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AN ELASTOHYDRODYNAMIC COUPLED PROBLEM BETWEEN A PIEZOVISCOUS REYNOLDS EQUATION AND A HINGED PLATE MODEL (*) (**)

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Abstract — *The elastohydrodynamic contact in journal-bearing devices with elastically deforming thin bearing is not a well known coupled problem in lubrication theory This work deals with the existence of solution of a mathematical model which rules the displacement of piezoviscous thin fluid films between an elastic surface and a rigid one The hydrodynamic part is governed by the Reynolds lubrication equation combined with the cavitation free boundary model of Elrod Adams and the Barus law for piezoviscous fluids The elastic deformation of one of the lubricated surfaces is taken into account by means of the hinged plate biharmonic equation, where the fluid pressure acts as an external force These deformations modify the width of the film in a direct way and another additional non linearity to the one of the free boundary can appear*

Finally, an iterative algorithm uncoupling the hydrodynamic part of the problem and the elastic one is suggested This method also includes finite element approximations and upwind techniques to discretize the lubrication model

Résumé — *Le contact élastohydrodynamique dans un palier avec coussinet mince et élastique est un problème couplé qui n'est pas très bien connu dans la théorie de la lubrification Ce travail traite de l'existence de solution d'un modèle mathématique du mouvement d'un film mince de fluide piézovisqueux qui se déplace entre le coussinet élastique et l'axe rigide La pression hydrodynamique du fluide vérifie l'équation de Reynolds avec un modèle de frontière libre du type Elrod Adams pour la cavitation et la loi de piézoviscosité de Barus La déformation est prise en compte par une équation biharmonique des plaques (hinged plate model) où la pression du fluide représente une force extérieure Ces déformations modifient l'épaisseur du film d'une façon directe et alors il faut ajouter une nouvelle non linéarité à celles dues à la piézoviscosité et à la frontière libre*

Finalement, un algorithme itératif qui découple les parties hydrodynamiques et élastiques du problème est proposé La méthode utilise des techniques décentrées et des approximations d'éléments finis pour la discrétisation du problème de départ

1. INTRODUCTION

The real importance of the mathematical models of lubrication processes lays on the great number of technical devices whose adequate running is based on the lubricated cylinder-cylinder, cylinder-plane, ball-plane or journal-bearing contacts (see [11]).

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Some industrial cases of special relevance are related to piezoviscous lubricating films in journal-bearing devices, with a bearing that may suffer any elastic deformation due to the lubricant pressure. From a mathematical point of view this fact leads to the analysis of an elastohydrodynamic model. This kind of devices consists on a cylindrical journal which rotates into a fixed cylindrical bearing, and it is separated from the journal by a very thin film of lubricating fluid which dumps heating and friction effects as well, the latter possible causers of damage in both contact surfaces. In order to get a proper performance of the device it is necessary to keep a fluid supply using an axial or circumferential slot in the journal-bearing pair (see the journal-bearing pair with circumferential supply in Figure 1).

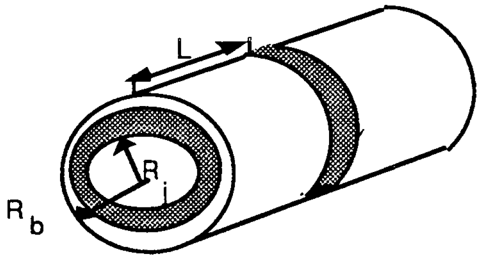


Figure 1. — Journal-bearing device.

The three main physical aspects to take into account which are coupled to each other and involved to the problem are : the fluid hydrodynamic displacement, the cavitation phenomenon inside the lubricant and the elastic behaviour of the contact surfaces.

A Newtonian fluid displacement, in laminar regime, follows Navier-Stokes equations. However when we study displacements in a preferential direction, that is, if a dimension (gap) is really small compared to the other two, Navier-Stokes equations are simplified in a great way leading us to the Reynolds bidimensional equation. This equation describes the behaviour of the pressure p of the lubricating fluid film for a gap h, a velocity field $u = (s, 0)$, a viscosity ν and a density ρ. Reynolds equation using two independent variables in the xy-plane takes the following form :

$$\frac{\partial}{\partial x}\left(\frac{\rho}{6\,\nu} h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{6\,\nu} h^3 \frac{\partial p}{\partial y}\right) = s \frac{\partial}{\partial x}(\rho h) . \tag{1.1}$$

The equation (1.1) is classically obtained by using heuristic reasoning (see [24], [17], [11], for example) and lately it has been deduced from Stokes equations (see [15], [5]) by using the more rigurous tool of asymptotic techniques.

The two features of the lubricating fluid are density and viscosity. The small fluctuation of density due to pressure applied to the usual lubricating fluids allows us to suppose an homogeneous fluid ($\rho = 1$). On the other hand, apart from the isoviscous case, the most used relation linking viscosity and pressure is the Barus law (see [11]):

$$\nu = \nu_0 \, e^{\alpha p} \tag{1.2}$$

where ν_0 represents viscosity at atmospheric pressure and α is the piezoviscosity coefficient, different for each lubricating fluid. Mathematically, considering piezoviscous laws gives place to a non-linearity in the diffusion part of Reynolds equation (1.1).

A pressure lower than saturation one causes the presence of air bubbles : this phenomenon is called cavitation. The cavitation leads to a fluid flow rupture and, mathematically, a free boundary problem is posed where we do not know *a priori* neither the lubricated zone (Ω^+) nor the cavitation region (Ω_0) (see *fig.* 2). Moreover, Reynolds equation (1.1) is no longer valid in the cavitation zone.

In [6] several mathematical models for cavitation applied to hydrodynamical problems are analyzed and the one called Reynolds model is the most used one. Essentially, this mathematical model considers the continuity of a given flux through the free boundary between both zones (fluid and cavitated) and leads to a variational inequality formulation. Unluckily, this model imposes a mathematical restriction on the location of the cavitated zone since it can only appear in the part of the domain where h is increasing. A more realistic model, here adopted, is the Elrod-Adams one because it avoids the previous restriction and also let us consider the starvation phenomenon appearing in several practical cases. This model is also based on a conservation law for the fluid flux through the free boundary but it introduces an additional unknown to the original problem : the saturation $\theta(x, y)$, which represents the lubricating fluid concentration in a neighbourhood of the point (x, y). In this way, concentration takes the value one for the fluid part and takes any other value between 0 and 1 for the cavitated region.

