M2AN - Modélisation mathématique et analyse numérique

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M2AN - Modélisation mathématique et analyse numérique, tome 31, nº 6 (1997), p. 733-763
<http://www.numdam.org/item?id=M2AN_1997__31_6_733_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHÉMATIQUE ET ANALYSE NUMERIQUE

(Vol 31, n° 6, 1997, p 733 à 763)

MODELING AND OPTIMIZATION OF NON-SYMMETRIC PLATES (*)

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Abstract — *In the first part of this paper we present a limit model for non-symmetric elastic plates using an asymptotic method from three dimensional elasticity and we prove existence and uniqueness of solution of this model In the second part we try the shape optimization of the place, obtaining the existence of an optimal profile and proposing its numerical solution by means of penalty methods*

Key words non-symmetric plates, asymptotic analysis, shape optimization, penalty methods
Mathematics Subject Classification 73K10, 73K40, 73C02, 35C20

Résumé — *Dans la première partie du travail nous présentons un modèle limite pour des plaques non symétriques en utilisant une méthode asymptotique en élasticité tridimensionnelle et nous montrons l'existence et l'unicité de solution pour le modèle ainsi obtenu Dans la deuxième partie nous proposons d'optimiser la forme de la plaque en obtenant l'existence d'un profil optimal et nous étudions les méthodes de pénalisation pour la solution pratique.*

1. INTRODUCTION

The use of asymptotic methods for obtention and mathematical justification of plate models in the framework of theory of elasticity has been shown to be a successful technique during last decades. First fundamental contributions in this direction, any of whose basic ideas already appear in Friedrichs-Dressler [31] and Goldenveizer [32], were obtained by Ciarlet-Destuynder [15, 16] with the justification of the classical bi-harmonic model of Kirchhoff-Love the bending of symmetric plates. The application of this method to different situations (linear and nonlinear elasticity, composite and anisotropic materials, static, dynamic and thermoelastic cases, homogenization and so on) has provided important contributions, among which, without attempting to be exhaustive, we mention the works of Caillerie [8], Ciarlet [12, 14], Ciarlet-Rabier [21], Destuynder [25, 26, 28], Raoult [42-44], Blanchard [4], Ciarlet-Kesavan [17], Blanchard-Ciarlet [5], Viaño [46], Kohn-Vogelius [37-39], Cioranescu-Saint Jean Paulin [23], Davet [24], Blanchard-Francfort [6],

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Ciarlet-Le Dret [18], Quintela Estevez [40, 41], Alvarez Vazquez-Quintela Estevez [2]. A complete analysis of plate models with exhaustive bibliographic references may be found in Ciarlet [14]. asymptotic methods have also reached an important development in the case of shells (Destuynder [27], Ciarlet-Paumier [20], Figueiredo [30], Ciarlet-Miara [19]) and rods (Trabucho-Viaño [45], Cimitiere et al. [22]).

All of the above works have been exclusively devoted to the study of symmetric plates. The case of non-symmetric plates has so far remained, as far as we know, completely unreported. The first aim of this paper is the obtention of a mechanical model for non-symmetric plates in the framework a linear elasticity. In Section 2 we describe the physical problem, we obtain the mathematical model by using asymptotic analysis of the three-dimensional problem and we prove existence and uniqueness of solution for such model.

In Section 3, we study the optimal design problem of minimizing the weight of the non-symmetric plate under some geometric and technological constraints and considering some bounds on the deflection. In structural optimization, these problems can be formulated as state constrained optimal control problems governed by an elliptic differential equation, the control being a small parameter appearing in the coefficients of the differential operator (*cf.* Casas [9, 10], Haug-Arora [34], Hlavacek-Bock-Lovisek [35, 36], Haslinger-Neittanmaki [33]). In Section 3 we pose such a problem and obtain the existence of, at least, an optimal profile.

In order to carry out the numerical solution of the optimal shape problem, in last section we propose the use of penalty techniques, obtaining necessary optimality conditions and some convergence results.

2. MODELING OF A NON-SYMMETRIC PLATE

Let ε be a positive real parameter and ω be a domain in $\mathbb{R}^2$ with coordinates axis $x_1\, x_2$. Let $h \in W^{2,\infty}(\omega)$ be a "thickness" function verifying:

$$h(x_1, x_2) \geqslant \delta > 0 \text{ for all } (x_1, x_2) \in \bar{\omega}\,. \tag{1}$$

We consider the non-symmetric elastic plate of thickness ε occupying the reference configuration $\bar{\Omega}^\varepsilon$ defined by:

$$\Omega^\varepsilon = \{(x_1, x_2, x_3^\varepsilon) : (x_1, x_2) \in \omega, 0 < x_3^\varepsilon < \varepsilon h(x_1, x_2)\}\,. \tag{2}$$

We denote by $n^\varepsilon = (n_i^\varepsilon)$ the outward normal vector to the boundary of Ω^ε, by $\partial_\alpha^\varepsilon v = \partial_\alpha v$ the derivative $\partial v / \partial x_\alpha$ and by $\partial_3^\varepsilon v$ the derivative $\partial v / \partial x_3^\varepsilon$. Here and along the whole work we use, as it is customary in elasticity theory, the summation convention on repeated indices, supposing the Latin indices range over $\{1, 2, 3\}$ and Greek ones over $\{1, 2\}$.

We try to study the physical problem corresponding to the mechanical behavior of a non-symmetric elastic plate, supposed to be clamped on the lateral surface and submitted to body and surface forces. We assume the constitutive material of the plate to be a homogeneous isotropic elastic material of Saint Venant-Kirchhoff's type with Young's modulus E and Poisson's ratio ν. Then, on the linearized elasticity framework, the displacement field u^ε and the Piola-Kirchhoff stress tensor σ^ε are the solution of the following problem (see Ciarlet [13]):

$$
\begin{aligned}
&- \partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \text{ in } \Omega^\varepsilon , \\
&u^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon , \\
&\sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \text{ on } \Gamma_\pm^\varepsilon = \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon ,
\end{aligned}
\tag{3}
$$

where:

$$
\begin{aligned}
&\Gamma_0^\varepsilon = \{ (x_1, x_2, x_3^\varepsilon) : (x_1, x_2) \in \partial\omega, 0 < x_3^\varepsilon < \varepsilon h(x_1, x_2) \} , \\
&\Gamma_+^\varepsilon = \{ (x_1, x_2, x_3^\varepsilon) : (x_1, x_2) \in \bar{\omega}, x_3^\varepsilon = \varepsilon h(x_1, x_2) \} , \\
&\Gamma_-^\varepsilon = \{ (x_1, x_2, x_3^\varepsilon) : (x_1, x_2) \in \bar{\omega}, x_3^\varepsilon = 0 \} , \\
&n^\varepsilon \equiv (n_i^\varepsilon) = (- \varepsilon\, \partial_1 h, - \varepsilon\, \partial_2 h, 1)/\sqrt{1 + \varepsilon^2 (\partial_1 h)^2 + \varepsilon^2 (\partial_2 h)^2} \text{ on } \Gamma_+^\varepsilon , \\
&n^\varepsilon \equiv (n_i^\varepsilon) = (0, 0, -1) \text{ on } \Gamma_-^\varepsilon .
\end{aligned}
\tag{4}
$$

In previous problem, the stress tensor obeys to the Hooke's law:

$$
\sigma_{ij}^\varepsilon = \frac{E}{1+\nu} \gamma_{ij}^\varepsilon(u^\varepsilon) + \frac{E\nu}{(1+\nu)(1-2\nu)} \gamma_{kk}^\varepsilon(u^\varepsilon)\, \delta_{ij} , \tag{5}
$$

where $\gamma^\varepsilon(u^\varepsilon) = (\gamma_{ij}^\varepsilon(u^\varepsilon))$ is the linearized strain tensor:

$$
\gamma_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2} (\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon) . \tag{6}
$$

This problem can be formulated in the following mixed variational form:

$$u^\varepsilon \in V(\Omega^\varepsilon), \quad \sigma^\varepsilon \in \Sigma(\Omega^\varepsilon):$$

$$\int_{\Omega^\varepsilon} \left(\frac{1+\nu}{E} \sigma_{ij}^\varepsilon - \frac{\nu}{E} \sigma_{kk}^\varepsilon \delta_{ij} \right) \tau_{ij}^\varepsilon \, dx^\varepsilon - \int_{\Omega^\varepsilon} \gamma_{ij}^\varepsilon(u^\varepsilon)\, \tau_{ij}^\varepsilon \, dx^\varepsilon = 0 \text{ for all } \tau^\varepsilon \in \Sigma(\Omega^\varepsilon), \tag{7}$$

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \gamma_{ij}^\varepsilon(v^\varepsilon)\, dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \, dx^\varepsilon + \int_{\Gamma_\pm^\varepsilon} g_i^\varepsilon v_i^\varepsilon \, da^\varepsilon \text{ for all } v^\varepsilon \in V(\Omega^\varepsilon), \tag{8}$$

where:

$$V(\Omega^\varepsilon) = \{ v^\varepsilon \equiv (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3 : v_i^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon \}, \tag{9}$$

$$\Sigma(\Omega^\varepsilon) = [L^2(\Omega^\varepsilon)]_S^9 = \{ \tau^\varepsilon \equiv (\tau_{ij}^\varepsilon) \in [L^2(\Omega^\varepsilon)]^9 : \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon \}. \tag{10}$$

We consider the reference symmetric plate of constant thickness 1 occupying the volume $\bar{\Omega}$ where:

$$\Omega = \{ (x_1, x_2, x_3) : (x_1, x_2) \in \omega, 0 < x_3 < 1 \} = \omega \times (0, 1), \tag{11}$$

and we define:

$$\Gamma_0 = \{ (x_1, x_2, x_3) : (x_1, x_2) \in \partial\omega, 0 < x_3 < 1 \} = \partial\omega \times (0, 1),$$

$$\Gamma_+ = \{ (x_1, x_2, x_3) : (x_1, x_2) \in \omega, x_3 = 1 \} = \omega \times \{1\},$$

$$\Gamma_- = \{ (x_1, x_2, x_3) : (x_1, x_2) \in \omega, x_3 = 1 \} = \omega \times \{0\}.$$

We can define the change of variable from Ω^ε to the fixed domain Ω:

$$\pi^\varepsilon : x \equiv (x_1, x_2, x_3) \in \bar{\Omega} \to \pi^\varepsilon(x_1, x_2, x_3) =$$

$$(x_1, x_2, \varepsilon x_3 \, h(x_1, x_2)) \equiv (x_1, x_2, x_3^\varepsilon) \equiv x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{12}$$

