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FINITE ELEMENT CONVERGENCE FOR THE DARWIN MODEL TO MAXWELL'S EQUATIONS (*)

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Abstract. — *In three dimensional polyhedral domains with a Lipschitz continuous boundary, we derive the* $H(\mathbf{curl}, \Omega)$ *and* $H(\mathbf{curl}, \operatorname{div}, \Omega)$ *variational formulations for the Darwin model of approximation to Maxwell's equations and prove the well-posedness of the variational systems. Then Nedelec's and standard finite element methods are used to solve two kinds of variational problems. Though symmetric bilinear forms in the variational systems fail to define full norms equivalent to the standard norms in the finite element subspaces of* $H(\mathbf{curl}, \Omega)$ *and* $H(\mathbf{curl}, \operatorname{div}, \Omega)$, *we can still prove the finite element convergence and obtain the error estimates, without requiring the physical domains to be convex.*

Résumé. — *Dans des domaines polyhédriques tridimensionnels de frontière Lipschitz continue, on calcule les formulations variationnelles dans* $H(\mathbf{rot}, \Omega)$ *et* $H(\operatorname{div}, \mathbf{rot}, \Omega)$ *du modèle de Darwin qui est une approximation des équations de Maxwell. On prouve que les problèmes variationnels sont bien posés, puis, une famille régulière de triangulations* $(\mathcal{T}^h)_h$ *étant donnée, on utilise les éléments finis de type Nédélec et de type standard pour discrétiser ces problèmes. On démontre la convergence des méthodes d'éléments finis et des estimations d'erreur sont obtenues. En particulier, dans le cas où l'on ne peut pas prouver l'équivalence des formes bilinéaires symétriques des problèmes variationnels et des normes usuelles indépendamment de h, on obtient ces résultats en utilisant une méthode légèrement modifiée de résolution des problèmes de point-selle.*

1. INTRODUCTION

It is known that there are more and more scientific problems which involve the solutions of Maxwell's equations, e.g., plasma physics, microwave devices, diffraction of electromagnetic waves. In many cases, the numerical resolution of the full system of Maxwell's equations may be very expensive in terms of the computational cost. However, for some problems, e.g. the simulation of charged particle beams when no high frequency phenomenon or no rapid current change occurs, it is possible to use some simplified model

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which approximates Maxwell system in some sense and can be solved more economically The Darwin model is such a simplified model which is obtained from Maxwell's equations by neglecting the transverse component of the displacement current, see Hewett-Nielson [12], Hewett-Boyd [11] and Nielson-Lewis [17] Degond-Raviart [8] considered how to choose the boundary conditions so that the Darwin model is mathematically well-posed and characterized the electric field and the magnetic field as the solutions of elliptic boundary value problems

In this paper, we are interested in the solutions of elliptic boundary value problems in the Darwin model by finite element methods To that aim, we will first derive appropriate variational formulations for the concerned problems and prove the well-posedness of the formulations, then propose the finite element methods for the variational problems and show the finite element convergence and derive the error estimates

The contents of the paper are arranged as follows Section 2 introduces some natural Hilbert spaces for Maxwell's equations and Green's formulae in the forms of $\nabla . \mathbf{u}$ and $\nabla \times \mathbf{u}$ as well as the formulation of general continuous and discrete saddle point problems and the uniqueness and existence of their solutions Sections 3 and 4 describe Maxwell's equations and their Darwin model of approximation Section 5 presents two systems, one of Dirichlet type and the other of Neumann type, on which we will focus for the numerical solutions by finite element methods In Section 6, the $H(\mathbf{curl}, \Omega)$ variational formulations for the Dirichlet and Neumann problems is derived, together with their finite element solutions (Nedelec's elements) and convergence Finally, in Sections 7 and 8 we address the $H(\mathbf{curl}, \operatorname{div}, \Omega)$ variational formulations for both Dirichlet and Neumann problems and their $H^1(\Omega)$ conforming finite element approximations and convergence

2. PRELIMINARIES

Throughout the paper, we assume that Ω is a **simply-connected** domain in $\mathbb{R}^3$ and its boundary $\Gamma = \partial\Omega$ is **Lipschitz-continuous** (*cf* Girault-Raviart [9] for a definition) Whenever finite element formulations are considered, we additionally assume that Ω is a polyhedron We denote by Γ_i, $0 \leq i \leq m$, the connected components of the boundary Γ, Γ_0 being the outer boundary In this section, we introduce a few natural Hilbert spaces related to the Maxwell's equations, and some basic formulae and lemmas to be used in the subsequent sections The most frequently used Hilbert spaces will be

$$H(\operatorname{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3, \nabla . \mathbf{v} \in L^2(\Omega)\},$$

$$H(\mathbf{curl}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3, \nabla \times \mathbf{v} \in (L^2(\Omega))^3\},$$

