{\xi}, \bar{\xi}) = 0$, then $\xi_{\alpha^1}^1 = \xi_{\alpha^2}^2 = 0$, which contradicts the second assumption of (3.73).

In addition, since the injection

$$\begin{aligned} \mathcal{V} &\rightarrow \mathcal{H} \\ (\xi_{\alpha^1}^1, \xi_{\alpha^2}^2) &\rightarrow (\xi_{\alpha^1}^1, \xi_{\alpha^2}^2, \xi_2^1(0)) \tag{3.75} \end{aligned}$$

is compact, the existence of the eigenvalues and eigenfunctions follows from classical spectral theory, see e.g. [25]. □

LEMMA 13: *If η is an eigenvalue of the limit problem (3.66) and ζ an associated eigenfunction then we have*

$$\zeta \in \mathcal{V}_\eta . \tag{3.76}$$

Proof: It suffices to take a test-function of the form $(0, 0, x_2)$ in equation (3.66). □

LEMMA 14: *We have $\eta_\ell(0) = \eta_\ell$, counting multiplicities and for each integer $\ell \geq 1$, the whole family $(\eta_\ell(\varepsilon))_{\varepsilon>0}$ converges to $(\eta_\ell(0))$, as $\varepsilon \rightarrow 0$. In addition if $\eta_\ell(0)$ is a simple eigenvalue of (3.66), then the whole family of flexural displacements converges to $\pm (\zeta^{1,\ell}, \zeta^{2,\ell})$.*

Proof: The proof is identical to that of [7]. □

Lemma 14 completes the proof of the theorem.

4. INTERPRETATION OF THE RESULTS

Let us give the strong formulation of the limit variational model.

PROPOSITION: *The flexural components of the limit displacement are solutions of the following system:*

$$\begin{cases} E \frac{d^2}{dx_1^2} \left(I_{\alpha^1 \beta^1}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2} \right) = \eta_\ell(0) \zeta_{\beta^1}^{1,\ell} & 0 < x_1 < 1, \\ E \frac{d^2}{dx_2^2} \left(I_{\alpha^2 \beta^2}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2} \right) = \eta_\ell(0) \zeta_{\beta^2}^{2,\ell} & 0 < x_2 < 1, \end{cases} \quad (4.1)$$

with

$$\zeta_{\alpha^1}^{1,\ell}(1) = \dot{\zeta}_{\alpha^1}^{1,\ell}(1) = 0, \quad (4.2)$$

$$\zeta_1^{2,\ell}(0) = 0, \quad \dot{\zeta}_2^{1,\ell}(0) = \dot{\zeta}_1^{2,\ell}(0),$$

$$\zeta_3^{1,\ell}(0) = \zeta_3^{2,\ell}(0), \quad \dot{\zeta}_3^{2,\ell}(0) = \frac{\eta_\ell(0)}{K^1} \int_0^1 x_2 \zeta_3^{2,\ell}(x_2) dx_2, \quad (4.3)$$

$$\begin{cases} EI_{\alpha^1 \beta^3}^1 \frac{d^3 \zeta_{\alpha^1}^{1,\ell}}{dx_1^3}(0) + EI_{\alpha^2 \beta^3}^2 \frac{d^3 \zeta_{\alpha^2}^{2,\ell}}{dx_2^3}(0) = 0, \\ EI_{\alpha^1 \beta^2}^1 \frac{d^3 \zeta_{\alpha^1}^{1,\ell}}{dx_1^3}(0) = \eta_\ell(0) \zeta_{\beta^2}^{1,\ell}(0), \end{cases} \quad (4.4)$$

$$\begin{cases} EI_{\alpha^1 \beta^2}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) = EI_{\alpha^2 \beta^1}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0), \\ EI_{\alpha^1 \beta^3}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) = 0, \\ EI_{\alpha^2 \beta^3}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0) = K^1 \dot{\zeta}_3^{2,\ell}(0), \end{cases} \quad (4.5)$$

and

$$\frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(1) = \frac{d^3 \zeta_{\alpha^2}^{2,\ell}}{dx_2^3}(1) = 0. \quad (4.6)$$

