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INTERPOLATION IN HARMONIC HILBERT SPACES (*) (**)

by Franz-J. DELVOS (1)

Abstract — *Babuska introduced the concept of periodic Hilbert space for studying universally optimal quadrature formulas Prager continued these investigations and discovered the relationship between optimal approximation of linear functionals on periodic Hilbert spaces and minimum norm interpolation (optimal periodic interpolation) In the case of a uniform mesh methods of periodic interpolation by translation are applicable and relations to periodic spline interpolation and approximation have been studied*

It is the objective of this paper to introduce the concept of harmonic Hilbert space as an extension of periodic Hilbert space to study interpolation problems on an infinite uniform mesh and for l^2 data This gives a unified variational approach to infinite interpolation with sinc-functions, splines, and holomorphic functions

Résumé — *Babuska a introduit le concept de l'espace hilbertien périodique pour étudier des formules de quadrature universellement optimales Prager a continué ces recherches et il a découvert la relation entre l'approximation optimale des fonctionnelles linéaires sur des espaces hilbertiens périodiques et l'interpolation basée sur des espaces hilbertiens périodiques et l'interpolation basée sur la minimalisation de la norme (interpolation périodique optimale) Dans le cas d'un réseau uniforme, on peut appliquer des méthodes d'interpolation périodique par translation , des relations avec l'interpolation spline périodique et l'approximation ont été étudiées*

Le but de cet article est l'introduction du concept de l'espace hilbertien harmonique en tant qu'extension de l'espace hilbertien pour étudier des problèmes d'interpolation dans un réseau uniforme infini et pour des dates l^2 Ceci établit un accès variationnel unifié à l'interpolation infinie par des fonctions sinc, spline et holomorphiques

Babuska introduced the concept of periodic Hilbert space for studying universally optimal quadrature formulas. Prager continued these investigations and discovered the relationship between optimal approximation of linear functionals on periodic Hilbert spaces and minimum norm interpolation (optimal periodic interpolation). In the case of a uniform mesh methods of periodic interpolation by translation are applicable and relations to periodic spline interpolation and approximation have been studied.

It is the objective of this paper to introduce the concept of harmonic Hilbert space as an extension of periodic Hilbert space to study interpolation problems

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on an infinite uniform mesh and for l_2-data. This gives a unified variational approach to infinite interpolation with sinc-functions, splines, and holomorphic functions.

1. PERIODIC HILBERT SPACES

To motivate the construction of harmonic Hilbert spaces we briefly recall the construction of periodic Hilbert spaces and thereby extend the approach of Babuska and Prager.

A periodic Hilbert space H_d is defined by a nonnegative even and summable sequence $(d_k)_{k \in \mathbf{Z}}$, the *defining sequence :*

$$d_{-k} = d_k, k \in \mathbf{N}, \quad d_k \geq 0, k \in \mathbf{Z}, \quad 0 < \sum_{k=-\infty}^{\infty} d_k < \infty .$$

Let $L^2_{2\pi}$ be the linear space of square summable periodic functions with inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt .$$

Then

$$H_d = \left\{ f \in L^2_{2\pi} : \sum_{k=-\infty}^{\infty} |(f, e_k)|^2 / d_k < \infty \right\}$$

is the *periodic Hilbert space* with respect to $d = (d_k)_{k \in \mathbf{Z}}$. The inner product is given by

$$(f, g)_d = \sum_{k=-\infty}^{\infty} (f, e_k)(e_k, g)/d_k$$

where $e_k(t) = \exp(ikt)$, $k \in \mathbf{Z}$.

Note that $d_k = 0$ implies $(f, e_k) = 0$ for any $f \in H_d$ and by definition $|(f, e_k)|^2 / d_k = 0$.

$C_{2\pi}$ denotes the Banach algebra of continuous complex-valued periodic functions equipped with the maximum norm $\|f\|_\infty = \max\{|f(t)| : 0 \leq t < 2\pi\}$.

We have

$$C_{2\pi} \subseteq L^2_{2\pi}, \quad \|f\| \leq \|f\|_\infty .$$

The Wiener algebra $A_{2\pi}$ consists of those functions from $C_{2\pi}$ having an absolutely convergent Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} (f, e_k)\, e^{ikx}.$$

$A_{2\pi}$ is also a Banach algebra with respect to the norm $\|f\|_a = \sum_{k=-\infty}^{\infty} |(f, e_k)|$.

The following relations hold :

$$A_{2\pi} \subseteq C_{2\pi}, \quad \|f\|_\infty \leqslant \|f\|_a.$$

It is an important fact that any periodic Hilbert space is contained in the Wiener algebra :

$$(1.1) \quad H_d \subseteq A_{2\pi} \subseteq C_{2\pi} \subseteq L^2_{2\pi}, \quad \|f\|_2 \leqslant \|f\|_\infty \leqslant \|f\|_a \leqslant \|f\|_d \cdot \left(\sum_{k=-\infty}^{\infty} d_k\right)^{1/2}.$$

The periodic Hilbert space H_d is related to the *generating function*

$$(1.2) \qquad \psi(x) = \sum_{k=-\infty}^{\infty} d_k\, e^{ikx}.$$

Any periodic Hilbert space is translation invariant

$$(f(\cdot - a), g(\cdot - a))_d = (f, g)_d, \quad (a \in \mathbf{R}).$$

Using the generating function we have

$$(1.3) \qquad f(x) = (f, \psi(\cdot - x))_d.$$

This shows that any periodic Hilbert space possesses a reproducing kernel (Young, p. 15).

We discuss now the problem of minimum norm interpolation in the reproducing kernel Hilbert space H_d. Let $0 \leqslant x_0 < \dots < x_{n-1} < 2\pi$ be such that $\langle \psi(\cdot - x_j) : 0 \leqslant j < n \rangle$ is an n-dimensional subspace of H_d. It is well known from theory of reproducing kernel Hilbert spaces that for a given function $f \in H_d$ there is a unique function

$$Q_n(f) = \sum_{j=0}^{n-1} c_j\, \psi(\cdot - x_j)$$

which interpolates f at $x_0, \ldots, x_{n-1}$ and which is also the unique function of minimum norm in the linear manifold of $g \in H_d$ satisfying $g(x_j) = f(x_j)$, $0 \leq j < n-1$. Since

$$(f - Q_n(f), \psi(\cdot - x_j))_d = 0, \quad 0 \leq j < n-1$$

the minimum norm interpolant $Q_n(f)$ is also the best approximation of f in the subspace $\langle \psi(\cdot - x_j) : 0 \leq j < n \rangle$. Thus the minimum norm interpolation projection is also uniquely characterized as the orthogonal projector Q_n satisfying

$$\operatorname{im}(Q_n) = \langle \psi(\cdot - x_j) : 0 \leq j < n \rangle,$$

$$\ker(Q_n) = \{ f \in H_d : f(x_j) = 0, 0 \leq j < n \}. \tag{1.4}$$

This relation between minimum interpolation and best approximation and the associated orthogonal projector will be of importance in infinite interpolation problems as discussed in Section 4.

