Sonia M. F. Garcia
Søren Jensen

The p version of mixed finite element methods for parabolic problems

THE *p* VERSION OF MIXED FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS (*) (¹)

by Sonia M F. GARCIA (²) and Søren JENSEN (³)

Abstract — *We investigate a parabolic problem from the point of view of stability and approximation properties of increasing order mixed (in space) finite element methods Previous estimates for the Raviart Thomas projection are proven to be sharp We analyze the effects of mixed finite element discretization in space to present transient error estimates (for semidiscrete mixed finite element methods) The results in this paper (submitted in Oct 1993) complement the results already published in [6 8]*

Résumé — *Un problème parabolique a été étudié d'un point de vue stabilité, ainsi que des propriétés d'approximation des méthodes des éléments finis mixtes (en espace) d'ordre croissant Il est montré que des estimations précédentes faites sur la projection Raviart-Thomas sont précises Nous avons pu analyser les effets de la discretization du méthode des éléments finis, en espace, afin de présenter des estimations d'erreurs transitoires (pour des méthodes des éléments finis mixtes semidiscrets) Les résultats de cet article (soumis en Oct 1993) complètent les résultats déjà publiés dans les references [6 8]*

1. INTRODUCTION

Consider the following parabolic initial, boundary value problem

$$c(x, t) u_t - \nabla \cdot (a(x, t) \nabla u(x, t) + b(x, t)) = f(x, t), (x, t) \in \Omega \times \mathbf{J},$$

$$u(x, 0) = u_0(x), x \in \Omega,\qquad (1.1)$$

$$u(x, t) = g(x, t), (x, t) \in \partial\Omega \times \mathbf{J}$$

(²) Department of Mathematics, U S Naval Academy, Annapolis, MD 21402
(³) Department of Mathematics, University of Maryland Baltimore County, Baltimore, MD 21250

Here, Ω is a convex, plane polygonal domain, $\mathbf{J} = (0, T)$, and we assume that a, b_1, b_2, $c \in C^{\infty}(\bar{\Omega} \times \bar{\mathbf{J}})$ and that $\inf\limits_{x \in \Omega, t \in \mathbf{J}} a(x, t) > 0$ as well as $\inf\limits_{x \in \Omega, t \in \mathbf{J}} c(x, t) > 0$. We assume that this problem is well-posed and in particular that a solution exists at each time t for

$$(f(\,.\,, t), u_0(\,.\,), g(\,.\,, t)) \in L^2(\Omega) \times H^2(\Omega) \times H^{3/2}(\partial\Omega)$$

with the regularity estimate

$$\|u(\,.\,, t)\|_2 \leqslant C(\|f(\,.\,, t)\|_0 + |g(\,.\,, t)|_{3/2})$$

where we have denoted the norms in the Sobolev spaces $H^m(\Omega)$ by $\|\,.\,\|_m$ and in $H^s(\partial\Omega)$ by $|\,.\,|_s$, respectively. We denote the ess sup norm in the space $L^{\infty}(\Omega)$ by $\|\,.\,\|_{\infty}$.

Let σ be a new variable denoting a « flux » :

$$\sigma := -\,(a(x, t)\,\nabla u(x, t) + b(x, t)\,u(x, t))$$

and define the spaces

$$V(\Omega) := L^2(\Omega), \text{ and,}$$

$$H(\operatorname{div}, \Omega) \overset{\text{def}}{=} ([C^{\infty}(\Omega)]^2)^{\text{closure under } \|\,.\,\|_{H(\operatorname{div})}},$$

where we take the closure with respect to the norm defined by

$$\|\chi\|^2_{H(\operatorname{div})} \overset{\text{def}}{=} \|\chi\|^2_{L^2} + \|\nabla \cdot \chi\|^2_{L^2}\,.$$

When it is clear from the context we will use V and H to denote $V(\Omega)$ and $H(\operatorname{div}, \Omega)$.

A solution pair $(u, \sigma)(\,.\,, t) \in V \times H$ of (1.1) may be thought of as a solution of the variational problem

$$(\sigma(x, t), \chi) + (a(x, t)\,\nabla u(x, t), \chi) + (b(x, t)\,u(x, t), \chi) = 0, \quad \forall \chi \in H,$$

$$(c(x, t)\,u_t(x, t), v) + (\operatorname{div} \sigma(x, t), v) = (f(x, t), v),$$

$$\forall v \in V,$$

$$u(x, 0) = u_0(x)\,, \qquad (1.2)$$

where $(\,.\,,\,.\,)$ is the usual $L^2(\Omega)$-inner product, and u furthermore is subject to the boundary conditions in (1.1). The second term in the first equation of (1.2) is meant in the dual sense (using that $\chi \in H$).

Let

$$1/a(x, t) = \alpha(x, t) \quad \text{and} \quad b(x, t)/a(x, t) = \beta(x, t) .$$

Simplifying the notation by dropping (x, t) for the time-dependent variables as well as dropping the (x) for u_0, (1.2) can be written as

$$(\alpha\sigma, \chi) - (u, \operatorname{div} \chi) + (\beta u, \chi) = \langle g, \chi \cdot \nu \rangle , \quad \forall \chi \in H , \tag{1.3}$$

$$(cu_t, v) + (\operatorname{div} \sigma, v) = (f, v) , \quad \forall v \in V , \tag{1.4}$$

$$u(\, . \, , 0) = u_0 , \tag{1.5}$$

where $\langle \, . \, , \, . \, \rangle$ is the L^2-inner product on the boundary of Ω, i.e.,

$$\langle w, v \rangle = \int_{\partial\Omega} wv \, ds$$

and ν is the unit outward normal vector to $\partial\Omega$.

This paper studies the behavior of a finite element method to be defined precisely in (1.6)-(1.8) using a fixed mesh as in [11]. As there we then pay particular attention to what happens as the polynomial degree of the finite elements is increased. This type of method is at times called a p version of the mixed finite element method, but could also be considered as a mixed, Galerkin spectral element method.

[6] has portrayed the h-version of the mixed finite element method for the quasilinear parabolic problem including the questions of uniqueness, existence, and convergence of the discrete solution as $h \downarrow 0$.

Now, let us define some discrete subspaces, denoting by N some measure of the dimension of these. Let $\mathcal{T}_N$ be a geometric partition of Ω into parallelograms which we term geometric « elements » E. Let $Q^{p, q}$ be the set of polynomials in two variables (x and y) of separate degree at most p (in x) and q (in y), respectively. If $p = q$, we simply write Q^p. For each $E \in \mathcal{T}_N$, let F_E be an affine, orientation preserving (i.e., $(\det (DF_E) > 0)$ mapping which maps E onto $R = (-1, 1)^2$, a reference square. We then define two spaces of piecewise polynomials :

$$V_N = \left\{ v \in L^2(\Omega) ; v|_E \circ F_E^{-1} \in Q^p, \forall E \in \mathcal{T}_N \right\}$$

$$H_N = \left\{ \chi \in H(\operatorname{div}) ; \chi|_E \circ F_E^{-1} \in Q^{p+1, p} \times Q^{p, p+1}, \forall E \in \mathcal{T}_N \right\}$$

Note that

$$\operatorname{div} H_N = V_N .$$

With the definitions already in place, also please note that a necessary and sufficient condition for the inclusion $H_N \subseteq H$ is that normal components (after convening on one common normal) are continuous across inter-element boundaries (whose common length is positive).

With the above definitions a discrete analogue to (1.2) is given by the following.