On the other hand, due to the elasticity of the bearing, high values of the pressure p can modify the gap h between the surfaces. A reasonable approximation to describe this elastic behaviour of the very thin bearing is to use the hinged plate biharmonic equation. In fact, it was Cimatti in [16] who first suggested this model in a coupled problem with a variational inequality formulation for cavitation in an isoviscous hydrodynamic lubricant. In [16], the existence and uniqueness of solution for the problem is stated. A mathematical and numerical study of the same problem but with the Elrod-Adams model for cavitation applied to isoviscous flows in journal-bearing devices is given in [7] and [19], respectively.

In this work, we generalize the results of [7] and [19] to piezoviscous fluids.

2. THE PROBLEM

Let $\Omega = (0, 2\pi) \times (0, L)$ be the domain of the problem ($L = 1$ is taken for simplicity). We define the sets that will appear in the equations as follows :

$$\begin{aligned} \Omega^+ &= \{(x, y) \in \Omega / p(x, y) > 0\} ; \\ \Omega_0 &= \{(x, y) \in \Omega / p(x, y) = 0\} ; \\ \Sigma &= \partial\Omega^+ \cap \Omega ; \\ \Gamma_{per} &= \{(x, y) \in \partial\Omega / x = 0 \text{ or } x = 2\pi\} ; \\ \Gamma_a &= \{(x, y) \in \partial\Omega / y = 1\} ; \\ \Gamma_b &= \{(x, y) \in \partial\Omega / y = 0\} \end{aligned} \tag{2.1}$$

where the boundary Γ_a denotes the supply groove (see *fig. 2*).

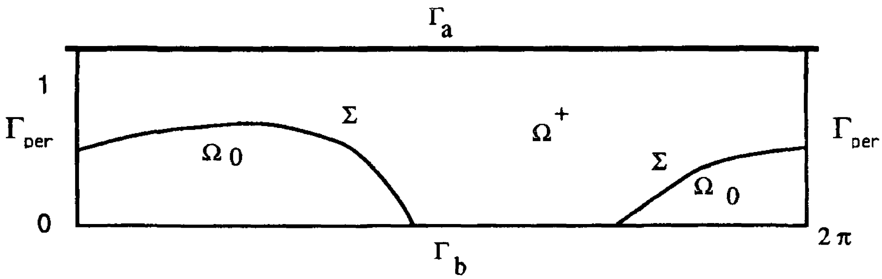


Figure 2. — Domain of the problem.

The dimensionless equations that describe the whole problem, considering cavitation and surfaces elasticity, are the following (see [7]) :

Find (p, θ, ω) such that

$$\frac{\partial}{\partial x}\left(\frac{(h_0+\omega)^3}{6\nu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{(h_0+\omega)^3}{6\nu}\frac{\partial p}{\partial y}\right) = s\,\frac{\partial(h_0+\omega)}{\partial x} ; \tag{2.2}$$

$$p > 0 \quad \text{and} \quad \theta = 1 \quad \text{in } \Omega^+$$

(2.3) $$\frac{\partial}{\partial x}(\theta(h_0+\omega))=0, \quad p=0 \quad \text{and} \quad 0 \leqslant \theta < 1 \text{ in } \Omega_0$$

(2.4) $$\frac{(h_0+\omega)^3}{6\,\nu}\frac{\partial p}{\partial n}=s(1-\theta)(h_0+\omega)\cos(n,i)\,; \quad p=0 \text{ on } \Sigma$$

(2.5) $$p=p_a \quad \text{on} \quad \Gamma_a\,; \quad p=0 \quad \text{on} \quad \Gamma_b\,;$$
$$p \quad \text{and} \quad \theta\; 2\,\pi-x \text{ periodic}$$

where the unknown quantities p, θ and ω are the fluid pressure and saturation, and the deformation of the bearing, respectively. The given constants of the problem are : the angular velocity of the journal s, the supply pressure p_a, the Barus law's coefficients ν_0 and α appearing in (1.2), and the gap between the rigid journal and bearing surfaces which is given by the function

(2.6) $$h_0(x)=c(1+\beta\cos x)$$

that depends on two parameters : the difference between the journal and the bearing radii c and the eccentricity $\beta \in (0,1)$. The involved sets in the equations are : the fluid region Ω^+, the cavitated zone Ω_0, the free boundary Σ between them, the supply boundary Γ_a, the boundary at ambient pressure Γ_b and the periodic boundary Γ_{per} where the point $(0,y)$ and $(2\,\pi,y)$ are physically identified in the device. Moreover, the vector n is the unitary normal one to the free boundary Σ pointing to Ω_0, and the vector i is the unitary one in the x-direction.

The function

(2.7) $$h(p)(x,y)=h_0(x)+\omega(p(x,y))$$

represents the total gap between the journal and the elastic bearing. It is the sum of the rigid gap (h_0) and the deformation (ω). This elastic component is obtained as solution of the fourth order elliptic problem :

(2.8) $$\eta\,\Delta^2\omega=p \quad \text{in} \quad \Omega$$

(2.9) $$\omega=\Delta\omega=0 \quad \text{on} \quad \Gamma_a \cup \Gamma_b\,;$$
$$\omega \quad \text{and} \quad \Delta\omega\; 2\,\pi-x \text{ periodic}$$

which corresponds to a hinged plate model. The flexure rigidity coefficient η is a given data.

Equations (2.2)-(2.4) and the boundary condition (2.5) constitute the Elrod-Adams cavitation model for the hydrodynamic lubrication. Elastohydrodynamic coupling is firstly caused by the fluid pressure dependence on the fluctuation of the gap (equation (2.2)) and secondly, by the effect of this fluid pressure in the elastic deformation (equation (2.8)).

The existence of solution of the problem (2.2)-(2.9) for the isoviscous case ($\nu = $ constant) has been proved in [7]. When the viscosity Barus law is used another non-linearity appears in the hydrodynamic problem and a generalized version of the mentioned result is not trivial. Recently, in [8] it has been demonstrated the existence of solution for a similar problem but replacing the elastic part, equations (2.8), (2.9), by an hertzian contact law. It is also in the hertzian contact domain that, in [20] and [18] it has been proved the existence of solution for elastohydrodynamic piezoviscous problems with mixed Dirichlet-Neumann boundary conditions.

The elastic part (equations (2.8)-(2.9)) will be treated by using a mixed formulation which involves the recursive solution of two second-order elliptic equations (see [22], for example) :

$$(2.10) \qquad -\Delta\psi = p \text{ in } \Omega\,; \quad \psi = 0 \text{ on } \Gamma_a \cup \Gamma_b\,; \quad \psi\ 2\pi - x \text{ periodic}$$

$$(2.11) \qquad -\eta\,\Delta\omega = \psi \text{ in } \Omega\,; \quad \omega = 0 \text{ on } \Gamma_a \cup \Gamma_b\,; \quad \omega\ 2\pi - x \text{ periodic}\,.$$

In the following paragraph the whole problem (2.2)-(2.7) and (2.10)-(2.11) is formulated in a variational form.