Proof: The solutions of the variational problem (3.51) are $\mathcal{C}^\infty([0, 1])$. Then, performing two integrations by parts in the right-hand side of (3.51), we obtain

$$\begin{aligned}
 & E \int_0^1 \frac{d^2}{dx_1^2} (I_{\alpha^1 \beta^1}^1 \check{\zeta}_{\alpha^1}^{1,\ell}(x_1)) \xi_{\beta^1}^1(x_1) dx_1 + E \int_0^1 \frac{d^2}{dx_2^2} (I_{\alpha^2 \beta^2}^2 \check{\zeta}_{\alpha^2}^{2,\ell}(x_2)) \xi_{\beta^2}^2(x_2) dx_2 \\
 & + K^1 \check{\zeta}_3^{2,\ell}(0) \xi_3^2(0) - \left[E \frac{d}{dx_1} (I_{\alpha^1 \beta^1}^1 \check{\zeta}_{\alpha^1}^{1,\ell}(x_1)) \xi_{\beta^1}^1(x_1) \right]_0^1 \\
 & - \left[E \frac{d}{dx_2} (I_{\alpha^2 \beta^2}^2 \check{\zeta}_{\alpha^2}^{2,\ell}(x_2)) \xi_{\beta^2}^2(x_2) \right]_0^1 \\
 & + [EI_{\alpha^1 \beta^1}^1 \check{\zeta}_{\alpha^1}^{1,\ell}(x_1) \xi_{\beta^1}^1(x_1)]_0^1 + [EI_{\alpha^2 \beta^2}^2 \check{\zeta}_{\alpha^2}^{2,\ell}(x_2) \xi_{\beta^2}^2(x_2)]_0^1 \\
 & = \eta_\ell(0) \int_0^1 \zeta_{\alpha^1}^{1,\ell}(x_1) \xi_{\alpha^1}^1(x_1) dx_1 + \eta_\ell(0) \\
 & \quad \times \int_0^1 \zeta_{\alpha^2}^{2,\ell}(x_2) \xi_{\alpha^2}^2(x_2) dx_2 + \eta_\ell(0) \zeta_2^{1,\ell}(0) \xi_2^1(0). \tag{4.7}
 \end{aligned}$$

Taking successively $\xi = (\xi_{\alpha^1}^1; 0) \in \mathcal{V}$ such that $\xi_2^1(0) = \xi_3^1(0) = 0$ and $\xi = (0; \xi_{\alpha^2}^2) \in \mathcal{V}_0$ such that $\xi_{\alpha^2}^2(1) = \xi_{\beta^2}^2(1) = 0$, we obtain (4.1).

Using the fact that $\xi \in \mathcal{V}$, we next obtain

$$\begin{aligned}
 & \left[EI_{\alpha^1 \beta^1}^1 \frac{d^3 \zeta_{\alpha^1}^{1,\ell}}{dx_1^3}(0) + EI_{\alpha^2 \beta^2}^2 \frac{d^3 \zeta_{\alpha^2}^{2,\ell}}{dx_2^3}(0) \right] \xi_3^1(0) \\
 & \quad + \left[EI_{\alpha^1 \beta^1}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) + EI_{\alpha^2 \beta^2}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0) \right] \xi_2^1(0) \\
 & \quad - EI_{\alpha^1 \beta^1}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}(0) \xi_3^1(0) - \left[EI_{\alpha^2 \beta^2}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(0) - K^1 \check{\zeta}_3^2(0) \right] \xi_3^2(0) \\
 & \quad + EI_{\alpha^2 \beta^2}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(1) \xi_{\beta^2}^2(1) + EI_{\alpha^2 \beta^2}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}(1) \xi_{\beta^2}^2(1) \\
 & \quad + EI_{\alpha^1 \beta^1}^1 \frac{d^3 \zeta_{\alpha^1}^{1,\ell}}{dx_1^3}(0) \xi_2^1(0) = \eta_\ell(0) \zeta_2^{1,\ell}(0) \xi_2^1(0). \tag{4.8}
 \end{aligned}$$

To obtain (4.4) we choose in (4.8) a test-function $\zeta \in \mathcal{V}_0$ such that $\zeta_{\alpha^2}^2(1) = \dot{\zeta}_{\alpha^2}^2(1) = \zeta_{\alpha^1}^1(0) = 0$ and we take successively $\zeta_2^1(0) = 0$ and $\zeta_3^1(0) = 0$. Then, choosing $\zeta \in \mathcal{V}$ such that $\zeta_{\alpha^2}^2(1) = \dot{\zeta}_{\alpha^2}^2(1) = 0$ and taking successively $\zeta_{\alpha^1}^1(0) = 0, \bar{\zeta}_2^1(0) = 0$ and $\dot{\zeta}_3^1(0) = 0$, we obtain (4.5). By the same way we obtain (4.6). \square