Find $(u_N, \sigma_N) \in V_N \times H_N$ such that

$$(\alpha\sigma_N, \chi) - (u_N, \operatorname{div} \chi) + (\beta u_N, \chi) = \langle g, \chi \cdot v \rangle, \quad \forall \chi \in H_N \qquad (1.6)$$

$$(cu_{N,t}, v) + (\operatorname{div} \sigma_N, v) = (f, v), \quad \forall v \in V_N, \qquad (1.7)$$

$$u_N(0) = U(0), \qquad (1.8)$$

where $U(0)$ is the first component of an elliptic mixed method projection of $(u, \sigma)(0)$ onto the finite dimensional space $V_N \times H_N$ to be defined precisely in (2.4)-(2.5). Assume until then that $U(0)$ is well-defined. If one were to introduce bases for H_N and V_N, one could rewrite (1.7) and (1.6) in matrix form :

$$C(\vec{u}_N)_t - D(\vec{\sigma}_N) = F,$$

$$A(\vec{\sigma}_N) + D^T(\vec{u}_N) + B(\vec{u}_N) = G,$$

with the initial value $(\vec{u}_N)(0)$ prescribed. The matrices A and C are symmetric and positive definite so that we may solve for $(\vec{\sigma}_N)$ in the second equation (uniquely in terms of $(\vec{u}_N)$). Now we interpret the first equation as a system of first order ordinary differential equations with a positive definite coefficient matrix of $(\vec{u}_N)_t$ which in turn determines $(\vec{u}_N)$ uniquely. Thus (u_N, σ_N) exists and is uniquely defined.

2. ESTIMATES AT FIXED TIME FOR ELLIPTIC MIXED AND OTHER USEFUL PROJECTIONS

The main tools for our work are an elliptic-mixed projection, the Raviart-Thomas projection for the p-version for the mixed method, and also the L^2-projection onto polynomial subspaces to be defined below.

The L^2-projection onto V_N, $P_N : L^2 \to V_N$, is uniquely given by

$$(P_N v - v, w) = 0, \quad w \in V_N,$$

where by [12], e.g. we have the following estimates,

$$\| P_N v - v \|_0 \leqslant C p^{-m} \| v \|_m, \quad m \geqslant 0, \tag{2.1}$$

for $v \in H^m(\Omega)$.

As in [16], [11] and of course [13] we will use the Raviart-Thomas projection of H onto H_N, $\Pi_N : H \to H_N$.

The definition of Π_N is, for any $\chi \in H$:

$$\langle (\Pi_N \chi - \chi) \cdot v_E, v \rangle_{\partial E_i} = 0, \quad 1 \leqslant i \leqslant 4, \quad \forall v : v \circ F_E^{-1} \in Q^p,$$

$$(\Pi_N \chi - \chi, \rho)_E = 0, \quad \forall \rho : \rho \circ F_E^{-1} \in Q^{p-1,p} \times Q^{p,p-1}. \tag{2.2}$$

$\Pi_N \chi$ exists and is unique by [13]. A consequence is the commutation property div $\Pi_N = P_N$ div, or precisely,

$$(\text{div } \Pi_N \chi, v) = (\text{div } \chi, v), \quad \forall v \in V_N, \forall \chi \in H.$$

The approximation properties of Π_N — as dependent on the parameter p — were given in [11] extending results in [16]:

PROPOSITION 2.1 [[11] Prop. 2.1] : *Let $\chi \in H$ and let $\Pi_N \chi$ be its Raviart-Thomas projection onto H_N given by (2.2). Then, if $\chi \in (H^r(\Omega))^2$, we have*

$$\| \Pi_N \chi - \chi \|_0 \leqslant C p^{1/2-r} \| \chi \|_r, \quad r > 1/2, \tag{2.3}$$

where $C > 0$ is a constant independent of p and χ but depending on r. Moreover, this estimate is sharp.

The inequality was proved in [16] for $r > 1$ and in [11] for $r > 1/2$. Concerning the sharpness of the inequality, the proof will depend on properties of the Legendre polynomials to be introduced in section 4. We will postpone the proof to Lemma 4.8 for that reason.

Define an elliptic-mixed-method projection (U, Σ) of the solution pair (u, σ) of the variational problem onto the finite dimensional space $V_N \times H_N$ to be the mapping $(U, \Sigma) : V \times H \times \mathbf{J} \to V_N \times H_N$ given by

$$(\alpha(\sigma - \Sigma), \chi) - (u - U, \text{div } \chi) + (\beta(u - U), \chi) = 0, \quad \forall \chi \in H_N, \tag{2.4}$$

$$(\text{div }(\sigma - \Sigma), v) + \lambda(u - U, v) = 0, \quad \forall v \in V_N, \tag{2.5}$$

where λ is chosen to be independent of $t \in \mathbf{J}$ such that

$$(\alpha\phi, \phi) + (\beta\varphi, \phi) + \lambda(\varphi, \varphi) \geqslant c^*(\| \phi \|_0^2 + \| \varphi \|_0^2).$$

LEMMA 2.2 : *There exists a unique solution pair* (U, Σ) *of (2.4)-(2.5). For any given* $t \in \bar{\mathbf{J}}$, *the mapping*

$$(u, \sigma)(\, . \, , t) \in V \times H \to (U, \Sigma) \in V_N \times H_N$$

is a projection.

Proof : Since the linear system (2.4)-(2.5) is square, we need only verify uniqueness in order to also have existence. Suppose there exists another pair (U', Σ') solving (2.4)-(2.5). Then take $\chi = \Sigma - \Sigma'$ in (2.4) and $v = U - U'$ in (2.5) and add the resulting equations to get :

$$(\alpha(\Sigma - \Sigma'), (\Sigma - \Sigma')) + (\beta(U - U'), (\Sigma - \Sigma')) + \lambda(U - U', U - U') = 0,$$

where then the coercivity above yields uniqueness. This also proves the mapping defined in (2.4)-(2.5) is a projection. $\qquad\square$

Let us ease the notation by setting,

$$\xi = U - u_N, \quad \text{and} \quad \zeta = \Sigma - \sigma_N, \tag{2.6}$$

so that $\|(\xi, \zeta)\|$ is the distance from the elliptic projection to the finite element solution, and

$$\eta = u - U, \quad \text{and} \quad \rho = \sigma - \Sigma, \tag{2.7}$$

so that $\|(\eta, \rho)\|$ is the distance from the elliptic projection to the exact solution.

In order to estimate these distances and the corresponding distances of (η_t, ρ_t) — their partial derivatives with respect to time, we need three lemmas modelled after [11], Lemma 3.1 and Thm. 4.1 which in turn are modelled after [5]. Lemma 2.3 yields a kind of (super-) stability result á la [5].

LEMMA 2.3 [[11], Lemma 3.1] : *Let* $p \in H$, $Q \in (L^2(\Omega))^2$, *and* $q \in L^2(\Omega)$. *If* $\tau \in V_N$ *satisfies*

$$(\alpha p, \chi) - (\operatorname{div} \chi, \tau) + (\beta \tau, \chi) = (Q, \chi), \quad \forall \chi \in H_N,$$

$$(\operatorname{div} p, v) + \lambda(\tau, v) = (q, v), \quad \forall v \in V_N, \tag{2.8}$$

then there exists a constant $C = C(\alpha, \beta, \lambda, \Omega)$, *such that,*

$$\|\tau\|_0 \leq C[p^{-1/2} \|p\|_0 + p^{-2} \|\operatorname{div} p\|_0 + \|Q\|_0 + \|q\|_0]. \tag{2.9}$$

The proof relies on a duality argument (originating in [10]) and coincides with the special case of the proof in [11] (with their $\theta = 2$).