3. WEAK FORMULATION OF THE COUPLED PROBLEM

In order to prove the existence of solution for the previous problem, we consider a suitable variational formulation by using the classical spaces $L^p(\Omega)$ and the Sobolev spaces $H^m(\Omega)$. In fact, we define the functional spaces :

$$(3.1) \qquad V_a = \{\phi \in H^1(\Omega)/\phi = p_a \text{ on } \Gamma_a,\ \phi = 0 \text{ on } \Gamma_b \text{ and } \phi\ 2\pi - x \text{ periodic}\}$$

$$(3.2) \qquad V_0 = \{\phi \in H^1(\Omega)/\phi = 0 \text{ on } \Gamma_a \cup \Gamma_b \quad \text{and} \quad \phi\ 2\pi - x \text{ periodic}\}$$

$$(3.3) \qquad W_0 = \{\phi \in H^2(\Omega)/\phi = 0 \text{ on } \Gamma_a \cup \Gamma_b \quad \text{and} \quad \phi\ 2\pi - x \text{ periodic}\}$$

The variational problem is the following one :

Problem (P) : *Find* $(p, \theta, \psi, \omega) \in V_a \times L^{\infty}(\Omega) \times W_0 \times W_0$ *such that*

$$\int_{\Omega} (h_0 + \omega(p))^3 e^{-\alpha p} \nabla p \nabla \phi \, d\Omega = 6 \nu_0 s \int_{\Omega} (h_0 + \omega(p)) \theta \frac{\partial \phi}{\partial x} d\Omega , \quad \forall \phi \in V_0 \tag{3.4}$$

$$\theta \in H(p) \tag{3.5}$$

$$\int_{\Omega} \nabla \psi \nabla z \, d\Omega = \int_{\Omega} pz \, d\Omega , \quad \forall z \in W_0 \tag{3.6}$$

$$\int_{\Omega} \nabla \omega \nabla z \, d\Omega = \frac{1}{\eta} \int_{\Omega} \psi z \, d\Omega , \quad \forall z \in W_0 \tag{3.7}$$

where $H(p)$ refers to the multivalued Heaviside operator :

$$H(p) = \left\{ \begin{array}{cl} 1 & \text{if } p > 0 \\ [0, 1] & \text{if } p = 0 \\ 0 & \text{if } p < 0 \end{array} \right\} . \tag{3.8}$$

Remark 3.1 : The geometrical gap function $h_0(x)$ is 2π-periodic in the x-direction and it is bounded by :

$$0 < h_1 = c(1 - \beta) \leqslant h_0 \leqslant c(1 + \beta) = h_2 . \tag{3.9}$$

A justification of the variational formulation (problem (P)) in the sense that a regular solution is also a solution to the continuous problem (equations (2.2)-(2.9)) can be seen in [2] and [4] applied to the lubrication problem defined by equations (3.4)-(3.5) with $\omega = 0$. The justification for the elastic problem is trivial considering that Ω is an open set whose boundary $\partial\Omega$ is Lipschitz and taking $z \in D(\Omega)$ (infinite class and compact support in Ω) as test functions in the equations (3.6) and (3.7). The natural periodicity condition of the functions

$$\frac{\partial p}{\partial x}, \frac{\partial \psi}{\partial x}, \frac{\partial \omega}{\partial x} \text{ on the boundaries } x = 0 \; (\Gamma^0_{per}) \quad \text{and} \quad x = 2\pi \; (\Gamma^{2\pi}_{per}) \tag{3.10}$$

is implicitly taken into account by the variational formulations in a weak sense. That is,

$$\int_{\Gamma^0_{per}} \frac{\partial u}{\partial x} z \, dy = \int_{\Gamma^{2\pi}_{per}} \frac{\partial u}{\partial x} z \, dy \tag{3.11}$$

for $u = p$, with $z \in V_0$ and $u = \psi$ or $u = \omega$, with $z \in W_0$.

The difficulty of finding a result that justifies the existence of solution to the problem (3.4)-(3.7) lays basically on the double non-linearity of the equations (3.4)-(3.5). As a first approach we consider a family of problems related to the regularization of the Heaviside operator.

4. REGULARIZED PROBLEM

Let H_ε be the set of lipschitz functions depending on the positive parameter ε, given by

$$H_\varepsilon(t) = \left\{ \begin{array}{lll} 1 & \text{if} & t \geqslant \varepsilon \\ \frac{t}{\varepsilon} & \text{if} & 0 \leqslant t \leqslant \varepsilon \\ 0 & \text{if} & t \leqslant 0 \end{array} \right\} \tag{4.1}$$

and let the regularized problem be posed as follows :

Problem (P_ε) : *To find* $(p_\varepsilon, \psi_\varepsilon, \omega_\varepsilon)$ *in* $V_a \times W_0 \times W_0$ *such that*

$$\int_\Omega (h_0 + \omega_\varepsilon)^3 \, e^{-\alpha p_\varepsilon} \nabla p_\varepsilon \nabla \phi \, d\Omega = 6 \, \nu_0 \, s \int_\Omega (h_0 + \omega_\varepsilon) \, H_\varepsilon(p_\varepsilon) \frac{\partial \phi}{\partial x} \, d\Omega \,, \quad \forall \phi \in V_0 \tag{4.2}$$

$$\int_\Omega \nabla \psi_\varepsilon \nabla z \, d\Omega = \int_\Omega p_\varepsilon \, z \, d\Omega \,, \quad \forall z \in W_0 \tag{4.3}$$

$$\int_\Omega \nabla \omega_\varepsilon \nabla z \, d\Omega = \frac{1}{\eta} \int_\Omega \psi_\varepsilon \, z \, d\Omega \,, \quad \forall z \in W_0 \tag{4.4}$$

As in the work [8], in order to prove the existence of solution for the regularized problem (4.2)-(4.4), we use a fixed point method and we pose the following problem :

Let be $B_R = \{v \in L^2(\Omega)/0 \leqslant v \leqslant R, a.e. \text{ in } \Omega\}$ and let T be the operator defined from B_R into $L^2(\Omega)$ by :

$$T(p) = q \tag{4.5}$$

where $q \in V_a$ is the final solution of the problem :

$$\int_\Omega \nabla \psi(p) \nabla z \, d\Omega = \int_\Omega p \, z \, d\Omega , \quad \forall z \in W_0 \tag{4.6}$$

$$\int_\Omega \nabla \omega(p) \nabla z \, d\Omega = \frac{1}{\eta} \int_\Omega \psi(p) \, z \, d\Omega , \quad \forall z \in W_0 \tag{4.7}$$

$$\int_\Omega (h_0 + \omega(p))^3 e^{-\alpha p} \nabla q \nabla \phi \, d\Omega = 6 \, \nu_0 \, s \int_\Omega (h_0 + \omega(p)) \, H_\varepsilon(q) \frac{\partial \phi}{\partial x} d\Omega , \quad \forall \phi \in V_0 \tag{4.8}$$

where the ε index has been dropped to simplify the notation.

