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# VIBRATIONS OF THIN ELASTIC STRUCTURES AND EXACT CONTROLLABILITY (1)

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Résumé. — *Nous considérons un corps élastique mince dans une direction, notre but est d'en contrôler exactement les vibrations en agissant sur les faces supérieure et inférieure et sur une partie de la frontière latérale Nous établissons l'existence de contrôles exacts, puis nous étudions leur comportement asymptotique quand l'épaisseur tend vers zéro Nous caractérisons la limite en fonction de la solution d'un problème de contrôlabilité exacte (avec contrôle interne) pour une plaque*

Abstract — *In this article, we address the question of exact controllability of the vibrations of three-dimensional elastic media which are thin in one direction Apart from proving the existence of exact controls, we examine their asymptotic behaviour as thickness parameter goes to zero We characterize the limit in terms of the solution of an exact controllability problem associated with the plate equation in two dimensions*

## 1. INTRODUCTION

In this work, we consider the vibrations of three-dimensional elastic bodies which are thin in one direction, say that of the $x_3$-axis. These bodies are assumed to be homogeneous but may not be isotropic. Let $e > 0$ denote the thickness of the body in that direction. We are interested in small values of $e$. The boundary of the body is divided into three disjoint pieces: the lateral part and the top-bottom surfaces. The mathematical model is an initial boundary value problem with mixed boundary conditions corresponding to the system of linear elasticity; we impose Dirichlet condition on the lateral part of the boundary while Neumann condition is taken on the top-bottom surfaces.

We consider the exact controllability of the vibrations of the system described above by acting on the boundary of the body. More precisely, we look for suitable controls acting through the boundary conditions mentioned above and a finite time $T_e$ such that these vibrations are killed at time $T_e$. In this work, we study the following two aspects: existence of exact controls and time of controllability for each $e > 0$ and behaviour of controls as $e \to 0$.

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The problem of exact controllability for distributed systems has been studied extensively by Lions [1988a], [1988b] in a very general set-up The procedure introduced by J L Lions to attack the problem is the so called Hilbert Uniqueness Method (HUM) combined with the method of multipliers Our plan in this paper is to follow HUM but of course with certain modifications adopted by us in our earlier work See Saint Jean Paulin & Vanninathan [1994] where the vibrations are modelled by the usual wave equation and not by the system of linear elasticity The model considered here is more realistic and more complicated because it is a system in which there is interaction between various components These complications demand further modification in the HUM

There is a vast literature on the movement of thin elastic bodies under the action of given volumic and surface force fields See Ciarlet & Destuynder [1979] The typical result one gets is the following when $e \to 0$, the displacement vector which represents the solution of the three-dimensional problem is described by a set of two-dimensional problems called plate problem The corresponding result for the vibrating bodies have been obtained by Raoult [1980], [1988] and Ciarlet & Kesavan [1981] Some of these results are beautifully presented in the book by Ciarlet [1990]

It is now time to comment on the nature of the results obtained here and the techniques followed As mentioned above, we follow the general lines of HUM However there are modifications Our domain is not smooth neither convex and we have mixed boundary conditions In such circumstances, it is known that HUM has to be combined with the regularity results of Grisvard [1989] and Nicaise [1992] (see Saint Jean Paulin & Vanninathan [1994]) Actually in the above cited work of Nicaise, the regularity result is proved only for isotropic medium One of the aims of our work is to show that the HUM solves the exact controllability problem and its asymptotic behaviour can be analyzed even in the non-isotropic case as long as the above mentioned regularity result is valid Since we are interested in the asymptotic behaviour of the exact controls as $e \to 0$, we need to establish various bounds independent of $e$ on the solution To this end, we choose the multipliers which are more suitable to thin bodies Next, with a view to obtain a two-dimensional problem at the limit, we choose suitable multipliers to obtain the so-called inverse inequality With all these preparations, we will be able to prove the existence of exact controls thereby slightly generalizing the results of Lions [1988a] from the isotropic case to the general case Our approach also gives the estimates on the controls which are used in analyzing their behaviour as $e \to 0$ First of all, it is established that the minimal time $T_e$ of exact controllability is bounded above independent of $e$ We fix one time $T$ (independent of $e$) at which we have exact controllability and work with it subsequently

Since the problem is linear, weak convergence is enough to pass to the limit To identify the limit, suitable test functions akin to thin bodies are needed To contruct them we utilize energy method and the asymptotic expansion. See Section 5.

Our results show that the controls on the top-bottom surfaces of the body tend to zero in a suitable topology. The third component of the lateral control also besomes zero at the limit. The behaviour of the first two components is described in terms of a problem of exact controllability in two dimensions associated with the vibrations of a plate. We remark that this two-dimensional problem has also a control in the entire interior of the domain and its presence is due to the boundary controls on the top-bottom surfaces in the original three-dimensional problem.

The article is organized as follows: we introduce the notations and pose the problem in the next section. Following HUM, we consider the associated forward problem with homogeneous boundary condition in § 3. Several estimates in the form of energy inequality, direct and inverse inequalities on this problem are derived. The exact controllability problem is then solved with the introduction of the backward Cauchy problem with non-homogeneous boundary conditions and the operator $\Lambda^e$. This is done in § 4. The existence result is proved in Theorem 4.2. The behaviour of its solution as $e \to 0$ is analyzed in § 5 which is divided into several paragraphs. Each one of them is devoted to the study of the asymptotic behaviour of various problems introduced in the previous sections. The final result is stated in Theorem 5.12. An expanded version giving all the details of proof of the results found here are given in the unpublished preprint Saint Jean Paulin & Vanninathan [1995].

When this article was in preparation, the preprint of the work of Figueiredo & Zuazua [1994] was brought to our attention. There are lot of similarities between these works. Both deal with the asymptotic behaviour of the controls when there is reduction in the dimension. The lateral control in their paper acts on the entire part $\Gamma_0$ whereas in our work it acts only on the subset $\gamma(z^0)$. Further the class of solutions which are shown to be exactly controllable is larger in our case. This is because the space $F^e$ is smaller than the space of initial conditions with finite energy. The results formulated in their work are valid only in the isotropic case whereas we have worked with non-isotropic medium assuming the validity of the regularity results described in Theorem 3.1. The inverse inequality established in § 3 shows that the minimal time of exact controllability in three-dimensions in $0(1)$ as $e \to 0$ whereas it is proved to be $0(e)$ in Figueiredo & Zuazua [1994].

Summation convention with respect to repeated indices is used throughout unless stated otherwise. Following the standard practice in the analysis of thin bodies, Latin indices $i, j, k$ etc. take values in $\{1, 2, 3\}$ and Greek indices $\alpha, \beta, \gamma$ etc. take values in $\{1, 2\}$. We invariably use, in the sequel, the subindices for the various components of vectors, tensors etc., and superindices for the parameters on which they depend.

## 2. NOTATIONS AND PROBLEM TO BE STUDIED

The thin three-dimensional elastic body whose vibrations interest us is constructed as follows: let $\omega$ be a bounded connected open set in the plane whose smooth boundary is denoted as $\gamma$. It is not assumed that $\omega$ is convex. Given the thickness parameter $\epsilon > 0$, we let

$$(2.1) \quad \Omega^e = \omega \times \left] -\frac{e}{2}, \frac{e}{2} \right[, \quad \Gamma_0^e = \gamma \times \left[ -\frac{e}{2}, \frac{e}{2} \right], \quad \Gamma_{\pm}^e = \omega \times \left\{ \pm \frac{e}{2} \right\}.$$

We see that $\Omega^e$ is the three-dimensional body whose boundary $\Gamma^e$ has been partitioned into three pieces $\Gamma_0^e$ (lateral part), $\Gamma_+^e$ (top part) and $\Gamma_-^e$ (bottom part):

$$\Gamma^e = \Gamma_0^e \cup \Gamma_+^e \cup \Gamma_-^e .$$

As mentioned in the introduction, we take control on a part of the lateral boundary $\Gamma_0^e$ apart from the entire top-bottom boundaries $\Gamma_{\pm}^e$. The required part of $\Gamma_0^e$ is defined as follows. Let $x^0$ be a point with $x_3^0 = 0$. We define

$$(2.2) \qquad m(x) = x - x^0, \quad \gamma(x^0) = \{ x \in \gamma \, ; m(x) \, . \, \nu(x) > 0 \},$$

where $\nu(x)$ is the unit exterior normal to $\Gamma^e$. We next set

$$(2.3) \quad \gamma_* = \gamma \backslash \gamma(x^0), \quad \Gamma^e(x^0) = \gamma(x^0) \times \left[ -\frac{e}{2}, \frac{e}{2} \right], \quad \Gamma_*^e = \Gamma_0^e \backslash \Gamma^e(x^0) .$$

The homogeneous elastic medium is represented by real constants $(a_{ijk\ell})$ with the following properties:

$$(2.4a) \qquad a_{ijk\ell} \, \xi_{ij} \, \xi_{k\ell} \geqslant c_0 \left( \sum_{i,j=1}^{3} |\xi_{ij}|^2 \right)$$

for all real symmetric matrices $(\xi_{ij})$. Here $c_0$ is a positive number. We also have

$$(2.4b) \qquad a_{ijk\ell} = a_{jik\ell} = a_{ij\ell k} = a_{k\ell ij} \quad \forall i, j, k, \ell \in \{1, 2, 3\} .$$

In the case of isotropic medium, we have

$$a_{ijk\ell} = \lambda \delta_{ij} \, \delta_{k\ell} + \mu(\delta_{ik} \, \delta_{j\ell} + \delta_{i\ell} \, \delta_{jk})$$

where $\lambda$ and $\mu$ are positive numbers called Lamé constants.

We are interested in the vibrations of the elastic medium described above. Let $y(x) = (y_1(x), y_2(x), y_3(x))$ denote the displacement vector. The corresponding strain tensor is denoted as follows:

$$\hat{e}_{ij} = \hat{e}_{ij}(y) = \frac{1}{2}\left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i}\right) \quad \forall i, j . \tag{2.5}$$

The system of equations which governs the vibrations of the medium $\Omega^e$ is:

$$e^2 \frac{\partial^2 y_i}{\partial t^2} - \frac{\partial}{\partial x_j}(a_{ijk\ell}\, \hat{e}_{k\ell}(y)) = 0 \quad \text{in} \quad Q^e, \text{ for } i = 1, 2, 3 , \tag{2.6a}$$

$$y = v \text{ on } \Sigma^e(x^0) \quad \text{and} \quad y = 0 \text{ on } \Sigma^e_* , \tag{2.6b}$$

$$a_{ijk\ell}\hat{e}_{k\ell}(y)\, \nu_j = w_i^{\pm} \quad \text{on} \quad \Sigma^e_{\pm}, \text{ for } i = 1, 2, 3 , \tag{2.6c}$$

$$y(0) = y^0 \quad \text{and} \quad \frac{\partial y}{\partial t}(0) = y^1 \text{ in } \Omega^e . \tag{2.6d}$$

We use the following notations:

$$\begin{cases} Q^e = \Omega^e \times ]0, T[, \quad \Sigma^e = \Gamma^e \times ]0, T[, \quad \Sigma^e_0 = \Gamma^e_0 \times ]0, T[ \\ \Sigma^e_{\pm} = \Gamma^e_{\pm} \times ]0, T[, \quad \Sigma^e(x^0) = \Gamma^e(x^0) \times ]0, T[, \quad \Sigma^e_* = \Gamma^e_* \times ]0, T[ . \end{cases} \tag{2.7}$$

The controls are $v$ on the part $\Gamma^e(x^0)$ of the lateral boundary through Dirichlet action and $w^{\pm}$ on the top-bottom surfaces through Neumann action. Note that there is no control on the part $\Gamma^e_*$.

We would like to analyze the exact controllability of the above system. Since the domain $\Omega^e$ is varying with $e$, it is difficult to analyze the functions defined on $\Omega^e$ and so let us first transform the problem (2.6) from the variable domain $\Omega^e$ to the fixed domain $\Omega = \omega \times ] - 1/2, 1/2[$. To this end, we define the following correspondence between points by affinity:

$$x = (x_1, x_2, x_3) \in \Omega^e \rightarrow z = (z_1, z_2, z_3) \in \Omega$$

where $z_\alpha = x_\alpha$ and $z_3 = e^{-1} x_3$. Under this change of independent variables, the system (2.6) would be transformed into another one on $\Omega$. We rewrite this in terms of the scaled displacement vector defined on $\Omega$ as follows:

$$y^e_\alpha(z) = e^{-2} y_\alpha(x), \quad y^e_3(z) = e^{-1} y_3(x) . \tag{2.8}$$

We then have the strain tensor $\hat{e}_{ij}(y^e)$ corresponding to $y^e$ defined on $\Omega$ :

$$\hat{e}_{ij}(y^e) = \frac{1}{2}\left(\frac{\partial y_i^e}{\partial z_j} + \frac{\partial y_j^e}{\partial z_i}\right) \quad \forall i,j . \tag{2.9}$$

We then see easily the relationships which exist between $\hat{e}_{ij}(y)$ and $\hat{e}_{ij}(y^e)$ :

$$\hat{e}_{\alpha\beta}(y)(x) = e^2 \hat{e}_{\alpha\beta}(y^e)(z), \quad \hat{e}_{\alpha 3}(y)(x) = e\hat{e}_{\alpha 3}(y^e)(z), \tag{2.10}$$

$$\hat{e}_{33}(y)(x) = \hat{e}_{33}(y^e)(z) .$$

The transformed system on $\Omega$ can therefore be written as follows:

$$e^3 \frac{\partial^2 y_\tau^e}{\partial t^2} - e \frac{\partial \hat{\sigma}_{\tau\eta}}{\partial z_\eta} - \frac{\partial \hat{\sigma}_{\tau 3}}{\partial z_3} = 0 \text{ in } Q, \text{ for } \tau = 1, 2 , \tag{2.11a}$$

$$e^2 \frac{\partial^2 y_3^e}{\partial t^2} - e \frac{\partial \hat{\sigma}_{3\eta}}{\partial z_\eta} - \frac{\partial \hat{\sigma}_{33}}{\partial z_3} = 0 \text{ in } Q , \tag{2.11b}$$

$$y^e = v^e \text{ on } \Sigma(z^0), \quad y^e = 0 \text{ on } \Sigma_* , \tag{2.11c}$$

$$\hat{\sigma}_{i3}\, \nu_3 = w_i^{\pm, e} \text{ on } \Sigma_{\pm}, \text{ for } i = 1, 2, 3 , \tag{2.11d}$$

$$y^e(0) = y^{0,e} \quad \text{and} \quad \frac{\partial y^e}{\partial t}(0) = y^{1,e} \text{ in } \Omega . \tag{2.11e}$$

Here we have used the following notations:

$$\hat{\sigma}_{ij} = \sigma_{ij}(y^e) \equiv a_{ij\alpha\beta}\hat{e}_{\alpha\beta} + 2\, e^{-1}\, a_{ij\alpha 3}\hat{e}_{\alpha 3} + e^{-2}\, a_{ij33}\hat{e}_{33} \tag{2.12}$$

which can also be written as

$$\hat{\sigma}_{ij} = a_{ijk\ell}\hat{\xi}_{k\ell} \tag{2.13}$$

if we define the tensor $(\hat{\xi}_{k\ell})$ as follows:

$$\hat{\xi}_{\alpha\beta} = \hat{e}_{\alpha\beta}, \quad \hat{\xi}_{\alpha 3} = e^{-1}\hat{e}_{\alpha 3}, \quad \hat{\xi}_{33} = e^{-2}\hat{e}_{33} . \tag{2.14}$$

Next, $y^{0,e}$ and $y^{1,e}$ are rescaled initial conditions according to the rule (2.8). Finally, the symbols $Q$, $\Sigma(x^0)$, $\Sigma_*$, $\Sigma_{\pm}$ and $\Sigma_0$ correspond to the domain $\Omega$

and are defined analogously as in (2.7). We decide to deal with the problem (2.11) directly without referring back to (2.6).