Remark: Let us draw a sketch of the flexural components of the limit displacements

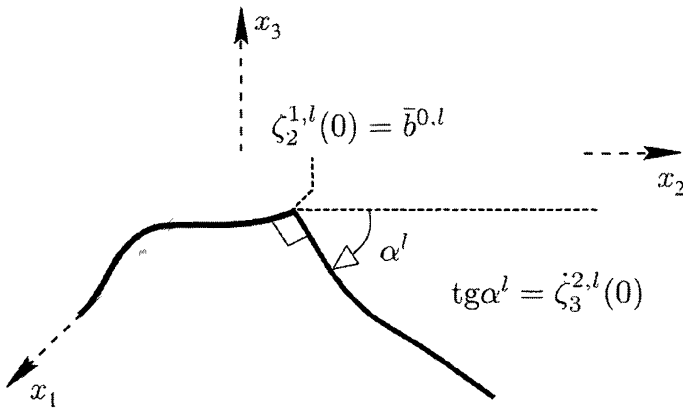


Figure 2.

We see that the junction end of the clamped rod follows the translations of the free rod in the x_2 -direction. The free rod undergoes a rotation whose axis is the clamped rod, which create a torsion in the clamped rod.

Remark: The bending moments are defined by

$$\begin{cases} m_2^{1,\ell} = \int_{\omega^1} (x_3 - 1/2) \sigma_{11}^{1,\ell}(0) dx_2 dx_3, \\ m_3^{1,\ell} = - \int_{\omega^1} (x_2 - 1/2) \sigma_{11}^{1,\ell}(0) dx_2 dx_3, \end{cases} \quad (4.9)$$

where

$$\sigma_{11}^{1,\ell}(0) = \lim_{\varepsilon \rightarrow 0} \sigma_{11}^{1,\ell}(\varepsilon), \quad (4.10)$$

and

$$\begin{cases} m_1^{2,\ell} = - \int_{\omega^2} (x_3 - 1/2) \sigma_{22}^{2,\ell}(0) dx_1 dx_3, \\ m_3^{2,\ell} = \int_{\omega^2} (x_1 - 1/2) \sigma_{22}^{2,\ell}(0) dx_1 dx_3, \end{cases} \tag{4.11}$$

where

$$\sigma_{22}^{2,\ell}(0) = \lim_{\varepsilon \rightarrow 0} \sigma_{22}^{2,\ell}(\varepsilon). \tag{4.12}$$

Due to formulas (3.28)-(3.33) and (3.49)-(3.50) we have

$$\begin{cases} m_2^{1,\ell} = - EI_{\alpha^1 3}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}, \\ m_3^{1,\ell} = - EI_{\alpha^1 2}^1 \frac{d^2 \zeta_{\alpha^1}^{1,\ell}}{dx_1^2}, \end{cases} \quad \text{and} \quad \begin{cases} m_1^{2,\ell} = EI_{\alpha^2 3}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}, \\ m_3^{2,\ell} = EI_{\alpha^2 1}^2 \frac{d^2 \zeta_{\alpha^2}^{2,\ell}}{dx_2^2}. \end{cases} \tag{4.13}$$

Consequently, we can rewrite equations (4.1)-(4.6) as follows

$$\begin{cases} \frac{d^2 m_3^{1,\ell}}{dx_1^2} = \eta_\ell(0) \zeta_2^{1,\ell}, \\ - \frac{d^2 m_2^{1,\ell}}{dx_1^2} = \eta_\ell(0) \zeta_3^{1,\ell}, \\ - \frac{d^2 m_3^{2,\ell}}{dx_2^2} = \eta_\ell(0) \zeta_1^{2,\ell}, \\ \frac{d^2 m_3^{2,\ell}}{dx_2^2} = \eta_\ell(0) \zeta_3^{2,\ell}, \end{cases} \tag{4.14}$$