THEOREM 4.1 : *For a given function $p \in B_R$, it exists a unique triplet $(q, \psi(p), \omega(p))$ that is the solution of (4.6)-(4.8). Moreover, the estimates*

$$\| \psi(p) \|_{L^\infty(\Omega)} \leqslant K_1(\Omega) R \tag{4.9}$$

$$\| \omega(p) \|_{L^\infty(\Omega)} \leqslant K_2(\Omega) R \tag{4.10}$$

$$\| \nabla q \|_{L^2(\Omega)} \leqslant \frac{|\Omega|^{1/2} e^{\alpha R}}{h_1^2} (6 \, \nu_0 \, s + p_a(h_2 + K_2(\Omega) R)^2) \tag{4.11}$$

$$\| q \|_{L^\infty(\Omega)} \leqslant p_a + \frac{e^{\alpha R} \, 6 \, \nu_0 \, s C_1(\Omega)}{h_1^2} \tag{4.12}$$

hold with $K_1(\Omega)$, $K_2(\Omega)$ and $C_1(\Omega)$ constants that only depend on the domain Ω.

Proof : For a given $p \in B_R$ the existence and uniqueness of the pair $(\psi(p), \omega(p))$ solution of the equations (4.6)-(4.7) comes from classical results for the Dirichlet problem (see [21] between others). Moreover, it must be taken into account that periodicity conditions involve the continuity of the normal derivatives on the lateral boundaries of the domain as it is mentioned in the previous section.

Moreover, the following estimates hold as well :

$$\|\psi\|_{H^2(\Omega)} \leqslant C_2(\Omega)\,\|p\|_{L^2(\Omega)} \tag{4.13}$$

$$\|\omega\|_{H^4(\Omega)} \leqslant C_3(\Omega)\,\|\psi\|_{H^2(\Omega)} \tag{4.14}$$

and from both :

$$\|\omega\|_{H^4(\Omega)} \leqslant C_4(\Omega)\,\|p\|_{L^2(\Omega)} \tag{4.15}$$

with $C_2(\Omega)$, $C_3(\Omega)$, $C_4(\Omega)$ constants that only depend on the domain Ω.

Using Sobolev spaces inclusions (see [1] for example), it holds

$$\|\omega\|_{L^\infty(\Omega)} \leqslant K_3(\Omega)\,\|\omega\|_{H^4(\Omega)} \tag{4.16}$$

where $K_3(\Omega)$ is a constant that only depends on the space dimension and the domain Ω.

Joining (4.15) and (4.16) and taking $K_2(\Omega) = K_3(\Omega)\,C_4(\Omega)\,|\Omega|^{1/2}$, we obtain the estimate :

$$\|\omega\|_{L^\infty(\Omega)} \leqslant K_3(\Omega)\,C_4(\Omega)\,\|p\|_{L^2(\Omega)} \leqslant K_2(\Omega)\,R \tag{4.17}$$

and the same for ψ to obtain (4.9).

On the other hand, the existence and uniqueness of solution for the equation (4.8) comes from [4], due to the inequality

$$0 < h_1\,e^{-\alpha R} \leqslant (h_0 + \omega(p))\,e^{-\alpha p} \leqslant h_2 + K_2\,R \tag{4.18}$$

and the use of Lax-Milgram theorem. The non-negative behaviour of the solution q is also proved in [4].

Finally, we can obtain the estimates (4.11) and (4.12). In the first case, we consider as test function in (4.8) the function $\phi = q - p_a\,y$ which belongs to V_0. Using (4.18) and the fact that p is upper bounded by R as an element of B_R, it holds that

$$\begin{aligned} &\int_\Omega (h_0 + \omega(p))^3\,e^{-\alpha p}\,|\nabla q|^2\,d\Omega \\ &\leqslant 6\,\nu_0\,s \int_\Omega (h_0 + \omega(p))\,\left|\frac{\partial q}{\partial x}\right|\,d\Omega \\ &+ p_a \int_\Omega (h_0 + \omega(p))^3\,e^{-\alpha p}\,\left|\frac{\partial q}{\partial y}\right|^2\,d\Omega \end{aligned} \tag{4.19}$$

so

$$\begin{aligned} (4.20) \qquad & h_1 e^{-\alpha R} \int_\Omega (h_0 + \omega(p))^2 \, |\nabla q|^2 \, d\Omega \\ & \leqslant 6 \, \nu_0 \, s \int_\Omega (h_0 + \omega(p)) \left| \frac{\partial q}{\partial x} \right| d\Omega \\ & \quad + p_a (h_2 + K_2(\Omega) \, R)^2 \int_\Omega (h_0 + \omega(p)) \left| \frac{\partial q}{\partial y} \right| d\Omega \\ & \leqslant (6 \, \nu_0 \, s + p_a (h_2 + K_2(\Omega) \, R)^2) \int_\Omega (h_0 + \omega(p)) \, |\nabla q| \, d\Omega \, . \end{aligned}$$

Applying Hölder inequality, we conclude

$$\begin{aligned} (4.21) \qquad & \left(\int_\Omega (h_0 + \omega(p))^2 \, |\nabla q|^2 \, d\Omega \right)^{1/2} \\ & \leqslant \frac{e^{\alpha R}}{h_1} (6 \, \nu_0 \, s + p_a \, (h_2 + K_2(\Omega) \, R)^2) \, |\Omega|^{1/2} \end{aligned}$$

and therefore the estimate (4.11) is proved.