The estimates for η, ρ, and $\operatorname{div} \rho$ are given in [11], here called Lemma 2.4.

LEMMA 2.4 [[11], Theorem 4.1] : *Assume that the solution u at any time in* **J** *is in* $H^2(\Omega)$. *Then there is a positive constant C, independent of p such that, for* $m \geq 2$

$$\|u - U\|_0 \leq Cp^{1-m}\|u\|_m, \tag{2.10}$$

$$\|\sigma - \Sigma\|_0 \leq Cp^{3/2-m}\|u\|_m, \tag{2.11}$$

$$\|\operatorname{div}(\sigma - \Sigma)\|_0 \leq Cp^{2-m}\|u\|_m. \tag{2.12}$$

Since the regularity requirements are less than those of [11] (where they required $u \in H^{7/2}$ used to bound some nonlinearities) we would like to present a proof.

Proof : Let

$$\theta = u - P_N u, \quad \mathsf{P} = \sigma - \Pi_N \sigma, \tag{2.13}$$

$$\tau = P_N u - U, \quad \delta = \Pi_N \sigma - \Sigma, \tag{2.14}$$

then

$$\eta = u - U = (u - P_N u) + (P_N u - U) = \theta + \tau, \tag{2.15}$$

$$\rho = \sigma - \Sigma = (\sigma - \Pi_N \sigma) + (\Pi_N \sigma - \Sigma) = \mathsf{P} + \delta. \tag{2.16}$$

To get the estimate for η we need only concentrate on one for τ. If we take $\tau = P_N u - U$, $\rho = \sigma - \Sigma$, $Q = (P_N u - u)\beta$, and $q = 0$ in (2.8), we recover a rewriting of (2.4) and (2.5). Hence, by the previous lemma,

$$\|\tau\|_0 \leq C[p^{-1/2}\|\rho\|_0 + p^{-2}\|\operatorname{div}\rho\|_0 + \|Q\|_0]$$

$$\leq C[p^{-1/2}(\|\mathsf{P}\|_0 + \|\delta\|_0) + p^{-2}(\|\operatorname{div}\mathsf{P}\|_0 + \|\operatorname{div}\delta\|_0) + \|\theta\|_0]$$

$$\leq C[p^{-1/2}(\mathrm{I} + \mathrm{II}) + p^{-2}(\mathrm{III} + \mathrm{IV}) + \mathrm{V}],$$

where each of the terms enumerated with an odd roman numeral satisfies

$$\mathrm{I} = \|\mathsf{P}\|_0 = \|\sigma - \Pi_N \sigma\|_0 \leq Cp^{1/2-r}\|\sigma\|_r \leq Cp^{3/2-m}\|u\|_m, \quad m > 3/2,$$

$$\mathrm{III} = \|\operatorname{div}\sigma - P_N \operatorname{div}\sigma\|_0 \leq Cp^{-r}\|\operatorname{div}\sigma\|_r \leq Cp^{2-m}\|u\|_m, \quad m \geq 2,$$

$$\mathrm{V} = \|\theta\|_0 = \|u - P_N u\|_0 \leq Cp^{-m}\|u\|_m, \quad m \geq 0.$$

For the term II, we state (2.4)-(2.5) once more in our recently introduced notation and split off some terms to the right hand side :

$$(\alpha\delta, \chi) - (\tau, \operatorname{div}\chi) + (\beta\tau, \chi) = -(\alpha P, \chi) - (\theta\beta, \chi), \quad \forall\chi \in H_N, \quad (2.17)$$

$$(\operatorname{div}\delta, v) + \lambda(\tau, v) = 0, \quad \forall v \in V_N. \quad (2.18)$$

Now let $\chi = \delta$ and $v = \tau$ in the sum equation (2.17) + (2.18) to obtain

$$c^*(\|\delta\|_0^2 + \|\tau\|_0^2) \leq (\alpha\delta, \delta) + (\beta\tau, \delta) + \lambda(\tau, \tau)$$

$$= -(\alpha P, \delta) - (\theta\beta, \delta)$$

$$\leq C(\|P\|_0 + \|\theta\|_0)\|\delta\|_0,$$

using coercivity once more so that

$$\mathrm{II} = \|\delta\|_0 \leq C(\|P\|_0 + \|\theta\|_0)$$

$$\leq Cp^{1/2-r}\|\sigma\|_r + Cp^{-m}\|u\|_m$$

$$\leq Cp^{3/2-m}\|u\|_m, \quad m > 3/2.$$

To obtain the last bound, one on IV, we take $v = \operatorname{div}\delta$ in (2.18),

$$\mathrm{IV} = \|\operatorname{div}\delta\|_0 \leq C\|\tau\|_0$$

$$\leq C(\|P\|_0 + \|\theta\|)^{1/2}(\|\delta\|_0)^{1/2}$$

$$\leq Cp^{3/2-m}\|u\|_m, \quad m > 3/2.$$

using the bounds above involving $\|\delta\|_0$.   □

Remark 2.5 : We note that (2.12) is sharp, given the regularity or shift inequalities : $\|\operatorname{div}\sigma\|_{m-2} \leq C\|\sigma\|_{m-1} \leq C\|u\|_m$ $(m \geq 2)$. Regarding (2.11), since we employ an estimate of a term like $\|P\|_0$ involving the Raviart-Thomas projection for which the estimate (2.3) will be proven to sharp, we could not expect a better bound that (2.11). One may actually formulate (2.10) differently with the highest order term on the right hand side being $Cp^{-r}\|\sigma\|_r$ — stemming from estimating the term I by (2.3), involved in our use of the super-stability lemma above (with only a half mitigating power of p) — (other terms being bounded by $(Cp^{-m}\|u\|_m)$, for which we can hope no better. That would require an improved version of Lemma 2.3.

We now give estimates for η_t, ρ_t, and $\operatorname{div}\rho_t$.

LEMMA 2.6 : *Assume that the solution u and its partial time derivative u_t are in $H^2(\Omega)$ for all $t \in \mathbf{J}$. Then there is a positive constant C, independent of p such that, for $m \geqslant 2$*

$$\| u_t - U_t \|_0 \leqslant Cp^{3/2 - m} (\| u \|_m + \| u_t \|_m) \, ,$$

$$\| \sigma_t - \Sigma_t \|_0 \leqslant Cp^{3/2 - m} (\| u \|_m + \| u_t \|_m) \, ,$$

$$\| \operatorname{div} (\sigma_t - \Sigma_t) \|_0 \leqslant Cp^{2 - m} (\| u \|_m + \| u_t \|_m) \, .$$

Proof : Rewriting (2.4) and (2.5) using the nomenclature introduced in (2.7) yields :

$$(\alpha\rho, \chi) - (\operatorname{div} \chi, \eta) + (\beta\eta, \chi) = 0 \, , \quad \forall \chi \in H_N,$$

$$(\operatorname{div} \rho, v) + \lambda(\eta, v) = 0 \, , \quad \forall v \in V_N.$$

Now, differentiate the above two equations with respect to time (taking χ and v to be independent of time) :

$$(\alpha\rho_t, \chi) - (\operatorname{div} \chi, \eta_t) + (\beta\eta_t, \chi) = - (\alpha_t \rho, \chi) - (\beta_t \eta, \chi) \, , \quad \forall \chi \in H_N,$$

$$(\operatorname{div} \rho_t, v) + \lambda(\eta_t, v) = 0 \, , \quad \forall v \in V_N,$$

noting that this operator $(\partial/\partial t)$ commutes with the spatial operators P_N, Π_N and div (e.g. $((P_N v)_t = P_N v_t)$ and that

$$(\operatorname{div} \chi, \theta_t) = (v, \theta_t) = 0 \, , \quad \forall \chi \in H_N \quad \forall v \in V_N,$$

$$(\operatorname{div} \mathsf{P}, v) = (\operatorname{div} \mathsf{P}_t, v) = 0 \, , \quad \forall v \in V_N.$$