Let us now ask the following question which is an exact controllability problem: under what conditions on the initial data in (2.11e), does there exist a time $T > 0$ such that we can find controls $v^e, w^{\pm,e}$ in such a way that the solution $y^e$ of (2.11) satisfies

$$y^e(\,.\,, T) = \frac{\partial y^e}{\partial t}(\,.\,, T) = 0 \quad \text{in } \Omega\,? \tag{2.15}$$

We follow HUM to answer this question and proceed to analyze the asymptotic behaviour of the controls provided by HUM as $e \to 0$. The principal results are given in Theorem 4.2 and Theorem 5.12.

Because of the particular geometry of the domain, it is natural to separate the $z_3$ variable from the rest. We write $\tilde{z} = (z_1, z_2) \in \omega$, and $z_3 \in I = ]-1/2, 1/2[$. For a function $g$ defined on $\Omega$, we denote by $m(g)$ its average with respect to $z_3$ variable:

$$m(g)(\tilde{z}) = \int_I g(\tilde{z}, z_3)\, dz_3 \quad \text{for} \quad \tilde{z} \in \omega\,.$$

Generally, $c$ stands for a constant that is independent of $e$.

### 3. FORWARD PROBLEM, ENERGY INEQUALITY, DIRECT INEQUALITY AND INVERSE INEQUALITY

The first step in HUM is to consider the homogeneous problem associated with (2.11), that is, we take the boundary controls to be zero. Thus we introduce the following forward Cauchy problem: Find $\theta^e = (\theta_i^e)$ satisfying

$$e^3 \frac{\partial^2 \theta_\tau^e}{\partial t^2} - e \frac{\partial \tilde{\sigma}_{\tau\eta}}{\partial z_\eta} - \frac{\partial \tilde{\sigma}_{\tau 3}}{\partial z_3} = f_\tau^e \text{ in } Q, \text{ for } \tau = 1, 2\,, \tag{3.1a}$$

$$e^2 \frac{\partial^2 \theta_3^e}{\partial t^2} - e \frac{\partial \tilde{\sigma}_{3\eta}}{\partial z_\eta} - \frac{\partial \tilde{\sigma}_{33}}{\partial z_3} = f_3^e \text{ in } Q\,, \tag{3.1b}$$

$$\theta^e = 0 \quad \text{on } \Sigma_0, \tag{3.1c}$$

(3.1d) $$\tilde{\sigma}_{ij} \nu_j = 0 \quad \text{on } \Sigma_\pm, \text{ for } i = 1, 2, 3,$$

(3.1e) $$\theta^e(0) = \theta^{0,e} \quad \text{and} \quad \frac{\partial \theta^e}{\partial t}(0) = \theta^{1,e} \text{ in } \Omega .$$

The tensor $(\tilde{\sigma}_{ij})$ is defined in terms of the strain tensor $(\tilde{e}_{ij})$ of $\theta^e$ as follows:

(3.2) $$\tilde{\xi}_{\alpha\beta} = \tilde{e}_{\alpha\beta}, \quad \tilde{\xi}_{\alpha 3} = e^{-1}\tilde{e}_{\alpha 3}, \quad \tilde{\xi}_{33} = e^{-2}\tilde{e}_{33},$$

(3.3) $$\tilde{\sigma}_{ij} = a_{ij\ell m}\tilde{\xi}_{\ell m} .$$

If $f^e = 0$, then we use $\varphi^{0,e}, \varphi^{1,e}, \varphi^e, (\xi_{ij})$ and $(\sigma_{ij})$ in the place of $\theta^{0,e}, \theta^{1,e}, \theta^e, (\tilde{\xi}_{ij})$ and $(\tilde{\sigma}_{ij})$. We will solve the above problem using the following Sobolev spaces:

(3.4) $$H^1_{\Gamma_0} = \{\psi \in H^1(\Omega) ; \psi = 0 \text{ on } \Gamma_0\}, \quad V = (H^1_{\Gamma_0})^3, \quad H = (L^2(\Omega))^3 .$$

The energy at time $t$ is defined by

(3.5) $$E(\theta^e ; t) = \frac{1}{2}\int_\Omega \left\{ e^2 \sum_{\tau=1}^{2} \left(\frac{\partial \theta^e_\tau}{\partial t}\right)^2 + \left(\frac{\partial \theta^e_3}{\partial t}\right)^2 \right\} dz + \frac{1}{2} a^e(\theta^e, \theta^e) .$$

where the bilinear form $a^e(\cdot,\cdot)$ is defined by

(3.6) $$a^e(\theta, \chi) = \int_\Omega a_{ijk\ell}\xi_{k\ell}(\theta)\, \xi_{ij}(\chi)\, dz, \quad \text{for all } \theta, \chi \in V .$$

We have then the following result giving the existence and regularity of the solution $\theta^e$:

THEOREM 3.1. *(a) We take the initial conditions $\theta^{0,e} \in V$ and $\theta^{1,e} \in H$. Let $f^e \in L^1(0, T; H)$. Then there exists a unique solution $\theta^e$ with*

$$\theta^e \in C^0([0,T] ; V) \cap C^1([0,T] ; H) \cap C^2([0,T] ; V') .$$

*(b) We have the following energy inequality:*

(3.7) $$E(\theta^e ; t) \leq c\left\{ E(\theta^e ; 0) + e^{-4}\left(\int_0^t \|f^e(s)\|_H \, ds\right)^2 \right\} .$$

*If $f^e = 0$, then we have the conservation of energy:*

$$E(\varphi^e ; t) = E(\varphi^e ; 0) .$$

*(c) Furthermore if* $f^e \in L^1(0, T; V)$, $\theta^{0,e} \in (H^2(\Omega))^3 \cap V$ *and* $\theta^{1,e} \in V$ *then the solution* $\theta^e$ *has the following regularity in the isotropic case:*

$$\theta^e \in C^0([0, T]; (H^s(\Omega))^3) \cap C^1([0, T]; V) \cap C^2([0, T]; H), \tag{3.8}$$

*for some* $s$ *with* $3/2 < s < 2$.

*Proof:* (a) is classical. See for instance Duvaut-Lions [1976]. For $(c)$, we refer the reader to the works of Grisvard [1989] and Nicaise [1992]. To prove (b), we multiply the equation (3.1a) by $e^{-1} \dfrac{\partial \theta_\tau^e}{\partial t}$ and the equation (3.1b) by $e^{-2} \dfrac{\partial \theta_3^e}{\partial t}$ respectively. A simple integration by parts leads us to the following energy identity:

$$\frac{dE}{dt}(\theta^e; t) = \int_\Omega \left\{ e^{-1} f_\tau^e \frac{\partial \theta_\tau^e}{\partial t} + e^{-2} f_3^e \frac{\partial \theta_3^e}{\partial t} \right\} dz. \tag{3.9}$$

The energy inequality (3.7) follows immediately from the above identity. ■

As mentioned in Introduction, whenever we consider non-isotropic system, we assume that the regularity result of Theorem 3.1 is valid.

The second step in HUM is to establish certain refined estimates on the solutions of the problem (3.1) which are of finite energy. These estimates, referred to as direct inequalities, give a bound on the first order derivatives of the solution on the boundary. These are deduced from an identity valid for arbitrary solutions of (3.1) with finite energy. This identity is first obtained for smooth solutions by the so-called multiplier method with the Rellich multipliers. It is then extended to arbitrary solutions with finite energy by a density argument using the regularity described in Theorem 3.1. This identity has been proved in Lions [1988a] in the isotropic case. Here we are dealing with the general case and, more importantly, we have to keep track of the small parameter $e$ to get estimates uniform in $e$. Thus, we multiply (3.1a) by $e^{-1} m_k \dfrac{\partial \theta_\tau^e}{\partial z_k}$ and (3.1b) by $e^{-2} m_k \dfrac{\partial \theta_3^e}{\partial z_k}$ (where $m_k$ are smooth) and integrate over $Q$. We obtain

THEOREM 3.2: *We suppose that* $f^e \in L^1(0, T; H)$, $\theta^{0,e} \in V$ *and* $\theta^{1,e} \in H$. *Then the solution* $\theta^e$ *of* (3.1) *satisfies*

$$\frac{1}{2}\int_{\Sigma_0} (a_{ij\ell m}\tilde{\xi}_{\ell m}\tilde{\xi}_{ij})(m_k\, \nu_k)\, d\sigma\, dt +$$

$$+\frac{1}{2}\int_{\Sigma_+} \left\{ \sum_{\tau=1}^{2} e^2\left(\frac{\partial \theta_\tau^e}{\partial t}\right)^2 + \left(\frac{\partial \theta_3^e}{\partial t}\right)^2 \right\} (m_k\, \nu_k)\, d\sigma\, dt -$$

$$-\frac{1}{2}\int_{\Sigma_\pm} (a_{ij\ell m}\tilde{\xi}_{\ell m}\tilde{\xi}_{ij})(m_k\, \nu_k)\, d\sigma\, dt = \left[\int_{\Omega} \left\{ e^2 \frac{\partial \theta_\tau^e}{\partial t}\frac{\partial \theta_\tau^e}{\partial z_k} + \frac{\partial \theta_3^e}{\partial t}\frac{\partial \theta_3^e}{\partial t} \right\} m_k\, dz \right]_0^T +$$

$$+\frac{1}{2}\int_{Q} \left\{ \sum_{\tau=1}^{2} \left| e^2 \frac{\partial \theta_\tau^e}{\partial t}\right|^2 + \left| e\frac{\partial \theta_3^e}{\partial t}\right|^2 \right\} \frac{\partial m_k}{\partial z_k}\, dz\, dt - \frac{1}{2}\int_{Q} (a_{ij\ell m}\tilde{\xi}_{\ell m}\tilde{\xi}_{ij}) \frac{\partial m_k}{\partial z_k}\, dz\, dt +$$

$$+\int_{Q} \left\{ \frac{\partial m_k}{\partial z_\eta}\tilde{\sigma}_{\tau\eta}\frac{\partial \theta_\tau^e}{\partial z_k} + e^{-1}\frac{\partial m_k}{\partial z_3}\tilde{\sigma}_{\tau 3}\frac{\partial \theta_\tau^e}{\partial z_k} + e^{-1}\frac{\partial m_k}{\partial z_\eta}\tilde{\sigma}_{3\eta}\frac{\partial \theta_3^e}{\partial z_k} + e^{-2}\frac{\partial m_k}{\partial z_3}\tilde{\sigma}_{33}\frac{\partial \theta_3^e}{\partial z_k} \right\} dz\, dt -$$

$$-\int_{Q} \left\{ e^{-1} f_\tau^e\, m_k \frac{\partial \theta_\tau^e}{\partial z_k} + e^{-2} f_3^e\, m_k\frac{\partial \theta_3^e}{\partial z_k} \right\} dz\, dt\,,$$

*where the functions* $m_k \in W^{1,\infty}(\Omega)$ *are arbitrary.* ■

Now, we consider the following functions which are adapted to the geometry of $\Omega$ and which were introduced in our earlier work. See Saint Jean Paulin & Vanninathan [1994].

$$(3.10)\qquad \begin{cases} m_k \text{ smooth, } m_1, m_2 \text{ are independent of } z_3\,, \\ \begin{cases} m_\alpha = \nu_\alpha \text{ on } \Gamma_0,\ \alpha = 1, 2 \quad \text{and} \quad m_3 = 0 \text{ on } \Gamma_\pm \\ \qquad \text{or} \\ m_\alpha = 0 \text{ on } \Gamma_0,\ \alpha = 1, 2 \quad \text{and} \quad m_3 = \nu_3 \text{ on } \Gamma_\pm\,. \end{cases} \end{cases}$$

This choice leads to the so-called direct estimates on the solution $\theta^e$ which are given in the following result:

THEOREM 3.3: *We take* $\theta^{0,e} \in V$, $\theta^{1,e} \in H$ *and* $f^e \in L^1(0,T;H)$. *Consider the solution* $\theta^e$ *of the problem* (3.1). *Then the following estimates hold for all times T with a constant c independent of e:*

$$\text{(3.11}a\text{)} \qquad \int_{\Sigma_0} (a_{ij\ell m}\tilde{\xi}_{ij}\tilde{\xi}_{\ell m})\, d\sigma\, dt \leqslant$$

$$\leqslant c\left[E(\theta^e;0) + \left(\int_0^T \left\{\sum_{\tau=1}^2 \|e^{-1}f_\tau^e\|_{L^2(\Omega)} + \|e^{-2}f_3^e\|_{L^2(\Omega)}\right\} dt\right)^2\right],$$

$$\text{(3.11}b\text{)} \qquad \left|\int_{\Sigma_\pm} \left\{e^2 \sum_{\tau=1}^2 \left|\frac{\partial\theta_\tau^e}{\partial t}\right|^2 + \left|\frac{\partial\theta_3^e}{\partial t}\right|^2 - a_{ij\ell m}\tilde{\xi}_{ij}\tilde{\xi}_{\ell m}\right\} d\sigma\, dt\right| \leqslant$$

$$\leqslant c\left[E(\theta^e;0) + \left(\int_0^T \left\{\sum_{\tau=1}^2 \|e^{-1}f_\tau^e\|_{L^2(\Omega)} + \|e^{-1}f_3^e\|_{L^2(\Omega)}\right\} dt\right)^2\right].$$