In order to obtain (4.12) we follow the classical techniques for $L^\infty(\Omega)$-estimates for elliptic variational equations (see [23] and [12], for example). In fact, for $k \geqslant p_a$, we take

$$(4.22) \qquad \xi = (q - k)^+ - (q + k)^- \in H_0^1(\Omega)$$

so that

$$\begin{aligned} (4.23) \qquad & \xi = 0 && \text{in } \{ |q| \leqslant k \} \\ & \xi = \pm (|q| - k) && \text{in } A(k) = \{ |q| > k \} \end{aligned}$$

Using $\phi = \xi$ as a test function in (4.8), the bound (3.9) and the Hölder inequality we deduce that

$$\begin{aligned} & h_1 e^{-\alpha R} \int_\Omega (h_0 + \omega(p))^2 \, |\nabla \xi|^2 \, d\Omega \\ (4.24) \qquad & \leqslant 6 \, \nu_0 \, s \int_{A(k)} (h_0 + \omega(p)) \left| \frac{\partial \xi}{\partial x} \right| d\Omega \\ & \leqslant 6 \, \nu_0 \, s \left(\int_{A(k)} (h_0 + \omega(p))^2 \left| \frac{\partial \xi}{\partial x} \right|^2 d\Omega \right)^{1/2} |A(k)|^{1/2} \end{aligned}$$

where $|A(k)|$ denotes the measure of the set $A(k)$. So,

$$\left(\int_{\Omega}(h_0+\omega(p))^2\,|\nabla\xi|^2\,d\Omega\right)^{1/2}\leqslant\frac{6\,\nu_0\,s\,e^{\alpha R}}{h_1}\,|A(k)|^{1/2}\tag{4.25}$$

and

$$\left(\int_{\Omega}|\nabla\xi|^2\,d\Omega\right)^{1/2}\leqslant\left(\frac{6\,\nu_0\,s\,e^{\alpha R}}{h_1^2}\right)|A(k)|^{1/2}.\tag{4.26}$$

For $g\geqslant k\geqslant p_a$ we have the inclusion $A(g)\subset A(k)$ and then

$$\begin{aligned}(g-k)^{r^*}|A(g)|&=\int_{A(g)}(g-k)^{r^*}\,d\Omega\\&\leqslant\int_{A(g)}(|q|-k)^{r^*}\,d\Omega\leqslant\int_{\Omega}(|\xi|)^{r^*}\,d\Omega.\end{aligned}\tag{4.27}$$

Choosing now $r^*>2$ and using the continuous inclusion of $H_0^1(\Omega)$ into $L^{r^*}(\Omega)$ it holds that

$$\begin{aligned}(g-k)^{r^*}|A(g)|&\leqslant C^{r^*}\left(\int_{A(g)}|\nabla\xi|^2\,d\Omega\right)^{\frac{r^*}{2}}\\&\leqslant C^{r^*}\left(\frac{6\,\nu_0\,s\,e^{\alpha R}}{h_1^2}\right)^{r^*}(|A(k)|)^{r^*/2}\end{aligned}\tag{4.28}$$

where C is the Sobolev constant of the inclusion that only depends on the domain Ω. As a result we obtain

$$|A(g)|\leqslant\left(\frac{6\,\nu_0\,s\,e^{\alpha R}\,C}{h_1^2(g-k)}\right)^{r^*}(|A(k)|)^{r^*/2}.\tag{4.29}$$

By using this estimate in the Lemma B.1 of Kinderlehrer-Stampacchia [23, pp. 63] we deduce (4.12) with $C_1(\Omega)$ given by

$$C_1(\Omega)=C(|\Omega|)^{1/2-1/r^*}\,2^{r^*/(r^*-2)}\tag{4.30}$$

as we wanted to prove. □

Once the existence and uniqueness of solution for the problem (4.6)-(4.8) has been studied, we proceed to analyze the existence of solution of the regularized problem (4.2)-(4.4) with the help of the following lemma :

LEMMA 4.1 : *The functional that assigns* $(h_0 + \omega(p)) \in L^2(\Omega)$ *to each* $p \in L^2(\Omega)$ *is continuous.*

Proof : Since h_0 does not depend on p, it is sufficient to verify the continuity for $\omega(p)$. For this, using the estimate (4.15) it holds the continuity of the functional of $L^2(\Omega)$ into $H^4(\Omega)$ and due to Sobolev inclusions we get the result as a particular case. □

THEOREM 4.2 : (Existence of solution for the regularized problem). *If parameters included in the problem (4.2)-(4.4) and in the estimates (3.9) and (4.12) verify :*

$$\frac{6\, v_0\, sC_1(\Omega)\, \alpha\, e^{\alpha p_a + 1}}{h_1^2} \leqslant 1 \tag{4.31}$$

then for each $\varepsilon > 0$ *there exists a solution* $(p_\varepsilon, \psi_\varepsilon, \omega_\varepsilon)$ *of (4.2)-(4.4) and it also holds the following estimates for* p_ε :

$$\|p_\varepsilon\|_{H^1(\Omega)} \leqslant C_5(\Omega) \tag{4.32}$$

$$\|p_\varepsilon\|_{L^\infty(\Omega)} \leqslant C_6(\Omega) \tag{4.33}$$

where $C_5(\Omega)$, $C_6(\Omega)$ *are constants that do not depend on* ε. *Moreover, the estimates (4.9), (4.10) are valid for* ψ_ε, ω_ε, *respectively.*

Proof : The existence of solution is obtained by using the Schauder fixed point theorem for the operator $T(p)$ defined in (4.5). In fact, the operator T of B_R into $L^2(\Omega)$ is continuous from Lemma 4.1 and the continuity of the functions

$$p \to e^{-\alpha p}\,; \quad p \to H_\varepsilon(p)\,. \tag{4.34}$$

On the other hand, the operator is compact as direct conclusion from the estimate (4.11) and the compact inclusion of $H^1(\Omega)$ into $L^2(\Omega)$. Finally in order to choose a positive real number R that holds that $T(B_R) \subset B_R$, we just have to select an R that verifies the inequality

$$p_a + \frac{e^{\alpha R}\, 6\, v_0\, sC_1(\Omega)}{h_1^2} \leqslant R \tag{4.35}$$

or equivalently

$$\frac{6\,\nu_0\, sC_1(\Omega)}{h_1^2} \leqslant (R - p_a)\, e^{-\alpha R}\,. \tag{4.36}$$

Nevertheless, as the maximum value of the function

$$\Phi(R) = (R - p_a)\, e^{-\alpha R} \tag{4.37}$$

is achieved in $R = p_a + \alpha^{-1}$ then the condition (4.31) ensures the existence of an R satisfying the required inequality (4.35).

Estimates (4.32) and (4.33) are easily obtained from (4.11) and (4.12), and in the same way the estimates (4.9) and (4.10) are gained for ψ_ε and ω_ε. □

5. EXISTENCE OF SOLUTION OF THE COUPLED PROBLEM

In order to complete the process, the existence of solution to the problem (3.4)-(3.7) is concluded as a limit of solutions in the regularized problems (4.2)-(4.4).