Then, for each function:

$$\Phi^\varepsilon : x^\varepsilon \in \bar{\Omega}^\varepsilon \to \Phi^\varepsilon(x^\varepsilon) \in \mathbb{R}$$

we denote by Φ the function:

$$\Phi : x \in \bar{\Omega} \to \Phi(x) \in \mathbb{R}$$

given by $\Phi = \Phi^\varepsilon \circ \pi^\varepsilon$, i.e.:

$$\Phi(x_1, x_2, x_3) = \Phi^\varepsilon(x_1, x_2, \varepsilon x_3 \, h(x_1, x_2)).$$

This function verifies the following properties, where dependence on variables x or x^{ε} are implicitly assumed and not displayed:

$$\partial_{\alpha} \Phi = \partial_{\alpha} \Phi^{\varepsilon} + \varepsilon x_3 \partial_{\alpha} h \partial_3 \Phi^{\varepsilon} \Leftrightarrow \partial_{\alpha} \Phi^{\varepsilon} = \partial_{\alpha} \Phi - x_3 h^{-1} \partial_{\alpha} h \partial_3 \Phi$$

$$\partial_3 \Phi = \varepsilon h \partial_3 \Phi^{\varepsilon} \Leftrightarrow \partial_3 \Phi^{\varepsilon} = \varepsilon^{-1} h^{-1} \partial_3 \Phi$$

$$\int_{\Omega^{\varepsilon}} \Phi^{\varepsilon} dx^{\varepsilon} = \varepsilon \int_{\Omega} \Phi h \, dx .$$

$$\int_{\Gamma_-^{\varepsilon}} \Phi^{\varepsilon} da^{\varepsilon} = \int_{\Gamma_-} \Phi da .$$

$$\int_{\Gamma_+^{\varepsilon}} \Phi^{\varepsilon} da^{\varepsilon} = \int_{\Gamma_+} \Phi h^*(\varepsilon) \, da ,$$

where:

$$h^*(\varepsilon) = [1 + \varepsilon^2(\partial_1 h)^2 + \varepsilon^2(\partial_2 h)^2]^{1/2}$$

$$= 1 + \frac{1}{2}\varepsilon^2[(\partial_1 h)^2 + (\partial_2 h)^2] - \frac{1}{8}\varepsilon^4[(\partial_1 h)^2 + (\partial_2 h)^2]^2 + \cdots$$

Now we scale the different fields appearing in the variational formulation (see Ciarlet-Destuynder [15]) and we define $u(\varepsilon)$ and $\sigma(\varepsilon)$ by:

$$u_{\alpha}(\varepsilon)(x) = u_{\alpha}^{\varepsilon}(x^{\varepsilon}), u_3(\varepsilon)(x) = \varepsilon u_3^{\varepsilon}(x^{\varepsilon}) ; \quad (13)$$

$$\sigma_{\alpha\beta}(\varepsilon)(x) = \sigma_{\alpha\beta}^{\varepsilon}(x^{\varepsilon}), \sigma_{\alpha 3}(\varepsilon)(x) = \varepsilon^{-1} \sigma_{\alpha 3}^{\varepsilon}(x^{\varepsilon}), \sigma_{33}(\varepsilon)(x) = \varepsilon^{-2} \sigma_{33}^{\varepsilon}(x^{\varepsilon}) . \quad (14)$$

Also we assume that the applied forces are such that:

$$f_{\alpha}^{\varepsilon}(x^{\varepsilon}) = f_{\alpha}(x), f_3^{\varepsilon}(x^{\varepsilon}) = \varepsilon f_3(x) ; \quad (15)$$

$$g_{\alpha}^{\varepsilon}(x^{\varepsilon}) = \varepsilon g_{\alpha}(x), g_3^{\varepsilon}(x^{\varepsilon}) = \varepsilon^2 g_3(x) , \quad (16)$$

where $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent on ε. Then, we obtain that $(u(\varepsilon), \sigma(\varepsilon))$ is the only solution of the following scaled variational problem posed in Ω :

$$u(\varepsilon) \in V(\Omega), \quad \sigma(\varepsilon) \in \Sigma(\Omega)$$

$$-\int_\Omega h\gamma_{ij}^h(u(\varepsilon))\,\tau_{ij}\,dx + \int_\Omega h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\sigma_{\gamma\gamma}(\varepsilon)\,\delta_{\alpha\beta}\right\}\tau_{\alpha\beta}\,dx$$

$$+\varepsilon^2\int_\Omega h\left\{2\,\frac{1+\nu}{E}\sigma_{\alpha 3}(\varepsilon)\,\tau_{\alpha 3} - \frac{\nu}{E}(\sigma_{33}(\varepsilon)\,\tau_{\alpha\alpha} + \sigma_{\alpha\alpha}(\varepsilon)\,\tau_{33})\right\}dx$$

$$+\varepsilon^4\int_\Omega h\,\frac{1}{E}\sigma_{33}(\varepsilon)\,\tau_{33}\,dx = 0, \text{ for all } \tau \in \Sigma(\Omega)\,, \tag{17}$$

$$\int_\Omega h\sigma_{ij}(\varepsilon)\,\gamma_{ij}^h(v)\,dx = \int_\Omega hf_i\,v_i\,dx + \int_{\Gamma_+} h^*(\varepsilon)\,g_i\,v_i\,da, \text{ for all } v \in V(\Omega)\,, \tag{18}$$

where:

$$V(\Omega) = \{v \equiv (v_i) \in [H^1(\Omega)]^3 : v_i = 0 \text{ on } \Gamma_0\}\,,$$

$$\Sigma(\Omega) = [L^2(\Omega)]_S^9\,,$$

and $\gamma^h(v) \equiv (\gamma_{ij}^h(v))$ is the generalized strain tensor defined by:

$$\gamma_{\alpha\beta}^h(v) = \frac{1}{2}\left[\partial_\alpha v_\beta + \partial_\beta v_\alpha - x_3 h^{-1}\partial_\alpha h\,\partial_3 v_\beta - x_3 h^{-1}\partial_\beta h\,\partial_3 v_\alpha\right],$$

$$\gamma_{\alpha 3}^h(v) = \frac{1}{2}\left[\partial_\alpha v_3 + h^{-1}\partial_3 v_\alpha - x_3 h^{-1}\partial_\alpha h\,\partial_3 v_3\right], \tag{19}$$

$$\gamma_{33}^h(v) = h^{-1}\partial_3 v_3\,.$$

In order to obtain the convergence of the scaled three-dimensional problem as ε tends to zero we are going to show several technical results. In what follows, $|\cdot|_{0,\Omega}$ and $\|\cdot\|_{m,\Omega}$ be the following space

$$E^h(\Omega) = \{v \equiv (v_i) \in [L^2(\Omega)]^3 : \gamma_{ij}^h(v) \in L^2(\Omega)\}$$

endowed with the following norm:

$$\|v\|_\Omega^h = \left\{|v|_{0,\Omega}^2 + \sum_{i,j=1}^3 |\gamma_{ij}^h(v)|_{0,\Omega}^2\right\}^{1/2}, \quad v \in E^h(\Omega)\,. \tag{20}$$

We have the following obvious result:

THEOREM 1: *$[H^1(\Omega)]^3 \subset E^h(\Omega)$ and the inclusion is continuous, that is, there exists a constant $C = C(h, \omega)$ such that:*

$$\|v\|_{\Omega}^{h} \leqslant C\|v\|_{1,\Omega} \text{ for all } v \in [H^1(\Omega)]^3 .$$

In fact, we prove below that $[H^1(\Omega)]^3$ and $E^h(\Omega)$ are the same.

THEOREM 2: *$[H^1(\Omega)]^3 = E^h(\Omega)$ and the norms $\| \cdot \|_{\Omega}^{h}$ and $\| \cdot \|_{1,\Omega}$ are equivalent.*

Proof: In order to prove this result we use the lemma of J. L. Lions (see [24]) stablishing that if a distribution $\omega \in H^{-1}(\Omega)$ is such that $\partial_k \omega \in H^{-1}(\Omega)$, then $\omega \in L^2(\Omega)$. To apply this result, we recall that from the assumptions on h we have:

$$h, h^{-1}, \partial_\alpha h, \partial_{\alpha\beta} h \in L^\infty(\omega) ,$$

and we note that for all $v \in E^h(\Omega)$ we have $v_i \in L^2(\Omega)$ and $\gamma_{ij}^h(v) \in L^2(\Omega)$. Consequently, $\partial_j v_i \in H^{-1}(\Omega)$ and $\partial_k \gamma_{ij}^h(v) \in H^{-1}(\Omega)$. Next we show that each of second partial derivatives $\partial_{jk} v_i$ can be written as a linear combination of distributions in $H^{-1}(\Omega)$. In fact, we have:

$$\partial_{33} v_3 = h\, \partial_3 \gamma_{33}^h(v) ,$$

$$\partial_{\alpha 3} v_3 = \partial_\alpha h \gamma_{33}^h(v) + h\, \partial_\alpha \gamma_{33}^h(v) ,$$

$$\partial_{33} v_\alpha = 2\, h\, \partial_3 \gamma_{\alpha 3}^h(v) - h\, \partial_{\alpha 3} v_3 + \partial_\alpha h\, \partial_3 v_3 + x_3\, \partial_\alpha h\, \partial_{33} v_3 ,$$

$$\partial_{\alpha 3} v_\alpha = h\, \partial_3 \gamma_{\alpha\alpha}^h(v) + \partial_\alpha h\, \partial_3 v_\alpha + x_3\, \partial_\alpha h\, \partial_{33} v_\alpha \ (\text{no sum on } \alpha) ,$$

$$\partial_{\alpha\alpha} v_\alpha = \partial_\alpha h \gamma_{\alpha\alpha}^h(v) + h\, \partial_\alpha \gamma_{\alpha\alpha}^h(v) + x_3\, \partial_{\alpha\alpha} h\, \partial_3 v_\alpha + x_3\, \partial_\alpha h\, \partial_{\alpha 3} v_\alpha (\text{no sum on } \alpha) ,$$

$$\partial_{\alpha\alpha} v_3 = h^{-1}[2\, \partial_\alpha h \gamma_{\alpha 3}^h(v) + 2\, h\, \partial_\alpha \gamma_{\alpha 3}^h(v) - \partial_\alpha h\, \partial_\alpha v_3 - \partial_{\alpha 3} v_\alpha$$