Note that in the finite interpolation problem the assumption $\dim(\operatorname{im}(Q_n)) = n$ implies that for any data y_j, $0 \leq j < n$ there is a unique minimum norm interpolant $g_n \in H_d$ satisfying $g_n(x_j) = y_j$, $0 \leq j < n$.

In the following we assume that $x_j = \frac{2\pi}{n} j$, $0 \leq j < n$, which implies that the space of interpolating functions $\operatorname{im}(Q_n) = \ker(Q_n)^{\perp}$ is translation invariant with respect to $\frac{2\pi}{n}$.

If the *fundamental Lagrange interpolant* γ_n of periodic minimum norm interpolation exists it is given by

$$\gamma_n(x) = \sum_{k=0}^{n-1} a_k \, \psi\left(x - \frac{2\pi}{n} k\right).$$

Then

$$\gamma_n(x) = \sum_{k=0}^{n-1} a_k \sum_{m=-\infty}^{\infty} d_m \, e_m\left(-\frac{2\pi}{n} k\right) e_m(x)$$

$$= \sum_{m=-\infty}^{\infty} d_m \left(\sum_{k=0}^{n-1} a_k \, e_m\left(-\frac{2\pi}{n} k\right)\right) e_m(x),$$

$$\gamma_n(x) = \sum_{m=-\infty}^{\infty} d_m A_m \, e_m(x),$$

$$A_m = \sum_{k=0}^{n-1} a_k \, e_m\left(-\frac{2\pi}{n} k\right), \quad A_{rn+m} = A_m, \ (r, m \in \mathbf{Z}).$$

Thus the interpolation conditions

$$\gamma_n\left(\frac{2\pi}{n}j\right)=\delta_{0,j}, \quad (0\leq j<n) \tag{1.5}$$

are valid if and only if

$$\sum_{j=0}^{n-1}\left(\sum_{s=-\infty}^{\infty}d_{j+sn}\right)A_j\,e_j\left(\frac{2\pi}{n}k\right)=\delta_{0,k}, \quad (0\leq k<n).$$

The properties of the discrete Fourier transform imply

$$A_j=\frac{1}{n}\frac{1}{\sum_{s=-\infty}^{\infty}d_{j+sn}}, \ (0\leq j<n).$$

Thus a necessary and sufficient condition for the existence of the *fundamental Lagrange interpolant* γ_n of periodic minimum norm interpolation is

$$\sum_{s=-\infty}^{\infty}d_{j+sn}=\frac{1}{n}\sum_{k=0}^{n-1}\psi\left(\frac{2\pi}{n}k\right)e_{-j}\left(\frac{2\pi}{n}k\right)\neq 0, \quad (0\leq j<n) \tag{1.6}$$

in which case we have

$$\gamma_n(x)=\frac{1}{n}\sum_{k=-\infty}^{\infty}\frac{d_k}{\sum_{s=-\infty}^{\infty}d_{k+sn}}e^{ikx}. \tag{1.7}$$

The minimum norm interpolant is then given by the

(1.8)

$$Q_n(f)(x)=\sum_{j=0}^{n-1}f\left(\frac{2\pi}{n}j\right)\gamma_n\left(x-\frac{2\pi}{n}j\right) \ (\textit{Lagrange representation formula}).$$

We conclude this brief revision of periodic Hilbert spaces by noting that the sequence

$$d_k=\max\{1,k^2\}^{-r}.$$

defines the periodic Sobolev space $W^r_{2\pi}$. The generating function is

$$\psi(t)=1+(-1)^r B_{2r}(t), \quad B_q(t)=\sum_{k\neq 0}(ik)^{-q}\exp(ikt)$$

$B_q(t)$ is the Bernoulli function of degree $q\in\mathbf{N}$. Minimum norm interpolation yields in this case periodic monosplines as interpolating functions (see Locher and Delvos).

As a second instance we consider the sequence

$$d_k = \max\{1, k^2\}^{-r} \frac{(1 - (-1)^k)}{2}.$$

Then the generating function is

$$\frac{(-1)^r}{2}(B_{2r}(x) - B_{2r}(x + \pi)) = (-1)^r E_{2r-1}(x)$$

which is the Euler function (antiperiodic extension of the Euler polynomial) of degree $2r - 1$.

Minimum norm interpolation in the corresponding periodic Hilbert space leads to interpolation of antiperiodic functions with antiperiodic splines (see Delvos 1991).

2. HARMONIC HILBERT SPACES

The objective of construction of harmonic Hilbert spaces is to extend minimum norm interpolation of periodic functions to non periodic functions being defined on the real line $\mathbf{R}$ on an infinite uniform mesh $\frac{\pi}{b}k$, $k \in \mathbf{Z}$, $b > 0$. In contrast to periodic minimum norm interpolation the problems of existence and uniqueness are enlarged by the question of *convergence of interpolation series* due to the infiniteness of the interpolation data. On the other hand the common idea behind both interpolation methods is the projection theorem in Hilbert spaces. The basic tool for defining harmonic Hilbert spaces $H_D(\mathbf{R})$ is the Fourier integral for which we mainly refer to Chandrasekharan, Garnir, and Katznelson.