their subspaces

$$H_0(\mathrm{div}\,;\Omega)=\{\mathbf{v}\in H(\mathrm{div}\,;\Omega)\,;\,\mathbf{v}\,.\,\mathbf{n}=0 \text{ on } \Gamma\},$$

$$H_0(\mathrm{div}0\,;\Omega)=\{\mathbf{v}\in H_0(\mathrm{div}\,;\Omega)\,;\,\nabla\,.\,\mathbf{v}=0 \text{ on } \Gamma\},$$

$$H_0(\mathbf{curl}\,;\Omega)=\{\mathbf{v}\in H(\mathbf{curl}\,;\Omega)\,;\,\mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma\},$$

and their intersection space

$$H(\mathbf{curl},\mathrm{div}\,;\Omega)=H(\mathrm{div}\,;\Omega)\cap H(\mathbf{curl}\,;\Omega)\,.$$

For the spaces $H(\mathrm{div}\,;\Omega)$, $H(\mathbf{curl}\,;\Omega)$ and $H(\mathbf{curl},\mathrm{div}\,;\Omega)$, we define the respective norms by

$$\|\mathbf{v}\|_{0,\mathrm{div}}=(\|\mathbf{v}\|_0^2+\|\nabla\,.\,\mathbf{v}\|_0^2)^{1/2},$$

$$\|\mathbf{v}\|_{0,\mathrm{curl}}=(\|\mathbf{v}\|_0^2+\|\nabla\times\mathbf{v}\|_0^2)^{1/2},$$

$$\|\mathbf{v}\|_{0,\mathrm{curl},\mathrm{div}}=(\|\mathbf{v}\|_0^2+\|\nabla\times\mathbf{v}\|_0^2+\|\nabla\,.\,\mathbf{v}\|_0^2)^{1/2}.$$

Here and in the sequel of the paper, $\|\,.\,\|_0$ will always mean the $(L^2(\Omega))^3$-norm (or $L^2(\Omega)$-norm if only scalar functions are involved). And in general, we will use $\|\,.\,\|_m$ and $|\,.\,|_m$ to denote the norm and semi-norm in the Sobolev space $(H^m(\Omega))^3$ (or $H^m(\Omega)$ if only scalar functions are involved). We refer to Adams [1] or Grisvard [10] for a definition of Sobolev spaces.

Green's formula. For $\mathbf{u}\in H(\mathrm{div}\,;\Omega)$ and $\phi\in H^1(\Omega)$, or $\mathbf{u}\in H(\mathbf{curl}\,;\Omega)$ and $\mathbf{w}\in(H^1(\Omega))^3$, we have

$$\int_\Omega \nabla\,.\,\mathbf{u}\,\phi\,dx=-\int_\Omega \mathbf{u}\,.\,\nabla\phi\,dx+\langle\mathbf{u}\,.\,\mathbf{n},\phi\rangle_\Gamma, \tag{2.1}$$

$$\int_\Omega(\nabla\times\mathbf{u})\,.\,\mathbf{w}\,dx=\int_\Omega \mathbf{u}\,.\,(\nabla\times\mathbf{w})\,dx+\langle\mathbf{n}\times\mathbf{u},\mathbf{w}\rangle_\Gamma. \tag{2.2}$$

Here $\langle\,.\,,\,.\,\rangle_\Gamma$ corresponds to the dual pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (or $(H^{-1/2}(\Gamma))^3$ and $(H^{1/2}(\Gamma))^3$).

DEFINITION 1 (SADDLE POINT PROBLEM) : *Let X and Q be two Hilbert spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Q$ respectively, $a(\cdot , \cdot)$ and $b(\cdot , \cdot)$ two continuous linear forms defined respectively on $X \times X$ and $X \times Q$, and $f(\cdot)$ and $g(\cdot)$ two continuous linear forms defined respectively on X and Q. Then the problem : find $(\mathbf{u}, p) \in (X, Q)$ such that*

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) , \quad \forall \mathbf{v} \in X , \tag{2.3}$$

$$b(\mathbf{u}, q) = g(q) , \quad \forall q \in Q , \tag{2.4}$$

is called a saddle point problem.

THEOREM 2.1 (BABUSKA-BREZZI) : *Let V be a closed subspace of X defined as*

$$V = \{ \mathbf{v} \in X ; \quad b(\mathbf{v}, q) = 0, \quad \forall q \in Q \} .$$

Assume that there exist two positive constants α and β such that

$$a(\mathbf{v}, \mathbf{v}) \geqslant \alpha \| \mathbf{v} \|_X^2, \quad \forall \mathbf{v} \in V , \quad (\textit{V-ellipticity})$$

and

$$\sup_{v \in X} \frac{b(\mathbf{v}, q)}{\| \mathbf{v} \|_X} \geqslant \beta \| q \|_Q, \quad \forall q \in Q . \quad (\textit{inf-sup condition})$$

Then there exists a unique solution to the saddle point problem (2.3)-(2.4).