THEOREM 5.1 : *If the condition (4.31) is verified, then there exists a solution to the problem (3.4)-(3.7).*

Proof: From (4.31) and (4.33) we can deduce the existence of a subsequence still denoted by $\{p_\varepsilon\}_\varepsilon$, such that

$$\exists p \in V_a \cap L^\infty(\Omega), \quad p_\varepsilon \to p \text{ in } H^1(\Omega) \text{ weakly and } \quad L^\infty(\Omega) \text{ weakly-* }. \tag{5.1}$$

Moreover, since $0 \leqslant H_\varepsilon(p_\varepsilon) \leqslant 1$, it holds that

$$\exists \theta \in L^\infty(\Omega), \quad H_\varepsilon(p_\varepsilon) \to \theta \text{ in } \quad L^\infty(\Omega) \text{ weakly-* }. \tag{5.2}$$

On the other hand, from (4.13), (4.15) and (4.17), we can obtain

$$\exists \psi \in W_0 \cap L^\infty(\Omega), \quad \psi_\varepsilon \to \psi \text{ in } H^2(\Omega) \text{ weakly and } \quad L^\infty(\Omega) \text{ weakly-*} \tag{5.3}$$

$$\exists \omega \in W_0 \cap L^\infty(\Omega), \quad \omega_\varepsilon \to \omega \text{ in } H^2(\Omega) \text{ weakly and } \quad L^\infty(\Omega) \text{ weakly-* }. \tag{5.4}$$

Finally, by using the Lemma 4.1, the continuity of the functions that appear in (4.34) and the previous convergences (5.1)-(5.4), it is possible to obtain $(p, \theta, \psi, \omega)$ solution of the equations (3.4), (3.6) and (3.7) by passing to the limit in the regularized problem (4.2)-(4.4). If we also consider that

$$(5.5) \quad 0 \leqslant \int_\Omega p_\varepsilon(1 - H_\varepsilon(p_\varepsilon))\, d\Omega$$

$$= \int_{[p_\varepsilon < \varepsilon]} p_\varepsilon(1 - H_\varepsilon(p_\varepsilon))\, d\Omega \leqslant \int_{[p_\varepsilon < \varepsilon]} \varepsilon\, d\Omega \leqslant \varepsilon|\Omega|$$

we pass to the limit in ε and the equation (3.5) is deduced in an easy way, completing with this the whole proof. □

6. A NUMERICAL PROCEDURE

In order to compute a numerical approximation for the solution of equations (3.4)-(3.7), we first propose an explicit iterative algorithm that uncouples the hydrodynamic and the elastic parts of the problem (P). In this way, we follow the same scheme than in the proof of the existence of solution as we have already done in [19] for isoviscous fluids. Thus, we pose :

- *Hydrodynamic part :*
  - Step 0 : Start with arbitrary p^0, θ^0 and $\omega^0 = 0$.
  - Step $n+1$: Let ψ^n, ω^n, p^n be given, compute $(p^{n+1}, \theta^{n+1}) \in V_a \times L^\infty(\Omega)$ solution of

$$(6.1) \quad \int_\Omega (h_0 + \omega^n)^3\, e^{-\alpha p^n}\, \nabla p^{n+1}\, \nabla\phi\, d\Omega$$

$$= 6\, \nu_0\, s \int_\Omega (h_0 + \omega^n)\, \theta^{n+1} \frac{\partial\phi}{\partial x}\, d\Omega, \quad \forall\phi \in V_0$$

$$(6.2) \qquad \theta^{n+1} \in H(p^{n+1})\,.$$

- *Elastic part :*

Compute $(\psi^{n+1}, \omega^{n+1}) \in W_0 \times W_0$ by solving recursively the linear problems

$$(6.3) \qquad \int_\Omega \nabla\psi^{n+1}\, \nabla z\, d\Omega = \int_\Omega p^{n+1}\, z\, d\Omega, \quad \forall z \in W_0$$

$$(6.4) \qquad \int_\Omega \nabla\omega^{n+1}\, \nabla z\, d\Omega = \frac{1}{\eta} \int_\Omega \psi^{n+1}\, z\, d\Omega, \quad \forall z \in W_0\,.$$

Each iteration of the hydrodynamic part in this algorithm requires the numerical solution of the non linear free boundary lubrication subproblem (6.1)-(6.2). In a similar problem, in [9] it has been suggested an alternative method that consists on introducing an artificial dependence on time in all functions appearing in (6.1)-(6.2) and the artificial velocity field $v(x, y, t)$

$$\phi^*(x, y, t) = \phi(x, y) \quad \forall t ; \quad v(x, y, t) = (-1, 0) \tag{6.5}$$

to rewrite the second part of (6.1) in terms of the total derivative

$$\frac{D\phi^*}{Dt} = \frac{\partial \phi^*}{\partial t} + v \cdot \nabla \phi^* = -\frac{\partial \phi}{\partial x} . \tag{6.6}$$

Next, this total derivative is discretized by using an « upwind » scheme of characteristics :

$$\frac{D\phi}{Dt}(x, y) \approx \frac{\phi(x, y) - \phi(X^k(x, y))}{k} \tag{6.7}$$

where k plays the role of an artificial time step and $X^k(x, y)$ represents the point $X(x, y, t ; t-k)$ (the distinctive * has been suppressed to simplify the notation). In this process $\tau \to X(x, y, t ; \tau)$ denotes the trajectory of a particle of fluid placed in the point (x, y) at time t. That is, X is the solution of the final value problem

$$\frac{dX}{d\tau}(x, y, t ; \tau) = v(X(x, y, t ; \tau), \tau) ; \quad X(x, y, t ; t) = (x, y) \tag{6.8}$$

(notice that due to periodicity X transforms Ω into itself).

Thus, a reasonable approach of (6.1)-(6.2) is given by the following k-dependent family of equations :

$$\begin{aligned}(6.9)\int_\Omega & (h_0 + \omega^n)^3 e^{-\alpha p^n} \nabla p^{n+1} \nabla \phi \\ & + \frac{6 v_0 s}{k} \int_\Omega (h_0 + \omega^n) \theta^{n+1} (\phi - \phi_0 X^k) = 0, \quad \forall \phi \in V_0\end{aligned}$$

$$\theta^{n+1} \in H(p^{n+1}) . \tag{6.10}$$

Then we propose an iterative algorithm in time until reaching stationary state to solve it. That is,

— Step 0 : Start with arbitraries $(p^{n+1})^0$, $(\theta^{n+1})^0$ (for example p^n, θ^n)

— Step $k+1$: Compute the numerical solution of the non linear problem

$$\int_{\Omega} (h_0 + \omega^n)^3 e^{-\alpha p^n} \nabla (p^{n+1})^{k+1} \nabla \phi + \frac{6 v_0 s}{k} \int_{\Omega} (h_0 + \omega^n) (\theta^{n+1})^{k+1} \phi = \frac{6 v_0 s}{k} \int_{\Omega} (h_0 + \omega^n) (\theta^{n+1})^k \circ X^{-k} \phi , \quad \forall \phi \in V_0 \tag{6.11}$$

$$(\theta^{n+1})^{k+1} \in H((p^{n+1})^{k+1}) \tag{6.12}$$

where X^{-k} represents the inverse of X^k. The periodicity of the boundary conditions allows us to introduce properly this inverse function.