Let $D \in L_1(\mathbf{R}) \cap L_\infty(\mathbf{R})$ be non negative and even, i.e. D is a density function of a distribution function. Let F be a measurable function such that

$$F/\sqrt{D} \in L_2(\mathbf{R}).$$

Since $\sqrt{D} \in L_2(\mathbf{R})$ an application of the Schwarz inequality yields

$$\|F\|_1 \leq \|F/\sqrt{D}\|_2(\|D\|_1)^{1/2}. \tag{2.1}$$

Thus the Fourier integral of F

$$f(x) = \int_{-\infty}^{\infty} F(t) \exp(itx)\, dt$$

is a function in $C_0(\mathbf{R})$ where $C_0(\mathbf{R})$ is the Banach algebra of continuous complex valued functions on $\mathbf{R}$ which vanish at infinity. The norm of $C_0(\mathbf{R})$ is

$$\|f\|_\infty = \max\{|f(x)| : x \in \mathbf{R}\}.$$

$C_0(\mathbf{R})$ corresponds to $C_{2\pi}$ in the periodic case. The analogon of the Wiener algebra $A_{2\pi}$ is the linear space $A(\mathbf{R})$ of functions which possess an absolutly integrable Fourier integral representation. The norm of $A(\mathbf{R})$ is given by

$$\|f\|_a = \int_{-\infty}^{\infty} |F(t)|\, dt.$$

We have the relations

$$A(\mathbf{R}) \subseteq C_0(\mathbf{R}), \quad \|f\|_\infty \leqslant \|f\|_a.$$

The *harmonic Hilbert space* $H_D(\mathbf{R})$ is the subspace of functions $f \in A(\mathbf{R})$ such that $F/\sqrt{D} \in L_2(\mathbf{R})$:

$$f(x) = \int_{-\infty}^{\infty} F(t) \exp(itx)\, dt, \quad \int_{-\infty}^{\infty} |F(t)|^2 / D(t)\, dt < \infty.$$

$H_D(\mathbf{R})$ is a Hilbert space with inner product

$$(f, g)_D = \int_{-\infty}^{\infty} F(t)\, \overline{G(t)} / D(t)\, dt.$$

D is the *defining function* of the harmonic Hilbert space $H_D(\mathbf{R})$ in analogy to the periodic Hilbert space H_d. D corresponds to the defining sequence d. Clearly, $H_D(\mathbf{R})$ is translation invariant:

$$(f(\cdot - a), g(\cdot - a))_D = (f, g)_D, \quad a \in \mathbf{R}.$$

We have the relations

(2.2)
$$H_D(\mathbf{R}) \subseteq A(\mathbf{R}) \subseteq C_0(\mathbf{R}), \quad \|f\|_\infty \leqslant \|f\|_a \leqslant \|f\|_D \cdot \left(\int_{-\infty}^{\infty} D(t)\, dt\right)^{1/2}.$$

The Fourier integral of D defines the *generating function* of $H_D(\mathbf{R})$

$$d(x) = \int_{-\infty}^{\infty} D(t) \exp(ixt)\, dt$$

which corresponds to the generating function ψ of the periodic Hilbert space H_d We have

$$\| d(\quad - a) \|_D^2 = \int_{-\infty}^{\infty} D(t)\, dt\,, \quad (a \in \mathbf{R})$$

As in the periodic case we have the important relation

(2 3) $$f(x) = (f, d(\quad - x))_D, \quad x \in \mathbf{R}$$

This shows that any harmonic Hilbert space is a *reproducing kernel Hilbert space*

PROPOSITION 2 1 *Let* $f \in H_D(\mathbf{R})$ *Then* $F \in L_2(\mathbf{R})$ *and*

(2 4) $$\|f\|_2 = \sqrt{2\pi}\, \|F\|_2 \leq \sqrt{2\pi \|D\|_\infty}\, \|f\|_D$$

Proof By definition we have $F/\sqrt{D} = G \in L_2(\mathbf{R})$ Since $\sqrt{D} \in L_\infty(\mathbf{R})$ it follows

$$G\sqrt{D} = F \in L_2(\mathbf{R})$$

Moreover we can conclude

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt = \int_{-\infty}^{\infty} D(t)\, |F(t)|^2 / D(t)\, dt \leq \|D\|_\infty \|f\|_D^2$$

By Plancherel's theorem we have

$$F(t) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-itx)\, dx\,, \quad 2\pi \int_{-\infty}^{\infty} |F(t)|^2\, dt = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

which completes the proof of Proposition 2 1

As a first example we treat *Paley-Wiener spaces* as harmonic Hilbert spaces Let

$$D = \chi_{[-b\ b]} \quad b > 0$$

In this case we introduce the special notation $PW_b = H_D(\mathbf{R})$ For any $f \in PW_b$ we have

$$f(x) = \int_{-b}^{b} F(t) \exp(ixt)\, dt\,, \quad F \in L_2(\mathbf{R})\,, \quad F(t) = 0\ (|t| > b)$$

The inner product of PW_b is given by

$$(f, g)_b = \int_{-b}^{b} F(t)\,\overline{G(t)}\,dt\,.$$

The generating function of PW_b is given by

$$h_b(x) = \int_{-b}^{b} \exp(ixt)\,dt = \frac{2\sin(bx)}{x} = 2\,b\,\mathrm{sinc}(bx)\,.$$

3. INTERPOLATION IN PALEY WIENER SPACES

We collect some well known facts on cardinal interpolation. By its definition PW_b is isomorphic and isometric to $L_2[-b, b]$. Given $f \in PW_b$ there is a unique $F \in L_2[-b, b]$ such that

$$f(x) = \int_{-b}^{b} F(t)\exp(ixt)\,dt\,.$$

Then we have

$$f(x) = (f, h_b(\,\cdot\, - x))_b, \quad x \in R\,, \quad h_b(x) = \frac{2\sin(bx)}{x}\,. \tag{3.1}$$

This implies $\left(h_b\left(\,\cdot - \frac{\pi k}{b}\right), \left(\,\cdot - \frac{\pi l}{b}\right)\right)_b = 2\,b\delta_{k,l}$, k, $l \in Z$, i.e., the functions $h_b\left(\,\cdot\, - \frac{\pi k}{b}\right)$, $k \in Z$, are orthogonal in PW_b. In particular we have

$$f\left(\frac{\pi k}{b}\right) = \int_{-b}^{b} F(t)\exp(i\pi kt/b)\,dt\,, \tag{3.2}$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{2b} f\left(\frac{-\pi k}{b}\right)\exp\left(i\frac{\pi k}{b}t\right) \sim F(t)\,. \tag{3.3}$$

Now Parseval's relation for $L_2[-b, b]$ yields

$$\int_{-b}^{b} F(t)\,G(t)^*\,dt = \sum_{k=-\infty}^{\infty} \frac{1}{2b} f\left(\frac{\pi k}{b}\right) g\left(\frac{\pi k}{b}\right)\,. \tag{3.4}$$

Choosing $g = h_b(\,\cdot - x)$ and taking into account (2.3) we obtain the formula of cardinal interpolation :

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{b}\right)\mathrm{sinc}\left(b\left(x - \frac{\pi k}{b}\right)\right), \quad \mathrm{sinc}(x) = \frac{\sin x}{x}\,. \tag{3.5}$$

This interpolation series converges in the norm of $C_0(\mathbf{R})$ in view of the estimate (2.2). Moreover relation (2.4) implies also convergence in $L_2(\mathbf{R})$.