Proof : See Babuska [3] and Brezzi [5], or Brezzi-Fortin [6] and Girault-Raviart [9]. □

DEFINITION 2 (DISCRETIZED SADDLE POINT PROBLEM) : *Let X_h and Q_h be two finite dimensional subspaces of X and Q respectively. Then the problem : find $(\mathbf{u}_h, p_h) \in (X_h, Q_h)$ such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = f(\mathbf{v}_h) , \quad \forall \mathbf{v}_h \in X_h , \tag{2.5}$$

$$b(\mathbf{u}_h, q_h) = g(q_h) , \quad \forall q_h \in Q_h , \tag{2.6}$$

is called a discretized saddle point problem.

Let Q_h^* be the dual space of Q_h, with the dual pairing $\langle \cdot , \cdot \rangle$ and equipped with the norm

$$\| \chi \|_{Q_h^*} = \sup_{q_h \in Q_h} \langle \chi, q_h \rangle / \| q_h \|_Q, \quad \forall \chi \in Q_h^* .$$

We define a linear operator $B_h : X_h \to Q_h^*$ by

$$\langle B_h \mathbf{v}_h, q_h \rangle = b(\mathbf{v}_h, q_h), \quad \forall \mathbf{v}_h \in X_h, \quad q_h \in Q_h$$

and a subset $V_h(\chi)$ of X_h for any $\chi \in Q_h^*$ by

$$V_h(\chi) = \{\mathbf{v}_h \in X_h \,;\, b(\mathbf{v}_h, q_h) = \langle \chi, q_h \rangle, \quad \forall q_h \in Q_h\} .$$

The following theorem will play a crucial role in our later error estimates for the finite element methods.

THEOREM 2.2 : *Suppose that there exists a positive constant* $\underline{\alpha}(h)$ *which may be depending on h such that*

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \underline{\alpha}(h)\, \|\mathbf{v}_h\|_X^2, \quad \forall \mathbf{v}_h \in V_h(0)\,, \quad (V_h(0)\text{-ellipticity})$$

and another positive constant $\underline{\beta}$ *independent of h such that*

$$\sup_{v_h \in X_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_X} \geq \underline{\beta} \|q_h\|_Q, \quad \forall q_h \in Q_h \quad (\textit{discrete inf-sup condition})\,.$$

Then $V_h(g) \neq \emptyset$ *and there exists a unique solution* $(\mathbf{u}_h, p_h)$ *to the discretized saddle point problem (2.5)-(2.6). Moreover, if we let* $(\mathbf{u}, p)$ *be the solution to the saddle point problem (2.3)-(2.4),*

(a) B_h *is an isomorphism from* $V_h(0)^{\perp}$ *(taken in* X_h*) onto* Q_h^* *; and*

$$\underline{\beta} \|\mathbf{v}_h\|_X \leq \|B_h \mathbf{v}_h\|_{Q_h^*}, \quad \forall \mathbf{v}_h \in V_h(0)^{\perp}\,.$$

(b) Let $b_0 > 0$ *be a constant such that* $|b(\mathbf{v}, q)| \leq b_0 \|\mathbf{v}\|_X \|q\|_Q$, $\forall (\mathbf{v}, q) \in X \times Q$. *Then*

$$\inf_{\mathbf{w}_h \in V_h(g)} \|\mathbf{u} - \mathbf{w}_h\|_X \leq \left(1 + \frac{b_0}{\underline{\beta}}\right) \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X\,.$$

(c) Let $\|\,.\,\|_a$ *be the* $a(\,.\,,.\,)$*-induced norm,* b_1 *and* a_0 *two positive constants satisfying* $|b(\mathbf{v}, q)| \leq b_1 \|\mathbf{v}\|_a \|q\|_Q$, $\forall (\mathbf{v}, q) \in X \times Q$; $|a(\mathbf{v}, \mathbf{w})| \leq a_0 \|\mathbf{v}\|_X \|\mathbf{w}\|_X$, $\forall \mathbf{v}, \mathbf{w} \in X$. *Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_a \leq 2 \inf_{\mathbf{w}_h \in V_h(g)} \|\mathbf{u} - \mathbf{w}_h\|_a + b_1 \inf_{q_h \in Q_h} \|p - q_h\|_Q\,.$$

(d) Let $a_0 > 0$ *be a constant defined in the above (c), then*

$$\|p - p_h\|_Q \leq \frac{a_0}{\underline{\beta}} \|\mathbf{u} - \mathbf{u}_h\|_X + \left(1 + \frac{b_0}{\underline{\beta}}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q\,.$$

Proof : $V_h(g) \neq \emptyset$ and the existence and uniqueness of the solutions to (2.5)-(2.6) were proved in Girault-Raviart [9] (Theorem 1.1, Chap. 2). We emphasize here that the constant $\underline{\alpha}(h)$ is not necessarily required to be independent of h for the existence and uniqueness. The conclusions in (a) follow directly by applying Lemma 4.1 in [9] (page 58) to the two spaces X_h and Q_h. Though (b)-(d) were in principle proved in Girault-Raviart [9], we still give a slightly different proof here to stress that the constant $\underline{\alpha}(h)$ does not need to appear in the error bounds of (b)-(d).