*Proof:* We apply the identity of Theorem 3.2 to the finite energy solution $\theta^e$ with multipliers $(m_k)$ specified by the first choice in (3.10). We get the following relation:

$$\text{(3.12)} \quad \frac{1}{2}\int_{\Sigma_0} (a_{ij\ell m}\tilde{\xi}_{ij}\tilde{\xi}_{\ell m})\, d\sigma\, dt = \left[\int_\Omega \left\{e^2 \frac{\partial\theta_\tau^e}{\partial t}\frac{\partial\theta_\tau^e}{\partial z_k} + \frac{\partial\theta_3^e}{\partial t}\frac{\partial\theta_3^e}{\partial z_k}\right\} m_k\, dz\right]_0^T +$$

$$+\frac{1}{2}\left\{e^2\sum_{\tau=1}^2\left(\frac{\partial\theta_\tau^e}{\partial t}\right)^2 + \left(\frac{\partial\theta_3^e}{\partial t}\right)^2\right\}\frac{\partial m_k}{\partial z_k}dz\, dt - \frac{1}{2}\int_Q (a_{ij\ell m}\tilde{\xi}_{ij}\tilde{\xi}_{\ell m})\frac{\partial m_k}{\partial z_k}dz\, dt$$

$$+\int_Q\left\{\frac{\partial m_k}{\partial z_\eta}\tilde{\sigma}_{\tau\eta}\frac{\partial\theta_\tau^e}{\partial z_k} + e^{-1}\frac{\partial m_3}{\partial z_3}\tilde{\sigma}_{\tau 3}\frac{\partial\theta_\tau^e}{\partial z_3} + e^{-1}\frac{\partial m_k}{\partial z_\eta}\tilde{\sigma}_{3\eta}\frac{\partial\theta_3^e}{\partial z_k} + e^{-2}\frac{\partial m_3}{\partial z_3}\tilde{\sigma}_{33}\frac{\partial\theta_3^e}{\partial z_3}\right\} dz\, dt$$

$$-\int_Q\left\{e^{-1}f_\tau^e m_k\frac{\partial\theta_\tau^e}{\partial z_k} + e^{-2}f_3^e m_k\frac{\partial\theta_3^e}{\partial z_k}\right\} dz\, dt.$$

In order to establish (3.11a), we observe that the last term in (3.12) containing $f^e$ is estimated by the integral terms involving $f^e$ in (3.11a). It remains now to examine each remaining term from the right side of (3.12) and to show that it is bounded by the energy. To this end, we observe, from the expression of the energy (see (3.5) and (3.6)), that the sequences $(\xi_{ij}), (\tilde{\sigma}_{ij})$ and $(\tilde{e}_{ij})$ are bounded by the energy in the space $L^\infty(0, T; L^2(\Omega))$. It follows from Korn's inequality, that $\{\theta^e\}$ is bounded in $L^\infty(0, T; V)$.

If we use all these informations in (3.12), we see that all terms in the right side of (3.12) are bounded by $E(\theta^e; 0)$ except the term in $f^e$ which is easy to treat and the two following terms which shall be treated separately.

$$(3.13) \qquad e^{-1} \int_Q \frac{\partial m_3}{\partial z_3} \tilde{\sigma}_{\tau 3} \frac{\partial \theta^e_\tau}{\partial z_3} dz\, dt \quad \text{and} \quad e^{-1} \int_Q \frac{\partial m_k}{\partial z_\eta} \tilde{\sigma}_{3\eta} \frac{\partial \theta^e_3}{\partial z_k} dz\, dt .$$

These terms were not present in the case of the scalar wave equation (cf. Saint Jean Paulin & Vanninathan [1994]) and they are special to the system under consideration. The idea is to use the equations (3.1) again to estimate them.

To estimate the first integral term in (3.13), for instance, we integrate by parts and using the fact that $\tilde{\sigma}_{\tau 3} = 0$ on $\Sigma_\pm$, we get

$$(3.14) \quad e^{-1} \int_Q \frac{\partial m_3}{\partial z_3} \tilde{\sigma}_{\tau 3} \frac{\partial \theta^e_\tau}{\partial z_3} dz\, dt =$$

$$= - e^{-1} \int_Q \frac{\partial^2 m_3}{\partial z_3^2} \tilde{\sigma}_{\tau 3}\, \theta^e_\tau\, dz\, dt - e^{-1} \int_Q \frac{\partial m_3}{\partial z_3} \frac{\partial \tilde{\sigma}_{\tau 3}}{\partial z_3}\, \theta^e_\tau\, dz\, dt .$$

To handle the first integral on the right side of (3.14), we introduce

$$\chi^e_\tau(z, t) = \int_{-1/2}^{z_3} \frac{\partial^2 m_3}{\partial z_3^2} (z_1, z_2, w, t)\, \theta^e_\tau(z_1, z_2, w, t)\, dw .$$

Observe that we have

$$\begin{cases} \chi^e_\tau = 0 \text{ on } \Sigma_0 , \\ \dfrac{\partial \chi^e_\tau}{\partial z_3} = \dfrac{\partial^2 m_3}{\partial z_3^2}\, \theta^e_\tau \text{ in } Q . \end{cases}$$

We use $\chi^e_\tau$ as a multiplier in the equation (3.1a) satisfied by $\theta^e_\tau$. We get

$$e^3 \int_Q \frac{\partial \chi^e_\tau}{\partial t} \frac{\partial \theta^e_\tau}{\partial t} dz\, dt - \left[ \int_\Omega e^3 \frac{\partial \theta^e_\tau}{\partial t} \chi^e_\tau\, dz \right]_0^T - e \int_Q \tilde{\sigma}_{\tau\eta} \frac{\partial \chi^e_\tau}{\partial z_\eta} dz\, dt + \int_Q f^e_\tau \chi^e_\tau\, dz\, dt =$$

$$= \int_Q \tilde{\sigma}_{\tau 3} \frac{\partial^2 m_3}{\partial z_3^2}\, \theta^e_\tau\, dz\, dt .$$

Multiplying this relation by $e^{-1}$, we see that the all terms on the left side are bounded the right side of (3.11a). This is how the first integral on the right side of (3.14) is handled. To estimate the second integral in the right side of (3.14), we use the equation (3.1a) directly after multiplying it by $e^{-1} \frac{\partial m_3}{\partial z_3} \theta^e_\tau$. We get

$$\int_\Omega e^{-1} \frac{\partial m_3}{\partial z_3} \frac{\partial \tilde{\sigma}_{\tau 3}}{\partial z_3} \theta^e_\tau \, dz \, dt =$$

$$= \int_Q \frac{\partial m_3}{\partial z_3} \left( e^2 \frac{\partial \theta^e_\tau}{\partial t^2} - \frac{\partial \tilde{\sigma}_{\tau\eta}}{\partial z_\eta} \right) \theta^e_\tau \, dz \, dt - e^{-1} \int_Q f^e_\tau \frac{\partial m_3}{\partial z_3} \theta^e_\tau \, dz \, dt ,$$

which yields after integration by parts

$$\int_\Omega e^{-1} \frac{\partial m_3}{\partial z_3} \frac{\partial \tilde{\sigma}_{\tau 3}}{\partial z_3} \theta^e_\tau \, dz \, dt = - \int_\Omega e^2 \frac{\partial m_3}{\partial z_3} \frac{\partial \theta^e_\tau}{\partial t} \frac{\partial \theta^e_\tau}{\partial t} \, dz \, dt +$$

$$+ \left[ \int_\Omega e^2 \frac{\partial m_3}{\partial z_3} \frac{\partial \theta^e_\tau}{\partial t} \theta^e_\tau \, dz \right]_0^T +$$

$$+ \int_Q \left( \frac{\partial^2 m_3}{\partial z_3 \, \partial z_\eta} \theta^e_\tau + \frac{\partial m_3}{\partial z_3} \frac{\partial \theta^e_\tau}{\partial z_\eta} \right) \tilde{\sigma}_{\tau\eta} \, dz \, dt - e^{-1} \int_Q f^e_\tau \frac{\partial m_3}{\partial z_3} \theta^e_\tau \, dz \, dt .$$

Thanks to the expression of the energy, we see that all the above integral terms are dominated by the right side of (3.11a) and thus we are also through with the second integral in the right side of (3.14). Thus the first integral in (3.13) is estimated as follows:

$$(3.15) \qquad e^{-1} \left| \int_Q \frac{\partial m_3}{\partial z_3} \tilde{\sigma}_{\tau 3} \frac{\partial \theta^e_\tau}{\partial z_3} \, dz \, dt \right| \leqslant \text{right side of } (3.11\text{a}),$$

with a constant $c$ independent of $e$.

To have a similar estimate on the second integral in (3.13), we rewrite it as

$$e^{-1} \int_Q \frac{\partial m_\alpha}{\partial z_\eta} \tilde{\sigma}_{3\eta} \left( \frac{\partial \theta^e_3}{\partial z_\alpha} + \frac{\partial \theta^e_\alpha}{\partial z_3} \right) dz \, dt - e^{-1} \int_Q \frac{\partial m_\alpha}{\partial z_\eta} \tilde{\sigma}_{3\eta} \frac{\partial \theta^e_\alpha}{\partial z_3} \, dz \, dt ,$$

which is equal to

$$2\int_Q \frac{\partial m_\alpha}{\partial z_\eta}\tilde{\sigma}_{3\eta}\tilde{\xi}_{\alpha 3}\,dz\,dt + e^{-1}\int_Q \frac{\partial^2 m_\alpha}{\partial z_\eta\,\partial z_3}\tilde{\sigma}_{3\eta}\,\theta^e_\alpha\,dz\,dt + e^{-1}\int_Q \frac{\partial m_\alpha}{\partial z_\eta}\frac{\partial\tilde{\sigma}_{3\eta}}{\partial z_3}\theta^e_\alpha\,dz\,dt\,.$$

The first integral is obviously bounded by the right side of (3.11a) and the second vanishes as $m_\alpha$ is chosen independent of $z_3$. The third integral can be treated in the same fashion as those on the right side of (3.14). This gives the following estimate:

$$(3.16)\qquad e^{-1}\left|\int_Q \frac{\partial m_\alpha}{\partial z_\eta}\tilde{\sigma}_{3\eta}\frac{\partial\theta^e_3}{\partial z_\alpha}\,dz\,dt\right| \leqslant \text{right side of (3.11a)}\,.$$

To finish the proof of (3.11a), it is now enough to use the estimates (3.15) and (3.16) in the identity (3.12).

Proof of (3.11b) is similar. Instead of the multipliers defined by the first choice in (3.10), one uses those defined by the second choice. ∎

The estimate (3.11a) implies, in particular, that $\{\tilde{\xi}_{ij}\}$ is bounded in $L^2(\Sigma_0)$ which, in turn, shows that $\{\tilde{e}_{ij}\}$ is bounded in $L^2(\Sigma_0)$. Since $\theta^e = 0$ on $\Sigma_0$, it follows then that

$$(3.17)\qquad \left\{\frac{\partial\theta^e}{\partial\nu}\right\}\ \text{is bounded in}\ L^2(\Sigma_0)\,.$$

This is an easy consequence of the following point-wise inequality which is valid on $\Sigma_0$:

$$(3.18)\qquad \sum_{i,j=1}^{3}\tilde{e}_{ij}\tilde{e}_{ij} = \frac{1}{2}\left[\sum_{i=1}^{3}\left(\frac{\partial\theta^e_i}{\partial\nu}\right)^2 + (\operatorname{div}\theta^e)^2\right]\ \text{on}\ \Sigma_0\,.$$

Let us now turn our attention to establishing the inverse inequality. It will give us an estimate of the energy norm of $\varphi^e$ in terms of another norm (classically denoted as $\|\ \|_F$) on the initial data. This norm is stronger than that defined by the quantities on the left side of the direct inequalities (3.11). The inverse inequality to be established in this section shows that this norm is stronger than the energy norm also. Another point about the inverse inequality is that while direct inequality is valid for all times $T > 0$, the inverse inequality is valid for sufficiently large times.

As in the proof of the direct inequality, we use the identity of Theorem 3.2 to establish the inverse inequality. The choice of multipliers is made as follows:

$$(3.19)\qquad m_k(z) = z_k - z^0_k,\quad k = 1, 2, 3,\quad \text{with}\quad z^0_3 = 0\,.$$

Following the proof of Theorem 5.1 in Saint Jean Paulin & Vanninathan [1994], we establish.