4. MINIMUM NORM INTERPOLATION IN HARMONIC HILBERT SPACES

The following result is important for establishing the existence of minimum norm interpolants for arbitrary l_2-data.

PROPOSITION 4.1 : *Assume that for* $b > 0$.

(4.1)
$$c_b := c_b(D) := \inf\{D(t) : 0 \leqslant |t| < b\} > 0. \quad (\textit{Existence condition})$$

then

(4.2)
$$PW_b \subseteq H_D(\mathbf{R}), \quad \|f\|_D \leqslant \|f\|_b / \sqrt{c_b}.$$

In particular for any l_2*-sequence* (a_k) *there is a function* $f \in H_D(\mathbf{R})$ *with* $f\left(\frac{\pi}{b} k\right) = a_k, \quad (k \in \mathbf{Z})$.

Proof : Let $f \in PW_b$. Then

$$f(x) = \int_{-b}^{b} F(t) \exp(ixt)\, dt, \quad F \in L_2(\mathbf{R}), \quad F(t) = 0\ (|t| > b).$$

In view of (4.1) we have $F/\sqrt{D} \in L_2(\mathbf{R})$ which implies (4.2) :

$$\|f\|_D^2 = \int_{-\infty}^{\infty} |F(t)|^2 / D(t)\, dt = \int_{-b}^{b} |F(t)|^2 / D(t)\, dt$$

$$\leqslant c_b^{-1} \int_{-b}^{b} |F(t)|^2\, dt = \|f\|_b^2 / c_b.$$

We will study now *minimum norm interpolation* at the uniform mesh $\frac{\pi}{b} k$, $k \in \mathbf{Z}$ in the harmonic Hilbert space $H_D(\mathbf{R})$ for l_2-data (a_k). As in the case of periodic Hilbert spaces it is natural to use the projection theorem for constructing minimum norm interpolants in harmonic Hilbert spaces.

We introduce the closed linear subspace of the harmonic Hilbert space $H_D(\mathbf{R})$ defined by

(4.4)
$$M_b = \left\{f \in H_D(\mathbf{R}) : f\left(\frac{\pi k}{b}\right) = 0, k \in \mathbf{Z}\right\}.$$

By the projection theorem there exists a unique orthogonal projection Q_b of $H_D(\mathbf{R})$ with

$$\ker(Q_b) = M_b. \tag{4.5}$$

Its orthogonal complement is given by

$$\operatorname{im}(Q_b) = \operatorname{clin}\left\{d\left(\cdot - \frac{\pi k}{b}\right) : k \in \mathbf{Z}\right\} \tag{4.6}$$

where $\operatorname{clin}\left\{d\left(\cdot - \frac{\pi k}{b}\right) : k \in \mathbf{Z}\right\}$ denotes the *closed linear span* of $\left\{d\left(\cdot - \frac{\pi k}{b}\right) : k \in \mathbf{Z}\right\}$ and $\operatorname{im}(Q_b)$ is the image (range) of Q_b. It follows from (4.6) that the space $\operatorname{im}(Q_b)$ is translation invariant with respect to $\frac{\pi}{b}$.

As in the periodic case this projector enables minimum norm interpolation in the harmonic Hilbert space $H_D(\mathbf{R})$ (see Young, p. 147).

PROPOSITION 4.2 : *Let $f \in H_D(\mathbf{R})$. Then $Q_b(f)$ is the unique function in $H_D(\mathbf{R})$ satisfying*

$$Q_b(f)\left(\frac{\pi k}{b}\right) = f\left(\frac{\pi k}{b}\right), \ (k \in \mathbf{Z}), \quad \|Q_b(f)\|_D \leq \|f\|_D. \tag{4.7}$$

In particular the existence condition $c_b(D) > 0$ guarantees the existence of unique minimum norm interpolants for arbitrary l_2-data.

Proof : Given $f \in H_D(\mathbf{R})$ the minimum norm interpolant $Q_b(f)$ is obtained as the best approximation of f in $M_b^{\perp}$. Then $f - Q_b(f) \in M_b$ which implies the interpolation properties

$$Q_b(f)\left(\frac{\pi k}{b}\right) = f\left(\frac{\pi k}{b}\right), \quad (k \in \mathbf{Z}).$$

Assume that $g \in H_D(\mathbf{R})$ satisfies $g\left(\frac{\pi k}{b}\right) = f\left(\frac{\pi k}{b}\right)$, $(k \in \mathbf{Z})$. This implies

$$g - f \in M_b = \ker(Q_b), \text{ i.e. } Q_b(g) = Q_b(f).$$

Moreover

$$\|g\|_D^2 = \|g - Q_b(g)\|_D^2 + \|Q_b(g)\|_D^2 = \|g - Q_b(f)\|_D^2 + \|Q_b(f)\|_D^2.$$

Thus $Q_b(f)$ is the unique function of minimum norm in the linear manifold

$$f + M_b = \left\{g \in H_D(R) : g\left(\frac{\pi k}{b}\right) = f\left(\frac{\pi k}{b}\right), (k \in \mathbf{Z})\right\}.$$

It is interesting to note that infinite minimum norm interpolation is obtained as a « limit » of finite minimum norm interpolation problems in the following way. Let

$$M_{b,m} = \left\{ f \in H_D(\mathbf{R}) : f\left(\frac{\pi k}{b}\right) = 0,\ |k| \leq m \right\},\ m \in \mathbf{N}\,.$$

As before let $Q_{b,m}$ be the unique orthogonal projector of $H_D(\mathbf{R})$ defined by

$$\ker\left(Q_{b,m}\right) = M_{b,m}.$$

Then

$$\operatorname{im}\left(Q_{b,m}\right) = \operatorname{lin}\left\{ d\left(\,\cdot\, - \frac{\pi k}{b}\right) : |k| \leq m \right\}.$$

Since

$$\ker\left(Q_{b,m}\right) \supseteq \ker\left(Q_{b,m+1}\right),\ m \in \mathbf{N}\,,\quad \bigcap_{m \in \mathbf{N}} \ker\left(Q_{b,m}\right) = \ker\left(Q_b\right)$$

it follows

$$\lim_{m \to \infty} \| Q_b(f) - Q_{b,m}(f) \|_D = 0\,,\quad f \in H_D(\mathbf{R})\,.$$

Since

$$Q_{b,m}(f) = \sum_{k=-m}^{m} c_{k,m}(f)\, d\left(\,\cdot\, - \frac{\pi}{b} k\right)$$

it is natural to ask whether $\left\{ d\left(\,\cdot\, - \frac{\pi}{b} r\right) : r \in \mathbf{Z} \right\}$ is a Schauder basis of $M_b^{\perp}$.