$$+ x_3\, \partial_{\alpha\alpha} h\, \partial_3 v_3 + x_3\, \partial_\alpha h\, \partial_{\alpha 3} v_3 \text{ (no sum on } \alpha) ,$$

$$\partial_{12} v_3 = h^{-1}[\partial_2 h\gamma^h_{13}(v) + h\, \partial_2 \gamma^h_{13}(v) + \partial_1 h\gamma^h_{23}(v) + h\, \partial_1 \gamma^h_{23}(v) - 2\, h\, \partial_3 \gamma^h_{12}(v)$$

$$- \frac{1}{2}\, \partial_2 h\, \partial_1 v_3 - \frac{1}{2}\, \partial_1 h\, \partial_2 v_3 + 2\, x_3\, \partial_{12} h\, \partial_3 v_3 + x_3\, \partial_1 h\, \partial_{23} v_3 + x_3\, \partial_2 h\, \partial_{13} v_3$$

$$- \partial_1 h\, \partial_3 v_2 - \partial_2 h\, \partial_3 v_1 - x_3\, \partial_1 h\, \partial_{33} v_2 - x_3\, \partial_2 h\, \partial_{33} v_1] ,$$

$$\partial_{23} v_1 = \partial_2 h\gamma^h_{13}(v) + h\, \partial_2 \gamma^h_{13}(v) - \frac{1}{2}\, \partial_2 h\, \partial_1 v_3$$

$$- \frac{1}{2}\, h\, \partial_{12} v_3 + x_3\, \delta_{12}\, \mathrm{h}\, \partial_3 v_3 + x_3\, \partial_1 h\, \partial_{23} v_3,$$

$$\partial_{13} v_2 = \partial_1 h\gamma^h_{23}(v) + h\, \partial_1 \gamma^h_{23}(v) - \frac{1}{2}\, \partial_1 h\, \partial_2 v_3$$

$$- \frac{1}{2}\, h\, \partial_{12} v_3 + x_3\, \partial_{12} h\, \partial_3 v_3 + x_3\, \partial_2 h\, \partial_{13} v_3 ,$$

$$\partial_{12} v_1 = h^{-1}[\partial_2 h\gamma^h_{11}(v) + h\, \partial_2 \gamma^h_{11}(v) + x_3\, \partial_{12} h\, \partial_3 v_1 + x_3\, \partial_1 h\, \partial_{23} v_1 - \partial_2 h\, \partial_1 v_1] ,$$

$$\partial_{12} v_2 = h^{-1}[\partial_1 h\gamma^h_{22}(v) + h\, \partial_1 \gamma^h_{22}(v) + x_3\, \partial_{12} h\, \partial_3 v_2 + x_3\, \partial_2 h\, \partial_{13} v_2 - \partial_1 h\, \partial_2 v_2] ,$$

$$\partial_{22} v_1 = h^{-1}[2\, \partial_2 h\gamma^h_{12}(v) + 2\, h\, \partial_2 \gamma^h_{12}(v) - h\, \partial_{12} v_2 + x_3\, \partial_{12} h\, \partial_3 v_2$$

$$+ x_3\, \partial_1 h\, \partial_{23} v_2 + x_3\, \partial_{22} h\, \partial_3 v_1 + x_3\, \partial_2 h\, \partial_{23} v_1 - \partial_2 h\, \partial_1 v_2 - \partial_2 h\, \partial_2 v_1] ,$$

$$\partial_{11} v_2 = h^{-1}[2\, \partial_1 h\gamma^h_{12}(v) + 2\, h\, \partial_1 \gamma^h_{12}(v) - \partial_{12} v_1 + x_3\, \partial_{11} h\, \partial_3 v_2$$

$$+ x_3\, \partial_1 h\, \partial_{13} v_2 + x_3\, \partial_{12} h\, \partial_3 v_1 + x_3\, \partial_2 h\, \partial_{13} v_1 - \partial_1 h\, \partial_2 v_1 - \partial_1 h\, \partial_1 v_2] .$$

Above relations imply that $\partial_{jk} v_i = \partial_k(\partial_j v_i) \in H^{-1}(\Omega)$ and therefore $\partial_j v_i \in L^2(\Omega)$. Hence we obtain that $v \in H^1(\Omega)$.

Thus, theorem 1 prove that the identity mapping from $[H^1(\Omega)]^3$ into $E^h(\Omega)$ is continuous. Then, by the open mapping theorem of Banach (Yosida [47]) the identity mapping from $E^h(\Omega)$ into $[H^1(\Omega)]^3$ is also continuous and, consequently, both norms are equivalent. □

We introduce the space of displacements of Kirchhoff-Love:

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) \in V(\Omega) : \gamma^h_{i3}(v) = 0\} . \tag{21}$$

We have the following characterization of this space:

THEOREM 3:

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) : v_3(x_1, x_2, x_3) = \zeta_3(x_1, x_2)\,,$$

$$v_\alpha(x_1, x_2, x_3) = \zeta_\alpha(x_1, x_2) - x_3\, h(x_1, x_2)\, \partial_\alpha \zeta_3(x_1, x_2)\,, \qquad (22)$$

$$\zeta_3 \in H^2_0(\omega), \zeta_\alpha \in H^1_0(\omega)\}\,.$$

Proof: Since $\gamma^h_{33}(v) = h^{-1}\,\partial_3 v_3 = 0$ we obtain that $v_3(x_1, x_2, x_3) = \zeta_3(x_1, x_2)$, where $\zeta_3 \in H^1_0(\omega)$. Using now $\gamma^h_{\alpha 3}(v) = 0$ we obtain that $\partial_3 v_\alpha = -h\,\partial_\alpha v_3 = -h\,\partial_\alpha \zeta_3$. Since the right hand of the expression is independent on x_3 we obtain the existence of functions ζ_α depending only on variables (x_1, x_2) such that:

$$v_\alpha(x_1, x_2, x_3) = \zeta_\alpha(x_1, x_2) - x_3\, h(x_1, x_2)\, \partial_\alpha \zeta_3(x_1, x_2)\,.$$

Regularity of $v \in V(\Omega)$ implies that $\zeta_\alpha \in H^1_0(\omega)$ and $\partial_\alpha \zeta_3 \in H^1_0(\omega)$, then $\zeta_3 \in H^2_0(\omega)$. □

Remark 1: We write (22) in the following abridged form:

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) : v_3 = \zeta_3, v_\alpha = \zeta_\alpha - x_3\, h\, \partial_\alpha \zeta_3,$$

$$\zeta_3 \in H^2_0(\omega), \zeta_\alpha \in H^1_0(\omega)\}$$

in such a way that the following mapping is an isomorphism:

$$j : \zeta \equiv (\zeta_1, \zeta_2, \zeta_3) \in [H^1_0(\omega)]^2 \times H^2_0(\omega)$$

$$\to j(\zeta) = (\zeta_1 - x_3\, h\, \partial_1 \zeta_3, \zeta_2 - x_3\, h\, \partial_2 \zeta_3, \zeta_3) \in V^h_{KL}(\Omega)\,. \quad \square \qquad (23)$$

THEOREM 4: *The seminorm* $|\,.\,|^h_\Omega$ *defined by:*

$$v \in V(\Omega) \to |v|^h_\Omega = \left\{ \sum_{i,j=1}^{3} |\gamma^h_{ij}(v)|^2_{0,\Omega} \right\}^{1/2}$$

is a norm over $V(\Omega)$ *equivalent to* $\|\,.\,\|_{1,\Omega}$.

Proof: In order to prove that $| \cdot |_{\Omega}^{h}$ is a norm we only need to show that if $v \in V(\Omega)$, $|v|_{\Omega}^{h} = 0$ then $v = 0$. In fact we have that $\gamma_{ij}^{h}(v) = 0$. Since $\gamma_{i3}^{h}(v) = 0$ implies that $v \in V_{KL}^{h}(\Omega)$, then v is of the following form:

$$v_3 = \zeta_3, \quad \zeta_3 \in H_0^2(\omega),$$

$$v_\alpha = \zeta_\alpha - x_3\, h\, \partial_\alpha \zeta_3, \quad \zeta_\alpha \in H_0^1(\omega).$$

Now, from conditions $\gamma_{\alpha\beta}^{h}(v) = 0$ we obtain

$$\partial_\alpha v_\beta + \partial_\beta v_\alpha - x_3\, h^{-1}\, \partial_\alpha h\, \partial_3 v_\beta - x_3\, h^{-1}\, \partial_\beta h\, \partial_3 v_\alpha = 0,$$

or equivalently:

$$\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha = 2\, x_3\, h\, \partial_{\alpha\beta} \zeta_3 .$$

Thus, since the left hand side of the above equality is only a function of (x_1, x_2), we have that $\partial_{\alpha\beta} \zeta_3 = 0$ and consequently ζ_3 is affine with respect to (x_1, x_2). Finally, ζ_3 must vanish because $\zeta_3 \in H_0^2(\omega)$.

The functions ζ_α verify then:

$$\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha = 0,$$

which implies that:

$$\zeta_1 = a_1 - bx_2, \quad \zeta_2 = a_2 + bx_1,$$

for some constants a_1, a_2 and b. But ζ_α must vanish because $\zeta_\alpha \in H_0^1(\omega)$. Hence $v = 0$.

In order to prove the equivalence of the norms only remains to show that there exists a constant $C = C(h, \omega)$ such that

$$\| v \|_{1,\Omega} \leqslant C |v|_{\Omega}^{h}, \quad \text{for all } v \in V(\Omega).$$

Asume the property to be false. Then there exists a sequence $\{v^k\}$ of functions in $V(\Omega)$ verifying:

$$\| v^k \|_{1,\Omega} = 1, \quad \forall k \geqslant 0,$$

$$|v^k|_{\Omega}^{h} \to 0 \quad \text{as} \quad k \to \infty .$$

Since $\{v^k\}$ is bounded in $[H^1(\Omega)]^3$, by the Rellich's theorem there exists a subsequence $\{v^\lambda\}$ that strongly converges in $[L^2(\Omega)]^3$ to a function v. Since

$|v^\lambda|_\Omega^h \to 0$ as $\lambda \to \infty$, this subsequence is a Cauchy sequence with respect to the norm $\| \cdot \|_{1,\Omega}$, which is equivalent to the norm $| \cdot |_\Omega^h$. As $[H^1(\Omega)]^3$ is a complete space, the subsequence $\{v^\lambda\}$ converges to v in $[H^1(\Omega)]^3$. Then,

$$|v|_\Omega^h = \lim_{\lambda \to \infty} |v^\lambda|_\Omega^h = 0 ,$$

and therefore $v = 0$. But this fact contradicts the equalities:

$$\|v^\lambda\|_{1,\Omega} = 1 , \quad \forall \lambda \geq 0 ,$$

and the proof is then complete. □

COROLLARY 1: *The mapping:*

$$v \in V_{KL}^h(\Omega) \to |v|_\Omega^h = \left\{ \sum_{\alpha,\beta=1}^{2} |\gamma_{\alpha\beta}^h(v)|_{0,\Omega}^2 \right\}^{1/2}$$

is a norm over $V_{KL}^h(\Omega)$ equivalent to the norm $\| \cdot \|_{1,\Omega}$.