THEOREM 3.4: *Let us consider the solution $\varphi^e$ of the system* (3.1) *with initial conditions $\varphi^{0,e} \in V$ and $\varphi^{1,e} \in H$. Then there are positive constants c and $T_*$ which are independent of e such that the following estimate holds for $T \geqslant T_*$:*

$$(3.20)\quad E(\varphi^e;0) \leqslant$$

$$\leqslant c\left\{\int_{\Sigma(z^0)} a_{ij\ell m}\,\xi_{ij}\,\xi_{\ell m}\,d\sigma\,dt + \int_{\Sigma_\pm}\left\{e^2\sum_{\tau=1}^{2}\left(\frac{\partial\varphi_\tau^e}{\partial t}\right)^2 + \left(\frac{\partial\varphi_3^e}{\partial t}\right)^2\right\}d\sigma\,dt + \right.$$

$$\left. + \int_{\Sigma_\pm}\left\{e^2\sum_{\tau=1}^{2}(\varphi_\tau^e)^2 + (\varphi_3^e)\right\}d\sigma\,dt\right\}.$$

*Remark 3.5:* In their paper, Figueiredo & Zuazua [1994] establish an equality stronger than (3.20) which shows that $T_* = O(e)$. However, we work with the inequality (3.20) in the sequel to define the norm $\|\ \|_F$ which will enable us to analyze exact controllability in a space larger than theirs. ■

## 4. BACKWARD PROBLEM AND EXACT CONTROLLABILITY

The next step in the HUM is to introduce the space $F$ and resolve the backward Cauchy problem with nonhomogeneous boundary data taken from $F$ as explained below. To define the space $F$, we use the inverse inequality derived in the previous section. Recall that this inequality described by (3.20) is valid for $T$ sufficiently large. We fix one such time $T > 0$, independent of $e$, which is possible. We consider the forward Cauchy problem (3.1) with the initial conditions $\{\varphi^0, \varphi^1\}$ and with $f^e = 0$. We define the following norm:

$$(4.1)\quad \|\{\varphi^0,\varphi^1\}\|_{F^e}^2 = \int_{\Sigma(z^0)} (\,a_{ij\ell m}\,\xi_{ij}\,\xi_{\ell m})\,d\sigma\,dt +$$

$$+ \int_{\Sigma_\pm}\left\{e^2\sum_{\tau=1}^{2}\left(\frac{\partial\varphi_\tau^e}{\partial t}\right)^2 + \left(\frac{\partial\varphi_3^e}{\partial t}\right)^2\right\}d\sigma\,dt + \int_{\Sigma_\pm}\left\{e^2\sum_{\tau=1}^{2}(\varphi_\tau^e)^2 + (\varphi_3^e)^2\right\}d\sigma\,dt\,.$$

The direct inequality in Section 3 shows that the first integral on the right side of (4.1) is bounded by the energy functional. The third one is also estimated by $E(\varphi^e;0)$. However, the second integral is not so. Hence, we take more regular initial conditions, namely $\{\varphi^0,\varphi^1\} \in (H^2(\Omega)^3 \cap V) \times V$. According to Theorem 3.1, we then have $\frac{\partial \varphi^e}{\partial t} \in C^0([0, T]; V)$ and hence the second term on the right side of (4.1) also makes sense. We define the space $F^e$ to be the completion of $(H^2(\Omega)^3 \cap V) \times V$ under the norm (4.1). We remark that even though the norm depends on $e$, the underlying space $F^e$ is independent of $e$ as long as $e$ remains positive. We sometimes drop the dependence of $F^e$ on $e$ to ease the notation; we do so especially in the duality bracket between $F^e$ and its dual $(F^e)'$. It follows from the inverse inequality that the following inclusion is continuous and dense:

$$(F^e, \| \|_{F^e}) \to (V \times H, \| \|_E). \tag{4.2}$$

Moreover the constant of continuity is independent of $e$: there is a positive constant $c$ such that

$$\|\{\varphi^0, \varphi^1\}\|_E \leqslant c \|\{\varphi^0, \varphi^1\}\|_{F^e}. \tag{4.3}$$

We start with $\{\varphi^0, \varphi^1\} \in F^e$ and we solve the forward homogeneous problem (3.1) for $\varphi^e$ with initial condition $\{\varphi^0, \varphi^1\}$. Then we introduce the backward system:

$$e^3 \frac{\partial^2 \psi^e_\tau}{\partial t^2} - e \frac{\partial \hat{\sigma}_{\tau\eta}}{\partial z_\eta} - \frac{\partial \hat{\sigma}_{\tau 3}}{\partial z_3} = 0 \text{ in } Q, \quad \tau = 1, 2 \tag{4.4a}$$

$$e^2 \frac{\partial^2 \psi^e_3}{\partial t^2} - e \frac{\partial \hat{\sigma}_{3\eta}}{\partial z_\eta} - \frac{\partial \hat{\sigma}_{33}}{\partial z_3} = 0 \text{ in } Q, \tag{4.4b}$$

$$\hat{\sigma}_{\tau 3} \nu_3 = e^2 \left\{ \frac{\partial^2 \varphi^e_\tau}{\partial t^2} - \varphi^e_\tau \right\} \quad \text{and} \quad \hat{\sigma}_{33} \nu_3 = e^3 \left\{ \frac{\partial^2 \varphi^e_3}{\partial t^2} - \varphi^e_3 \right\} \text{ on } \Sigma_\pm, \tag{4.4c}$$

$$\psi^e_\tau = \begin{cases} \frac{\partial \varphi^e_\tau}{\partial \nu^2} \text{ on } \Sigma(z^0), \tau = 1, 2, \\ 0 \text{ on } \Sigma_*, \tau = 1, 2, \end{cases} \quad \text{and} \quad \psi^e_3 = \begin{cases} \frac{\partial \varphi^e_3}{\partial \nu} \text{ on } \Sigma(z^0), \\ 0 \text{ on } \Sigma_*, \end{cases} \tag{4.4d}$$

$$\psi^e(T) = 0 \quad \text{and} \quad \frac{\partial \psi^e}{\partial t}(T) = 0 \text{ on } \Omega. \tag{4.4e}$$

The tensors $(\hat{\sigma}_{ij})$, $(\hat{\xi}_{ij})$ and $(\hat{e}_{ij})$ are associated with $\psi^e$ in the same manner as $(\tilde{\sigma}_{ij})$, $(\tilde{\xi}_{ij})$ and $(\tilde{e}_{ij})$ are associated with $\theta^e$. Finally, we remark that in (4.4c), the time derivatives are taken in the sense of duality between $H^1(0, T; L^2(\Gamma_\pm))$ and its dual. (cf Lions [1988a] p. 209). To obtain a weak formulation of the system (4.4), we multiply (3.1a) by $e^{-1}\psi_\tau^e$ and (3.1b) by $e^{-2}\psi_3^e$; also, we multiply (4.4a) by $(-e^{-1}\theta_\tau^e)$ and (4.4b) by $(-e^{-2}\theta_3^e)$ and add all these equations. This leads us to the problem:

Find $\psi^e$ and the initial values $\{\psi^{0,e}, \psi^{1,e}\}$ such that

$$\{(e^2\psi_\tau^{1,e}, \psi_3^{1,e}), (-e^2\psi_\tau^{0,e}, -\psi_3^{0,e})\} \in (F^e)'$$

and, for all solutions $\theta^e \in F^e$,

$$_{F'}\langle\{(e^2\psi_\tau^{1,e}, \psi_3^{1,e}), (-e^2\psi_\tau^{0,e}, -\psi_3^{0,e})\}, \{\theta^0, \theta^1\}\rangle_F =$$

$$(4.5)\quad = \int_Q \{e^{-1} f_\tau^e \psi_\tau^e + e^{-2} f_3^e \psi_3^e\}\, dz\, dt + \int_{\Sigma(z^0)} \tilde{\sigma}_{ij}\, \xi_{ij}\, d\sigma\, dt +$$

$$+ e^2 \int_{\Sigma_\pm} \left\{\frac{\partial \varphi_\tau^e}{\partial t}\frac{\partial \theta_\tau^e}{\partial t} + \varphi_\tau^e \theta_\tau^e\right\} d\sigma\, dt + \int_{\Sigma_\pm} \left\{\frac{\partial \varphi_3^e}{\partial t}\frac{\partial \theta_3^e}{\partial t} + \varphi_3^e \theta_3^e\right\} d\sigma\, dt\,.$$

Existence and uniqueness of $\psi^e$, $\psi^{0,e}$ and $\psi^{1,e}$ follow immediately by duality arguments.

Following HUM, we now introduce the operator $\Lambda^e \in \mathscr{L}(F^e, (F^e)')$ defined by

$$(4.6)\qquad \Lambda^e\{\varphi^0, \varphi^1\} = \{(e^2\psi_\tau^{1,e}, \psi_3^{1,e}), (-e^2\psi_\tau^{0,e}, -\psi_3^{0,e})\}\,.$$

Some properties of $\Lambda^e$ to be used in the sequel are listed in the result below.

THEOREM 4.1: *(a) $\Lambda^e$ is a continuous linear operator whose norm is bounded independently of e. (b) $\Lambda^e$ is an isomorphism from $F^e$ onto $(F^e)'$. The norm of its inverse is bounded independently of e.*

*Proof:* The proof follows standard arguments (see Lions [1988a]) and is based on the following relations:

$$(4.7)\quad _{F'}\langle\Lambda^e\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\}\rangle_F = \int_{\Sigma(z^0)} \sigma_{ij}\, \xi_{ij}\, d\sigma\, dt +$$

$$+ e^2 \int_{\Sigma_+} \sum_{\tau=1}^{2} \left\{\left(\frac{\partial \varphi_\tau^e}{\partial t}\right) + (\varphi_\tau^e)^2\right\} d\sigma\, dt + \int_{\Sigma_\pm} \left\{\left(\frac{\partial \varphi_3^e}{\partial t}\right) + (\varphi_3^e)^2\right\} d\sigma\, dt\,,$$

$$(4.8)\qquad _{F'}\langle\Lambda^e\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\}\rangle_F = \|\{\varphi^0, \varphi^1\}\|_{F^e}^2\,.$$ ■

We are now in a position to show the exact controllability of the problem (2.12). We choose $y^{0,e}$ and $y^{1,e}$ in such a way that

$$\{(e^2 y_\tau^{1,e}, y_3^{1,e}), (-e^2 y_\tau^{0,e}, y_3^{0,e})\} \in (F^e)'. \tag{4.9}$$

Since $\Lambda^e$ is an isomorphism, we solve

$$\Lambda^e\{\varphi^{0,e}, \varphi^{1,e}\} = \{(e^2 y_\tau^{1,e}, y_3^{1,e}), (-e^2 y_\tau^{0,e}, -y_3^{0,e})\}, \tag{4.10a}$$

$$\{\varphi^{0,e}, \varphi^{1,e}\} \in F^e. \tag{4.10b}$$

Next, we solve the forward problem (3.1) for $\varphi^e$ with the initial conditions $\{\varphi^{0,e}, \varphi^{1,e}\}$. We define

$$v_\tau^e = \frac{\partial \varphi_\tau^e}{\partial \nu}, \tau = 1, 2 \quad \text{and} \quad v_3^e = e\frac{\partial \varphi_3^e}{\partial \nu} \text{ on } \Sigma(z^0), \tag{4.11}$$

$$w_\tau^{\pm,e} = e^3\left\{\frac{\partial^2 \varphi_\tau^e}{\partial t^2} - \varphi_\tau^e\right\}, \tau = 1, 2 \text{ and } w_3^{\pm,e} = e^2\left\{\frac{\partial^2 \varphi_3^e}{\partial t^2} - \varphi_3^e\right\} \text{ on } \Sigma_\pm, \tag{4.12}$$

as controls in the problem (2.11). With these choices, we observe that the problem (2.11) coincides with (4.4) and so $y^e = \psi^e$. In particular, it shows that the system is at rest at time $T$ which means that we have exact controllability of the problem (2.11) with the controls chosen according to (4.11) and (4.12). We state this as a separate result:

THEOREM 4.2: *We fix $T > 0$ such that the inverse inequality* (3.20) *holds. Then the problem* (2.11) *with initial conditions* $(y^{0,e}, y^{1,e})$ *chosen such that* (4.9) *is satisfied is exactly controllable at time $T$ with the controls $v^e, w^{\pm,e}$ defined according to* (4.10)-(4.12). *Moreover, these controls have the following regularity properties:*

$$v^e \in L^2(\Sigma(z^0)) \quad \text{and} \quad w_i^{\pm,e} \in [H^1(0,T; L^2(\Gamma_\pm))]', i = 1, 2, 3 \quad \blacksquare \tag{4.13}$$

## 5. BEHAVIOUR WHEN THE THICKNESS PARAMETER IS SMALL

In this section, we let the thickness parameter $e$ tend to zero and we shall analyze the behaviour of the exact controllability problem which has been solved in the previous section. Since HUM identifies the exact controllability problem for $y^e$ with the backward problem for $\psi^e$, our goal is to pass to the limit in each term of the weak formulation of the backward problem. All our efforts here are directed towards achieving this. Since this problem is driven by the solution $\varphi^e$ of the homogeneous forward problem, the main step is to analyze the behaviour of $\varphi^e$ as $e$ varies. Indeed, we are led to study, more generally, the behaviour of solutions $\theta^e$ with source terms as these appear as test functions in the weak formulation of the backward problem. This is carried out in § 5.1 where the weak convergence of $\theta^e$ in the natural energy space is first analyzed. In the context of exact controllability, this is not sufficient because crucial role is played by the smaller space $F^e$. The emphasis is therefore to produce test solutions $\theta^e$ which behave nicely with respect to $F^e$-topology. We present two techniques to build such solutions: (a) energy method, (b) method of asymptotic expansions. While (a) is useful to prove the convergence as well as to identify the limit of the backward problem, (b) serves us well to identify the limit if we already know that there is convergence.

This section is divided into several paragraphs. In § 5.1, we describe the behaviour of the homogeneous forward problem. Next, we pass to the limit in the nonhomogeneous backward problem in § 5.2. These results are subsequently used to pass to the limit in the exact controllability problem and this is done in § 5.4 and § 5.5 while § 5.3 is devoted to the description of the limiting two-dimensional exact controllability problem.

### 5.1. Behaviour of the forward homogeneous problem

We study the behavior of the solution $\theta^e$ of the system (3.1) as $e \to 0$. This subject matter is classical and the ideas are essentially developed by Ciarlet & Destuynder [1979], Ciarlet [1990], Raoult [1980], [1984], [1988] and Caillerie [1982].

As is well-known in the theory of thin plates, the limit behaviour is characterized by two-dimensional problems. Let us start by defining them. The first one is an evolution equation associated with a scalar fourth order operator defined on $\omega$.

$$(5.1a) \qquad \frac{\partial^2 \theta^{\#}}{\partial t^2} + \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^4 \theta^{\#}}{\partial z_\alpha \, \partial z_\beta \, \partial z_\tau \, \partial z_\eta} = f^{\#} \text{ in } \omega \times (0, T),$$

$$(5.1b) \qquad \theta^{\#} = 0 \quad \text{and} \quad \frac{\partial \theta^{\#}}{\partial \nu} = 0 \text{ on } \gamma \times (0, T),$$

$$(5.1c) \qquad \theta^{\#}(\tilde{z}, 0) = \theta^{0,\#}(\tilde{z}) \quad \text{and} \quad \frac{\partial \theta^{\#}}{\partial t}(\tilde{z}, 0) = \theta^{1,\#}(\tilde{z}) \text{ in } \omega .$$

The coefficients $(b_{\alpha\beta\tau\eta})$ appearing above are defined by

$$(5.2) \qquad b_{\alpha\beta\tau\eta} = a_{\alpha\beta\tau\eta} - a_{\alpha\beta i 3} \, d_{ij} \, a_{j3\tau\eta}$$

where the matrix $(d_{ij})$ is nothing but the inverse of $(a_{i3j3})$.