To estimate η_t, it suffices to bound τ_t, since $\eta_t = \tau_t + \theta_t$ and θ_t can be bounded using (2.1). Observe that

$$(\alpha\delta_t, \chi) - (\operatorname{div} \chi, \tau_t) + (\beta\tau_t, \chi) = - (\alpha\mathsf{P}_t, \chi) - (\alpha_t \rho, \chi)$$

$$- (\beta_t \eta, \chi) - (\beta\theta_t, \chi) \, , \quad \forall \chi \in H_N,$$

$$(\operatorname{div} \delta_t, v) + \lambda(\tau_t, v) = 0 \, , \quad \forall v \in V_N. \tag{2.19}$$

Take $\chi = \delta_t$ and $v = \tau_t$ in these last two equations and add them up to get :

$$(\alpha\delta_t, \delta_t) + \lambda \| \tau_t \|_0^2 + (\beta\tau_t, \delta_t) = - (\alpha\mathsf{P}_t, \delta_t) - (\alpha_t \rho, \delta_t)$$

$$- (\beta_t \eta, \delta_t) - (\beta\theta_t, \delta_t) \, .$$

Using coercivity we get

$$\|\delta_t\|_0 + \|\tau_t\|_0 \leq C(\|\mathsf{P}_t\|_0 + \|\rho\|_0 + \|\eta\|_0 + \|\theta_t\|_0) .$$

It also follows from (2.19) and the fact that $\operatorname{div}\delta_t \in V_N$ that

$$\|\operatorname{div}\delta_t\|_0 \leq \lambda \|\tau_t\|_0.$$

Now,

$$\|\eta_t\|_0 \leq \|\theta_t\|_0 + \|\tau_t\|_0 = \|u_t - P_N u_t\|_0 + \|\tau_t\|_0,$$

$$\|\rho_t\|_0 \leq \|\mathsf{P}_t\|_0 + \|\delta_t\|_0 = \|\sigma_t - \Pi_N \sigma_t\|_0 + \|\delta_t\|_0,$$

$$\|\operatorname{div}\rho_t\|_0 \leq \|\operatorname{div}(\sigma - \Pi_N \sigma)_t\|_0 + \lambda \|\tau_t\|_0.$$

These inequalities suffice to prove the assertion of Lemma 2.6.   □

3. TRANSIENT ERROR ANALYSIS

In this section we first present the L^2 and L^∞-error estimates for $\zeta = \Sigma - \sigma_N$ and $\xi = U - u_N$.

PROPOSITION 3.1 : *Let u and $u_t \in L^2(H^m)$ for $m \geq 2$ where u is the solution of (1.1), and let (u_N, σ_N) and (U, Σ) be as defined in (1.6-1.8) and (2.4-2.5), respectively. Let $(\xi, \zeta) = (U, \Sigma) - (u_N, \sigma_N)$. Then there is a positive constant C, independent of p such that, for $m \geq 2$,*

$$\|\xi\|_{L^\infty(L^2)} + \|\zeta\|_{L^\infty(L^2)} \leq Cp^{3/2-m}(\|u\|_{L^2(H^m)} + \|u_t\|_{L^2(H^m)}) .$$

Proof : Subtracting (2.4, 2.5) and (1.6, 1.7) from (1.3, 1.4) yields :

$$(\alpha\zeta, \chi) - (\xi, \operatorname{div}\chi) + (\beta\xi, \chi) = 0, \quad \chi \in H_N, \tag{3.1}$$

$$(c(u_t - u_{N,t}), v) + (\operatorname{div}\zeta, v) + \lambda(\xi, v) = \lambda(\eta, v) + \lambda(\xi, v), v \in V_N, \tag{3.2}$$

take

$$\chi = \Sigma - \sigma_N = \zeta, \quad \text{and } v = U - u_N = \xi,$$

then (3.1) becomes

$$(\alpha\zeta, \zeta) - (\xi, \operatorname{div}\zeta) + (\beta\xi, \zeta) = 0,$$

$$(c\xi_t, \xi) + (\operatorname{div}\zeta, \xi) + \lambda(\xi, \xi) = -(c\eta_t, \xi) + \lambda(\eta, \xi) + \lambda(\xi, \xi),$$

add them up to get

$$\frac{1}{2}\frac{d}{dt}(c\xi, \xi) + (\alpha\zeta, \zeta) + (\beta\xi, \zeta) + \lambda(\xi, \xi) =$$

$$\frac{1}{2}(c_t\xi, \xi) - (c\eta_t, \xi) + \lambda(\eta, \xi) + \lambda(\xi, \xi).$$

Bound the terms on the right hand side by the Cauchy-Buniakowsky-Schwarz inequality, integrate with respect to time, observing that $u_N(0) = U(0)$, and employ the Gronwall inequality to get

$$\|\xi(t)\|_0^2 + \|\zeta\|_{L^2(L^2)}^2 + \|\xi\|_{L^2(L^2)}^2 \leq C(\|\eta_t\|_{L^2(L^2)}^2 + \|\eta\|_{L^2(L^2)}^2). \quad (3.3)$$

where C then depends on t.

Next, we give an estimate for $\|\zeta(t)\|_0^2$. Use (3.2) with $v = \xi_t$ to obtain

$$(c\xi_t, \xi_t) + (\operatorname{div}\zeta, \xi_t) + \lambda(\xi, \xi_t) = -(c\eta_t, \xi_t) + \lambda(\eta, \xi_t) + \lambda(\xi, \xi_t) \quad (3.4)$$

and differentiate throughout (3.1) with respect to t:

$$(\alpha\zeta_t, \chi) - (\xi_t, \operatorname{div}\chi) + \left(\frac{\partial}{\partial t}(\beta\xi), \chi\right) = -(\alpha_t\zeta, \chi) \quad (3.5)$$

(with χ independent of time). Now choose $\chi = \zeta(\bar{t})$ in the sum of (3.4) and (3.5), then a short calculation yields the following identity at an arbitrary $\bar{t}$:

$$(c\xi_t, \xi_t) + \frac{1}{2}\frac{d}{dt}\{(\alpha\zeta, \zeta) + \lambda(\xi, \xi) + (\beta\xi, \zeta)\} =$$

$$- (c\eta_t, \xi_t) + \lambda(\eta, \xi_t) + \lambda(\xi, \xi_t) - \frac{1}{2}\{(\alpha_t\zeta, \zeta) +$$

$$+ (\beta\xi, \zeta_t) - (\partial(\beta\xi)/\partial t, \zeta)\}. \quad (3.6)$$

We proceed to bound each term on the right-hand-side of (3.6) in turn:

$$|(c\eta_t, \xi_t)| \leq C_\epsilon\|\eta_t\|_0^2 + \epsilon\|\xi_t\|_0^2$$

with the two terms containing a factor λ majorizable in a very similar fashion (also using (3.3)), and further in (3.6),

$$|(\alpha_t \zeta, \zeta)| \leqslant \|\alpha_t\|_\infty \|\zeta\|_0^2$$

$$\left| \int_0^t (\beta \xi, \zeta_t)\, d\bar{t} \right| = \left| \left(\int_0^t \beta \xi \zeta_t\, d\bar{t}, 1 \right) \right|$$

$$= \left| -\left(\int_0^t (\beta \xi)_t \zeta\, d\bar{t}, 1 \right) + [(\beta \xi, \zeta)]_0^t \right|$$

$$= \left| -\int_0^t ((\beta \xi)_t, \zeta)\, d\bar{t} + (\beta \xi, \zeta)(t) \right|, \text{ where}$$

$$|(\beta \xi, \zeta)(t)| \leqslant C_\epsilon \|\xi(t)\|_0^2 + \epsilon \|\zeta(t)\|_0^2, \text{ and lastly}$$

$$|(\partial(\beta \xi)/\partial t, \zeta)| \leqslant C_\epsilon \|\zeta\|_0^2 + \epsilon \|\xi_t\|_0^2 + C\|\xi\|_0^2.$$

In the second estimation we used integration in time over the interval $(0, t)$ — which we will perform on all the terms of (3.6) in a moment — here we did it by parts and used that $\xi(0) = (U - u_N)(0) = 0$ by (1.8).