Note that in the case of cardinal interpolation in the Paley-Wiener space PW_b we have the particularly simple situation

$$Q_{b,m}(f)\,(x) = \sum_{k=-m}^{m} f\left(\frac{\pi k}{b}\right) \operatorname{sinc}\left(b\left(x - \frac{\pi k}{b}\right)\right),$$

$$Q_b(f)\,(x) = f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{b}\right) \operatorname{sinc}\left(b\left(x - \frac{\pi k}{b}\right)\right).$$

It follows from Proposition 1 that the *sinc-function* $\sigma_b(x) = \sin(bx)/(bx)$ is an element of the harmonic Hilbert space $H_D(\mathbf{R})$ as well as its translates : $\sigma_b\left(\,\cdot\, - \frac{\pi}{b} k\right) \in H_D(\mathbf{R}),\ (k \in \mathbf{Z})$.

Let $g_b \in H_D(\mathbf{R})$ denote the unique *fundamental Lagrange minimum norm interpolant :*

$$Q_b(\sigma_b) = g_b\,. \tag{4.9}$$

Then there is a unique function $G_b \in L_2(\mathbf{R}) \cap L_1(\mathbf{R})$ satisfying

$$g_b(x) = \int_{-\infty}^{\infty} G_b(t)\exp(itx)\,dt\,,\quad \int_{-\infty}^{\infty} |G_b(t)|^2/D(t)\,dt < \infty\,. \tag{4.10}$$

The determination of G_b is connected with the *2b-periodization* of D :

$$D_{2b}(t) := \sum_{k=-\infty}^{\infty} D(t + 2\,bk)\,. \tag{4.11}$$

PROPOSITION 4.3 : *The 2b-periodization of D is in $L_1[0, 2b]$ with*

$$D_{2b}(t) \geqslant c_b(D) > 0\,, \tag{4.12}$$

$$1/D_{2b} \in L_\infty[0, 2b]\,. \tag{4.13}$$

Proof: Let $D_{2b,n}(t) = \sum_{k=-n}^{n} D(t+2\,bk)$. Then

$$c_b(D) \leqslant D_{2b,n}(t) \leqslant D_{2b,n+1}(t)\,,\quad \int_0^{2b} D_{2b,n}(t)\,dt \leqslant \int_{-\infty}^{\infty} D(t)\,dt < \infty\,.$$

By Levi's theorem (Garnir, p. 38) we have $D_{2b,n}(t) \uparrow D_{2b}(t)$ almost everywhere with $D_{2b} \in L_1[0, 2b]$ and $D_{2b}(x) \geqslant c_b(D)$.

PROPOSITION 4.4 : *Assume in addition that* $D_{2b} \in L_\infty(\mathbf{R})$ (Convergence condition). *Then the estimate*

$$c_b(D) \sum_{k=-m}^{m} |b_k|^2 \leqslant \frac{1}{2b}\left\| \sum_{k=-m}^{m} b_k\, d\left(\,\cdot\, - \frac{\pi}{b}k\right)\right\|_D^2 \tag{4.14}$$

$$\leqslant \|D_{2b}\|_\infty \sum_{k=-m}^{m} |b_k|^2,\quad (m \in \mathbf{Z})\,,$$

holds. In particular $\operatorname{im}(Q_b) = M_b^\perp$ *is isomorphic to* l_2 *and* $\left\{d\left(\cdot - \frac{\pi}{b}k\right) : k \in \mathbf{Z}\right\}$ *forms a* Schauder basis *of* $\operatorname{im}(Q_b) = M_b^\perp$. *For any* $f \in H_D(\mathbf{R})$ *there is a unique* $(b_k) \in l_2$ *such that*

$$(4.15)\quad Q_b(f)(x) = \sum_{k=-\infty}^{\infty} b_k\, d\left(x - \frac{\pi}{b}k\right) \quad (\textit{dual interpolation series})$$

converges in the norm of $H_D(\mathbf{R})$ *and therefore also in the uniform norm and in the least square norm by (2.2) and (2.4).*

Proof: Consider $s(x) = \sum_{k=-m}^{m} b_k\, d\left(x - \frac{\pi}{b}k\right) \in \operatorname{im}(Q_b)$. Then

$$s(x) = \int_{-\infty}^{\infty} D(t)\left[\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right]\exp(ixt)\,dt,$$

$$\|s\|_D^2 = \int_{-\infty}^{\infty} D(t)\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt$$

$$= \int_0^{2b} D_{2b}(t)\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt,$$

$$c_b(D)\int_0^{2b}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt \leqslant \|s\|_D^2$$

$$\leqslant \|D_{2b}\|_\infty \int_0^{2b}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt,$$

$$c_b(D)\,2b\sum_{k=-m}^{m}|b_k|^2 \leqslant \left\|\sum_{k=-m}^{m} b_k\, d\left(\cdot - \frac{\pi}{b}k\right)\right\|_D^2 \leqslant \|D_{2b}\|_\infty\, 2b\sum_{k=-m}^{m}|b_k|^2$$

which completes the proof of Proposition 4.4.