The solution of the above system will characterize the behaviour of the third component $\theta_3^e$ as $e$ tends to zero. To describe the behaviour of the first two components $\theta_\alpha^e$, we need the following stationary system involving a second order operator: Find $(\Theta_\alpha^{\#})$ satisfying

$$(5.3a) \qquad -\frac{\partial}{\partial z_\eta}(b_{\tau\eta\alpha\beta}\, e_{\alpha\beta}(\Theta^{\#})) = g_\tau^{\#} \text{ in } \omega, \quad \tau = 1, 2,$$

$$(5.3b) \qquad \Theta^{\#} = 0 \text{ on } \gamma .$$

Let us now proceed to state the conditions under which we will prove weak convergence of $\theta^e$. We suppose that

$$(5.4) \qquad E(\theta^e; 0) \leqslant c,$$

where $c > 0$ is independent of $e$ and

$$(5.5) \qquad \{\theta_3^{0,e}, \theta_3^{1,e}\} \rightharpoonup \{\theta_3^{0,*}, \theta_3^{1,*}\} \text{ in } H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ weak}.$$

It is an obvious consequence of (5.4) that $\theta_3^{0,*}$ is independent of $z_3$. Regarding the source terms $f_i^e$, we stipulate that

$$(5.6) \qquad \{e^{-1} f_\alpha^e\} \rightharpoonup f_\alpha^* \text{ in } H^1(0, T; L^2(\Omega)) \text{ weak},$$

$$(5.7) \qquad \{e^{-2} f_3^e\} \rightharpoonup f_3^* \text{ in } L^2(0, T; L^2(\Omega)) \text{ weak}.$$

We are now in a position to announce the first result on the limiting behaviour of $\theta^e$ as $e \to 0$.

THEOREM 5.1: *Under the hypotheses* (5.4)-(5.7), *the solution* $\theta^e$ *satisfies*

$$(5.8) \qquad \{\theta^e\} \rightharpoonup \theta^* \text{ in } L^\infty(0, T; V) \text{ weak } *$$

$$(5.9) \qquad \left\{e\frac{\partial \theta_\alpha^e}{\partial t}, \frac{\partial \theta_3^e}{\partial t}\right\} \rightharpoonup \left\{0, \frac{\partial \theta_3^*}{\partial t}\right\} \text{ in } L^\infty(0, T; L^2(\Omega))^2 \text{ weak } * .$$

*Further the limit* $(\theta_i^*)$ *is characterized as follows:*

*(i) The component* $\theta_3^*$ *is independent of* $z_3$ *and satisfies*

$$(5.10) \qquad \theta_3^* \in C^0([0, T]; H_0^2(\omega)) \cap C^1([0, T]; L^2(\omega)) \cap C^2([0, T]; H^{-2}(\omega)).$$

*It is the solution of the problem (5.1) with*

$$f^{\#} = m(f_3^*) + \frac{\partial}{\partial z_r} m(z_3 f_\tau^*) , \tag{5.11}$$

$$\theta^{0,\#} = m(\theta_3^{0,*}) = \theta_3^{0,*} \quad \text{and} \quad \theta^{1,\#} = m(\theta_3^{1,*}) . \tag{5.12}$$

*(ii) The first two components $(\theta_\alpha^*)$ are of the form*

$$\theta_\alpha^* = \Theta_\alpha^* - z_3 \frac{\partial \theta_3^*}{\partial z_\alpha} , \tag{5.13}$$

*where $(\Theta_\alpha^*)$ is the solution of problem (5.3) with*

$$g_\tau^{\#} = m(f_\tau^*) . \tag{5.14}$$

*Moreover, $\Theta_\alpha^*$ is independent of $z_3$ and $\Theta_\alpha^* \in H_0^1(\omega)$. Its t-dependence comes only from that of $m(f_\tau^*)$ through the problem (5.3) wherein t plays the role of a parameter.*

*Regarding the behaviour of the stress tensor $(\tilde{\sigma}_{ij})$ associated with $\theta^e$, we have the following:*

$$\tilde{\sigma}_{ij} \textit{ converges to } \tilde{\sigma}_{ij}^* \textit{ in } L^\infty(0, T; L^2(\Omega)) \textit{ weak } * , \tag{5.15}$$

$$\tilde{\sigma}_{i3}^* = 0 \quad \textit{and} \quad \tilde{\sigma}_{\alpha\beta}^* = b_{\alpha\beta\tau\eta}\, e_{\tau\eta}(\theta^*) . \tag{5.16}$$

■

Above results establish the weak convergence of $\{\theta^e\}$. Now we ask the following question: under what conditions on the initial data and on the source terms do we have

$$\begin{cases} \{\theta^e\} \text{ converges strongly in the energy norm} \\ \left\{ e \dfrac{\partial \theta_\tau^e}{\partial t} \right\} \text{ and } \left\{ \dfrac{\partial \theta_3^e}{\partial t} \right\} \text{ converge strongly in } L^2(\Sigma_\pm) . \end{cases} \tag{5.17}$$

In the literature, one can see some results on the strong convergence of $\{\theta^e\}$ in the energy norm; see for instance Ciarlet [1990] pp. 109-110. Since we want the strong convergence of the time derivative of $\theta^e$ on the boundary, the canonical idea is to consider the system satisfied by $\frac{\partial \theta^e}{\partial t}$. The invariance of the system (3.1) with respect to time translation implies that $\frac{\partial \theta^e}{\partial t}$ also satisfies a system similar to (3.1) and so an application of Theorem 5.1 for this new system will imply the desired convergence (5.17).

THEOREM 5.2: *We consider problem* (3.1) *where the initial conditions* $\theta^{0,e}$, $\theta^{1,e}$ *and the source terms* $f_\iota^e$ *satisfy the assumptions of Theorem 5.1 and also*

$$\left\{ e^{-1}\frac{\partial f_\tau^e}{\partial t}, e^{-2}\frac{\partial f_3^e}{\partial t} \right\} \text{ bounded in } H^1(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Omega)), \tag{5.18a}$$

$$\{ e^{-2} f_\iota^e \} \text{ bounded in } L^\infty(0,T;L^2(\Omega)), \tag{5.18b}$$

$$\{ e^{-1}\xi_\eta(z,0)\} \text{ bounded in } L^2\left(-\frac{1}{2},\frac{1}{2};H^1(\omega)\right), \tag{5.19a}$$

$$\{ e^{-2}\xi_\eta(z,0)\} \text{ bounded in } H^1\left(-\frac{1}{2},\frac{1}{2};L^2(\omega)\right), \tag{5.19b}$$

$$\{\theta_3^{1,e}\} \text{ bounded in } H^1_{\Gamma_0}(\Omega), \tag{5.20a}$$

$$\left\{ \frac{\partial \theta_\alpha^{1,e}}{\partial z_\beta} + \frac{\partial \theta_\beta^{1,e}}{\partial z_\alpha} \right\} \text{ bounded in } L^2(\Omega), \tag{5.20b}$$

$$\left\{ e^{-1}\left( \frac{\partial \theta_\alpha^{1,e}}{\partial z_3} + \frac{\partial \theta_3^{1,e}}{\partial z_\alpha} \right), e^{-2}\frac{\partial \theta_3^{1,e}}{\partial z_3} \right\} \text{ bounded in } L^2(\Omega)^2. \tag{5.20c}$$

*Then we have*

$$\{\theta^e\} \rightharpoonup \theta^* \text{ in } W^{1,\infty}(0,T;V) \text{ weak } *, \tag{5.21}$$

$$\left\{ e\frac{\partial^2 \theta_\alpha^e}{\partial t^2} \right\} \rightharpoonup 0 \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak } *, \tag{5.22}$$

$$\left\{ \frac{\partial^2 \theta_3^e}{\partial t^2} \right\} \rightharpoonup \frac{\partial^2 \theta_3^*}{\partial t^2} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak } *, \tag{5.23}$$

*where the limit* $\theta^*$ *is characterized in Theorem 5.1 with the additional conclusion that*

$$\Theta_\alpha^* = 0 \quad \alpha = 1,2. \tag{5.24}$$ ■

By a standard application of compactness criteria of Rellich type and Lions type (see Lions [1969]), we can deduce the strong convergence of $\{\theta^e\}$ in $F^e$-norm in the sense of (5.17). We announce this as a separate result.

THEOREM 5.3: *Under the hypotheses of Theorem 5.2, we have*

$$\{\theta^e\} \rightarrow \theta^* \text{ in } L^\infty(0,T;V)\,, \tag{5.25}$$

$$\left\{e\frac{\partial\theta^e_\alpha}{\partial t}, \frac{\partial\theta^e_3}{\partial t}\right\} \rightarrow \left\{0, \frac{\partial\theta^*_3}{\partial t}\right\} \text{ in } L^\infty(0,T;L^2(\Omega))^2\,, \tag{5.26}$$

$$\left\{e\frac{\partial\theta^e_\alpha}{\partial t}, \frac{\partial\theta^e_3}{\partial t}\right\} \rightarrow \left\{0, \frac{\partial\theta^*_3}{\partial t}\right\} \text{ in } L^2(\Sigma_\pm)^2\,, \tag{5.27}$$

$$\{\tilde{\sigma}_{ij}\} \rightarrow \tilde{\sigma}^*_{ij} \quad \text{in } L^\infty(0,T;L^2(\Omega))\,. \tag{5.28}$$ ■

In our asymptotic analysis of the backward problem, we will be using the above result in passing to the limit in the integrals over $\Sigma_\pm$ occurring in the weak formulation (4.5). The integral over $\Sigma_0$ will be handled by the following result which uses strong energy convergence of $\{\theta^e\}$ and the weak convergence properties of $\{\varphi^e\}$.

THEOREM 5.4: *Let $\theta^e$ be the solution of system* (3.1) *satisfying the hypotheses of Theorem* 5.2. *Let $\varphi^e$ be the solution of* (3.1) *where the initial conditions satisfy the hypotheses of Theorem* 5.1 *with $f^e = 0$. Then the following holds:*

$$\int_{\Sigma_0} a_{ijk\ell}\tilde{\xi}_{ij}\,\xi_{k\ell}\,d\sigma\,dt \rightarrow \int_{\Sigma_0} a_{ijk\ell}\tilde{\xi}^*_{ij}\,\xi^*_{k\ell}\,d\sigma\,dt \tag{5.29}$$

*where $(\tilde{\xi}_{ij})$ and $(\xi_{ij})$ are the tensors associated with $\theta^e$ and $\varphi^e$ respectively.*

*Proof:* Even though the technique of the proof is similar to that of the direct inequality, it does not follow from it immediately. The proof consists of three steps. In the first step, we get an identity expressing the left side of (5.29) in terms of quantities which are bounded by energy norms of $\theta^e$ and $\varphi^e$. The next step consists in passing to the limit in various terms in the above identity using the strong convergence properties of $\theta^e$ and the weak convergence properties of $\varphi^e$. The last one is concerned with the identification of limits.

*First step:* We choose the multipliers $(m_k)$ according to the first relation in (3.10) with $m_3 = 0$. Let $\varphi^e$ be the solution of the system (3.1). We multiply the equation (3.1a) by $e^{-1} m_k \frac{\partial \varphi_\tau^e}{\partial z_k}$ and (3.1b) by $e^{-2} m_k \frac{\partial \varphi_3^e}{\partial z_k}$ and add them up. Similarly, we multiply the system of equations satisfied by $\varphi^e$ by $e^{-1} m_k \frac{\partial \theta_\tau^e}{\partial z_k}$ and $e^{-2} m_k \frac{\partial \theta_3^e}{\partial z_k}$ and add. We obtain

$$\int_Q \left\{ e^{-1} f_\tau^e m_k \frac{\partial \varphi_\tau^e}{\partial z_k} + e^{-2} f_3^e m_k \frac{\partial \varphi_3^e}{\partial z_k} \right\} dz\, dt =$$

$$= \left[ e^2 \int_\Omega m_p \frac{\partial \theta_\tau^e}{\partial t} \frac{\partial \varphi_\tau^e}{\partial z_p} dz \right]_0^T + \left[ \int_\Omega m_p \frac{\partial \theta_3^e}{\partial t} \frac{\partial \varphi_3^e}{\partial z_p} dz \right]_0^T$$

$$+ \left[ e^2 \int_\Omega m_p \frac{\partial \theta_\tau^e}{\partial z_p} \frac{\partial \varphi_\tau^e}{\partial t} dz \right]_0^T + \left[ \int_\Omega m_p \frac{\partial \theta_3^e}{\partial z} \frac{\partial \varphi_3^e}{\partial t} dz \right]_0^T$$

(5.30)

$$+ \int_\Omega \frac{\partial m_p}{\partial z_p} \left\{ e^2 \frac{\partial \theta_\tau^e}{\partial t} \frac{\partial \varphi_\tau^e}{\partial t} + \frac{\partial \theta_3^e}{\partial t} \frac{\partial \varphi_3^e}{\partial t} \right\} dz\, dt - \int_Q \frac{\partial m_p}{\partial z_p} \tilde{\sigma}_{ij} \xi_{ij}\, dz\, dt$$

$$+ \int_Q \left\{ \tilde{\sigma}_{\tau\eta} \frac{\partial \varphi_\tau^e}{\partial z_p} \frac{\partial m_p}{\partial z_\eta} + e^{-1} \tilde{\sigma}_{3\eta} \frac{\partial \varphi_3^e}{\partial z_p} \frac{\partial m_p}{\partial z_\eta} \right\} dz\, dt$$

$$+ \int_Q \left\{ \sigma_{\tau\eta} \frac{\partial \theta_\tau^e}{\partial z_p} \frac{\partial m_p}{\partial z_\eta} + e^{-1} \sigma_{3\eta} \frac{\partial \theta_3^e}{\partial z_p} \frac{\partial m_p}{\partial z_\eta} \right\} dz\, dt - \int_{\Sigma_0} \tilde{\sigma}_{ij} \xi_{ij}\, d\sigma\, dt .$$