Proof: If is a immediate consequence of the fact that $\gamma_{i3}^h(v) = 0$ for all $v \in V_{KL}^h(\Omega)$. □

COROLLARY 2: *The mapping $\| \cdot \|_{KL}$ defined by (see (22)):*

$$v \in V_{KL}^h(\Omega) \to \|v\|_{KL} = \{ \|\zeta_1\|_{1,\omega}^2 + \|\zeta_2\|_{1,\omega}^2 + \|\zeta_3\|_{2,\omega}^2 \}^{1/2}$$

is a norm over $V_{KL}^h(\Omega)$ equivalent to the norm $| \cdot |_\Omega^h$ and, consequently, to the norm $\| \cdot \|_{1,\Omega}$.

Proof: The isomorphism $j : [H_0^1(\omega)]^2 \times H_0^2(\omega) \to V_{KL}^h(\Omega)$ defined by (23) is continuous if we consider in $V_{KL}^h(\Omega)$ the norm $| \cdot |_\Omega^h$. By the open mapping theorem of Banach (Yosida [47]) j^{-1} is also continuous, from which the result is obtained. □

The first stage in order to obtain convergence of the sequence $(u(\varepsilon), \sigma(\varepsilon))$ consists of following *a priori* estimates:

THEOREM 5: *There exist $\varepsilon_0 > 0$ and a constant $C = C(h, \omega)$ such that for all $0 < \varepsilon < \varepsilon_0$ we have:*

$$\|u(\varepsilon)\|_{1,\Omega} \leq C , \quad |\sigma_{\alpha\beta}(\varepsilon)|_{0,\Omega} \leq C ,$$

$$|\varepsilon\sigma_{\alpha 3}(\varepsilon)|_{0,\Omega} \leq C , \quad |\varepsilon^2 \sigma_{33}(\varepsilon)|_{0,\Omega} \leq C .$$

Proof: If we take $\tau = \sigma(\varepsilon)$ and $v = u(\varepsilon)$ in the scaled variational problem we have:

$$\int_\Omega h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\sigma_{\gamma\gamma}(\varepsilon)\,\delta_{\alpha\beta}\right\}\sigma_{\alpha\beta}(\varepsilon)\,dx$$

$$+ \varepsilon^2\int_\Omega h\left\{2\,\frac{1+\nu}{E}\sigma_{\alpha 3}(\varepsilon)\,\sigma_{\alpha 3}(\varepsilon) - 2\,\frac{\nu}{E}\sigma_{\alpha\alpha}(\varepsilon)\,\sigma_{33}(\varepsilon)\right\}dx$$

$$+ \varepsilon^4\int_\Omega h\,\frac{1}{E}\sigma_{33}(\varepsilon)\,\sigma_{33}(\varepsilon)\,dx = \int_\Omega h\gamma_{ij}^h(u(\varepsilon))\,\sigma_{ij}(\varepsilon)\,dx\,,$$

$$\int_\Omega h\sigma_{ij}(\varepsilon)\,\gamma_{ij}^h(u(\varepsilon))\,dx = \int_\Omega hf_i\,u_i(\varepsilon)\,dx + \int_{\Gamma_+} h^*(\varepsilon)\,g_i\,u_i(\varepsilon)\,da\,.$$

As a direct consequence of the second equality we have:

$$\int_\Omega h\sigma_{ij}(\varepsilon)\,\gamma_{ij}^h(u(\varepsilon))\,dx \leqslant c_1\|u(\varepsilon)\|_{1,\Omega}\,.$$

If we define the element $\tilde{\sigma}(\varepsilon)$ in $\Sigma(\Omega)$ by:

$$\tilde{\sigma}_{\alpha\beta}(\varepsilon) = \sigma_{\alpha\beta}(\varepsilon)\,,\quad \tilde{\sigma}_{\alpha 3}(\varepsilon) = \varepsilon\sigma_{\alpha 3}(\varepsilon)\,,\quad \tilde{\sigma}_{33}(\varepsilon) = \varepsilon^2\sigma_{33}(\varepsilon)\,,$$

we have:

$$\int_\Omega h\gamma_{ij}^h(u(\varepsilon))\,\sigma_{ij}(\varepsilon)\,dx =$$

$$\int_\Omega h\left\{\frac{1+\nu}{E}\tilde{\sigma}_{ij}(\varepsilon) - \frac{\nu}{E}\tilde{\sigma}_{kk}(\varepsilon)\,\delta_{ij}\right\}\tilde{\sigma}_{ij}(\varepsilon)\,dx \geqslant c_2|\tilde{\sigma}(\varepsilon)|_{0,\Omega}^2$$

and therefore:

$$|\tilde{\sigma}(\varepsilon)|_{0,\Omega}^2 \leqslant c_3\|u(\varepsilon)\|_{1,\Omega}\,.$$

By other hand we have:

$$|u(\varepsilon)|_\Omega^h \leqslant c_4\sup_{\tau\in\Sigma(\Omega)}\frac{\left|\int_\Omega h\tau_{ij}\,\gamma_{ij}^h(u(\varepsilon))\,dx\right|}{|\tau|_{0,\Omega}}$$

but we know that for ε_0 given and for $0 < \varepsilon < \varepsilon_0$:

$$\int_\Omega h\tau_{ij}\,\gamma^h_{ij}(u(\varepsilon))\,dx = \varepsilon^2 \int_\Omega h\frac{1}{E}\tilde{\sigma}_{33}(\varepsilon)\,\tau_{33}\,dx$$

$$+ \varepsilon \int_\Omega 2\,h\frac{1+\nu}{E}\tilde{\sigma}_{\alpha 3}(\varepsilon)\,\tau_{\alpha 3}\,dx - \int_\Omega h\frac{\nu}{E}\tilde{\sigma}_{33}(\varepsilon)\,\tau_{\alpha\alpha}\,dx + \varepsilon^2\int_\Omega h\frac{\nu}{E}\tilde{\sigma}_{\alpha\alpha}(\varepsilon)\,\tau_{33}\,dx$$

$$+ \int_\Omega h\left\{\frac{1+\nu}{E}\tilde{\sigma}_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\tilde{\sigma}_{\gamma\gamma}(\varepsilon)\,\delta_{\alpha\beta}\right\}\tau_{\alpha\beta}\,dx \leqslant c_5|\tilde{\sigma}(\varepsilon)|_{0,\Omega}|\tau|_{0,\Omega}\,.$$

Then, as a consequence of theorem 4:

$$\|u(\varepsilon)\|_{1,\Omega} \leqslant c_6|\tilde{\sigma}(\varepsilon)|_{0,\Omega}\,.$$

Thus we obtain

$$|\tilde{\sigma}(\varepsilon)|_{0,\Omega} \leqslant c_7, \quad \|u(\varepsilon)\|_{1,\Omega} \leqslant c_8$$

and the proof is complete. □

COROLLARY 3: *The sequences* $\{u(\varepsilon)\}_{\varepsilon>0}$ *and* $\{\sigma(\varepsilon)\}_{\varepsilon>0}$ *verify the following weak convergences:*

$$u(\varepsilon) \rightharpoonup u \quad \text{in } V(\Omega)\,,$$

$$\sigma_{\alpha\beta}(\varepsilon) \rightharpoonup \sigma_{\alpha\beta} \quad \text{in } L^2(\Omega)\,,$$

$$\varepsilon\sigma_{\alpha 3}(\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega)\,,$$

$$\varepsilon^2\sigma_{33}(\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega)\,,$$

where u *is the unique element in* $V^h_{KL}(\Omega)$ *solution of the problem:*

$$\int_\Omega \frac{E}{1-v^2}h\{(1-v)\,\gamma^h_{\alpha\beta}(u)\,\gamma^h_{\alpha\beta}(v) + v\gamma^h_{\alpha\alpha}(u)\,\gamma^h_{\beta\beta}(v)\}\,dx$$

$$= \int_\Omega hf_i\,v_i\,dx + \int_{\Gamma_\pm} g_i\,v_i\,da, \text{for all } v \in V^h_{KL}(\Omega)\,, \tag{24}$$

and $\sigma_{\alpha\beta}$ *is given by:*

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\{(1-\nu)\,\gamma^h_{\alpha\beta}(u) + \nu\gamma^h_{\mu\mu}(u)\,\delta_{\alpha\beta}\}\,. \tag{25}$$

Proof: As a direct consequence of previous theorem we obtain the existence of subsequences, still denoted in the same way, and elements $u \in V(\Omega)$, $\sigma_{\alpha\beta}, \chi_{\alpha 3}, \chi_{33} \in L^2(\Omega)$ such that:

$$\begin{aligned} u(\varepsilon) &\rightharpoonup u \quad \text{in } V(\Omega)\,, \\ \sigma_{\alpha\beta}(\varepsilon) &\rightharpoonup \sigma_{\alpha\beta} \quad \text{in } L^2(\Omega)\,, \\ \varepsilon\sigma_{\alpha 3}(\varepsilon) &\rightharpoonup \chi_{\alpha 3} \quad \text{in } L^2(\Omega)\,, \\ \varepsilon^2\,\sigma_{33}(\varepsilon) &\rightharpoonup \chi_{33} \quad \text{in } L^2(\Omega)\,. \end{aligned}$$

Taking $v = (v_1, v_2, 0) \in V(\Omega)$ in the scaled variational problem (18) we have:

$$\gamma^h_{\alpha\beta}(v) = \frac{1}{2}\left[\partial_\alpha v_\beta + \partial_\beta v_\alpha - x_3 h^{-1}\,\partial_\alpha h\,\partial_3 v_\beta - x_3 h^{-1}\,\partial_\beta h\,\partial_3 v_\alpha\right],$$

$$\gamma^h_{\alpha 3}(v) = \frac{1}{2} h^{-1}\,\partial_3 v_\alpha\,, \quad \gamma^h_{33}(v) = 0\,.$$

Then, multiplying by ε and passing to the limit we obtain:

$$2\int_\Omega h\chi_{\alpha 3}\,\gamma^h_{\alpha 3}(v)\,dx = 0\,, \quad \text{for all } v = (v_1, v_2, 0) \in V(\Omega)\,,$$

or equivalently:

$$\int_\Omega \chi_{\alpha 3}\,\partial_3 v_\alpha\,dx = 0, \text{ for all } v_\alpha \in H^1(\Omega),\ v_\alpha = 0 \text{ on } \Gamma_0\,,$$

then, $\chi_{\alpha 3} = 0$.