PROPOSITION 4.5 : *The fundamental minimum norm Lagrange interpolant is given by*

$$(4.16)\qquad g_b(x) = \sum_{k=-\infty}^{\infty} a_k(b)\, d\left(x - \frac{\pi}{b}k\right) \textit{ where}$$

$$1/(2b)D_{2b}(t) \sim \sum_{k=-\infty}^{\infty} a_k(b)\exp\left(ik\frac{\pi}{b}t\right).$$

Moreover we have

$$g_b(x) = \frac{1}{2b} \int_{-\infty}^{\infty} \frac{D(t)}{D_{2b}(t)} \exp(itx)\, dt . \tag{4.17}$$

Proof: By Proposition 4 we know

$$g_b(x) = \sum_{k=-\infty}^{\infty} a_k(b)\, d\left(x - \frac{\pi}{b} k\right), \quad \sum_{k=-\infty}^{\infty} |a_k(b)|^2 < \infty .$$

Since convergence is in the norm of $H_D(\mathbf{R})$ we obtain

$$g_b(x) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k(b) \exp\left(-ik\frac{\pi}{b} t\right) D(t) \exp(ixt)\, dt ,$$

$$g_b(x) = \int_{-\infty}^{\infty} A(t)\, D(t) \exp(ixt)\, dt ,$$

$$A(t) \sim \sum_{k=-\infty}^{\infty} a_k(b) \exp\left(-ik\frac{\pi}{b} t\right) .$$

In view of the orthogonality properties of the finite Fourier transform

$$g_b\left(\frac{\pi}{b} r\right) = \int_0^{2b} A(t)\, D_{2b}(t) \exp\left(i\frac{\pi}{b} rt\right) dt = \delta_{0,r}, \quad (r \in \mathbf{Z}) ,$$

we get

$$A(t) = \frac{1}{2bD_{2b}(t)} \sim \sum_{k=-\infty}^{\infty} a_k(b) \exp\left(-ik\frac{\pi}{b} t\right) .$$

Since $D(-t) = D(t)$ it follows $D_b(-t) = D_b(t)$ which completes the proof of Proposition 5.

Note that the Fourier integral representation (4.17) of the fundamental minimum norm interpolant g_b corresponds to the Fourier series representation (1.7) of γ_n in the periodic case.

PROPOSITION 4.6 : *Assume that* $c_b(D) > 0$ *and* $D_{2b} \in L_\infty(\mathbf{R})$. *Then the estimate*

$$(4.18) \qquad \frac{1}{\|D_{2b}\|_\infty} \sum_{k=-m}^{m} |b_k|^2 \leqslant 2b \left\| \sum_{k=-m}^{m} b_k\, g_b\left(\cdot - \frac{\pi}{b} k\right) \right\|_D^2$$

$$\leqslant \frac{1}{c_b(D)} \sum_{k=-m}^{m} |b_k|^2, \quad (m \in \mathbf{Z}),$$

holds. Thus $\left\{ g_b\left(\cdot - \frac{\pi}{b} k\right) : k \in \mathbf{Z}\right\}$ *is a Schauder basis of* $\operatorname{im}(Q_b) = M_b^\perp$ *which is biorthogonal to the Schauder basis* $\left\{ d\left(\cdot - \frac{\pi}{b} r\right) : r \in \mathbf{Z}\right\}$ *of* $\operatorname{im}(Q_b) = M_b^\perp$:

$$(4.19) \qquad \left(g_b\left(\cdot - \frac{\pi}{b} k\right), d\left(\cdot - \frac{\pi}{b} r\right)\right)_D = \delta_{k,r}, \quad k, r \in \mathbf{Z}.$$

For any $f \in H_D(\mathbf{R})$ *the interpolation series*

$$(4.20) \quad Q_b(f)(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{b}\right) g_b\left(x - \frac{\pi k}{b}\right) \quad (\textit{Lagrange interpolation series})$$

converges in the norm of $H_D(\mathbf{R})$ *and therefore also in the uniform norm and in the last square norm by (2.2) and (2.4).*

Proof: As in the proof of Proposition 4.4 consider

$$s(x) = \sum_{k=-m}^{m} b_k\, g_b\left(x - \frac{\pi}{b} k\right) \in \operatorname{im}(Q_b).$$

Then we have in view of (4.17)

$$s(x) = \int_{-\infty}^{\infty} G_b(t) \left[\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b} t\right)\right] \exp(ixt)\, dt$$

$$= \frac{1}{2b} \int_{-\infty}^{\infty} D(t) \frac{1}{D_{2b}(t)} \left[\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b} t\right)\right] \exp(ixt)\, dt.$$

This implies

$$\|s\|_D^2 = \frac{1}{(2b)^2}\int_{-\infty}^{\infty} D(t)\frac{1}{D_{2b}(t)^2}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt$$

$$= \frac{1}{(2b)^2}\int_0^{2b}\frac{1}{D_{2b}(t)}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt ,$$

$$\|D_{2b}\|_\infty^{-1}\int_0^{2b}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt \leqslant (2b)^{+2}\|s\|_D^2$$

$$\leqslant c_b(D)^{-1}\int_0^{2b}\left|\sum_{k=-m}^{m} b_k \exp\left(-ik\frac{\pi}{b}t\right)\right|^2 dt ,$$

$$\|D_{2b}\|_\infty^{-1}\sum_{k=-m}^{m}|b_k|^2 \leqslant 2b\left\|\sum_{k=-m}^{m} b_k g_b\left(\cdot - \frac{\pi}{b}k\right)\right\|_D^2 \leqslant c_b(D)^{-1}\sum_{k=-m}^{m}|b_k|^2 .$$

Passing to the limit in (4.18) we obtain from (4.20)

PROPOSITION 4.7 : *Assume* $c_b(D) > 0$ *and* $D_{2b} \in L_\infty[0, 2b]$. *Then for any* $f \in H_D(\mathbf{R})$

$$\sum_{k=-\infty}^{\infty}\left|f\left(\frac{\pi}{b}k\right)\right|^2 < 2b\|D_{2b}\|_\infty\|f\|_D^2, \tag{4.21}$$

$$\sum_{k=-\infty}^{\infty}\left|d\left(y+\frac{\pi}{b}k\right)\right|^2 < 2b\|D_{2b}\|_\infty\|D\|_1 \quad (y \in \mathbf{R}) . \tag{4.22}$$

Remark : It follows from Proposition 4.7 that for any $f \in H_D(\mathbf{R})$ its cardinal interpolant

$$T_b(f)(x) := \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{b}\right)\operatorname{sinc}\left(b\left(x-\frac{\pi k}{b}\right)\right)$$

exists and that the cardinal interpolation projector T_b is a continuous operator from $H_D(\mathbf{R})$ into $L_2(\mathbf{R})$.