*Second step:* It is observed that all integrals in relation (5.30) are bounded with respect to energy norms of $\theta^e$ and $\varphi^e$. Thus we are in a position to pass to the limit. Further, it is easy to identify the limit of all the terms except the ones which contain evaluation at time $T$. To obtain the later ones, we establish the following convergence and use the fact that $\overset{*}{\varphi}_3$ is independent of $z_3$ :

(5.31*a*) $$m\left( \frac{\partial \varphi_3^e}{\partial z_\alpha} \right)( \cdot , t) \rightharpoonup \frac{\partial \overset{*}{\varphi}_3}{\partial z_\alpha}( \cdot , t) \quad \text{in } L^2(\omega) \text{ weak } \forall t ,$$

(5.31*b*) $$m\left( \frac{\partial \varphi_3^e}{\partial t} \right)( \cdot , t) \rightharpoonup \frac{\partial \overset{*}{\varphi}_3}{\partial t}( \cdot , t) \quad \text{in } L^2(\omega) \text{ weak } \forall t .$$

Using all these, passing to the limit in (5.30) and simplifying the right hand side, we get

$$\lim_{e \to 0} \int_{\Sigma_0} a_{ijk\ell} \tilde{\xi}_{k\ell} \, \xi_{ij} \, d\sigma \, dt = \left[ \int_{\omega} m_p \left\{ \frac{\partial \theta_3^*}{\partial t} \frac{\partial \varphi_3^*}{\partial z_p} + \frac{\partial \theta_e^*}{\partial z_p} \frac{\partial \varphi_3^*}{\partial t} \right\} d\tilde{z} \right]_0^T$$

$$+ \int_{\omega \times (0,T)} \frac{\partial m_p}{\partial z_p} \left\{ \frac{\partial \theta_3^*}{\partial t} \frac{\partial \varphi_3^*}{\partial t} - \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^2 \theta_3^*}{\partial z_\alpha \, \partial z_\beta} \frac{\partial^2 \varphi_3^*}{\partial z_\tau \, \partial z_\eta} \right\} d\tilde{z} \, dt$$

$$(5.32) \quad + \frac{1}{6} \int_{\omega \times (0,T)} \frac{\partial m_p}{\partial z_p} b_{\alpha\beta\tau\eta} \left\{ \frac{\partial^2 \theta_3^*}{\partial z_\alpha \, \partial z_\beta} \frac{\partial^2 \varphi_3^*}{\partial z_\tau \, \partial z_p} + \frac{\partial^2 \theta_3^*}{\partial z_\tau \, \partial z_p} \frac{\partial^2 \varphi_3^*}{\partial z_\alpha \, \partial z_\beta} \right\} d\tilde{z} \, dt$$

$$- \int_{\omega \times (0,T)} \left\{ m(f_3^*) + \frac{\partial}{\partial z_\tau} m(z_3 f_\tau^*) \right\} m_p \frac{\partial \varphi_3^*}{\partial z_p} d\tilde{z} \, dt$$

$$+ \frac{1}{12} \int_{\omega \times (0,T)} \frac{\partial^2 m_p}{\partial z_\tau \, \partial z_\eta} b_{\alpha\beta\tau\eta} \left\{ \frac{\partial \theta_3^*}{\partial z_p} \frac{\partial^2 \varphi_3^*}{\partial z_\alpha \, \partial z_\beta} + \frac{\partial^2 \theta_3^*}{\partial z_\alpha \, \partial z_\beta} \frac{\partial \varphi_3^*}{\partial z_p} \right\} d\tilde{z} \, dt .$$

*Third step:* It remains to identify the right side of the above relation. To this end, we multiply the equation (5.1) satisfied by $\theta_3^*$ with $m_p \frac{\partial \varphi_3^*}{\partial z_p}$ and the analogous equation satisfied by $\varphi_3^*$ by $m_p \frac{\partial \theta_3^*}{\partial z_p}$ and proceed as in the first step. The final result is that the right side of (5.32) is found to be equal to

$$\frac{1}{12} \int_{\gamma \times (0,T)} b_{\alpha\beta\tau\eta} \frac{\partial \theta_3^*}{\partial z_\alpha \, \partial z_\beta} \frac{\partial^2 \varphi_3^*}{\partial z_\tau \, \partial z_\eta} d\sigma \, dt ,$$

which coincides with the right side of (5.29). This completes the proof. ■

The method of proof of the above result shows clearly that we have the convergence of the energy of the waves observed through $\Sigma_0$ in the following sense:

THEOREM 5.5: *Under the hypotheses of Theorem 5.2 the following holds:*

$$\lim_{e \to 0} \int_{\Sigma_0} a_{ijk\ell} \tilde{\xi}_{ij} \tilde{\xi}_{k\ell} \, d\sigma \, dt = \int_{\omega \times (0,T)} a_{ijk\ell} \tilde{\sigma}_{ij}^* \tilde{\xi}_{k\ell}^* \, d\sigma \, dt . \qquad ■$$

So far we have seen sufficient conditions on the data $\{f_i^e\}$ and $\{\theta^{0,e}, \theta^{1,e}\}$ so that the energy norm and the $F^e$-norm of $\theta^e$ are bounded. The purpose of our next result is to analyze the dependence of the solution on the Neumann boundary data on $\Sigma_{\pm}$.

THEOREM 5.6: *Let us consider the solution* $\theta^e$ *of the system* (3.1) *where* $f^e = 0$, $\theta^{0,e} = 0$, $\theta^{1,e} = 0$ *and* (3.1d) *is replaced by*

$$\tilde{\sigma}_{i3}\, \nu_3 = g_i^{\pm,e} \text{ on } \Sigma_\pm . \tag{5.33}$$

*If*

$$\{e^{-1} g_\tau^{\pm,e}\}, \{e^{-2} g_3^{\pm,e}\} \text{ are bounded in } H^1(0, T; L^2(\Gamma_\pm)), \tag{5.34}$$

*then*

$$\{\theta^e\} \text{ is bounded in } L^\infty(0, T; V), \tag{5.35a}$$

$$\left\{e \frac{\partial \theta_\tau^e}{\partial t}\right\} \text{ and } \left\{\frac{\partial \theta_3^e}{\partial t}\right\} \text{ are bounded in } L^\infty(0, T; L^2(\Omega)). \tag{5.35b}$$

*Further if*

$$\{e^{-1} g_\tau^{\pm,e}\} \text{ and } \{e^{-2} g_3^{\pm,e}\} \text{ are bounded in } H^2(0, T; L^2(\Gamma_\pm)), \tag{5.36}$$

*then*

$$\left\{\frac{\partial \theta^e}{\partial t}\right\} \text{ is bounded in } L^\infty(0, T, V), \tag{5.37a}$$

$$\left\{e \frac{\partial^2 \theta_\tau^e}{\partial t^2}\right\} \text{ and } \left\{e \frac{\partial^2 \theta_3^e}{\partial t^2}\right\} \text{ are bounded in } L^\infty(0, T; L^2(\Omega)). \tag{5.37b}$$

*In particular, under the hypothesis* (5.36) *it follows that*

$$e \frac{\partial \theta_\tau^e}{\partial t} \to 0 \quad \text{in } L^2(\Sigma_\pm), \tag{5.38}$$

$$\left\{\frac{\partial \theta_3^e}{\partial t}\right\} \text{ remains in a compact subset of } L^2(\Sigma_\pm). \tag{5.39}$$

*Proof:* As before, the conclusions (5.38), (5.39) will follow from (5.37*a*), (5.37*b*) by an application of Lemma of Lions. Since $\frac{\partial \theta^e}{\partial t}$ satisfies a system similar to that of $\theta^e$ with Neumann data $\frac{\partial g^{\pm,e}}{\partial t}$ instead of $g^{\pm,e}$, we note that

(5.37*a*), (5.37*b*) will be a consequence of (5.35*a*), (5.35*b*). The proof of these later assertions is based on the following energy identity which is obtained, as usual, by multiplying the system by $\frac{\partial \theta^e}{\partial t}$ and integrating by parts:

$$E(\theta^e ; t) = e^{-1} \int_0^t \int_{\Gamma_\pm} g_\tau^{\pm, e} \frac{\partial \theta_\tau^e}{\partial s} \, d\sigma \, ds + e^{-2} \int_0^t \int_{\Gamma_\pm} g_3^{\pm, e} \frac{\partial \theta_3^e}{\partial s} \, d\sigma \, ds . \tag{5.40}$$

Gronwall type arguments now apply and show that

$$E(\theta^e ; t) \leqslant c \quad \forall t \in [0, T] . \tag{5.41}$$

Thanks to the definition of $E(\theta^e ; t)$, the properties (5.35*a*), (5.35*b*) follow from the above estimate. ■

### 5.2. Behaviour of the nonhomogeneous backward problem

Our next aim is to analyze the behaviour of the backward Cauchy problem (4.4). To this end, we start with initial conditions $\{\varphi^{0, e}, \varphi^{1, e}\}$ which satisfy the requirements of Theorem 5.1 and such that

$$\left\{ e \frac{\partial \varphi_\tau^e}{\partial t}, \frac{\partial \varphi_3^e}{\partial t} \right\} \rightharpoonup \left\{ 0, \frac{\partial \varphi_3^*}{\partial t} \right\} \quad \text{in } L^2(\Sigma_\pm)^2 \text{ weak} , \tag{5.42}$$

$$\{\xi_{ij}, \sigma_{ij}\} \rightharpoonup \{\xi_{ij}^*, \sigma_{ij}^*\} \quad \text{in } L^2(\Sigma(z^0))^2 \text{ weak} , \tag{5.43}$$

where $\varphi^*$, $(\sigma_{ij}^*)$ and $(\xi_{ij}^*)$ are defined in Theorem 5.1 with $f^e = 0$.

Our next result aims at analyzing the consequence of these convergence properties of $\varphi^e$ on the solution of the backward problem (4.4).

It is clear, from the weak formulation (4.5), that the solution consists of three unknowns: the interior solution $\psi^e$ and the two initial data $\psi^{0, e}$ and $\psi^{1, e}$. Since the problem is linear, the analysis of the behaviour of these unknowns can be separated by choosing test functions $\theta^e$ such that $f_i^e = 0$ and then initial data $\theta^{0, e} = 0$, $\theta^{1, e} = 0$ separately.

Our first task is to analyze the behaviour of the interior solution $\psi^e$ and this corresponds to taking $\theta^{0, e} = \theta^{1, e} = 0$ in (4.5).

THEOREM 5.7: *Let $\varphi^e$ be the solution of* (3.1) *where the initial conditions satisfy* (5.42), (5.43) *and the requirements of Theorem 5.1. The solution $\psi^e$ of the backward problem* (4.4) *has the following behaviour:*

(5.44*a*) $$e\psi_\tau^e \to 0 \quad in\ H^{-2}(0, T; L^2(\Omega)), \quad \tau = 1, 2,$$

(5.44*b*) $$m(\psi_3^e) \rightharpoonup \psi_3^e \quad in\ H^{-1}(0, T; L^2(\omega))\ weak,$$

*where $\psi_3^*$ is the solution of*

(5.45*a*) $$\frac{\partial^2 \psi_3^*}{\partial t^2} + \frac{1}{12} b_{\alpha\beta\gamma\delta} \frac{\partial^4 \psi_3^*}{\partial z_\alpha \partial z_\beta \partial z_\gamma \partial z_\delta} = 2\left(\frac{\partial^2 \psi_3^*}{\partial t^2} - \varphi_3^*\right) in\ \omega \times (0, T),$$

(5.45*b*) $$\psi_3^* = 0 \quad on\ \gamma \times (0, T),$$

(5.45*c*) $$\frac{\partial \psi_3^*}{\partial \nu} = \begin{cases} \dfrac{\partial^2 \varphi_3^*}{\partial \nu^2} & on\ \gamma(z^0) \times (0, T), \\ 0 & on\ \gamma_* \times (0, T), \end{cases}$$

(5.45*d*) $$\psi_3^*(T) = \frac{\partial \psi_3^*}{\partial t}(T) = 0 \quad in\ \omega,$$

*where $\dfrac{\partial^2 \varphi_3^*}{\partial \nu^2}$ is defined as in Nečas [1967].*

*Proof:* The idea is to use the weak formulation (4.5) with the test solutions $\theta^e$ which correspond to data $f_\tau^e = 0$ to prove (5.44*b*) and then $f_3^e = 0$ to show (5.44*a*). For instance, to prove (5.44*b*), we choose, in addition, $f_3^e$ such that

(5.46) $$\{e^{-2} f_3^e\} \rightharpoonup f_3^* \quad in\ H^1(0, T; L^2(\Omega))\text{weak}.$$

This immediately implies that $\{\psi_3^e\}$ is bounded in $H^{-1}(0, T; L^2(\Omega))$. To identify the limit, we take $f_3^e = e^2 f_3^*$ and pass to the limit in (4.5) using Theorems 5.3 and 5.4. ∎

Let us now study of the behavior of the initial values $\{\psi^{0,e}, \psi^{1,e}\}$. This will be done by means of test solutions $\theta^e$ which correspond to data $f_i^e = 0$ and $\{\theta^{0,e}, \theta^{1,e}\}$ suitably chosen. The results are summarized in the following

THEOREM 5.8: *Let $\varphi^e$ be the solution of* (3.1) *where the initial conditions satisfy* (5.42), (5.43) *and the requirements of Theorem 5.1. Then the initial values $\{\psi^{0,e}, \psi^{1,e}\}$ of the backward problem* (4.4) *satisfy:*

(i) $e^2 m(\psi_\tau^{0,e}) \to 0 \quad in\ H^{-1}(\omega)$,

(ii) $e^2 \frac{\partial}{\partial z_\tau} m(z_3\, \psi_\tau^{0,e}) + m(\psi_3^{0,e}) \to \psi_3^{0,*}$ *in* $H^{-2}(\omega)$,

(iii) $e^4 \frac{\partial}{\partial z_\tau} m(z_3\, \psi_\tau^{1,e}) + e^2\, m(\psi_3^{1,e}) \to 0$ *in* $H^{-3}(\omega)$.