If we take $v = (0, 0, v_3) \in V(\Omega)$ in the scaled problem (18) we have:

$$\gamma^h_{\alpha\beta}(v) = 0\,, \quad \gamma^h_{\alpha 3}(v) = \frac{1}{2}\left[\partial_\alpha v_3 - x_3 h^{-1}\,\partial_\alpha h\,\partial_3 v_3\right], \quad \gamma^h_{33}(v) = h^{-1}\,\partial_3 v_3\,.$$

Then, multiplying by ε^2 and passing to the limit we obtain:

$$\int_\Omega h\chi_{33}\,\gamma^h_{33}(v)\,dx = 0, \text{ for all } v = (0, 0, v_3) \in V(\Omega)\,,$$

or equivalently:

$$\int_\Omega \chi_{33}\,\partial_3 v_3\,dx = 0, \text{ for all } v_3 \in H^1(\Omega),\ v_3 = 0 \text{ on } \Gamma_0\,,$$

then, $\chi_{33} = 0$.

Passing directly to the limit in scaled equation (17) we obtain:

$$\int_\Omega h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right\}\tau_{\alpha\beta}\,dx = \int_\Omega h\gamma^h_{ij}(u)\,\tau_{ij}\,dx, \text{ for all } \tau \in \Sigma(\Omega)\,.$$

Thus we have $\gamma^h_{i3}(u) = 0$, that is, $u \in V^h_{KL}(\Omega)$ and

$$\gamma^h_{\alpha\beta}(u) = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\,.$$

Relation (25) is obtained by inverting the previous equality.

At last, taking $v \in V^h_{KL}(\Omega)$ in the scaled variational problem (18) and passing to the limit we obtain:

$$\int_\Omega h\sigma_{\alpha\beta}\,\gamma^h_{\alpha\beta}(v)\,dx = \int_\Omega hf_i\,v_i\,dx + \int_{\Gamma_\pm} g_i\,v_i\,da, \text{ for all } v \in V^h_{KL}(\Omega)\,.$$

If we substitute in this equation the expression obtained for $\sigma_{\alpha\beta}$ we have the desired limit problem (24).

Finally, as a consequence of Lax-Milgram theorem, the limit problem (24) has an unique solution $u \in V^h_{KL}(\Omega)$, because the bilinear form is $V^h_{KL}(\Omega)$-elliptic:

$$\int_\Omega \frac{E}{1-\nu^2} h\{(1-\nu)\,\gamma^h_{\alpha\beta}(v)\,\gamma^h_{\alpha\beta}(v) + \nu\gamma^h_{\alpha\alpha}(v)\,\gamma^h_{\beta\beta}(v)\}\,dx$$

$$\geqslant \frac{E\delta}{1+\nu}\sum_{\alpha,\beta=1}^{2} |\gamma^h_{\alpha\beta}(v)|^2_{0,\Omega} = \frac{E\delta}{1+\nu}|v|^{h^2}_{\Omega}, \textit{for all } v \in V^h_{KL}(\Omega)\,. \quad \square \qquad (26)$$

COROLLARY 4: *We have the following strong convergences:*

$$u(\varepsilon) \to u \quad \text{in } V(\Omega)\,,$$

$$\sigma_{\alpha\beta}(\varepsilon) \to \sigma_{\alpha\beta} \quad \text{in } L^2(\Omega)\,,$$

$$\varepsilon\sigma_{\alpha3}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega)\,,$$

$$\varepsilon^2\sigma_{33}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega)\,.$$

Proof: If we define the elements $\tilde{\sigma}(\varepsilon)$ and σ^* in $\Sigma(\Omega)$ given by:

$$\tilde{\sigma}_{\alpha\beta}(\varepsilon) = \sigma_{\alpha\beta}(\varepsilon)\,,\quad \tilde{\sigma}_{\alpha3}(\varepsilon) = \varepsilon\sigma_{\alpha3}(\varepsilon)\,,\quad \tilde{\sigma}_{33}(\varepsilon) = \varepsilon^2\sigma_{33}(\varepsilon)\,,$$

$$\sigma^*_{\alpha\beta} = \sigma_{\alpha\beta},\quad \sigma^*_{\alpha3} = 0\,,\quad \sigma^*_{33} = 0\,,$$

we have to prove that $\bar{\sigma}(\varepsilon) = (\tilde{\sigma}(\varepsilon) - \sigma^{*}) \to 0$ in $\Sigma(\Omega)$.

From the scaled problem (17), (18) and the limit problem (24), (25) we have:

$$\int_{\Omega} h\left\{\frac{1+\nu}{E}\bar{\sigma}_{ij}(\varepsilon) - \frac{\nu}{E}\bar{\sigma}_{kk}(\varepsilon)\,\delta_{ij}\right\}\bar{\sigma}_{ij}(\varepsilon)\,dx = \int_{\Omega} hf_i\,u_i(\varepsilon)\,dx$$

$$+ \int_{\Gamma_{\pm}} h^{*}(\varepsilon)\,g_i\,u_i(\varepsilon)\,da + \int_{\Omega} h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\,\delta_{\alpha\beta}\right\}\sigma_{\alpha\beta}\,dx$$

$$-2\int_{\Omega} h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\sigma_{\gamma\gamma}(\varepsilon)\,\delta_{\alpha\beta}\right\}\sigma_{\alpha\beta}\,dx + 2\,\varepsilon^2\int_{\Omega} h\,\frac{\nu}{E}\sigma_{33}(\varepsilon)\,\sigma_{\alpha\alpha}\,dx\,.$$

If we pass to the limit as $\varepsilon \to 0$ we obtain:

$$\lim_{\varepsilon\to 0}\int_{\Omega} h\left\{\frac{1+\nu}{E}\bar{\sigma}_{ij}(\varepsilon) - \frac{\nu}{E}\bar{\sigma}_{kk}(\varepsilon)\,\delta_{ij}\right\}\bar{\sigma}_{ij}(\varepsilon)\,dx = \int_{\Omega} hf_i\,u_i\,dx$$

$$+ \int_{\Gamma_{\pm}} g_i\,u_i\,da - \int_{\Omega} h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\,\delta_{\alpha\beta}\right\}\sigma_{\alpha\beta}\,dx\,.$$

But, from the characterization of $\sigma_{\alpha\beta}$, we have:

$$\int_{\Omega} h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\,\delta_{\alpha\beta}\right\}\sigma_{\alpha\beta}\,dx = \int_{\Omega} hf_i\,u_i\,dx + \int_{\Gamma_{\pm}} g_i\,u_i\,da\,,$$

and therefore:

$$\lim_{\varepsilon\to 0}|\bar{\sigma}(\varepsilon)|^2_{0,\Omega} \leqslant c_1\lim_{\varepsilon\to 0}\int_{\Omega} h\left\{\frac{1+\nu}{E}\bar{\sigma}_{ij}(\varepsilon) - \frac{\nu}{E}\bar{\sigma}_{kk}(\varepsilon)\,\delta_{ij}\right\}\bar{\sigma}_{ij}(\varepsilon)\,dx = 0\,.$$

Moreover, we know that:

$$\|u(\varepsilon) - u\|_{1,\Omega} \leqslant c_2 \sup_{\tau\in\Sigma(\Omega)} \frac{\left|\int_{\Omega} h\tau_{ij}\,\gamma^h_{ij}(u(\varepsilon) - u)\,dx\right|}{|\tau|_{0,\Omega}}$$

and we have:

$$\int_{\Omega} h\tau_{ij}\, \gamma_{ij}^{h}(u(\varepsilon) - u)\, dx = \varepsilon^{4} \int_{\Omega} h \frac{1}{E} \sigma_{33}(\varepsilon)\, \tau_{33}\, dx$$

$$+ \varepsilon^{2} \int_{\Omega} h\left\{ 2\, \frac{1+\nu}{E}\, \sigma_{\alpha 3}(\varepsilon)\, \tau_{\alpha 3} - \frac{\nu}{E}\left[\sigma_{33}(\varepsilon)\, \tau_{\alpha\alpha} + \sigma_{\alpha\alpha}(\varepsilon)\, \tau_{33}\right]\right\} dx$$

$$+ \int_{\Omega} h\left\{ \frac{1+\nu}{E}\, (\sigma_{\alpha\beta}(\varepsilon) - \sigma_{\alpha\beta})\, \frac{\nu}{E}\, (\sigma_{\gamma\gamma}(\varepsilon) - \sigma_{\gamma\gamma})\, \delta_{\alpha\beta}\right\} \tau_{\alpha\beta}\, dx\,.$$

Then we have that for all $\tau \in \Sigma(\Omega)$:

$$\left| \int_{\Omega} h\tau_{ij}\, \gamma_{ij}^{h}(u(\varepsilon) - u)\, dx \right| \leqslant c_3\{ \varepsilon^{2} |\varepsilon^{2}\, \sigma_{33}(\varepsilon)|_{0,\Omega} |\tau_{33}|_{0,\Omega}$$

$$+ \varepsilon |\varepsilon \sigma_{\alpha 3}(\varepsilon)|_{0,\Omega} |\tau_{\alpha 3}|_{0,\Omega} + |\varepsilon^{2}\, \sigma_{33}(\varepsilon)|_{0,\Omega} |\tau_{\alpha\alpha}|_{0,\Omega}$$

$$+ \varepsilon^{2} |\sigma_{\alpha\alpha}(\varepsilon)|_{0,\Omega} |\tau_{33}|_{0,\Omega} + |\sigma_{\alpha\beta}(\varepsilon) - \sigma_{\alpha\beta}|_{0,\Omega} |\tau_{\alpha\beta}|_{0,\Omega} \}\,,$$

and therefore:

$$\left| \int_{\Omega} h\tau_{ij}\, \gamma_{ij}^{h}(u(\varepsilon) - u)\, dx \right| \leqslant \Phi(\varepsilon)\, |\tau|_{0,\Omega}\,,$$

where $\lim_{\varepsilon \to 0} \Phi(\varepsilon) = 0$. Thus, $u(\varepsilon) \to u$ in $V(\Omega)$. □