We first prove (b). For any $\mathbf{v}_h \in X_h$, obviously $B_h(\mathbf{u}_h - \mathbf{v}_h) \in Q_h^*$. Thus from (a) there exists a $\mathbf{z}_h \in V_h(0)^\perp$ such that $B_h \mathbf{z}_h = B_h(\mathbf{u}_h - \mathbf{v}_h)$ and

$$\underline{\beta} \| \mathbf{z}_h \|_X \leq \| B_h(\mathbf{u}_h - \mathbf{v}_h) \|_{Q_h^*} . \tag{2.7}$$

But it follows from (2.4) and (2.6) that for any $q_h \in Q_h$,

$$\langle B_h(\mathbf{u}_h - \mathbf{v}_h), q_h \rangle = b(\mathbf{u}_h - \mathbf{v}_h, q_h) = g(q_h) - b(\mathbf{v}_h, q_h) =$$
$$= b(\mathbf{u}, q_h) - b(\mathbf{v}_h, q_h) = b(\mathbf{u} - \mathbf{v}_h, q_h) ,$$

combining this with (2.7) shows

$$\underline{\beta} \| \mathbf{z}_h \|_X \leq b_0 \| \mathbf{u} - \mathbf{v}_h \|_X . \tag{2.8}$$

Now set $\mathbf{w}_h = \mathbf{z}_h + \mathbf{v}_h$, using $B_h \mathbf{z}_h = B_h(\mathbf{u}_h - \mathbf{v}_h)$ we obtain for any $q_h \in Q_h$ that

$$b(\mathbf{w}_h, q_h) = \langle B_h \mathbf{z}_h, q_h \rangle + \langle B_h \mathbf{v}_h, q_h \rangle = \langle B_h \mathbf{u}_h - B_h \mathbf{v}_h, q_h \rangle + \langle B_h \mathbf{v}_h, q_h \rangle =$$
$$= b(\mathbf{u}_h, q_h) = g(q_h) ,$$

this implies $\mathbf{w}_h \in V_h(g)$, and we get from (2.8) and the triangle's inequality that

$$\| \mathbf{u} - \mathbf{w}_h \|_X \leq \| \mathbf{u} - \mathbf{v}_h \|_X + \| \mathbf{z}_h \|_X \leq \left(1 + \frac{b_0}{\underline{\beta}} \right) \| \mathbf{u} - \mathbf{v}_h \|_X ,$$

which proves (b).

Next we show (c). For any $\mathbf{w}_h \in V_h(g)$, let $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$, then $\mathbf{v}_h \in V_h(0)$ and we see from (2.5) that

$$a(\mathbf{v}_h, \mathbf{v}_h) = f(\mathbf{v}_h) - a(\mathbf{w}_h, \mathbf{v}_h) .$$

Let $\mathbf{v}_h \in X_h$, then substituting $\mathbf{v} = \mathbf{v}_h$ in (2.3) gives

$$a(\mathbf{v}_h, \mathbf{v}_h) = a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p) .$$

Using $b(\mathbf{v}_h, q_h) = 0$ for any $q_h \in Q_h$, we obtain

$$a(\mathbf{v}_h, \mathbf{v}_h) = a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - q_h) , \quad \forall q_h \in Q_h .$$

Applying the Cauchy-Schwarz inequality to it and using the definition of b_1 imply

$$\|\mathbf{v}_h\|_a \leqslant \|\mathbf{u} - \mathbf{w}_h\|_a + b_1 \|p - q_h\|_Q .$$

Now the triangle's inequality and $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h$ yield

$$\|\mathbf{u} - \mathbf{u}_h\|_a \leqslant 2\|\mathbf{u} - \mathbf{w}_h\|_a + b_1 \|p - q_h\|_Q ,$$

which proves (c).