Integrating (3.6) with respect to time from 0 to t and applying coercivity we get

$$\|\xi_t\|_{L^2(L^2)}^2 + \|\xi(t)\|_0^2 + \|(\zeta(t))\|_0^2 \leqslant$$

$$C_\epsilon \|\eta_t\|_{L^2(L^2)}^2 + \epsilon C \|\xi_t\|_{L^2(L^2)}^2 + C_\epsilon \|\xi\|_{L^2(L^2)}^2 + C\|\zeta(0)\|_{L^2}^2$$

$$+ C_\epsilon \|\eta\|_{L^2(L^2)}^2 + (\|\alpha_t\|_{L^\infty(L^\infty)} + C_\epsilon)\|\zeta\|_{L^2(L^2)}^2 + C_\epsilon \|\xi(t)\|_0^2 + \epsilon \|\zeta(t)\|_0^2.$$

Note that $\|\zeta(0)\|_0^2 = 0$ due to the facts $\xi(0) = 0$ and that we may take $\chi = \zeta(0)$ in (3.1). Using (3.3) we get

$$\|\xi\|_{L^\infty(L^2)}^2 + \|\zeta\|_{L^\infty(L^2)}^2 \leqslant C(\|\eta\|_{L^2(L^2)}^2 + \|\eta_t\|_{L^2(L^2)}^2). \tag{3.7}$$

Using previous lemmas in (3.7), one obtains the inequality stated in the proposition. $\qquad\square$

We may now state and prove our main result regarding estimates for the semi-discrete problem in time :

THEOREM 3.2 : *Let u and $u_t \in L^2(H^m)$ for $m \geq 2$ and let (u, σ) and (u_N, σ_N) be as defined in (1.1) and (1.6-1.8), respectively. Then there is a positive constant C, independent of p such that, for $m \geq 2$*

$$\| u - u_N \|_{L^\infty(L^2)} + \| \sigma - \sigma_N \|_{L^\infty(L^2)} \leq C p^{3/2 - m} (\| u \|_{L^2(H^m)} + \| u_t \|_{L^2(H^m)}) .$$

Proof : The proof follows directly from the previous lemmas.

Remark 3.3 : For the equation in (1.1) of parabolic type, one may well hope that parabolic smoothing will occur so that $u(\, . \, , t)$ and $u_t(\, . \, , t)$ will be rather smooth (read lie in $H^m(\Omega)$ for $m \gg 2$). If this is indeed the case, the error estimate in Thm. 3.2 will yield superior rate of convergence for the error as compared to an h version with a fixed p once an initial transient has passed by.

4. p-STABILITY OF MIXED METHOD SPATIAL DISCRETIZATIONS AND A LOWER BOUND FOR THE RAVIART-THOMAS PROJECTION

In this section we delve a little more into the discretization of the elliptic problem. Since it is important for existence, uniqueness and error estimations for the elliptic problem to have elements that satisfy the general requirements in the saddle-point framework of [1] and [2], we will prove certain norm estimates for a right inverse of the divergence operator in $\mathscr{B}(L^2, \mathbf{H}(\operatorname{div}))$ acting between the discrete subspaces. We shall then use this to prove stability of the Brezzi-Douglas-Marini subspaces and the very same technique is valid for the Raviart-Thomas case. We conclude with a proof that one cannot in general improve on (2.3).

Let (for the purposes of simple L^2-orthogonality as before) Ω be partitioned into parallelograms $E \in \mathscr{T}_N$ affinely mapped onto $R = (-1, 1)^2$ by F_E. We introduce the finite dimensional subspaces of V and H :

$$V_N^{BDM} = \left\{ v \in L^2(\Omega) : v|_E \circ F_E^{-1} \in \mathscr{P}^{p-1}, \forall E \in \mathscr{T}_N \right\}$$

$$H_N^{BDM} = \left\{ \chi \in \mathbf{H}(\operatorname{div}, \Omega) : \chi|_E \circ F_E^{-1} \in (\mathscr{P}^p \times \mathscr{P}^p) \oplus \right.$$
$$\left. \operatorname{span} \left\{ (x^{p+1}, - x^p y)^T, (xy^p, - y^{p+1})^T \right\} \right\},$$

where $\mathscr{P}^p$ is the set of polynomials (in two variables x and y) of total degree at most p.

In order to have hope for quasi-optimal error estimates for a mixed finite element method based on these spaces, for a saddle point formulation of the spatial problem, it is necessary to investigate the question of stability : Does there exist a sequence of positive numbers $\gamma^{\mathrm{BDM}}(p)$ satisfying

$$\min_{v_N \in V_N^{\mathrm{BDM}}} \max_{\tau_N \in H_N^{\mathrm{BDM}}} \frac{(v_N, \nabla \cdot \tau_N)_\Omega}{\|v_N\|_{L^2} \|\tau_N\|_{H(\mathrm{div})}} = \gamma^{\mathrm{BDM}}(p) \qquad (4.1)$$

that is bounded uniformly from below ? Note the formal resemblance with the inf-sup constant present in p-discretizations of Stokes problem and that we again desire to invert the divergence operator but now in $\mathscr{B}(L^2, \mathbf{H}(\mathrm{div}))$ (in lieu of $\mathbf{H}^1$ as was the case in [8]).

Let $(\nabla \cdot)^{-1} : V_N^{\mathrm{BDM}} \to H_N^{\mathrm{BDM}}$, $p \geq 1$, be the operator which associates to any $q \in V_N^{\mathrm{BDM}}$ one of the — clearly many — choice of $\chi \in H_N^{\mathrm{BDM}}$ of minimal $\| \cdot \|_N$-norm satisfying $\nabla \cdot \chi = q$.

PROPOSITION 4.1 : *The operator* $(\nabla \cdot)^{-1} : V_p^{\mathrm{BDM}} \to H_p^{\mathrm{BDM}}$, $p \geq 1$, *considered as an operator from a subspace of* $L^2(R)$ *to a subspace of* $\mathbf{H}(\mathrm{div})$ *satisfies*

$$\|(\nabla \cdot)^{-1}\|_{\mathscr{B}(L^2, \mathbf{H}(\mathrm{div}))} \leq C$$

with a positive constant C *independent of* p.

Before we prove this proposition, allow us to note some properties of Legendre polynomials ; *cf.* [14] or [17].