Finally, the non linearity behaviour of $H((p^{n+1})^{k+1})$ can be solved at each step of the previous algorithm by using a duality type algorithm analyzed in [10]. This algorithm is based on the definition of a new function r by :

$$r \in H(p) - \delta p , \quad \text{with } \delta \text{ an arbitrary positive real number} , \tag{6.13}$$

and the equivalence result of the expression (6.13) with the identity :

$$r = (H - \delta I)_\lambda (p + \lambda r) \tag{6.14}$$

where $(H - \delta I)_\lambda$ is the Yosida approximation with parameter λ of the operator $(H - \delta I)$, I being the identity operator (see [10] for details).

In this way, the final algorithm searchs the solution of the linear equation

$$\int_{\Omega} (h_0 + \omega^n)^3 e^{-\alpha p^n} \nabla ((p^{n+1})^{k+1})^{j+1} \nabla \phi + \frac{6 v_0 s \delta}{k} \int_{\Omega} (h_0 + \omega^n)((p^{n+1})^{k+1})^{j+1} \phi = \frac{6 v_0 s}{k} \int_{\Omega} (h_0 + \omega^n) (\theta^{n+1})^k \circ X^{-k} \phi - \frac{6 v_0 s}{k} \int_{\Omega} (h_0 + \omega^n) r^j \phi, \quad \forall \phi \in V_0 \tag{6.15}$$

in each iteration and updates the Lagrange multiplier r^{j+1} by using the expression

$$r^{j+1} = (H - \delta I)_\lambda (((p^{n+1})^{k+1})^{j+1} + \lambda r^j) . \tag{6.16}$$

The convergence of this algorithm is analyzed in [10] choosing λ, δ such that $\lambda\delta \leqslant 0.5$.

The space discretization of (6.15) is made with Lagrange triangular finite elements of degree one and classical results of error estimates about finite element approximations for elliptic problems are valid (see [13]). However, when equation (6.15) is coupled with equations (2.8)-(2.9) the error estimate for the mixed finite element solution of equations (6.3)-(6.4) obtained in [14] is not valid for Lagrange finite elements of degree one to approximate W_0. As a result, it would be interesting to study the nature and behaviour of finite element approximations of degree one for ω in the global discretization scheme developed here. Other different choices such as the use of quadratic elements for ω in (6.4) and elements of degree one for ψ in equation (6.3) may further improve the results of approximation, but this is nothing but a conjecture (see [3]).

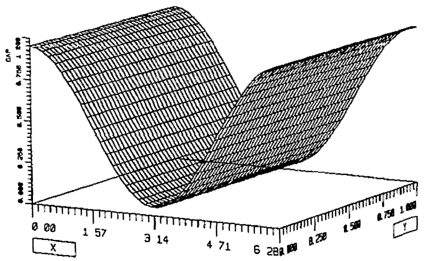
Summarizing, the advantages of this characteristics-duality iterative algorithm are reflected in the matrix system obtained in (6.15) that neither depends on k nor on j, it has a symmetric positive definite matrix and consequently it is also computed and factorized only once at each iteration of the fixed point algorithm. Moreover, the trajectories X can be automatically computed from the velocity field v only once.

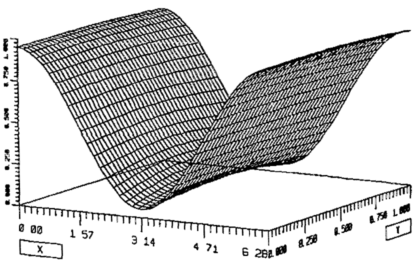
7. NUMERICAL EXAMPLES

The numerical method developed in the previous paragraph has been applied to solve some representative academic examples, those that better characterize the physical nature of the problem. So, we have tested the evolution of the solution for different flexure rigidity coefficients of the bearing and a very large parameter of eccentricity to show the physically expected behaviour of the device : the capability of greater deformations leads to lower maxima for the pressure.

The domain $\Omega = (0, 2\pi) \times (0, 1)$ is discretized using a uniform mesh with 800 triangular elements and 459 nodes which corresponds to 50 and 8 divisions in the x and y directions respectively. The physical data appearing in equations (1.2), (2.2), (2.5) and (2.6) are chosen to be $v_0 = 1/6$, $\alpha = 0.025$, $s = 1$, $p_a = 1$, $c = 0.5$ and $\beta = 0.9$, respectively. The parameters involved in the numerical algorithm were set to $k = \Delta x = 0.12566$, $\delta = 1$ and $\lambda = 0.5$.

Figures 3 and 4 show the numerical approximation of the gap $(h_0 + \omega)$ and the pressure p respectively for the rigid bearing case in the upper graphic and several flexure rigidity coefficients η in the elastic bearing. Notice that figure 4 represents the pressure $p(x, y)$ cuts for y constant and equal to $1 - i\,\Delta x$, $i = 0, 1, \ldots, 8$ where the straight horizontal line corresponds to the supply pressure $p_a = 1$, at the boundary $i = 0$.





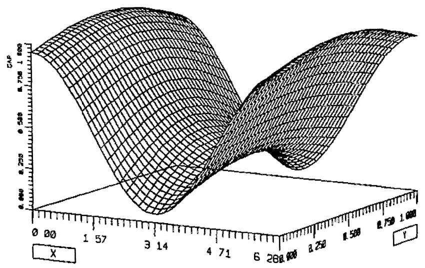


Figure 3. — Journal-bearing gap for $\eta = 1\,000$ (rigid case), $\eta = 0.5$ and $\eta = 0.05$ (from top to bottom).

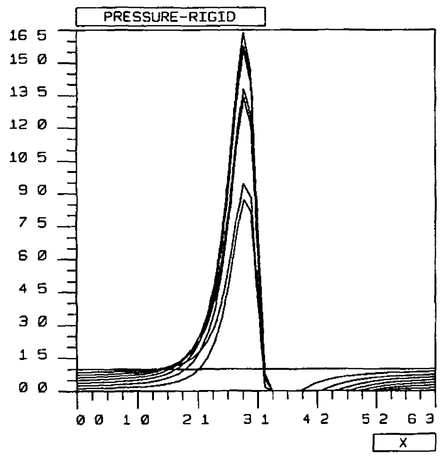
The numerical tests point out that as soon as the deformability of the bearing increases (that is η) the values of the pressure are lower in the contact region. Moreover, figure 3 shows that the gap between the journal and the bearing follows a monotone behaviour with respect to η.