Finally we show (d). For any $\mathbf{v}_h \in X_h$, $q_h \in Q_h$, we derive by (2.3) and (2.5) that

$$b(\mathbf{v}_h, p_h - q_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - q_h) .$$

This with the discrete inf-sup condition leads to

$$\|p_h - q_h\|_Q \leqslant \frac{1}{\underline{\beta}} \sup_{\mathbf{v}_h \in X_h} \frac{1}{\|\mathbf{v}_h\|_X} \{a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - q_h)\}$$
$$\leqslant \frac{1}{\underline{\beta}} \{a_0 \|\mathbf{u} - \mathbf{u}_h\|_X + b_0 \|p - q_h\|_Q\} ,$$

now (d) follows by applying the triangle's inequality. □

Remark 2.1 : The minor difference between Theorem 2.2 and the classical version (*cf.* Brezzi [5] and Girault-Raviart [9]) is that the former allows the constant $\underline{\alpha}(h)$ to be dependent on h. The classical error estimate in (c) is of the form :

$$\|\mathbf{u} - \mathbf{u}_h\|_X \leqslant \left(1 + \frac{a_0}{\underline{\alpha}(h)}\right) \inf_{\mathbf{w}_h \in V_h(g)} \|\mathbf{u} - \mathbf{w}_h\|_X + \frac{b_0}{\underline{\alpha}(h)} \inf_{q_h \in Q_h} \|p - q_h\|_Q .$$

Thus if $\underline{\alpha}(h)$ depends on h, no convergence or error estimate could be derived in norm $\|\cdot\|_X$ from this classical form. Our new version is more helpful in this case. It is crucial in obtaining our later finite element error estimates in the subsequent sections.

3. MAXWELL'S EQUATIONS

Let us now briefly recall the physical background of the problem we aim at solving numerically. Let $T > 0$ be a given number, then in the space-time domain $\Omega \times (0, T)$, Maxwell's equations in vacuum are of the following form :

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J} , \tag{3.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 , \tag{3.2}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho , \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = 0 , \tag{3.4}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ denote the electric field and the magnetic field respectively, while $\rho = \rho(\mathbf{x}, t)$ and $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ denote the charge and current densities They satisfy the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{3 5}$$

The constants c, ε_0 and μ_0 are the light velocity, the electric permittivity and magnetic permeability of vacuum, respectively They are related by

$$\varepsilon_0 \mu_0 c^2 = 1$$

To complete the above system, we have to add some boundary and initial conditions Let us assume that Γ is a perfect conductor Then we add the following boundary conditions on $\Gamma \times (0, T)$

$$\mathbf{E} \times \mathbf{n} = 0, \quad \frac{\partial}{\partial t} \mathbf{B} \cdot \mathbf{n} = 0 \tag{3 6}$$

and the initial conditions

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0, \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0, \quad \mathbf{x} \in \Omega \tag{3 7}$$

Here the initial data $\mathbf{E}_0$, $\mathbf{B}_0$ satisfy the constraints

$$\nabla \cdot \mathbf{E}_0 = \frac{1}{\varepsilon_0} \rho(\mathbf{x}, 0) \quad \text{in } \Omega, \tag{3 8}$$

$$\nabla \cdot \mathbf{B}_0 = 0 \quad \text{in } \Omega, \tag{3 9}$$

$$\mathbf{E}_0 \times \mathbf{n} = 0 \quad \text{on } \Gamma \tag{3 10}$$

For the well-posedness of the above Maxwell's equations, we have the following theorem which stems from the classical variational theory developped by Lions and Magenes [14]

THEOREM 3 1 *We assume that ρ and $\mathbf{J}$ satisfy the charge conservation equation (3 5) and*

$$\rho \in C^1([0, T], L^2(\Omega)),$$

$$\mathbf{J} \in C^1([0, T], (L^2(\Omega))^3) \cap C^0([0, T], H(\mathrm{div}, \Omega))$$

and $\mathbf{E}_0, \mathbf{B}_0 \in H(\mathbf{curl}, \operatorname{div}; \Omega)$ *satisfy the constraints (3.8)-(3.10). Then the problem (3.1)-(3.7) has a unique solution* $(\mathbf{E}, \mathbf{B})$ *satisfying*

$$\mathbf{E},\mathbf{B} \in C^1([0, T]; (L^2(\Omega))^3) \cap C^0([0, T]; H(\mathbf{curl}, \operatorname{div}; \Omega)).$$

4. DARWIN MODEL OF APPROXIMATION TO MAXWELL'S EQUATIONS

To derive the Darwin model of approximation to Maxwell's equations, one decomposes the electric field $\mathbf{E}$ into the sum of its transverse component $\mathbf{E}_T$ and longitudinal component $\mathbf{E}_L$, where $\mathbf{E}_T$ is divergence free and $\mathbf{E}_L$ is curl free. Then the Darwin model is derived by substituting $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_T$ into (3.1) and neglecting the transverse component $\partial \mathbf{E}_T/\partial t$ of the displacement current. In fact, if we add specified boundary conditions to determine $\mathbf{E}_L$ and $\mathbf{E}_T$ uniquely and denote by $\mathbf{E}^D = \mathbf{E}_L^D + \mathbf{E}_T^D$ and $\mathbf{B}^D$ the resulting Darwin approximations to the electric and magnetic fields respectively, then after some reformulations, we come to the Darwin model which has the following characteristics (*cf.* Degond-Raviart [8]) :