Denote by ℓ_n the n-th Legendre polynomial (with point value $+1$ at $+1$) of degree n and L^2-norm :

$$\int_{-1}^{1} \ell_n^2 \, dx = \frac{2}{2n+1}. \qquad (4.2)$$

By induction and L^2-orthogonality,

$$\int_{-1}^{x} \ell_n(t) \, dt = \frac{1}{2n+1} (\ell_{n+1} - \ell_{n-1})(x), \text{ for } n \geq 1. \qquad (4.3)$$

These two identities yield the following inequality :

$$\left\| \sum_{m, n \in \mathcal{S}} \alpha_{mn} \left(\int_{-1}^{x} \ell_m \right) \ell_n(y) \right\|_{L^2(R)}^2 \leq C \sum_{m, n \in \mathcal{S}} \alpha_{mn}^2 \frac{1}{(m+1)^3} \frac{1}{n+1} \qquad (4.4)$$

if the right-hand-side converges (e.g., if $\mathcal{I}$ if finite) and where C is independent of $\mathcal{I}$.

Proof of Proposition 4.1 : Fix an $E \in \mathcal{T}_N$ and denote, for simplicity's sake, $q^{(E)} = q|_E \circ F_E^{-1}$, as an example. For any given $q^{(E)} \in \mathcal{P}^{p-1}$, there exist $\{\alpha_{mn}\}$ such that

$$q^{(E)} = \sum_{\substack{m, n \geq 0}}^{m+n<p} \alpha_{mn}\, \ell_m(x)\, \ell_n(y)\,. \tag{4.5}$$

let

$$\chi^{(E)} = \begin{bmatrix} \sum_{m>n} \alpha_{mn} \left(\int_{-1}^{x} \ell_m \right) \ell_n(y) \\[2ex] \sum_{m \leq n} \alpha_{mn} \ell_m(x) \left(\int_{-1}^{y} \ell_n \right) \end{bmatrix}.$$

Then $\nabla \cdot \chi^{(E)} = q^{(E)}$. So

$$\| \nabla \cdot \chi^{(E)} \|_0 = \| q^{(E)} \|_0\,,$$

$$\| q^{(E)} \|_0^2 \simeq \sum_{m, n} \alpha_{mn}^2\, \frac{1}{(m+1)}\, \frac{1}{(n+1)}\,,$$

and, from (4.2)-(4.4),

$$\| \chi_1^{(E)} \|_0^2 \leq C \sum_{\substack{m>n}}^{m+n<p} \alpha_{mn}^2\, \frac{1}{(m+1)^3}\, \frac{1}{(n+1)} \leq C \| q^{(E)} \|_0^2$$

with a similar estimation for $\chi_2^{(E)}$ proving the uniform bound sought in the case of only one element. Note that $\chi^{(E)} \cdot \nu = 0$ for $x = \pm 1$ and also for $y = -1$. At $y = +1$, $\chi^{(E)} \cdot \nu = \alpha_{00}$, a constant. Further, if we exclude the one term α_{00} in $q^{(E)}$ and hence in $\chi_2^{(E)}$, also $\chi^{(E)} \cdot \nu = 0$ for $y = +1$.

Accordingly, in the case of multiple elements, where we have to construct $\chi \in H$ with normal components continuous across inter-element boundaries, we split

$$q = q_0 + q_+ \quad \text{with} \quad q_0^{(E)} = \alpha_{00}\, \ell_0\, \ell_0$$

and $q_+^{(E)}|_E$ of the form (4.5) with $(m, n) \neq (0, 0)$. For q_+ we see χ_+ constructed as above, when pieced together does indeed belong to $\mathbf{H}(\mathrm{div})$

(since $\chi_+^{(E)} \cdot v = 0$ for $x = \pm 1$ and $y = \pm 1$). For q_0 which is discontinuous and piecewise constant, there exists — by the following lemma, e.g. — a $\chi_0 \in \{v \in H : v^{(E)} \in [Q^2]^2\}$ such that $\nabla \cdot \chi_0 = q_0$ and $\chi_0 \cdot v$ is continuous across inter-element boundaries. Also, $\|\chi_0\|_H \leq C \|q\|_0$. In conclusion, $\chi_0 + \chi_+$ satisfies the requirements.   □

The next Lemma is actually a simple consequence of the well-known (among Stokes people) divergence stability of $([Q^2]^2, Q^0)$ on parallelogram elements, cf. Example 11.1 in [9]. We include a very simple proof sufficing for our purposes.

LEMMA 4.2 : *Let q_0 be discontinuous and piecewise constant on each element E of $\mathcal{T}_N$ — a partitioning of Ω into parallelograms. Then there exists a $\chi_0 \in \{v \in H : v^{(E)} \in [Q^2]^2\}$ such that $\nabla \cdot \chi_0 = q_0$ and $\chi_0 \cdot n$ is continuous across inter-element boundaries. Furthermore, $\|\chi_0\|_H \leq C \|q_0\|_0$.*

Proof : We meet the objective by a simple counting argument. $[Q^2]^2$ has $18 (= 2 \times 3^2)$ degrees of freedom for which we use at most $6 (= 4 \times 3 \times 1/2)$ to ensure the continuity of $\chi_0 \cdot n$ on the four sides of E whereas div $\chi_0 = q_0$ requires 8 linear equations. Choosing a χ_0 of minimal H-norm, ensures the norm bound with a C independent of p. Note that $\mathcal{Q}^2 \subset \mathcal{P}^4$.   □

COROLLARY 4.3 : *With the above, there exists $\gamma > 0$ independent of p such that*

$$\gamma^{\mathrm{BDM}}(p) > \gamma$$

i.e., the Brezzi-Douglas-Marini elements are p-stable.

Proof : It follows directly from the norm estimate in the previous proposition and the form of the inf-sup constant.   □

Remark 4.4 : This establishes that the Brezzi-Douglas-Marini elements are stable also with respect to p and improves on the statement of a lower bound $\gamma^{\mathrm{BDM}}(p) \geq c_\epsilon p^{-\epsilon}$ proved earlier in [16], Thm. 4.1. Our reason for not pursuing BDM-elements further in this paper — despite the comparatively smaller number of degrees of freedom — is that we do not know how to improve on the projection estimate in [16], Lemma 3.3 where one loses a whole power in p (as opposed to only 1/2 in (2.3)) relative to the L^2 projection.

The case of Raviart-Thomas elements, [13] may be handled in a very similar way.

PROPOSITION 4.5 : *The operator $(\nabla \cdot)^{-1} : V_N \to H_N, p \geq 1$, considered as an operator from a subspace of L^2 to a subspace of $\mathbf{H}(\mathrm{div})$ satisfies*

$$\|(\nabla \cdot)^{-1}\|_{\mathscr{B}(L^2, H(\mathrm{div}))} \leq C$$

with a positive constant C independent of p.

COROLLARY 4.6 : *With the above, there exists $\gamma > 0$ independent of p such that*

$$\gamma(p) > \gamma$$

i.e., the Raviart-Thomas elements are p-stable.

Remark 4.7 : This is the same result as in [16], Thm. 4.1, but with an alternate (perhaps simpler) proof.

LEMMA 4.8 : *There exist $\chi_* \in H \cap H^r$ with $r > 1$ for which their Raviart-Thomas projection onto H_N given by (2.2), $\Pi_N \chi^*$, satisfy the following estimate :*

$$\| \Pi_N \chi^* - \chi_* \|_0 \geq c_* p^{3/2 - 2r} (\log p)^{1/2}, \tag{4.6}$$

where $c_* > 0$ is a constant independent of p but depending on χ_*.