with

$$\zeta_{\alpha^1}^{1,\ell}(1) = \dot{\zeta}_{\alpha^1}^{1,\ell}(1) = 0, \tag{4.15}$$

$$\zeta_1^{2,\ell}(0) = 0, \quad \dot{\zeta}_2^{1,\ell}(0) = - \dot{\zeta}_1^{2,\ell}(0), \quad \zeta_3^{1,\ell}(0) = \zeta_3^{2,\ell}(0),$$

$$\dot{\zeta}_3^{2,\ell}(0) = \frac{\eta_\ell(0)}{K^1} \int_0^1 x_2 \zeta_3^{2,\ell}(x_2) dx_2, \tag{4.16}$$

$$\begin{cases} \frac{dm_3^{1,\ell}}{dx_1}(0) = \eta_\ell(0) \zeta_2^{1,\ell}(0), \\ -\frac{dm_2^{1,\ell}}{dx_1}(0) + \frac{dm_1^{2,\ell}}{dx_2}(0) = 0, \end{cases} \quad (4.17)$$

$$\begin{cases} m_3^{1,\ell}(0) = m_3^{2,\ell}(0), \\ m_2^{1,\ell}(0) = 0, \\ m_1^{2,\ell}(0) = \eta_\ell(0) \int_0^1 x_2 \zeta_3^{2,\ell}(x_2) dx_2 \end{cases} \quad (4.18)$$

and

$$m_{\beta^2}^{2,\ell}(1) = \frac{dm_{\beta^2}^{2,\ell}}{dx_2}(1) = 0. \quad (4.19)$$

Equations (4.14) are the usual equations of the eigenvalue problem for rods. Relations (4.15), (4.16) and (4.19) are respectively the clamping conditions, the limit junction relations and the free end conditions. Relations (4.17) express the balance of forces at the junction. The first equation indicates that the clamped rod exerts a force on the free rod at the junction equal to the global acceleration in the x_2 -direction. The second equation shows that the resultant of the vertical forces at the junction vanishes. The first relation of (4.18) shows the transmission of the vertical components of the moments at the junction. The last two relations of (4.18) show that the horizontal component of the bending moments of the clamped rod is completely free while that of the free rod is equal to the moments in x_1 -direction of the inertia forces exerted by the free rod.

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REFERENCES

- [1] I. AGANOVIČ and Z. TUTEK, 1986, A justification of the one-dimensional linear model of elastic beam, *Math. Meth. Appl. Sci.*, **8**, p. 1-14.