5. APPLICATIONS

In view of the poor decay properties of the function sinc (x) Schoenberg has proposed cardinal spline interpolation. We will show how Schoenberg's interpolation scheme may be considered as minimum norm interpolation in an appropriate Hilbert space $H_D(\mathbf{R})$. This is obtained by choosing

$$D(t) = \operatorname{sinc}\left(\frac{t}{2}\right)^{2q}, \quad q \in \mathbf{N} . \tag{5.1}$$

Then

$$d(x) = 2\pi M_{2q}(x) = \int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t}{2}\right)^{2q} \exp(ixt)\, dt \tag{5.2}$$

is the central cardinal B-spline of degree $2q-1$ and support $[-q, q]$ up to a constant. Clearly we have $c_b(D) > 0$, $(0 < b < 2\pi)$.

Moreover the Weierstrass M-test implies that the periodization

$$D_{2b}(t) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{1}{2}(t + 2bk)\right)^{2q}$$

satisfies the hypothesis of Proposition 4.4. Thus we obtain

PROPOSITION 5.1 : *For any* $(a_k) \in l_2$ *there is a unique cardinal splinee s in* $L_2(\mathbf{R})$ *of degree* $2q-1$ *satisfying the interpolation conditions* $s\left(\frac{\pi}{b}k\right) = a_k$, $k \in \mathbf{Z}$. *Moreover there is a unique sequence* $(b_k) \in l_2$ *such that* $s(x) = \sum_{k=-\infty}^{\infty} b_k M_{2q}\left(x - \frac{\pi}{b}k\right)$.

The symmetric case $b = \pi$ was established in Schoenberg in a more general setting. Our approach gives an interpretation of cardinal spline interpolation for l_2-data as a minimum norm interpolation problem in an appropriate harmonic Hilbert space.

Our next example is concerned with a classical interpolation problem for holomorphic functions. The defining function is

$$D(t) = \frac{t}{\sinh t}. \tag{5.3}$$

The Fourier integral of this kernel is given by (see Oberhettinger, p. 37)

$$d(x) = \frac{\pi}{4}/\cosh\left(\frac{\pi}{2}x\right)^2. \tag{5.4}$$

In this case the harmonic Hilbert space is

$$H_D(\mathbf{R}) = \left\{ f : f(x) = \int_{-\infty}^{\infty} F(t) \exp(ixt)\, dt, \int_{-\infty}^{\infty} |F(t)|^2 \frac{\sinh t}{t}\, dt < \infty \right\}.$$

Then f is the restriction of the holomorphic function in the strip $\left\{ z : |\mathcal{J} z| < \frac{1}{2} \right\}$ defined by

$$f(z) = \int_{-\infty}^{\infty} F(t) \exp(izt)\, dt$$

(see Epstein, Greenstein, and Miranker, p. 4). The harmonic Hilbert space is isomorphic and isometric to the Bergmann space of holomorphic functions being square integrable over the strip $\left\{z : |\mathcal{I}z| < \frac{1}{2}\right\}$. We have $c_b(D) = b/\sinh b > 0$, $(0 < b)$.

The Weierstraß M-test implies that the periodization

$$D_{2b}(t) = \sum_{k=-\infty}^{\infty} (t + 2bk)/\sinh(x + 2bk) > 0$$

satisfies the hypotheses of Proposition 4.4. Thus we obtain.

PROPOSITION 5.2: *For any* $(a_k) \in l_2$ *there is a unique function* $s \in H_D(\mathbf{R})$ *of minimum norm satisfying* $s\left(\frac{\pi}{b}k\right) = a_k$, $k \in \mathbf{Z}$. *Moreover there is a unique sequence* $(b_k) \in l_2$ *such that*

$$s(x) = \sum_{k=-\infty}^{\infty} b_k/\cosh\left(\frac{\pi}{2}\left(x - \frac{\pi}{b}k\right)\right)^2.$$

Note that our approach to the existence result of Epstein, Greenstein, and Miranker gives in addition an expansion theorem with respect to a Schauder basis.

6. BACK TO PERIODIC INTERPOLATION

The harmonic Hilbert space $H_D(\mathbf{R})$ is determined by the functions

$$d(x) = \int_{-\infty}^{\infty} D(t)\exp(ixt)\,dt, \quad D(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d(x)\exp(-ixt)\,dx.$$

Existence and expansions of minimum norm interpolants depend on the periodization

$$D_{2b}(t) = \sum_{k=-\infty}^{\infty} D(t + 2bk).$$

Therefore it is natural to consider the periodization

$$d_{2a}(x) = \sum_{k=-\infty}^{\infty} d(x + 2ak), \quad a > 0.$$

Then

$$\frac{1}{2a}\int_0^{2a} d_{2a}(x)\exp\left(-i\frac{\pi}{a}rx\right)dx=\frac{1}{2a}\int_0^{2a}\sum_{k=-\infty}^{\infty} d(x+2ak)\exp\left(-i\frac{\pi}{a}rx\right)dx$$

$$=\frac{1}{2a}\int_0^{2a}\sum_{k=-\infty}^{\infty} d(x+2ak)\exp\left(-i\frac{\pi}{a}r(x+2ak)\right)dx$$

$$=\frac{1}{2a}\int_{-\infty}^{\infty} d(x)\exp\left(-i\frac{\pi}{a}rx\right)dx=\frac{\pi}{a}D\left(-\frac{\pi}{a}r\right),$$

i.e., we formally have obtained (*Poisson summation formula*)

$$d_{2a}(x)=\frac{\pi}{a}\sum_{k=-\infty}^{\infty} D\left(-\frac{\pi}{a}k\right)\exp\left(i\frac{\pi}{a}kx\right).$$

We consider now the periodic interpolation problem at $\frac{2\pi}{n}j$, $0\leqslant j<n$, with translates of

$$\psi(x)=\frac{a}{\pi}d_{2a}\left(\frac{a}{\pi}x\right)=\sum_{k=-\infty}^{\infty} D\left(-\frac{\pi}{a}k\right)\exp(ikx).$$

Thus ψ may be considered as the generating function of the periodic Hilbert space H_d with defining sequence

$$d_k=D\left(-\frac{\pi}{a}k\right)=D\left(\frac{\pi}{a}k\right),\quad k\in\mathbf{Z}.$$

In view of (1.7) the fundamental periodic Lagrange interpolant γ_n exists if and only if

$$\sum_{s=-\infty}^{\infty} d_{j+sn}=\sum_{s=-\infty}^{\infty} D\left(\frac{\pi}{a}(j+sn)\right)\neq 0,\quad (0\leqslant j<n).$$

The first example is concerned with *infinite rational* and *periodic rational interpolation :*

$$D(t)=\exp(-|t|),\quad d(x)=\frac{2}{1+x^2},$$

$$d_{2\pi}(x)=\sum_{k=-\infty}^{\infty}\exp(-|k|)\exp(ikx)=\frac{\sinh(1)}{\cosh(1)-\cos(x)}$$

This generating function has been considered in Locher [1981] and Delvos [1990].