THEOREM 4.1 : *Under the assumptions of Theorem 3.1, the Darwin approximation* $(\mathbf{E}_L^D + \mathbf{E}_T^D, B^D)$ *is determined uniquely by the following systems :*

(i) $\mathbf{E}_L^D = \mathbf{E}_L = -\nabla\phi \in C^2([0, T]; (L^2(\Omega))^3) \cap C^1([0, T];$ $H(\mathbf{curl}, \operatorname{div}; \Omega))$, *where* $\phi = \phi(\,.\,, t)$, *for all* $t \in [0, T]$, *is the solution of*

$$-\Delta\phi = \frac{1}{\varepsilon_0}\rho \text{ in } \Omega\,; \quad \phi = \alpha_i \text{ on } \Gamma_i, \quad 0 \leqslant i \leqslant m$$

with $\alpha = (\alpha_i)_{0 \leqslant i \leqslant m}$ *being the solution of the differential system*

$$C\frac{d\alpha}{dt} = \frac{1}{\varepsilon_0}\int_\Omega \mathbf{J} \,.\, \nabla\chi \, dx, \quad \alpha(0) = \alpha_0\,,$$

where $C = (c_{ij})_{0 \leqslant i,j \leqslant m}$ *is the capacitance matrix defined by* $c_{ij} = \langle \partial\chi_i/\partial n, 1\rangle_{\Gamma_j}$, $\chi = (\chi_i)_{0 \leqslant i \leqslant m}$ *is the solution of*

$$\Delta\chi_i = 0 \text{ in } \Omega\,; \quad \chi_i = \delta_{ij} \text{ on } \Gamma_j\,,$$

and α_0 *depends on* $\mathbf{E}_{0L}$, *i.e. the initial value of* $\mathbf{E}_L$.

(ii) *the function* $\mathbf{B}^D \in C^1([0,T], H(\mathbf{curl}, \operatorname{div}, \Omega))$ *is for all* $t \in [0,T]$ *the unique solution of*

$$\begin{aligned} -\Delta \mathbf{B}^D &= \mu_0 \nabla \times \mathbf{J} && \text{in } \Omega, \\ \nabla \cdot \mathbf{B}^D &= 0 && \text{in } \Omega, \\ \mathbf{B}^D \cdot \mathbf{n} &= \mathbf{B}_0 \cdot \mathbf{n} && \text{on } \Gamma, \\ (\nabla \times \mathbf{B}^D) \times \mathbf{n} &= \mu_0 \mathbf{J} \times \mathbf{n} && \text{on } \Gamma, \end{aligned}$$

(iii) *the function* $\mathbf{E}_T^D \in C^0([0,T], H(\mathbf{curl}, \operatorname{div}, \Omega))$ *is for all* $t \in [0,T]$ *the unique solution of*

$$\begin{aligned} \Delta \mathbf{E}_T^D &= \frac{\partial}{\partial t} \nabla \times \mathbf{B}^D && \text{in } \Omega, \\ \nabla \cdot \mathbf{E}_T^D &= 0 && \text{in } \Omega, \\ \mathbf{E}_T^D \times \mathbf{n} &= 0 && \text{on } \Gamma, \\ \langle \mathbf{E}_T^D \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} &= 0, && 1 \leq i \leq m \end{aligned}$$

Here and afterwards, $\langle \cdot, \cdot \rangle_{\Gamma_i}$ represents the dual pairing between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$ Degond-Raviart [8] proved that the Darwin model approximates the Maxwell system up to the second order for the magnetic field, and to the third order for the electric field, in terms of the supposedly small dimensionless parameter $\eta = \bar{v}/c$, where $\bar{v}$ is a characteristic velocity Physical cases in which η is small are studied numerically in [2]

5. TWO DECOUPLED SYSTEMS FROM THE DARWIN MODEL

In this paper, we are interested in solving the following two kinds of boundary value problems which come from the Darwin model discussed in Section 4 The first is the Dirichlet problem

$$(5\ 1) \quad \begin{cases} \Delta \mathbf{E} = \nabla \times \mathbf{B}_1 & \text{in } \Omega, \\ \nabla \cdot \mathbf{E} = 0 & \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Gamma, \\ \langle \mathbf{E} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, & 1 \leq i \leq m, \end{cases}$$

where $\mathbf{B}_1 \in H(\mathbf{curl}, \Omega)$

The second is the Neumann problem :