*Here* $\psi_3^{0,*}$ *is the initial value of the limit* $\psi_3^*$ *of the third component* $m(\psi_3^e)$ *obtained in Theorem 5.7.*

*Proof:* For instance, let us indicate the proof of (ii). We choose

$$(5.47) \qquad \begin{cases} f_i^e = \theta_\tau^{0,e} = 0,\ \theta_3^{1,e} \text{ independent of } z_3, \\ \{\theta_3^{1,e}\} \rightharpoonup \theta_3^{1,*} \text{ in } H_0^2(\omega) \text{ weak and } \theta_\tau^{1,e} = -\, z_3 \dfrac{\partial \theta_3^{1,e}}{\partial z_\tau}. \end{cases}$$

With this choice, we pass to the limit in (4.5) and obtain

$$(5.48) \quad \lim_{F'} \left\langle \{(e^2\, \psi_\tau^{1,e}, \psi_3^{1,e}), (-\, e^2\, \psi_\tau^{0,e}, -\, \psi_3^{0,e})\}, \right.$$

$$\left. \left\{(0,0), \left(-\, z_3 \frac{\partial \theta_3^{1,e}}{\partial z_\tau}, \theta_3^{1,e}\right)\right\} \right\rangle_F =$$

$$= \int_{\gamma(z^0) \times (0,T)} \frac{1}{12}\, b_{\alpha\beta\tau\eta} \frac{\partial^2 \theta_3^*}{\partial z_\alpha\, \partial z_\beta} \frac{\partial^2 \varphi_3^*}{\partial z_\tau\, \partial z_\eta}\, d\sigma\, dt +$$

$$+ 2 \int_{\omega \times (0,T)} \left( \frac{\partial \varphi_3^*}{\partial t} \frac{\partial \theta_3^*}{\partial t} + \varphi_3^*\, \theta_3^* \right) d\tilde{z}\, dt\,.$$

This, when combined with the weak formulation of (5.45), yields

$$\left\langle e^2\, z_3 \frac{\partial \psi_\tau^{0,e}}{\partial z_\tau} + \psi_3^{0,e}, \theta_3^{1,e} \right\rangle \to \langle \psi_3^{0,*}, \theta_3^{1,*} \rangle\,.$$

This establishes (ii). ∎

Having estimated the solution $\psi^e$ of the backward problem (4.4) in the interior and its initial values at $t = 0$, we turn our attention to estimating its trace on the boundary $\Gamma_\pm$. It is true that the weak formulation (4.5) does not contain these traces explicitly because the test solutions $\theta^e$ chosen satisfied $\tilde{\sigma}_{i3} = 0$ on $\Sigma_\pm$. Thus, in order to achieve our objective, we take test solutions $\theta^e$ satisfying the system (3.1) where $f^e = 0$, $\theta^{0,e} = 0$, $\theta^{1,e} = 0$ and (3.1*d*) is replaced by (5.33).

Multiplying the system in $\theta^e$ by $\psi^e$ and system (4.4) by $\theta^e$ and integrating by parts, we arrive at the following formulation:

$$\int_{\Sigma(z^0)} \tilde{\sigma}_{ij}\,\xi_{ij}\,d\sigma\,dt + e^2 \int_{\Sigma_\pm} \left\{ \frac{\partial \varphi_\tau^e}{\partial t}\frac{\partial \theta_\tau^e}{\partial t} + \varphi_\tau^e\,\theta_\tau^e \right\} d\sigma\,dt$$
$$+ \int_{\Sigma_\pm} \left\{ \frac{\partial \varphi_3^e}{\partial t}\frac{\partial \theta_3^e}{\partial t} + \varphi_3^e\,\theta_3^e \right\} d\sigma\,dt$$
$$+ e^{-1}\int_{\Sigma_\pm} g_\tau^{\pm,e}\,\psi_\tau^e\,d\sigma\,dt + e^{-2}\int_{\Sigma_\pm} g_3^{\pm,e}\,\psi_3^e\,d\sigma\,dt = 0\,.$$

The following result is easily obtained using the above formulation and Theorem 5.6.

THEOREM 5.9: *Let $\psi^e$ be the solution of the backward problem* (4.4) *driven by $\varphi^e$ satisfying the convergences* (5.42) *and* (5.43). *Then*

$$\int_{\Sigma_\pm} \{e^{-1}\,g_\tau^{\pm,e}\,\psi_\tau^e + e^{-2}\,g_3^{\pm,e}\,\psi_3^e\}\,d\sigma\,dt$$

*is bounded whenever* $\{e^{-1}\,g_\tau^{\pm,e}\}$ *and* $\{e^{-2}\,g_3^{\pm,e}\}$ *are bounded in* $H^2(0,T;L^2(\Gamma_\pm))$. ∎

## 5.3. The limiting two-dimensional exact controllability problem

The purpose of this section is to introduce and analyze the exact controllability problem which is the limit of the three-dimensional problem posed in $\Omega$ as $e$ tends to zero. This limit problem is two-dimensional and posed in $\omega$. As observed earlier, the limit problem involves an interior control apart from a boundary one. The analysis of this problem is done in Lions [1988a] in the isotropic case and so we merely state the results in the general case.

Let us fix one interior control $\tilde{w}$ and one boundary control $\tilde{v}$ and consider the following problem:

(5.49*a*) $$\frac{\partial^2 \tilde{y}}{\partial t^2} + \frac{1}{12}\,b_{\alpha\beta\tau\eta}\,\frac{\partial^4 \tilde{y}}{\partial z_\alpha\,\partial z_\beta\,\partial z_\tau\,\partial z_\eta} = \tilde{w} \quad \text{in } \omega \times (0,\tilde{T})\,,$$

(5.49*b*) $$\tilde{y} = 0 \quad \text{on } \gamma \times (0,\tilde{T})\,,$$

(5.49*c*) $$\frac{\partial \tilde{y}}{\partial \nu} = \begin{cases} \tilde{v} & \text{on } \gamma(z^0) \times (0,\tilde{T}), \\ 0 & \text{on } \gamma_* \times (0,\tilde{T})\,, \end{cases}$$

(5.49*d*) $$\tilde{y}(0) = \tilde{y}^0 \quad \text{and} \quad \frac{\partial \tilde{y}}{\partial t}(0) = \tilde{y}^1 \text{ in } \omega\,.$$

The problem is to find suitable controls $\tilde{v}$ , $\tilde{w}$ and time $\tilde{T}$ such that the state of the system (5.49) is driven to rest at time $\tilde{T}$. Given initial conditions $\{\tilde{\theta}^0, \tilde{\theta}^1\} \in H_0^2(\omega) \times L^2(\omega)$ and the source term $\tilde{f} \in L^1(0, \tilde{T}; L^2(\omega))$, we seek the solution $\tilde{\theta}$ of the following forward problem:

$$(5.50a) \qquad \tilde{\theta} \in C^0([0, \tilde{T}]; H_0^2(\omega)), \quad \frac{\partial \tilde{\theta}}{\partial t} \in C^0([0, \tilde{T}]; L^2(\omega)),$$

$$(5.50b) \qquad \frac{\partial^2 \tilde{\theta}}{\partial t^2} + \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^4 \tilde{\theta}}{\partial z_\alpha \partial z_\beta \partial z_\tau \partial z_\eta} = \tilde{f} \quad \text{in } \omega \times (0, \tilde{T}),$$

$$(5.50c) \qquad \tilde{\theta}(0) = \tilde{\theta}^0 \quad \text{and} \quad \frac{\partial \tilde{\theta}}{\partial t}(0) = \tilde{\theta}^1 \text{ in } \omega .$$

The associated energy functional is the following:

$$(5.51) \qquad \tilde{E}(\tilde{\theta}; t) = \frac{1}{24} \int_\omega b_{\tau\eta\alpha\beta} \frac{\partial^2 \tilde{\theta}}{\partial z_\tau \partial z_\eta} \frac{\partial^2 \tilde{\theta}}{\partial z_\alpha \partial z_\beta} d\tilde{z} + \frac{1}{2} \int_\omega \left| \frac{\partial \tilde{\theta}}{\partial t} \right|^2 d\tilde{z} .$$

Following energy methods, the above problem is easily seen to admit a unique solution $\tilde{\theta}$ which satisfies the following energy inequality:

$$(5.52) \qquad \|\tilde{\theta}\|_{L^\infty(0, \tilde{T}; H_0^2(\omega))} + \left\|\frac{\partial \tilde{\theta}}{\partial t}\right\|_{L^\infty(0, \tilde{T}; L^2(\omega))} \leqslant$$
$$\leqslant c\{ \|\{\tilde{\theta}^0, \tilde{\theta}^1\}\|_{\tilde{E}} + \|\tilde{f}\|_{L^1(0, \tilde{T}; L^2(\omega))}\}$$

Here by the norm $\| \cdot \|_{\tilde{E}}$ we understand

$$(5.53) \qquad \|\{\tilde{\theta}^0, \tilde{\theta}^1\}\|_{\tilde{E}}^2 = \frac{1}{24} \int_\omega b_{\tau\eta\alpha\beta} \frac{\partial^2 \tilde{\theta}^0}{\partial z_\tau \partial z_\eta} \frac{\partial^2 \tilde{\theta}^0}{\partial z_\alpha \partial z_\beta} d\tilde{z} + \frac{1}{2} \int_\omega |\tilde{\theta}^1|^2 d\tilde{z} .$$

If $\tilde{f} \equiv 0$, the corresponding solution is denoted by $\tilde{\varphi}$ with initial condition $\{\tilde{\varphi}^0, \tilde{\varphi}^1\}$ and we have the energy conservation.

In the next step of HUM, one establishes the so-called direct and inverse inequalities for $\tilde{\varphi}$ :

$$(5.54) \qquad \int_{\gamma \times (0, \tilde{T})} b_{\tau\eta\alpha\beta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\tau \partial z_\eta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\alpha \partial z_\beta} d\sigma \, dt \leqslant c \|\{\tilde{\varphi}^0, \tilde{\varphi}^1\}\|_{\tilde{E}} ,$$

$$(5.55) \qquad \|\{\tilde{\varphi}^0, \tilde{\varphi}^1\}\|_{\tilde{E}}^2 \leqslant c \int_{\gamma(z^0) \times (0, \tilde{T})} b_{\tau\eta\alpha\beta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\tau \partial z_\eta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\alpha \partial z_\beta} d\sigma \, dt .$$

We take $\tilde{T} \geqslant T$ where $T > 0$ is the fixed time as in § 3 and § 4.

The third step introduces the space $\tilde{F}$ which is the closure of $\mathscr{D}(\Omega) \times \mathscr{D}(\Omega)$ under the norm defined by

$$(5.56) \quad \|\{\tilde{\varphi}^0, \tilde{\varphi}^1\}\|_{\tilde{F}}^2 = \int_{\gamma(z^0) \times (0, \tilde{T})} b_{\tau\eta\alpha\beta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\tau \partial z_\eta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\alpha \partial z_\beta} d\sigma \, dt + $$

$$+ 2 \int_{\omega \times (0, \tilde{T})} \left\{ \left( \frac{\partial \tilde{\varphi}}{\partial t} \right)^2 + (\tilde{\varphi})^2 \right\} d\tilde{z} \, dt ,$$

We see easily that the norms $\| \cdot \|_{\tilde{E}}$ and $\| \cdot \|_{\tilde{F}}$ are equivalent and so

$$(5.57) \qquad \tilde{F} = H_0^2(\omega) \times L^2(\omega) .$$

Taking $\{\tilde{\varphi}^0, \tilde{\varphi}^1\} \in \tilde{F}$ and $\tilde{\varphi}$ to be the corresponding solution, we consider the following backward non-homogeneous problem:

$$(5.58a) \quad \frac{\partial^2 \tilde{\psi}}{\partial t^2} + \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^4 \tilde{\varphi}}{\partial z_\alpha \partial z_\beta \partial z_\tau \partial z_\eta} = 2 \left( \frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \tilde{\varphi} \right) \text{ in } \omega \times (0, \tilde{T}) ,$$

$$(5.58b) \qquad \tilde{\psi} = 0 \quad \text{on } \gamma \times (0, \tilde{T}) ,$$

$$(5.58c) \qquad \frac{\partial \tilde{\psi}}{\partial \nu} = \begin{cases} \dfrac{\partial^2 \tilde{\varphi}}{\partial \nu^2} & \text{on } \gamma(z^0) \times (0, \tilde{T}), \\ 0 & \text{on } \gamma_* \times (0, \tilde{T}), \end{cases}$$

$$(5.58d) \qquad \tilde{\psi}(\tilde{T}) = 0 \quad \text{and} \quad \frac{\partial \tilde{\psi}}{\partial t}(\tilde{T}) = 0 \text{ in } \omega .$$

A weak formulation of (5.58) is as follows:

$$_{\tilde{F}}\left\langle \{\tilde{\theta}^0, \tilde{\theta}^1\}, \left\{ \frac{\partial \tilde{\psi}}{\partial t}(0), - \tilde{\psi}(0) \right\} \right\rangle_{\tilde{F}'} =$$

$$(5.59) \quad = \int_0^{\tilde{T}} \int_\omega \tilde{f} \, \tilde{\psi} \, d\tilde{z} \, dt + \int_0^{\tilde{T}} \int_{\gamma(z^0)} \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^2 \tilde{\varphi}}{\partial z_\alpha \partial z_\beta} \frac{\partial^2 \tilde{\theta}}{\partial z_\tau \partial z_\eta} d\sigma \, dt$$

$$+ 2 \int_0^{\tilde{T}} \int_\omega \left\{ \frac{\partial \tilde{\varphi}}{\partial t} \frac{\partial \tilde{\theta}}{\partial t} + \tilde{\varphi} \tilde{\theta} \right\} d\tilde{z} \, dt ,$$

where $\tilde{\theta}$ is a solution of (5.50) with $\tilde{f} \in L^2(0, \tilde{T}; L^2(\omega))$ and $\{\tilde{\theta}^0, \tilde{\theta}^1\} \in \tilde{F}$. Indeed, (5.59) gives

$$b_{\tau\eta\beta\rho} \frac{\partial \tilde{\psi}}{\partial z_\rho} \nu_\beta = b_{\tau\eta\beta\rho} \frac{\partial^2 \tilde{\varphi}}{\partial z_\beta \partial z_\rho} \quad \text{on } \gamma(z^0) \quad \forall \tau, \eta$$

which is equivalent to (5.58*c*) on $\gamma(z^0)$ because $b_{\tau\eta\beta\rho}\,\nu_\beta\,\nu_\rho \neq 0$ for some $\tau, \eta$. As usual, duality arguments establish the existence and uniqueness of a solution $\tilde{\psi}$ such that

$$(5.60)\qquad \tilde{\psi} \in L^\infty(0, \tilde{T}; L^2(\omega)), \quad \tilde{\psi}(0) \in L^2(\omega), \quad \frac{\partial \tilde{\psi}}{\partial t}(0) \in H^{-2}(\omega)\,.$$