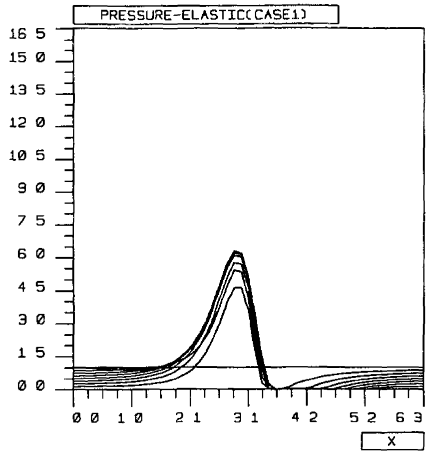
An L^∞-relative error in pressure and gap is taken to be less than $5 \cdot 10^{-3}$ as stopping test. So, the total number of iterations in pressure of the algorithm oscillates between 103 for the rigid case and 398 for the elastic one with $\eta = 0.05$.

8. CONCLUSIONS

In this work, we pose an elastohydrodynamic problem in a journal-bearing device. The proposed mathematical model involves the coupling between the hydrodynamic free-boundary model of Elrod-Adams and the fourth order hinged plate equation.

The consideration of piezoviscous lubricants implies a generalization in relation to the previous works in the literature. The theoretical proof of existence requires the use of additional techniques based on $L^\infty(\Omega)$ estimates. Moreover, the constructive character of the proof provides an idea for the numerical algorithm which has been used to compute the approximated solution.





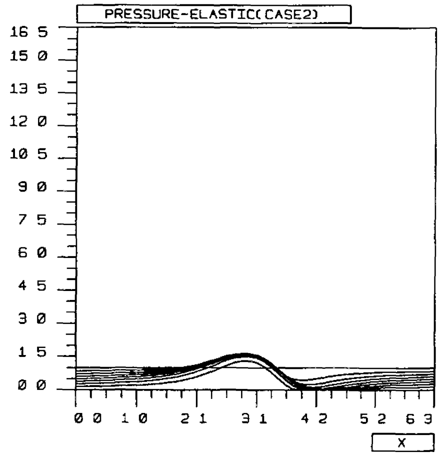


Figure 4. — Bearing pressure for $\eta = 1\,000$ (rigid case), $\eta = 0\,5$ and $\eta = 0\,05$ (from top to bottom).

In order to test the good performance of the complex global algorithm several academic examples have been solved. The numerical results obtained agree with the qualitative expected behaviour for different running conditions of the device.

One interesting possible research line which is now being implemented includes in the model a load restriction. In this way, the eccentricity of the journal-bearing represents an additional unknown that is computed from the balance between the imposed load and the hydrodynamic load.

REFERENCES

[1] R. A. ADAMS, 1975, Sobolev Spaces. *Academic Press.*

[2] S. ALVAREZ, 1986, Problemas de Frontera Libre en Teoría de Lubrificación. *Thesis, University Complutense of Madrid, Spain.*

[3] S. BALASUNDARAM, P. K. BHATTACHARYYA, 1982, A mixed finite element method for the Dirichlet problem of fourth order elliptic operators with variable coefficients. In : Finite Element Methods in Flow Problems (Ed. T. Kawai) *Tokyo Univ. Press.*

[4] G. BAYADA, M. CHAMBAT, 1984, Existence and uniqueness for a lubrication problem with nonregular conditions on the free boundary. *Boll. UMI.* **6,** 3B, 543-557.

[5] G. BAYADA, M. CHAMBAT, 1986, The transition between the Stokes equation and the Reynolds equation : a mathematical proof. *Appl. Math. Opt.,* **14,** 73-93.

[6] G. BAYADA, M. CHAMBAT, 1986, Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique. *J. of Theor. and Appl. Mech.,* **5,** 703-729.

[7] G. BAYADA, J. DURANY, C. VÁZQUEZ, 1995, Existence of solution for a lubrication problem in elastic journal bearing devices with thin bearing. *Math. Meth. in the Appl. Sc.,* **18,** 255-266.

[8] G. BAYADA, M. EL ALAOUI, C. VÁZQUEZ, 1996, Existence of solution for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation. *Eur. J. of Appl. Math.*, **7,** 63-73.

[9] A. BERMÚDEZ, J. DURANY, 1989, Numerical solution of cavitation problems in lubrication. *Comp. Meth. in Appl. Mech. and Eng.,* **75,** 457-466.

[10] A. BERMÚDEZ, C. MORENO, 1981, Duality methods for solving variational inequalities. *Comp. Math. with Appl.,* **7,** 43-58.

[11] A. CAMERON, 1981, Basic Lubrication Theory. *Ellis Horwood.*

[12] M. CHIPOT, 1984, Variational Inequalities and Flow on Porous Media. *Appl. Math. Sc. Series 52. Springer-Verlag.*

[13] P. CIARLET, 1978, The Finite Element Method for Elliptic Problems. *North-Holland.*

[14] P. CIARLET, P. A. RAVIART, 1974, A mixed finite element method for the biharmonic equation. In : Mathematical Aspects of Finite Elements in Partial Differential Equations. (Ed. C. de Boor), *Academic Press,* 125-145.

[15] G. CIMATTI, 1983, How the Reynolds equation is related to the Stokes equation. *Appl. Math. Opt.,* **10,** 223-248.

[16] G. CIMATTI, 1986, Existence and uniqueness for nonlinear Reynolds equations. *Int. J. Eng. Sc.,* **24,** N.5, 827-834.

[17] D. DOWSON, C. M. TAYLOR, 1979, Cavitation in bearings. *Ann. Rev. Fluid Mech.,* 35-66.

[18] J. DURANY, G. GARCÍA, C. VÁZQUEZ, 1996, A mixed Dirichlet-Neumann problem for a nonlinear Reynolds equation in elastohydrodynamic piezoviscous lubrication. *Proc. of the Edinb. Math Soc.*, **39**, 151-162

[19] J. DURANY, G GARCÍA, C VÁZQUEZ, 1996, Numerical computation of free boundary problems in elastohydrodynamic lubrication *Appl. Math. Mod*, **20**, 104-113.

[20] J. DURANY, C. VÁZQUEZ, 1994, Mathematical analysis of an elastohydrodynamic lubrication problem with cavitation *Appl Anal* Vol. 53, N. 1, 135-142.

[21] V. GIRAULT, P. A. RAVIART, 1979, Finite Element Approximation of the Navier-Stokes Equations. *Lecture Notes in Mathematics, 749, Springer*

[22] R GLOWINSKI, O PIRONNEAU, 1979, Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem *SIAM Review*, **21**, 167-212.

[23] D KINDERLEHRER, G STAMPACCHIA, 1980, An Introduction to Variational Inequalities and their Applications *Academic Press.*

[24] O REYNOLDS, 1886, On the theory of lubrication and its applications to M. Beauchamp Tower's experiments *Phil. Trans. Roy Soc London*, A117, 157-234