Remark 2: We can explicit the limit problem (24) by taking:

$$u = (\xi_1 - x_3\, h\, \partial_1\, \xi_3,\, \xi_2 - x_3\, h\, \partial_2\, \xi_3,\, \xi_3)\,, \tag{27}$$

$$v = (\zeta_1 - x_3\, h\, \partial_1\, \zeta_3,\, \zeta_2 - x_3\, h\, \partial_2\, \zeta_3,\, \zeta_3)\,, \tag{28}$$

for $\xi_\alpha, \zeta_\alpha \in H_0^1(\omega)$ and ξ_3, $\zeta_3 \in H_0^2(\omega)$ and denoting $g_i^+(x_1, x_2) = g_i(x_1, x_2, 1)$, $g_i^-(x_1, x_2) = g_i(x_1, x_2, 0)$. Then the limit problem takes the following form:

$$\frac{E}{1-\nu^2}\left\{\int_\omega h[(1-\nu)\,\gamma_{\alpha\beta}(\xi)\,\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi)\,\gamma_{\beta\beta}(\zeta)]\right.$$

$$-\int_\omega \frac{h^2}{2}[(1-\nu)\,(\gamma_{\alpha\beta}(\xi)\,\partial_{\alpha\beta}\zeta_3 + \gamma_{\alpha\beta}(\zeta)\,\partial_{\alpha\beta}\xi_3) +$$

$$+\nu(\gamma_{\alpha\alpha}(\xi)\,\partial_{\beta\beta}\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\,\partial_{\beta\beta}\xi_3)]$$

$$\left.+\int_\omega \frac{h^3}{3}[(1-\nu)\,\partial_{\alpha\beta}\xi_3\,\partial_{\alpha\beta}\zeta_3 + \nu\,\partial_{\alpha\alpha}\xi_3\,\partial_{\beta\beta}\zeta_3]\right\}$$

$$=\int_\omega h\left[\int_0^1 f_i\right]\zeta_i - \int_\omega h^2\left[\int_0^1 x_3 f_\alpha\right]\partial_\alpha\zeta_3 + \int_\omega [g_i^- + g_i^+]\,\zeta_i - \int_\omega h g_\alpha^+\,\partial_\alpha\zeta_3,$$

$$\text{for all } \zeta_\alpha \in H_0^1(\omega), \zeta_3 \in H_0^2(\omega)\,. \quad \square$$

Remark 3: If we assume the asymptotic expansion:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon^2(u^2, \sigma^2) + \cdots$$

and substitute this expression into the scaled variational problem we obtain that the first term of the asymptotic expansion (u^0, σ^0) must verify

$$u^0 = u\,, \quad \sigma^0_{\alpha\beta} = \sigma_{\alpha\beta}\,,$$

and the components σ^0_{i3} are given by:

$$\sigma^0_{\alpha 3} = x_3\,\sigma^0_{\alpha\beta}\,\partial_\beta h - \int_0^{x_3}\partial_\beta(h\sigma^0_{\alpha\beta}) - g_\alpha^- - \int_0^{x_3} hf_\alpha\,,$$

$$\sigma^0_{33} = x_3\,\sigma^0_{\alpha 3}\,\partial_\alpha h - \int_0^{x_3}\partial_\alpha(h\sigma^0_{\alpha 3}) - g_3^- - \int_0^{x_3} hf_3. \quad \square$$

Remark 4: We can return to the reference configuration Ω^ε taking into accoung that u is a limit approximation of $u(\varepsilon)$. Then if we do the inverse change of variable and the inverse scaling we obtain that with the functions:

$$\xi^\varepsilon_\alpha = \xi_\alpha\,, \quad \xi^\varepsilon_3 = \varepsilon^{-1}\,\xi_3\,,$$

the limit displacement field $u^{0\varepsilon}$ in Ω^ε is given by:

$$u_\alpha^{0\varepsilon}(x^\varepsilon) = u_\alpha(x) = \xi_\alpha(x_1, x_2) - x_3\, h(x_1, x_2)\, \partial_\alpha\, \xi_3(x_1, x_2)$$

$$= \xi_\alpha(x_1, x_2) - \varepsilon^{-1}\, x_3^\varepsilon\, \partial_\alpha\, \xi_3(x_1, x_2) = \xi_\alpha^\varepsilon(x_1, x_2) - x_3^\varepsilon\, \partial_\alpha\, \xi_3^\varepsilon(x_1, x_2)\,,$$

$$u_3^{0\varepsilon}(x^\varepsilon) = \varepsilon^{-1}\, u_3(x) = \varepsilon^{-1}\, \xi_3(x_1, x_2) = \xi_3^\varepsilon(x_1, x_2)\,.$$

Denoting by $g_\iota^{\varepsilon+}(x_1, x_2) = g_\iota^\varepsilon(x_1, x_2, \varepsilon h(x_1, x_2))$ and $g_\iota^{\varepsilon-}(x_1, x_2) = g_\iota^\varepsilon(x_1, x_2, 0)$ we have $g_\alpha^{\varepsilon\pm} = \varepsilon g_\alpha^{\pm}$ and $g_3^{\varepsilon\pm} = \varepsilon^2\, g_3^{\pm}$. Then, we obtain that $\xi_\alpha^\varepsilon \in H_0^1(\omega)$ and $\xi_3^\varepsilon \in H_0^2(\omega)$ are the unique solution of the following problem:

$$\frac{E}{1-\nu^2}\left\{\int_\omega \varepsilon h[(1-\nu)\,\gamma_{\alpha\beta}(\xi^\varepsilon)\,\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi^\varepsilon)\,\gamma_{\beta\beta}(\zeta)]\right.$$

$$-\int_\omega \frac{\varepsilon^2 h^2}{2}\,[(1-\nu)\,(\gamma_{\alpha\beta}(\xi^\varepsilon)\,\partial_{\alpha\beta}\,\zeta_3 + \gamma_{\alpha\beta}(\zeta)\,\partial_{\alpha\beta}\,\xi_3^\varepsilon) +$$

$$+\ \nu(\gamma_{\alpha\alpha}(\xi^\varepsilon)\,\partial_{\beta\beta}\,\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\,\partial_{\beta\beta}\,\xi_3^\varepsilon)]$$

$$\left.+\int_\omega \frac{\varepsilon^3 h^3}{3}\,[(1-\nu)\,\partial_{\alpha\beta}\,\xi_3^\varepsilon\,\partial_{\alpha\beta}\,\zeta_3 + \nu\partial_{\alpha\alpha}\,\xi_3^\varepsilon\,\partial_{\beta\beta}\,\zeta_3]\right\}$$

$$= \int_\omega \left[\int_0^{\varepsilon h} f_\iota^\varepsilon + g_\iota^{\varepsilon-} + g_\iota^{\varepsilon+}\right]\zeta_\iota - \int_\omega \left[\int_0^{\varepsilon h} x_3^\varepsilon f_\alpha^\varepsilon + \varepsilon h g_\alpha^{\varepsilon+}\right]\partial_\alpha\,\zeta_3\,,$$

for all $\zeta_\alpha \in H_0^1(\omega)$, $\zeta_3 \in H_0^2(\omega)$. □

3. OPTIMIZATION OF A NON-SYMMETRIC PLATE

We consider an homogeneous, isotropic, elastic non-symmetric plate occupying the reference configuration:

$$\Omega^h = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega,\ 0 < x_3 < h(x_1, x_2)\}$$

clamped in its lateral surface and submitted to body forces of density $f^h \equiv (f_i^h)$ on Ω^h and surface forces of density $g^{h+} \equiv (g_i^{h+})$ on Γ_+^h and $g^{h-} \equiv (g_i^{h-})$ on Γ_-^h where:

$$\Gamma_+^h = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega, x_3 = h(x_1, x_2)\},$$

$$\Gamma_-^h = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega, x_3 = 0\}.$$

We denote by $u^h \equiv (u_i^h) : \bar{\Omega}^h \to \mathbb{R}^3$ the displacement field corresponding to the above conditions. In the next we identify functions u^h, f^h and $g^{h\pm}$ with the functions $u(h) : \bar{\Omega} \to \mathbb{R}^3$, $f(h) : \Omega \to \mathbb{R}^3$ and $g^{\pm}(h) : \Gamma_{\pm} \to \mathbb{R}^3$ such that:

$$u(h)(x_1, x_2, x_3) = u^h(x_1, x_2, x_3\, h(x_1, x_2)),$$

$$f(h)(x_1, x_2, x_3) = f^h(x_1, x_2, x_3\, h(x_1, x_2)),$$

$$g^+(h)(x_1, x_2, 1) = g^{h+}(x_1, x_2, h(x_1, x_2)),$$

$$g^-(h)(x_1, x_2, 0) = g^{h-}(x_1, x_2, 0).$$

As it was shown in previous section the displacement field $u(h)$ can be approximated by the only solution of the following variational problem:

$$u(h) \in V_{KL}^h(\Omega),$$

$$a_h(u(h), v) = l_h(v), \text{ for all } v \in V_{KL}^h(\Omega),$$

where for all $u, v \in V_{KL}^h(\Omega)$ we define:

$$a_h(u, v) = \int_\Omega \frac{E}{1-\nu^2} h\{(1-\nu)\, \gamma_{\alpha\beta}^h(u)\, \gamma_{\alpha\beta}^h(v) + \nu \gamma_{\alpha\alpha}^h(u)\, \gamma_{\beta\beta}^h(v)\}\, dx, \tag{29}$$

$$l_h(v) = \int_\Omega h f_i(h)\, v_i\, dx + \int_{\Gamma_+} g_i^+(h)\, v_i\, da + \int_{\Gamma_-} g_i^-(h)\, v_i\, da. \tag{30}$$

In what follows we assume that applied loads f^h and $g^{h\pm}$ are such that $f(h)$ and $g^{\pm}(h)$ are independent on h:

$$f(h)(x_1, x_2, x_3) = f(x_1, x_2, x_3),$$

$$g^+(h)(x_1, x_2, 1) = g^+(x_1, x_2, 1),$$

$$g^-(h)(x_1, x_2, 0) = g^-(x_1, x_2, 0).$$

Then we have:

$$l_h(v) = \int_{\Omega} hf_i\, v_i\, dx + \int_{\Gamma_+} g_i^+\, v_i\, da + \int_{\Gamma_-} g_i^-\, v_i\, da\,. \tag{31}$$