$$\text{(5.2)} \qquad \begin{cases} -\Delta \mathbf{B} = \nabla \times \mathbf{f} & \text{in } \Omega , \\ \nabla . \mathbf{B} = 0 & \text{in } \Omega , \\ \mathbf{B} . \mathbf{n} = \mathbf{B}_0 . \mathbf{n} & \text{on } \Gamma , \\ (\nabla \times \mathbf{B}) \times \mathbf{n} = \mathbf{f} \times \mathbf{n} & \text{on } \Gamma , \end{cases}$$

where $\mathbf{f} \in H(\mathbf{curl} ; \Omega)$, $\nabla . \mathbf{B}_0 = 0$ and $\mathbf{B}_0 \in H(\mathbf{curl}, \mathrm{div} ; \Omega)$. And from Theorem 4.1, we know

LEMMA 5.1 : *The Dirichlet and Neumann problems (5.1) and (5.2) both have unique solutions* $\mathbf{E}$ *and* $\mathbf{B}$ *respectively. And*

$$\mathbf{E} \in H(\mathbf{curl}, div ; \Omega), \quad \mathbf{B} \in H(\mathbf{curl}, div ; \Omega) .$$

6. $H(\mathbf{curl} ; \Omega)$ FORMULATION

This section will first address the derivation of the $H(\mathbf{curl} ; \Omega)$ variational formulations for the Dirichlet and Neumann problems (5.1) and (5.2), then derive finite element methods based on the variational problems. Finally the error estimates of the finite element methods will be given.

6.1. The Dirichlet problem

To derive the variational formulation of (5.1), we start with the first equation of (5.1). Using the identity

$$\text{(6.1)} \qquad \nabla \times (\nabla \times \mathbf{u}) = -\Delta \mathbf{u} + \nabla(\nabla . \mathbf{u}) ,$$

and $\nabla . \mathbf{E} = 0$ in Ω, we get for any $\mathbf{v} \in H_0(\mathbf{curl} ; \Omega)$ from the first equation of (5.1) that

$$\text{(6.2)} \qquad \int_\Omega (\nabla \times \mathbf{E}) . (\nabla \times \mathbf{v}) \, dx = \int_\Omega \mathbf{B}_1 . (\nabla \times \mathbf{v}) \, dx .$$

In order to replace $\nabla . \mathbf{E} = 0$ and $\langle \mathbf{E} . \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$ in (5.1), we introduce the following space

$$\text{(6.3)} \qquad H_c^1(\Omega) = \{ q \in H^1(\Omega) ;$$
$$q = 0 \text{ on } \Gamma_0 \quad \text{and} \quad q = c_i \text{ on } \Gamma_i, c_i \in \mathbb{R}, 1 \leq i \leq m \} .$$

Remark 6.1 : For any function $q \in H_c^1(\Omega)$, we have $\nabla q \times \mathbf{n} = 0$ on Γ. Moreover, the semi-norm $|\,.\,|_1$ is a norm on $H_c^1(\Omega)$, because of the boundary condition imposed on Γ_0.

Now, multiplying $\nabla . \mathbf{E} = 0$ by any function q in $H_c^1(\Omega)$ and integrating over Ω by parts yield

$$\begin{aligned} 0 &= -\int_\Omega \nabla q . \mathbf{E}\, dx + \langle \mathbf{E} . \mathbf{n}, q \rangle_\Gamma \\ &= -\int_\Omega \nabla q . \mathbf{E}\, dx + \sum_{i=0}^{m} q_{|\Gamma_i} \langle \mathbf{E} . \mathbf{n}, 1 \rangle_{\Gamma_i} \\ &= -\int_\Omega \nabla q . \mathbf{E}\, dx . \end{aligned} \tag{6.4}$$

Therefore we derive the variational formulation for the Dirichlet problem (5.1) :
find $\mathbf{E} \in H_0(\mathbf{curl}\,; \Omega)$ such that

$$\int_\Omega (\nabla \times \mathbf{E}) . (\nabla \times \mathbf{v})\, dx = \int_\Omega \mathbf{B}_1 . (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in H_0(\mathbf{curl}\,; \Omega)\,, \tag{6.5}$$

$$\int_\Omega \mathbf{E} . \nabla q\, dx = 0, \quad \forall q \in H_c^1(\Omega)\,, \tag{6.6}$$

which is equivalent to the problem :
find $(\mathbf{E}, p) \in H_0(\mathbf{curl}\,; \Omega) \times H_c^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega (\nabla \times \mathbf{E}) . (\nabla \times \mathbf{v})\, dx + \int_\Omega \mathbf{v} . \nabla p\, dx &= \\ &= \int_\Omega \mathbf{B}_1 . (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in H_0(\mathbf{curl}\,; \Omega)\,, \end{aligned} \tag{6.7}$$

$$\int_\Omega \mathbf{E} . \nabla q\, dx = 0, \quad \forall q \in H_c^1(\Omega)\,. \tag{6.8}$$

The equivalence between (6.5)-(6.6) and (6.7)-(6.8) is easily proved by taking $\mathbf{v} = \nabla p$ in (6.7) for any $p \in H_c^1(\Omega)$, and we then have $p = 0$.