Proof : Let

$$\chi_* = \begin{bmatrix} \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n} \ell_m(x) \ell_n(y) \\ 0 \end{bmatrix}$$

and let

$$\Pi_N \chi^* = \begin{bmatrix} \displaystyle\sum_{m=0}^{p+1} \sum_{n=0}^{p} \tilde{\alpha}_{m,n} \ell_m(x) \ell_n(y) \\ \displaystyle\sum_{m=0}^{p} \sum_{n=0}^{p+1} \tilde{\beta}_{m,n} \ell_m(x) \ell_n(y) \end{bmatrix}.$$

From [11], identities (2.5)-(2.7), it follows easily that $\tilde{\beta}_{m,n} = 0$ for all m, n and that

$$\tilde{\alpha}_{m,n} = \alpha_{m,n}, \qquad 0 \leq m \leq p-1, \qquad 0 \leq n \leq p,$$

$$\tilde{\alpha}_{p,n} = \sum_{m=0}^{\infty} \alpha_{2m+p,n}, \quad \tilde{\alpha}_{p+1,n} = \sum_{m=0}^{\infty} \alpha_{2m+p+1,n}, \quad 0 \leq n \leq p.$$

One then may compute that

$$\| \chi_* - \Pi_N \chi^* \|_0^2 = \sum_{m=p}^{p+1} \sum_{n=0}^{p} \frac{4(\alpha_{m,n} - \tilde{\alpha}_{m,n})^2}{(2m+1)(2n+1)}$$

$$+ \sum_{m=p+2}^{\infty} \sum_{n=0}^{p} \frac{4(\alpha_{m,n})^2}{(2m+1)(2n+1)}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=p+1}^{\infty} \frac{4(\alpha_{m,n})^2}{(2m+1)(2n+1)}$$

$$= I + II + III .$$

As in [11] (or [16]), II & III can be bounded better than (2.3) :

$$II + III \leqslant Cp^{-2r} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha_{m,n})^2(1 + m^2 + n^2)^r}{(2m+1)(2n+1)} , \quad r \geqslant 0$$

$$\leqslant Cp^{-2r} \| \chi \|_r^2,$$

by [3]. We may verify that

$$I = \frac{4}{2p+1} \sum_{n=0}^{p} \frac{\left(\sum_{m=1}^{\infty} \alpha_{2m+p,n} \right)^2}{2n+1} + \frac{4}{2p+3} \sum_{n=0}^{p} \frac{\left(\sum_{m=1}^{\infty} \alpha_{2m+p+1,n} \right)^2}{2n+1}$$

Now let us choose, for a given $r > 1$,

$$\alpha_{m,n}^* = (1 + m^2 + n^2)^{-r}(1 + 2m) , \quad m, n \geqslant 0 .$$

A straightforward calculation confirms that $\chi_* = \sum_{m,n}^{\infty} \alpha_{m,n}^* \ell_m \ell_n \in H^r$. With this choice, we also readily verify that

$$\sum_{m=1}^{\infty} \alpha^*_{2m+p,\,n} = \sum_{m=1}^{\infty} (1 + (2m+p)^2 + n^2)^{-r} (1 + 2(2m+p))$$

$$= \sum_{\iota = p+2,\, \iota - p \text{ even}}^{\infty} (1 + i^2 + n^2)^{-r} (1 + 2i) \quad [\text{with } 0 \leqslant n \leqslant p]$$

$$\geqslant \sum_{\iota = p+2,\, \iota - p \text{ even}}^{\infty} (1 + i^2 + p^2)^{-r} (2i)$$

$$\geqslant \int_{p+4}^{\infty} (1 + x^2 + p^2)^{-r} (2x)\, dx$$

$$\geqslant \int_{1 + (p+4)^2 + p^2}^{\infty} t^{-r}\, dt$$

$$\geqslant cp^{2-2r},$$

so that

$$I \geqslant cp^{-1} p^{4-4r} \sum_{n=0}^{p} \frac{1}{2n+1} \geqslant c_* p^{3-4r} \log p,$$

which proves the assertion of the lemma.   □

COROLLARY 4.8 : *The estimate (2.3) is sharp, i.e. there cannot be an estimate in its stead of the form :*

$$\| \Pi_N \chi - \chi \|_0 \text{ " } \leqslant \text{ " } C_\chi p^{\kappa - r}, \quad r > 1, \quad \kappa < 1/2,$$

with C_χ independent of p that is valid for general $\chi \in H^r$, $r > 1$.

Proof : Such an estimate would, when combined with the lower estimate in the previous lemma, yield $(\log p)^{1/2} p^{(3/2 - \kappa) - r} \leqslant C_\chi$. Since $\kappa < 1/2$, $3/2 - \kappa > 1$ and we could choose r to satisfy $1 < r < 3/2 - \kappa$ to obtain a contradiction.   □

In this final portion of the section we describe two ways in which one could extend the results in this paper : one is that it is possible to treat curved domains and also having triangles in the geometric decomposition of the domain as long as they border the boundary. Another is a new mixed method projection which we will describe by ways of an example.

Remark 4.9 : Suppose $\tilde{K}$ is a triangle bordering $\partial\Omega$, which may be mapped affinely onto the triangle T below by the affine mapping $F_{\tilde{K}}$
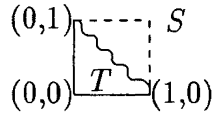
with the vertices as given and one curved side lying entirely within the unit square $S = [0, 1]^2$. The curved side of $\tilde{K}$ is assumed to be a part of $\partial\Omega$.

For our discrete spaces, we now insist that

$$\chi|_{\tilde{K}} \circ F_{\tilde{K}}^{-1} \in Q^{p+1,p} \times Q^{p,p+1} \quad \text{and}$$

$$v|_{\tilde{K}} \circ F_{\tilde{K}}^{-1} \in Q^p$$

as well as the continuity of normal components of functions χ across inter-element boundaries with $\tilde{K}$. (Note that this is richer than the usual Raviart-Thomas triangular elements.)

We extend the construction of $(\nabla \cdot)^{-1}$ given in Prop. 4.1 to such an element by,

for arbitrary $q_{\tilde{K}} \in Q^p$, let

$$\chi_{\tilde{K}} = \left(\begin{array}{c} \int_0^x q(s, t)\, ds \\ 0 \end{array} \right) = \left(\begin{array}{c} \chi_1 \\ 0 \end{array} \right).$$

Observe that $\chi_{\tilde{K}} \in Q^{p+1,p} \times Q^{p,p+1}$, $\nabla \cdot \chi_{\tilde{K}} = q_{\tilde{K}}$ and that $\chi_{\tilde{K}} \cdot v = 0$ on the two straight edges of $T : (0, 1) \times \{0\}$ and $\{0\} \times (0, 1)$. Thus $\chi_{\tilde{K}}$, as constructed here, will match with χ_E, as constructed earlier in Prop. 2.1 having vanishing normal components on inter-element boundaries. Also,

$$\|\nabla \cdot \chi\|_{0,T} = \|q\|_{0,T}$$

and, by the Poincaré-Friedrichs inequality,

$$\|\chi\|_{0,T} = \|\chi_1\|_{0,T} \leqslant \|\chi_{1,x}\|_{0,T} = \|q\|_{0,T}$$

guaranteeing that we may extend the uniform bound on the operator norm of $(\nabla \cdot)^{-1}$ to include curved boundary triangles.