Our next example is concerned with *classical cardinal interpolation* and *trigonometric interpolation :*

$$D(t) = \chi_{\left[-\left(m+\frac{1}{2}\right),\left(m+\frac{1}{2}\right)\right]}(t)\,, \quad d(x) = \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\frac{x}{2}}\,,$$

$$d_{2\pi}(x) = \sum_{k=-m}^{m} \exp(ikx) = \frac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}\,.$$

The final example is related to *spline interpolation :*

$$D(t) = \operatorname{sinc}\left(\frac{t}{2}\right)^{2q}, \quad d(x) = 2\pi M_{2q}(x)\,,$$

$$d_{2\left(q+\frac{1}{2}\right)}(x) = \frac{\pi}{\left(q+\frac{1}{2}\right)} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi}{2\left(q+\frac{1}{2}\right)}k\right)^{2q} \exp\left(i\frac{\pi}{q+\frac{1}{2}}kx\right),$$

$$\psi(x) = \frac{q+\frac{1}{2}}{\pi} d_{2\left(q+\frac{1}{2}\right)}\left(\frac{q+\frac{1}{2}}{\pi}x\right) = \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi}{2\left(q+\frac{1}{2}\right)}k\right)^{2q} \exp(ikx)\,.$$

7. GENERALIZED HARMONIC HILBERT SPACES

We conclude with some remarks concerning the relation between probability theory and reproducing kernel Hilbert spaces and give a unified approach to periodic and harmonic Hilbert spaces which was initiated by the referee.

The generating function d of the harmonic Hilbert space $H_D(\mathbf{R})$ is also the characteristic function of the probability density D. Due to Bochner's theorem (Chandrasekharan, p. 151) d is a positive definite function and therefore defines a reproducing kernel Hilbert space which in our case coincides with $H_D(\mathbf{R})$. Since Bochner's theorem holds also for probability measures (distributions) $dV(t)$ there is a unique reproducing kernel Hilbert space $H_{dV}(\mathbf{R})$ associated with the probability measure $dV(t)$. Its reproducing kernel is then obtained as the Fourier-Stieltjes transform of $dV(t)$. Note that no longer $H_{dV}(\mathbf{R}) \subseteq C_0(\mathbf{R})$ but

$$H_{dV}(\mathbf{R}) \subseteq C_b(\mathbf{R})$$

where $C_b(\mathbf{R})$ is the algebra of bounded and uniformly continuous functions on $\mathbf{R}$.

We indicate how to extend our construction to the more general situation. Let $dV(t)$ be a probability measure. Assume

$$f(x) = \int_{\mathbf{R}} \exp(ixt)\, dV_f(t) \text{ (Fourier – Stieltjes transform)}$$

where V_f is a function of bounded variation (finite signed measure).

We define the *generalized* harmonic Hilbert space $H_{dV}(\mathbf{R})$ as the linear subspace of $f \in C_b(\mathbf{R})$ with Radon-Nikodym derivative

$$\frac{dV_f}{dV} = F_V \in L_2(\mathbf{R}, dV) .$$

The inner product is now defined by

$$(f, g)_{dV} = \int_{\mathbf{R}} F_V(t)\, \overline{G_V(t)}\, dV(t) .$$

We have

$$H_{dV}(\mathbf{R}) \subseteq C_b(\mathbf{R}) , \quad \|f\|_\infty \leqslant \|f\|_{dV} \left(\int_{\mathbf{R}} dV(t) \right)^{1/2} .$$

Then $H_{dV}(\mathbf{R})$ is a reproducing kernel Hilbert space with reproducing kernel

$$d(x) = \int_{\mathbf{R}} \exp(ixt)\, dV(t) .$$

As a first instance of a generalized harmonic Hilbert space we consider the harmonic Hilbert space $H_{dV}(\mathbf{R})$ which is obtained by choosing $dV(t) = D(t)\, dt$. Then

$$dV_j(t) = F_V(t)\, dV(t) = F_V(t)\, D(t)\, dt$$

which implies $F(t) = F_V(t)\, D(t)$ (usual Fourier transform of f) and

$$(f, g)_{dV} = \int_{\mathbf{R}} F_V(t)\, \overline{G_V(t)}\, dV(t)$$

$$= \int_{\mathbf{R}} F(t)/D(t)\, \overline{G(t)/D(t)}\, D(t)\, d(t) = (f, g)_D .$$

Next we show that the periodic Hilbert space H_d is obtained as a special generalized harmonic Hilbert space by choosing the lattice distribution

$$V(t) = V(2\pi k + \pi), \quad 2\pi k < t < 2\pi(k+1), \quad k \in \mathbf{Z},$$

$$d_k = V(2\pi k + \pi) - V(2\pi k - \pi), \quad k \in \mathbf{Z}.$$

Then we have

$$(f, g)_{dV} = \int_{\mathbf{R}} F_V(t)\, \overline{G_V(t)}\, dV(t) = \sum_{k \in \mathbf{Z}} F_V(2\pi k)\, \overline{G_V(2\pi k)}\, d_k.$$

As in the continuous case we have

$$(f, e_k) = F_V(2\pi k)\, d_k,$$

$$(f, g)_{dV} = \sum_{k \in \mathbf{Z}} (f, e_k)/d_k\, \overline{(g, e_k)/d_k}\, d_k = (f, g)_d.$$

Thus the Fourier-Stieltjes transform as well as the Radon-Nikodym theorem provide appropriate tools to construct both periodic Hilbert spaces and harmonic Hilbert spaces. In particular replacing the lattice distribution by a general discrete distribution leads to Hilbert spaces of almost-periodic functions whose elements are given by nonharmonic Fourier series of the type

$$f(x) = \sum_{k \in \mathbf{Z}} a_k \exp(i\lambda_k x)$$

where $\{\lambda_k : k \in \mathbf{Z}\}$ are the jumps (spectrum) of the discrete distribution.

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