- [2] A. BERMUDEZ and J. M. VIANO, 1984, Une justification des équations de la thermoélasticité des poutres à section variable par des méthodes asymptotiques, *RAIRO, Analyse Numérique*, **18**, 347-376.
- [3] M. BERNADOU, S. FAYOLLE and F. LÉNÉ, 1989, Numerical analysis of junctions between plates, *Comput. Methods Appl. Mech. Engrg.*, **74**, 307-326.
- [4] F. BOURQUIN and P. G. CIARLET, 1989, Modeling and justification of eigenvalue problems for junctions between elastic structures, *J. Funct. Anal.*, **87**, 392-427.
- [5] P. G. CIARLET, 1990, *Plates and Junctions in Elastic Multi-Structures. An Asymptotic Analysis* RMA 14, Masson, Paris.
- [6] P. G. CIARLET and P. DESTUYNDER, 1979, A justification of two-dimensional linear plate model, *J. Mécanique*, **18**, 315-344.
- [7] P. G. CIARLET and S. KESAVAN, 1981, Two-dimensional approximation of three-dimensional eigenvalue problems in plate theory, *Comp. Meth. Appl. Mech. Eng.*, **26**, 145-172.
- [8] P. G. CIARLET, H. LE DRET and R. NZENGWA, 1989, Junctions between three-dimensional and two-dimensional linearly elastic structures, *J. Math. Pures Appl.*, **68**, 261-295.
- [9] A. CIMETIÈRE, G. GEYMONAT H. LE DRET, A. RAOULT and Z. TUTEK, 1988, Asymptotic theory and analysis for displacement and stress distributions in nonlinear elastic straight slender rods, *J. Elasticity*, **19**, 111-161.
- [10] S. FAYOLLE, 1987, Sur l'analyse numérique de raccords de poutres et de plaques, Thèse de 3^e cycle, Université Pierre et Marie Curie, 1987.
- [11] G. GEYMONAT, F. KRASUCKI and J. J. MARIGO, 1987, Stress distribution in anisotropic elastic composite beams, in: P. G. Ciarlet and E. Sanchez-Palencia, eds. Applications of Multiple Scalings in Mechanics, RMA 4, Masson, Paris, pp. 118-133.
- [12] I. GRUAIS, 1993, Modélisation de la jonction entre une poutre et une plaque en élasticité linéarisée, *RAIRO Analyse Numérique*, **27**, 77-105.
- [13] I. GRUAIS, 1993, Modeling of the junction between a plate and a rod in nonlinear elasticity, *Asymptotic Anal.*, **7**, 179-194.
- [14] N. KERDID, 1993, Comportement asymptotique quand l'épaisseur tend vers zéro du problème de valeurs propres pour une poutre mince encastree, en élasticité linéaire, *C. R. Acad. Sci. Paris*, t. **316**, Série I, 755-758.
- [15] N. KERDID, 1995, Modélisation des vibrations d'une multi-structure forme de deux poutres, *C. R. Acad. Sci. Paris*, t. **321**, Série I, 1641-1646.
- [16] N. KERDID, 1995, Étude de problèmes de jonctions de poutres en élasticité linéaire, Thèse de Doctorat, Université Pierre et Marie Curie.
- [17] H. LE DRET, 1990, Modeling of folded plate, *Comput. Mech.*, **5**, 401-416.
- [18] H. LE DRET, 1989, Folded plates revisited, *Comput. Mech.*, **5**, 345-365.
- [19] H. LE DRET, 1989, Modelling of the junction between two rods, *J. Math. Pures Appl.*, **68**, 365-397.
- [20] H. LE DRET, 1990, Vibration of a folded plate, *Modélisation Mathématique et Analyse Numérique*, **24**, 501-521.

- [21] H. LE DRET, 1994, Elastodynamics for multiplate structures, in: H. Brezis and J. L. Lions eds, *Nonlinear Partial Differential Equations and their Applications*, Collège de France Seminar, Vol. XI, pp. 151-180.
- [22] H. LE DRET, 1991, *Problèmes variationnels dans les multi-domaines. Modélisation des jonctions et applications*, RMA 19, Masson, Paris.
- [23] V. LODS, Modeling and junction of an eigenvalue problem for a plate inserted in a three-dimensional support, *Modélisation Mathématique et Analyse Numérique*, to appear.
- [24] A. RAOULT, 1992, Asymptotic modeling of the elastodynamics of a multi-structure, *Asymptotic Anal.*, **6**, 73-108.
- [25] P. A. RAVIART and J. M. THOMAS, 1983, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson, Paris.
- [26] J. SANCHEZ-HUBERT and E. SANCHEZ-PALENCIA, 1991, Couplage flexion-torsion-traction dans les poutres anisotropes à section hétérogène, *C. R. Acad. Sci. Paris*, **t. 312**, Série, 337-344.
- [27] L. TRABUCHO and J. M. VIAÑO, 1987, Derivation of generalized models for linear elastic beams by asymptotic expansion methods, in: P. G. Ciarlet and E. Sanchez-Palencia, eds. *Applications of multiple Scalings in Mechanics*, RMA 4, Masson, Paris, pp. 302-315.
- [28] L. TRABUCHO and J. M. VIAÑO, 1988, A derivation of generalized Saint Venant's torsion theory from three dimensional elasticity by asymptotic expansion methods, *Applicable Analysis*, **31**, 129-148.
- [29] L. TRABUCHO and J. M. VIAÑO, 1990, A new approach of Timoshenko's beam theory by the asymptotic expansion method, *Mathematical Modelling and Numerical Analysis*, **24**, 651-680.
- [30] L. TRABUCHO and J. M. VIAÑO, 1989, Existence and characterisation of higher order terms in an asymptotic expansion method for linearized elastic beams, *Asymptotic Anal.*, **2**, 223-255.
- [31] L. TRABUCHO and J. M. VIAÑO, 1995, Mathematical Modeling of Rods, in: P. G. Ciarlet and J. L. Lions, eds, *Handbook of Numerical Analysis*, Vol. IV, North-Holland, Amsterdam, pp. 487-969.