The optimal design problem deals with minimizing the weight of the non-symmetric plate Ω^h, or equivalently its volume (since the material is homogeneous), which we denote by:

$$J(h) = \int_{\omega} h\,.$$

The plate must be designed in such a way that allowed deflection be less than a given bound. As $u_3 \in H_0^2(\omega)$ we impose:

$$\| u_3 \|_{\infty,\,\omega} \leqslant e\,.$$

We will impose to the thickness some constraints of technological type:

i) First, the plate is designed in order to be constructed later, and this construction would not be possible if the plate is too thick or too thin. This fact leads us to impose a constraint of the following form:

$$0 < a \leqslant h(x) \leqslant b < \infty, \text{ for all } x \in \omega\,.$$

ii) By other hand, we also need that the variation of the thickness be slow and progressive. Then, we require:

$$\| \partial_\alpha h \|_{\infty,\,\omega} \leqslant c\,, \quad \| \partial_{\alpha\beta} h \|_{\infty,\,\omega} \leqslant d\,.$$

These technological constraints leads us to work in the set of feasible thickness:

$$U_{ad} = \{ h \in W^{2,\infty}(\omega) : a \leqslant h(x) \leqslant b,\ \forall x \in \omega,\ \| \partial_\alpha h \|_{\infty,\,\omega} \leqslant c,\ \| \partial_{\alpha\beta} h \|_{\infty,\,\omega} \leqslant d \}$$

that is a convex, closed and bounded subset of $W^{2,\infty}(\omega)$.

If we define the set:

$$K_h = \{ v \in V_{KL}^h(\Omega) : \| v_3 \|_{\infty,\,\omega} \leqslant e \}$$

that is a convex, closed subset of $V_{KL}^h(\Omega)$, we can formulate the following optimal design problem:

$$(P) \quad \text{Find } \bar{h} \in U_{ad} \text{ such that } u(\bar{h}) \in K_{\bar{h}}\,,$$

$$J(\bar{h}) = \inf \{ J(h) : h \in U_{ad},\, u(h) \in K_h \}\,.$$

Our first aim is to study existence of solution for problem (P). In order to do so, we consider J as a function defined in space $W^{1,\infty}(\omega)$:

$$J : h \in W^{1,\infty}(\omega) \to J(h) = \int_\omega h \in \mathbb{R}$$

and we define the set:

$$B = \{ h \in U_{ad} : u(h) \in K_h \} ,$$

that we assume to be non empty. Then, problem (P) can be written in the following form:

$$(P) \quad \text{Find } \bar{h} \in B \text{ such that } J(\bar{h}) = \inf \{ J(h) : h \in B \} .$$

Since J is continuous, for existence of solution of problem (P) it should be enough to prove that B is compact in $W^{1,\infty}(\omega)$.

In order to prove this fact, we write problem (P) in a more suitable form. We recall that the definition of Ω^h implicitly assumes that h is in the subset $A(\omega)$ of the space $W^{1,\infty}(\omega)$ given by:

$$A(\omega) = \{ h \in W^{2,\infty}(\omega) : h(x) \geqslant \delta(h) > 0, \text{ for all } x \in \omega \} .$$

We denote by $\mathscr{L}_2(V^h_{KL}(\Omega))$ the space of continuous bilinear forms on $V^h_{KL}(\Omega)$ and by $\mathscr{L}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)')$ the space of continuous linear forms from $V^h_{KL}(\Omega)$ on its dual. We also introduce the functions:

$$F : h \in A(\omega) \to F(h) = a_h \in \mathscr{L}_2(V^h_{KL}(\Omega))$$

$$G : a \in \mathscr{L}_2(V^h_{KL}(\Omega)) \to G(a) = A \in \mathscr{L}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)') ,$$

where:

$$A(y)(z) = a(y, z), \text{ for all } y, z \in V^h_{KL}(\Omega) .$$

Of course, G is an isometric isomorfism.

To each bilinear form $a_h = F(h)$ we associate the operator $A_h = G(a_h) = G(F(h))$. Since a_h is $V^h_{KL}(\Omega)$-elliptic for all $h \in A(\omega)$ (see (26)) from Lax-Milgram lemma it follows that for each $l_h \in V^h_{KL}(\Omega)'$ there exists an unique element $u(h) \in V^h_{KL}(\Omega)$ such that $A_h(u(h)) = l_h$. Then, A_h is an isomorphism from $V^h_{KL}(\Omega)$ onto $V^h_{KL}(\Omega)'$ and we have the function:

$$u : h \in A(\omega) \to u(h) = A_h^{-1}(l_h) \in V^h_{KL}(\Omega) .$$

THEOREM 6: *Function* $u : h \in A(\omega) \subset W^{1,\infty}(\omega) \to u(h) \in V^h_{KL}(\Omega) \subset [H^1(\Omega)]^3$ *is of class* C^∞.

Proof: Let $\mathcal{I}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)')$ be the subspace of $\mathcal{L}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)')$ of invertible applications. We define the mappings:

$$\Lambda : A \in \mathcal{I}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)') \to \Lambda(A) = A^{-1} \in \mathcal{I}(V^h_{KL}(\Omega)', V^h_{KL}(\Omega))$$

$$S : (M, l) \in \mathcal{L}(V^h_{KL}(\Omega)', V^h_{KL}(\Omega)) \times V^h_{KL}(\Omega)' \to S(M, l) = M(l) \in V^h_{KL}(\Omega)$$

$$T : h \in A(\omega) \to T(h) = l_h \in V^h_{KL}(\Omega)'.$$

Then, the result is obtained from the following equality, where functions S, Λ, G, F and T are of class C^∞:

$$u(h) = S((\Lambda \circ G \circ F)(h), T(h)). \quad \square$$

Remark 5: For h, $k \in A(\omega)$ we define:

$$m_{h,k} = DT(h)(k) \in V^h_{KL}(\Omega)',$$

$$a_{h,k} = DF(h)(k) \in \mathcal{L}_2(V^h_{KL}(\Omega)).$$

Then we have from (31):

$$m_{h,k} = \int_\Omega k f_i v_i .$$

In the same way from (29) for all u, $v \in V^h_{KL}(\Omega)$ given by (27), (28) we obtain:

$$a_{h,k}(u, v) = \frac{E}{1-\nu^2} \left\{ \int_\omega k[(1-\nu)\gamma_{\alpha\beta}(\xi)\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi)\gamma_{\beta\beta}(\zeta)] \right.$$

$$- \int_\omega hk[(1-\nu)(\gamma_{\alpha\beta}(\xi)\partial_{\alpha\beta}\zeta_3 + \gamma_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\xi_3) +$$

$$+ \nu(\gamma_{\alpha\alpha}(\xi)\partial_{\beta\beta}\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\partial_{\beta\beta}\xi_3)]$$

$$\left. + \int_\omega h^2 k[(1-\nu)\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\zeta_3 + \nu\,\partial_{\alpha\alpha}\xi_3\partial_{\beta\beta}\zeta_3] \right\}.$$

Let $A_{h,k} = G(a_{h,k}) \in \mathscr{L}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)')$. Then we have:

$$p = Du(h)(k) = A_h^{-1}[m_{h,k} - A_{h,k}(u(h))] ,$$

or equivalently:

$$a_h(p,q) = m_{h,k}(q) - a_{h,k}(u(h),q) , \quad \forall q \in V^h_{KL}(\Omega) .$$

This equation will be helpful for the numerical computation of $Du(h)(k)$ in the next section. □

THEOREM 7: *U_{ad} is compact in $W^{1,\infty}(\omega)$.*

Proof: By Rellich's theorem the imbedding of $W^{2,p}(\omega)$ in $C^1(\bar{\omega})$ is compact for $2 < p \leqslant \infty$. (Adams [1].) Since ω is bounded, U_{ad} is convex, closed and bounded in $W^{2,p}(\omega)$, and, consequently, weakly compact in $W^{2,p}(\omega)$. Thus, it is compact in $C^1(\bar{\omega})$. Since the imbedding of $C^1(\bar{\omega})$ in $W^{1,\infty}(\omega)$ is continuous, then we have that U_{ad} is compact in $W^{1,\infty}(\omega)$. □

THEOREM 8: *B is compact in $W^{1,\infty}(\omega)$.*

Proof: Let $K = \{\phi \in H^2_0(\omega) : \|\phi\|_{\infty,\omega} \leqslant e\}$. Since imbedding of $H^2(\omega)$ in $L^\infty(\omega)$ is continuous we have that K is a closed and convex set in $H^2_0(\omega)$. Moreover we have:

$$B = \{h \in U_{ad} : u_3(h) \in K\} ,$$

and consequently:

$$B = u_3^{-1}(K) \cap U_{ad} , \tag{32}$$

where:

$$u_3 : h \in A(\omega) \subset W^{1,\infty}(\omega) \to u_3(h) \in H^2_0(\omega) .$$

Using the following notation for $u(h) \in V^h_{KL}(\Omega)$:

$$u_\alpha(h)(x_1,x_2,x_3) = \xi_\alpha(h)(x_1,x_2) - x_3\, h(x_1,x_2)\, \partial_\alpha \xi_3(h)(x_1,x_2)$$

$$u_3(h)(x_1,x_2,x_3) = \xi_3(h)(x_1,x_2), \xi_\alpha(h) \in H^1_0(\omega), \xi_3(h) \in H^2_0(\omega)$$

we have the following descomposition for mapping u_3 :

$$h \in A(\omega) \subset W^{1,\infty}(\omega) \to u(h) \in V^h_{KL}(\Omega)$$

$$\to (\xi_1(h), \xi_2(h), \xi_3(h)) \in [H^1_0(\omega)]^2 \times H^2_0(\omega) \to \xi_3(h) \in H^2_0(\omega) .$$

From theorem 6 and corollary 2 we deduce that u_3 is a continuous function and then $u_3^{-1}(K)$ is closed. Now, from (32) and previous theorem we obtain that B is compact in $W^{1,\infty}(\omega)$. □

COROLLARY 5: *If l_h is of the form (31) then problem (P) has at least one solution.*

4. PENALTY METHODS

In order to obtain the numerical solution of the problem (P) we will use penalty techniques, defining a family of problems (P_δ) approximating the problem (P) and having an easier numerical solution. Besides, we will obtain necessary optimality conditions for the penalty problem (P_δ).