Next we introduce the operator $\tilde{\Lambda} \in \mathscr{L}(\tilde{F}, \tilde{F}')$ by

$$(5.61)\qquad \tilde{\Lambda}\{\tilde{\varphi}^0, \tilde{\varphi}^1\} = \left\{\frac{\partial \tilde{\psi}}{\partial t}(0), -\tilde{\psi}(0)\right\}.$$

Using the fact that $\tilde{\Lambda}$ is an isomorphism onto, we can solve the exact controllability problem (5.49). Indeed, we have

THEOREM 5.10: *We consider the two-dimensional exact controllability problem* (5.49) *with*

$$\tilde{y}(\tilde{T}) = 0 \quad \text{and} \quad \frac{\partial \tilde{y}(\tilde{T})}{\partial t} = 0 \text{ in } \omega\,.$$

*We take initial conditions to be such that* $\{\tilde{y}^1, -\tilde{y}^0\} \in H^{-2}(\omega) \times L^2(\omega)$. *Then for every* $\tilde{T} > 0$ *there are controls*

$$\tilde{w} \in [H^1(0, \tilde{T}; L^2(\omega))]' \quad \text{and} \quad \tilde{v} \in L^2(\gamma(z^0) \times (0, \tilde{T}))$$

*such that problem* (5.49) *is exactly controllable. They are given by*

$$(5.62)\qquad \tilde{\Lambda}\{\tilde{\varphi}^0, \tilde{\varphi}^1\} = \{\tilde{y}^1, -\tilde{y}^0\}\,.$$

$$(5.63a)\qquad \tilde{w} = 2\left(\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \tilde{\varphi}\right) \quad \text{in } \omega \times (0, T)$$

$$(5.63b)\qquad \tilde{v} = \frac{\partial^2 \tilde{\varphi}}{\partial t^2} \quad \text{on} \quad \gamma(z^0) \times (0, \tilde{T})\,.$$

■

## 5.4. Method of asymptotic expansion

Let us recall that we used the energy method to obtain bounds and identify the limit of some sequences e.g. $m(\psi_3^\varepsilon)$ in Theorem 5.7. On the other hand, this method is not successful in identifying the limit of $m(\psi_3^{1,\varepsilon})$ even if it exists. We plan to overcome such difficulties using the method of asymptotic expansion. Thus both methods seem indispensable to treat the problem on hand.

The method of asymptotic expansion is quite classical and this has been applied to treat thin elastic bodies in Destuynder [1980], Ciarlet & Destuynder [1979] and Raoult [1984]. The basic idea is to propose an Ansatz of the form

$$\theta^e = \theta^{(0)} + e\theta^{(1)} + \dots$$

for the solution of (3.1) and similar expressions for the stress and strain tensors in terms of asymptotic expansions of data. Substituting these expansions into (3.1), we can identify each term by induction. Using this method, we prove

THEOREM 5.11: *Consider the backward problem* (4.4) *for* $\psi^e$ *driven by* $\varphi^e$ *which satisfies* (5.42), (5.43) *and the hypotheses of Theorem 5.1. If* $\{m(\psi_3^{1,e})\}$ *converges in the sense of distribution, then the limit is nothing else than* $\psi_3^{1,*}$ *which is the initial value of* $\psi_3^*$ *solution of the backward problem* (5.45). *Analogous result holds for* $\{m(\psi_3^{0,e})\}$.

*Proof:* By the method of asymptotic expansions, we can produce $\theta^e = \sum_{k=0}^{4} e^k \theta^{(k)}$ which satisfies (3.1*a,b*) with $e^{-1} f_\tau^e = O(e)$, $e^{-2} f_3^e = O(e)$ and (3.1*c*). Further, (3.1*d*) is replaced by $\tilde{\sigma}_{ij} \nu_j = O(e^3)$ on $\Sigma_\pm$. Finally

$$\theta_\tau^e(0) = 0, \quad \theta_3^e(0) = \theta_3^{0,*} \in \mathcal{D}(\omega), \quad \frac{\partial \theta^e}{\partial t}(0) = 0 .$$

We can establish that $\theta^e$ constructed above satisfies

$$\left\{ e \frac{\partial \theta_\tau^e}{\partial t}, \frac{\partial \theta_3^e}{\partial t} \right\} \to \left\{ 0, \frac{\partial \theta_3^e}{\partial t} \right\} \quad \text{in } L^2(\Sigma_\pm)^2$$

where $\theta_3^*$ is the solution of

(5.64*a*) $$\frac{\partial^2 \theta_3^*}{\partial t^2} + \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^4 \theta_3^*}{\partial z_\alpha \, \partial z_\beta \, \partial z_\tau \, \partial z_\eta} = 0 \quad \text{in } \omega \times (0, T) ,$$

(5.64*b*) $$\theta_3^* = 0 \quad \text{and} \quad \frac{\partial \theta_3^*}{\partial \nu} = 0 \quad \text{on } \gamma \times (0, T) ,$$

(5.64*c*) $$\theta_3^*(0) = \theta_3^{0,*} \quad \text{and} \quad \frac{\partial \theta_3^*}{\partial t}(0) = 0 \quad \text{in } \omega .$$

With the help of all the results obtained above, we can pass to the limit in (4.5) and obtain

$$\lim \langle \psi_3^{1,e}, \theta_3^{0,*} \rangle = \int_{\gamma(z^0) \times (0,T)} \frac{1}{12} b_{\alpha\beta\tau\eta} \frac{\partial^2 \theta_3^*}{\partial z_\alpha \partial z_\beta} \frac{\partial^2 \varphi_3^*}{\partial z_\tau \partial z_\eta} d\sigma\, dt + \\ + 2 \int_0^T \int_\omega \left\{ \frac{\partial \varphi_3^*}{\partial t} \frac{\partial \theta_3^*}{\partial t} + \varphi_3^* \theta_3^* \right\} d\tilde{z}\, dt .$$

But, according to the weak formulation of (5.45), the right hand side of the above relation is equal to $\langle \psi_3^{1,*}, \theta_3^{0,*} \rangle$. Since $\theta_3^{0,*} \in \mathcal{D}(\omega)$ is arbitrary, this finishes the proof. ■

### 5.5. Behaviour of the exact controllability problem

After having studied the forward and the backward problems in detail in the previous paragraphs, we are now in a position to pass to the limit in the three-dimensional exact controllability problem (2.11), (2.15). The aim here is to describe the behaviour of the exact controls obtained in Theorem 4.2. To this end, we take the initial conditions satisfying

(5.65*a*) $\{(e^2 y_\tau^{1,e}, y_3^{1,e}), (-e^2 y_\tau^{0,e}, y_3^{0,e})\}$ is bounded $(F^e)'$,

(5.65*b*) $\{m(y_3^{1,e})\}$ converges in $H^{-3}(\omega)$ weak ,

(5.65*c*) $\{m(y_3^{0,e})\}$ converges in $H^{-2}(\omega)$ weak .

The hypothesis (5.65*a*) implies in conjunction with Theorem 4.1 that

(5.66) $$\|\{\varphi^{0,e}, \varphi^{1,e}\}\|_{F^e} \leq C .$$

Thus the convergence (5.42), (5.43) is valid for a *subsequence* of $e \to 0$. Applying the results of Theorem 5.11, we get

(5.67*a*) $m(\psi_3^{0,e}) \rightharpoonup \psi_3^{0,*}$ in $H^2(\omega)$ weak ,

(5.67*b*) $m(\psi_3^{1,e}) \rightharpoonup \psi_3^{1,*}$ in $H^{-3}(\omega)$ weak .

We remind the reader that $\{\psi_3^{0,*}, \psi_3^{1,*}\}$ is the initial value of $\psi_3^*$ solution (5.45). Because of our assumption (5.65*b, c*), we conclude that the above convergence

(5.67) takes place for the entire sequence and the limits are uniquely determined. On the other hand, comparing problem (5.45) with (5.58) and using the definition of the operator $\tilde{\Lambda}$, we draw the conclusion that

$$\tilde{\Lambda}\{\varphi_3^{0,*}, \varphi_3^{1,*}\} = \{\psi_3^{1,*}, -\psi_3^{0,*}\}. \tag{5.68}$$

Since $\tilde{\Lambda}$ is an isomorphism, this implies that the initial conditions $\{\varphi_3^{0,*}, \varphi_3^{1,*}\}$ for $\varphi_3^*$ are uniquely determined. Hence $\varphi_3^*$ is uniquely determined and as a consequence we deduce that the same holds regarding the first two components $(\varphi_\tau^*)$ as well. Combining this with (5.66), we see that the whole sequence $\{\varphi^{0,e}, \varphi^{1,e}\}$ converges weakly in $F^e$ in the sense of (5.42), (5.43). Thus the various convergence results on the backward problem established in § 5.2 are available to us for the entire sequence.

THEOREM 5.12: *Let us consider the exact controllability problem* (2.11), (2.15) *wherein the initial conditions satisfy* (5.65). *Then*

(i) $m(y_3^e) \rightharpoonup y_3^*$ *in* $H^{-1}(0, T; L^2(\omega))$ *weak where* $y_3^*$ *is the solution of the backward problem* (5.45) *driven by* $\varphi_3^*$.

(ii) $ey_\tau^e \to 0$ *in* $H^{-2}(0, T; L^2(\Omega))$, $\tau = 1, 2$.

(iii) *The exact controls provided by Theorem 4.2 have the following asymptotic behaviour:*

$$v_\tau^e \rightharpoonup -z_3 \frac{\partial^2 \varphi_3^*}{\partial \nu^2} \quad \textit{in } L^2(\Sigma(z^0)) \textit{ weak}, \tau = 1, 2, \tag{5.69}$$

$$v_3^e = 0(e) \quad \textit{in } L^2(\Sigma(z^0)), \tag{5.70}$$

$$w_i^{\pm, e} = 0(e^2) \quad \textit{in } [H^1(0, T; L^2(\Gamma_\pm))]', \quad i = 1, 2, 3. \tag{5.71}$$

*Proof:* The conclusions (i) and (ii) follow directly from Theorem 5.7. From (5.66), it follows that $\left\{ e \frac{\partial \varphi_\tau^e}{\partial t} \right\}$ and $\left\{ \frac{\partial \varphi_3^e}{\partial t} \right\}$ are bounded in $L^2(\Sigma_\pm)$. Relation (5.71) results from this fact. Again it is a consequence of (5.66) that $\{\xi_{ij}\}$ and hence $\{e_{ij}\}$ is a bounded sequence in $L^2(\Sigma(z^0))$. Now (5.70) is an easy consequence (see (3.18)). Let us finally take up (5.69). We have the following relations on $\Sigma_0$ since $\varphi^e = 0$ on it:

$$2\,\xi_{\tau\eta} = \frac{\partial \varphi_\tau^e}{\partial \nu} \nu_\eta + \frac{\partial \varphi_\eta^e}{\partial \nu} \nu_\tau ,$$

$$2\,\xi_{\tau\eta}\, \nu_\eta = \frac{\partial \varphi_\tau^e}{\partial \nu} + (\operatorname{div}_{\bar{z}} \varphi^e)\, \nu_\tau, \quad \xi_{\tau\eta}\, \nu_\tau\, \nu_\eta = \operatorname{div}_{\bar{z}} \varphi^e .$$

Hence the following conclusions are easily drawn:

$$\operatorname{div}_{\bar z} \varphi^e \rightharpoonup \xi^*_{\theta\eta}\, \nu_\theta\, \nu_\eta \quad \text{in } L^2(\Sigma(z^0)) \text{ weak},$$

$$\frac{\partial \varphi^e_\tau}{\partial \nu} \rightharpoonup 2\, \xi^*_{\tau\eta}\, \nu_\eta - \xi^*_{\theta\eta}\, \nu_\theta\, \nu_\eta \quad \text{in } L^2(\Sigma(z^0)) \text{ weak}.$$

Using the expression

$$\xi^*_{\tau\eta} = -\, z_3 \frac{\partial^2 \varphi^*_3}{\partial z_\tau\, \partial z_\eta}$$

and the boundary conditions $\varphi^*_3 = \frac{\partial \varphi^*_3}{\partial \nu} = 0$ on $\gamma \times (0, T)$, we easily deduce that

$$\frac{\partial \varphi^e_\tau}{\partial \nu} \rightharpoonup -\, z_3 \frac{\partial^2 \varphi^e_3}{\partial \nu^2}\, \nu_\tau \quad \text{in } L^2(\Sigma(z^0)) \text{ weak}.$$

This is nothing but (5.69). ■

*Remark 5.13:* The following conditions are enough to guarantee (5.65*a*):

(5.72*a*) $\quad \{ y_3^{0,e},\, e^3\, y_\tau^{0,e} \}$ bounded in $L^2(\Omega)^2$,

(5.72*b*) $\quad \{ e^2\, y_\tau^{1,e} \}$ bounded in $L^2\left(-\frac{1}{2}, \frac{1}{2}; H^{-1}(\omega)\right)$,

(5.72*c*) $\quad y_3^{1,e} = e^{-2} \frac{\partial g^e}{\partial z_3}$ with $\{ g^e \}$ bounded in $L^2(\Omega)$.

In the last relation, the derivative in taken in the sense of duality between $H^1\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left[H^1\left(-\frac{1}{2}, \frac{1}{2}\right)\right]'$.

*Remark 5.14:* Though the controls $w^{\pm, e}$ at the top-bottom surfaces vanish at the limit, they are responsible for the appearance of the interior control $\tilde{w}$ in the limit problem.

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