We conclude this remark by showing that one may extend the Raviart-Thomas projection to this case and that the error-estimate (2.3) for the Raviart-Thomas projection remains valid. Note that there exists an extension operator E such that $\forall \chi \in H^r(T)$, $\exists \chi \in H^r(S)$ with

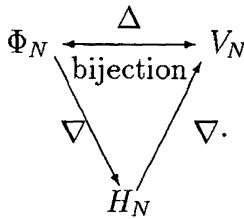$$\| E\chi \|_{r,S} \leqslant C \| \chi \|_{r,T},$$

cf. Theorem 5 in Stein's monograph [15]. Then we define the extended Raviart-Thomas projection to be $\tilde{\Pi}_N \chi = \Pi_N E\chi |_T$ so that

$$\| \tilde{\Pi}_N \chi - \chi \|_{0,T} = \| \Pi_N E\chi - E\chi \|_{0,T}$$

$$\leqslant \| \Pi_N E\chi - E\chi \|_{0,S}$$

$$\leqslant Cp^{1/2-r} \| E\chi \|_{r,S} \quad [\text{by (2.3)}]$$

$$\leqslant Cp^{1/2-r} \| \chi \|_{r,T}$$

so that boundary triangles with one curved side (along $\partial\Omega$) are allowed.

Example 4.10 : We now construct another mixed method projection Λ_N with some of the same properties as the Raviart-Thomas projection but with better error estimates. We will take — as an example — the instance of one element : let Ω be a triangle or a rectangle and let $a = 1$ and $b = g = 0$.

Let $\Phi_N \subseteq Q^{p+1} \cap H_0^1(\Omega)$ and define the discrete spaces $V_N = \Delta\Phi_N$ and $H_N = \nabla\Phi_N$. Clearly

$$\Phi_N \xleftarrow{\ \Delta\ } V_N$$

$$\underset{\text{bijection}}{\diagdown} \qquad \diagup$$

$$\nabla \diagdown \qquad \diagup \nabla.$$

$$H_N$$

Suppose $u \in L^2$ and let $\sigma = \nabla u \in H(\text{div})$, so that $\text{div}\,\sigma = \Delta u \in L^2$ and $P_N \,\text{div}\,\sigma = P_N \Delta u \in V_N$ and take P_N to be the L^2 projection onto V_N. Let ϕ solve

$$\Delta\phi = P_N \Delta u \text{ in } \Omega \tag{4.7}$$

$$\phi = 0 \text{ on } \partial\Omega \tag{4.8}$$

Then define

$$\Lambda_N \sigma \overset{\text{def}}{=} \nabla \phi \tag{4.9}$$

and we observe the crucial commutative property :

$$\text{div } \Lambda_N \sigma = P_N \text{ div } \sigma, \quad \forall \sigma \in H(\text{div}).$$

By elliptic estimates, we have the shift inequalities :

$$\|\nabla \phi\|_s \leqslant \|\phi\|_{s+1} \leqslant C \|\Delta \phi\|_{s-1} \quad \text{for} \quad s = 0, 1.$$

This immediately implies that the new projection is stable in H^1 (but, due to the inverse inequality, not necessarily in L^2) :

$$\|\Lambda_N \sigma\|_1 = \|\nabla \phi\|_1 \leqslant C \|P_N \Delta u\|_0 \leqslant C \|\Delta u\|_0 \leqslant C \|\sigma\|_1$$

as well as the error estimates :

$$\|\sigma - \Lambda_N \sigma\|_1 \leqslant \|\nabla u - \nabla \phi\|_1$$

$$\leqslant \|u - \phi\|_2$$

$$\leqslant \|\text{div } \sigma - P_N \text{ div } \sigma\|_0$$

$$\leqslant C p^{-r} \|\text{div } \sigma\|_r$$

using that

$$\|v - P_N v\|_0 = \|\Delta(\psi - \phi)\|_0 \leqslant \|\psi - \phi\|_2 \leqslant C p^{-r} \|\psi\|_{r+2} \leqslant C p^{-r} \|v\|_r.$$

We also get the quasi-optimal L^2 estimate which improves upon (2.3) :

$$\|\sigma - \Lambda_N \sigma\|_0 \leqslant \|\nabla u - \nabla \phi\|_0$$

$$\leqslant \|u - \phi\|_1$$

$$\leqslant \|\text{div } \sigma - P_N \text{ div } \sigma\|_{-1}$$

$$\leqslant C p^{-r-1} \|\text{div } \sigma\|_R$$

$$\leqslant C p^{-s} \|\sigma\|_s$$

using $s = r + 1$ and the quasi-optimality result :

$$\| v - P_N v \| = \sup_{w \in H_0^1} \frac{(v - P_N v, w)}{\| w \|_1}$$

$$= \sup_{w \in H_0^1} \inf_{w_N \in V_N} \frac{(v - P_N v, w - w_N)}{\| w \|_1}$$

$$\leqslant \sup_{w \in H_0^1} \inf_{w_N \in V_N} \frac{\| v - P_N v \|_0 \| w - w_N \|_0}{\| w \|_1}$$

$$\leqslant C p^{-1} \| v - P_N v \|_0$$

as there exists $w_N \in V_N$ so that $\| w - w_N \|_0 \leqslant C p^{-1} \| w \|_1$.

REFERENCES

[1] I. BABUŠKA, 1971, Error bounds for the finite element method, *Numer. Math.*, **16,** 322-333.

[2] F. BREZZI, 1974, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *RAIRO* **8-R.2,** 129-151.

[3] M. R. DORR, 1984, The approximation theory for the p-version of the finite element method, *SIAM J. Numer. Anal.*, **21,** 1180-1207.

[4] J. DOUGLAS Jr. and J. E. ROBERTS, 1982, Mixed finite element methods for second order elliptic problems, *Mat. Applic. Comp.*, **1,** 91-103.

[5] J. DOUGLAS Jr. and J. E. ROBERTS, 1985, Global estimates for mixed methods for second order elliptic equations, *Math. Comp.*, **44,** 39-52.

[6] S. M. F. GARCIA, 1994, Improved Error Estimates for Nonlinear Parabolic Equations — Continuous Time Case, *Numer. Methods in PDEs,* **10,** 129-147.

[7] S. Jensen, 1992, p-version of the mixed finite element method for Stokes-like problems, *Comp. Meth. Appl. Mech. Eng.,* **101,** 27-41.

[8] S. JENSEN and M. VOGELIUS, 1990, Divergence stability in connection with the p version of the finite element method, *RAIRO, Modélisation Math. Anal. Numér.,* **24-6,** 737-764.

[9] C. JOHNSON, *Numerical solutions of partial differential equations by the finite element methods,* Cambridge University Press, 1987.

[10] C. JOHNSON and V. THOMÉE, 1981, Error estimates for some mixed finite element methods for parabolic type problems, *RAIRO Anal. Numér.,* **15,** 41-78.

[11] F. A. MILNER and M. SURI, 1992, Mixed Finite Element Methods for Quasilinear Second Order Elliptic Problems : the p-version. *RAIRO, Modélisation Math. Anal. Numér.,* **24-7,** 913-931.

[12] A. QUARTERONI, 1984, Some results of Bernstein and Jackson type for polynomial approximation in L^p spaces, *Jap. J. Appl. Math.*, **1**, 173-181.

[13] P.-A. RAVIART and J. M. THOMAS, 1977, A Mixed Finite Element Method for 2-nd Order Elliptic Equations, in Mathematical Aspects of Finite Element Methods, *Lecture Notes in Mathematics*, **606**, ed. I. Galligani and E. Magenes, Springer, 292-315.

[14] G. SANSONE, *Orthogonal Functions*, Dover, Mineola, NY 1991 (orig. Interscience, 1959).

[15] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series vol. 30, Princeton University Press, NJ, 1970.

[16] M. SURI, 1990, On the stability and convergence of higher order mixed finite element methods for second order elliptic problems, *Math. Comp.*, **54**, 1-19.

[17] G. SZEGÖ, *Orthogonal Polynomials*, AMS Colloq. Publ. **23**, 1933.