For $\delta > 0$ we define the function:

$$J_\delta : h \in A(\omega) \to J_\delta(h) = J(h) + \frac{1}{2\delta} \| u(h) - P_{K_h}(u(h)) \|_{KL}^2 \in \mathbb{R} \quad (33)$$

where $P_{K_h} : V_{KL}^h(\Omega) \to K_h$ denotes the projection of the Hilbert space $V_{KL}^h(\Omega)$ over the closed and convex subset K_h, which is a continuous function (Barbu-Precupanu [3], Cea [11]). Thus, J_δ is also a continuous function.

We consider the following problem:

$$(P_\delta) \quad \text{Find } h_\delta \in U_{ad} \text{ such that } J_\delta(h_\delta) = \inf \{ J_\delta(h) : h \in U_{ad} \}.$$

Due to the compacity of U_{ad} and the continuity of J_δ, the problem (P_δ) has, at least, one solution.

If for each $\delta > 0$ we note h_δ a solution of the problem (P_δ) then we have the following result, similar to the one obtained in Casas [9]:

THEOREM 9: a) *If $\delta_1 < \delta_2$ then:*

i)

$$J_{\delta_2}(h_{\delta_2}) \leq J_{\delta_1}(h_{\delta_1}) \leq \inf_{h \in B} J(h). \quad (34)$$

ii)

$$\| u(h_{\delta_1}) - P_{K_{h\delta_1}}(u(h_{\delta_1})) \|_{KL} \leq \| u(h_{\delta_2}) - P_{K_{h\delta_2}}(u(h_{\delta_2})) \|_{KL}.$$

iii)

$$J(h_{\delta_2}) \leq J(h_{\delta_1}) \leq \inf_{h \in B} J(h). \quad (35)$$

b) *The family* $\{h_\delta : \delta > 0\}$ *is bounded in* $W^{2,\infty}(\omega)$ *and there exist sequences* $\{\delta_n\}$, $\{h_{\delta_n}\}$ *such that if* $n \to \infty$ *then:*

$$\delta_n \to 0 ,$$

$$h_{\delta_n} \to \bar{h} \text{ in } W^{1,\infty}(\omega) .$$

Each one of these limits $\bar{h}$ *is a solution of the problem* (P).

c) *It is verified that:*

$$\lim_{\delta \to 0} J(h_\delta) = \lim_{\delta \to 0} J_\delta(h_\delta) = \inf_{h \in B} J(h) .$$

Proof: a) If $\delta_1 < \delta_2$ we have that $J_{\delta_2}(h) \leqslant J_{\delta_1}(h)$, $\forall h \in A(\omega)$, then: $J_{\delta_2}(h_{\delta_2}) \leqslant J_{\delta_2}(h_{\delta_1}) \leqslant J_{\delta_1}(h_{\delta_1})$. Besides, for all $h \in B$, $J_{\delta_1}(h) = J(h)$. Thus, $J_{\delta_1}(h_{\delta_1}) \leqslant J_{\delta_1}(h) = J(h)$, $\forall h \in B$, and i) follows.

By the other hand, since $J_{\delta_1}(h_{\delta_1}) \leqslant J_{\delta_1}(h_{\delta_2})$ and $J_{\delta_2}(h_{\delta_2}) \leqslant J_{\delta_2}(h_{\delta_1})$ we have:

$$\left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) \| u(h_{\delta_1}) - P_{K_{h\delta_1}}(u(h_{\delta_1})) \|_{KL}^2 \leqslant \left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) \| u(h_{\delta_2}) - P_{K_{h\delta_2}}(u(h_{\delta_2})) \|_{KL}^2$$

and we deduce ii).

Finally, since $J_{\delta_2}(h_{\delta_2}) \leqslant J_{\delta_2}(h_{\delta_1})$ we have from ii) that:

$$J(h_{\delta_2}) - J(h_{\delta_1})$$

$$\leqslant \frac{1}{2\,\delta_2} \left(\| u(h_{\delta_1}) - P_{K_{h\delta_1}}(u(h_{\delta_1})) \|_{KL}^2 - \| u(h_{\delta_2}) - P_{K_{h\delta_2}}(u(h_{\delta_2})) \|_{KL}^2 \right) \leqslant 0 .$$

Thus,

$$J(h_{\delta_2}) \leqslant J(h_{\delta_1}) \leqslant J_{\delta_1}(h_{\delta_1}) \leqslant \inf_{h \in B} J(h) .$$

b) It has been shown that U_{ad} is compact in $W^{1,\infty}(\omega)$ and bounded in $W^{2,\infty}(\omega)$. Since $\{h_\delta : \delta > 0\} \subset U_{ad}$, there exist $\bar{h} \in W^{1,\infty}(\omega)$ and sequences $\{\delta_n\} \to 0$ and $\{h_{\delta_n}\} \to \bar{h}$ in $W^{1,\infty}(\omega)$, as $n \to \infty$.

By continuity of u we have that $u(h_{\delta_n}) \to u(\bar{h})$ in $[H^1(\Omega)]^3$ and, from (34):

$$\| u(h_{\delta_n}) - P_{K_{h\delta_n}}(u(h_{\delta_n})) \|_{KL}^2 \leqslant 2\,\delta_n \left(\inf_{h \in B} J(h) - J(h_{\delta_n}) \right) \leqslant 2\,\delta_n \inf_{h \in B} J(h) .$$

Then, for $n \to \infty$, we obtain that $\| u(\bar{h}) - P_{K\bar{h}}(u(\bar{h})) \|_{KL} = 0$, and, consequently $\bar{h} \in B$. Thus, from (35) and continuity of J we have that $\bar{h}$ is a solution of the problem (P).

c) From (35) $J(h_\delta)$ increases as $\delta \to 0$, and it is bounded, then there exists $\lim_{\delta \to 0} J(h_\delta)$. From b) there exists a sequence that converges to $\inf_{h \in B} J(h)$. Thus, $\lim_{\delta \to 0} J(h_\delta) = \inf_{h \in B} J(h)$. Finally, taking limit as $\delta \to 0$ in $J(h_\delta) \leqslant J_\delta(h_\delta) \leqslant \inf_{h \in B} J(h)$, we obtain desired equalities. □

As an inmediate consequence, we obtain the following convergence result:

COROLLARY 6: *If problem (P) has an unique solution $\bar{h}$ and for each $\delta > 0$, h_δ denotes a solution of the problem (P_δ), then as $\delta \to 0$ we have:*

$$h_\delta \to \bar{h} \text{ in } W^{1,\infty}(\omega) . \qquad \square$$

In order to obtain optimality conditions for the problem (P_δ) we will prove the following result:

THEOREM 10: *The function $J_\delta : A(\omega) \to \mathbb{R}$ is of class C^1 and we have:*

$$DJ_\delta(h)(k) = J(k) + a_{h,k}(u(h), z_\delta) - m_{h,k}(z_\delta) ,$$

where $u(h)$ and z_δ verify:

$$A_h(u(h)) = l_h , \tag{36}$$

$$p_\delta = \frac{I - P_{K_h}}{\delta}(u(h)) , \tag{37}$$

$$A_h^*(z_\delta) + p_\delta = 0 , \tag{38}$$

where $A_h^* : V_{KL}^h(\Omega) \to (V_{KL}^h(\Omega))'$ is the adjoint operator of A_h.

Proof: We consider:

$$\phi_\delta : v_{KL}^h(\Omega) \to \phi_\delta(v) = \frac{1}{2\delta} \| v - P_{K_h}(v) \|_{KL}^2 \in \mathbb{R} .$$

It is known that ϕ_δ is of class C^1 (*cf.* Barbu-Precupanu [3], Brezis [7]) and its derivative is:

$$D\phi_\delta(v)(w) = \left(\frac{I - P_{K_h}}{\delta}(v), w \right)_{KL} ,$$

where $(\cdot, \cdot)_{KL}$ denotes the inner product in $V_{KL}^h(\Omega)$.

J is a linear operator and, consequently, of class C^∞ with derivative:

$$DJ(h)(k) = J(k) .$$

As we have shown in theorem 6 and remark 5, u is C^∞ with derivative:

$$Du(h)(k) = A_h^{-1}[m_{h,k} - A_{h,k}(u(h))] .$$

Thus, $J_\delta = J + \phi_\delta \circ u$ is of class C^1 with derivative:

$$DJ_\delta(h)(k) = J(k) + a_{h,k}(u(h), z_\delta) - m_{h,k}(z_\delta) . \quad \square$$

Remark 6: Condition (38) must be seen as:

$$A_h(v)(z_\delta) + (p_\delta, v)_{KL} = 0 , \quad \forall v \in V_{KL}^h(\Omega) . \quad \square$$

A point $h_\delta \in U_{ad}$ is said to be a stationary point for the problem (P_δ) if and only if it verifies:

$$DJ_\delta(h_\delta)(h - h_\delta) \geq 0 , \quad \forall h \in U_{ad} .$$

Since U_{ad} is a convex set, we have that if $h_\delta \in U_{ad}$ is a solution of (P_δ) then h_δ is a stationary point for (P_δ) (*cf.* Cea [11]). This fact supplies us the following necessary optimality condition:

THEOREM 11: *$h_\delta \in U_{ad}$ is a stationary point for (P_δ) if and only if there exist p_δ, u_δ, $z_\delta \in V_{KL}^h(\Omega)$ verifying:*

$$A_{h_\delta}(u_\delta) = l_{h_\delta} ,$$

$$p_\delta = \frac{I - P_{K_{h\delta}}}{\delta}(u_\delta) ,$$

$$A_{h_\delta}^*(z_\delta) + p_\delta = 0 ,$$

$$J(h) - J(h_\delta) + a_{h_\delta, h - h_\delta}(u_\delta, z_\delta) - m_{h_\delta, h - h_\delta}(z_\delta) \geq 0 , \quad \forall h \in U_{ad} .$$

ACKNOWLEDGMENTS

This work is part of the Human Capital and Mobility Program "Shells: Mathematical Modeling and Analysis, Scientific Computing" of the Commission of the European Communities (Contract No. ERBCHRXCT 940536) and also of Project "Análisis asintótico y simulación numérica en vigas elásticas" of Dirección General de Investigación Científica y Técnica (DGICYT) of Spain (PB92-0396). The authors are grateful to Prof. E. Casas for many helpful discussions which improved the content of this work.

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