Remark 6.2 : We stated above that $p = 0$ in the system (6.7)-(6.8) follows by taking $\mathbf{v} = \nabla p$ in (6.7). This assumes ∇p is in $H(\mathbf{curl}\,; \Omega)$. Indeed, in the sense of distributions, one has, for all ϕ belonging to the space of infinitely differentiable functions with compact support in Ω (called $\mathscr{D}(\Omega)$),

$$\langle \nabla \times (\nabla p), \phi \rangle = \langle p, \nabla . (\nabla \times \phi) \rangle = 0\,.$$

Thus $\nabla \times (\nabla p) = 0$ in the sense of distributions, i.e. in $\mathscr{D}'(\Omega)$. Now, this implies in turn that $\nabla \times (\nabla p) = 0$ almost everywhere. In other words, $\nabla \times (\nabla p) = 0$ in $(L^2(\Omega))^3$ and ∇p is in $H(\mathbf{curl}\,; \Omega)$. In the subsequent sections, we will see many similar cases where such regularity results are then implicitly assumed.

We can now show the following theorem for the well-posedness of the system (6.7)-(6.8) :

THEOREM 6.1 : *There exists a unique solution* $(\mathbf{E}, p)$ *to (6.7)-(6.8) with* $p = 0$. *Moreover* $\mathbf{E}$ *is the solution of Dirichlet problem (5.1) and therefore* $\mathbf{E} \in H(\mathbf{curl}, div\,; \Omega)$.

Proof: We first apply Theorem 2.1 for the existence of a unique solution $(\mathbf{E}, p)$ to the system (6.7)-(6.8). We can introduce the spaces and linear functionals used in Theorem 2.1 as follows :

$$X = H_0(\mathbf{curl}\,; \Omega), \quad Q = H_c^1(\Omega),$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})\, dx, \quad f(\mathbf{v}) = \int_\Omega \mathbf{B}_1 \cdot (\nabla \times \mathbf{v})\, dx,$$

$$b(\mathbf{v}, q) = \int_\Omega \mathbf{v} \cdot \nabla q\, dx, \quad g(q) = 0,$$

for any $\mathbf{u}, \mathbf{v} \in X$ and $q \in Q$. Then the closed subspace V of X is :

$$V = \{\mathbf{v} \in X\,; \quad b(\mathbf{v}, q) = 0, \quad \forall q \in Q\},$$

which, by Green's formula (2.1), may be written as

$$V = \{\mathbf{v} \in X\,; \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \langle \mathbf{v} \cdot \mathbf{n}, 1\rangle_{\Gamma_i} = 0, \quad 1 \leqslant i \leqslant m\}.$$

We now claim that there exists a constant $C > 0$ such that

$$\|\mathbf{v}\|_0 \leqslant C \|\nabla \times \mathbf{v}\|_0, \quad \forall \mathbf{v} \in V. \tag{6.9}$$

Otherwise we have a sequence $\{\mathbf{v}_n\} \in H(\mathbf{curl}\,; \Omega)$ such that $\nabla \cdot \mathbf{v}_n = 0$ in Ω, $\mathbf{v}_n \times \mathbf{n} = 0$ on Γ and

$$\|\nabla \times \mathbf{v}_n\|_0 < 1/n, \quad \|\mathbf{v}_n\|_0 = 1. \tag{6.10}$$

Thus by the compact imbedding theorem (*cf* Weber [20]), $\{\mathbf{v}_n\}$ has a convergent subsequence still denoted by $\{\mathbf{v}_n\}$ in $(L^2(\Omega))^3$ This combining with (6 10) implies that $\{\mathbf{v}_n\}$ converges to some $\mathbf{v}$ in $H(\mathbf{curl}, \Omega)$ with $\|\mathbf{v}\|_0 = 1$, and

$$\nabla \times \mathbf{v} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{v} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma,$$

which ensures the existence of a function $q \in H^1(\Omega)$ (*cf* Girault Raviart [9], Theorem 2 9 of Chap 1) such that q is constant over each component Γ_i and $\mathbf{v} = \nabla q$ Now using $\mathbf{v} \in V$, and hence $b(\mathbf{v}, q) = 0$, we have $\|\mathbf{v}\|_0 = 0$, this contradicts with $\|\mathbf{v}\|_0 = 1$ Therefore (6 9) holds, and so does the V-ellipticity of $a(\cdot,\cdot)$

For the inf-sup condition note first that if $q \in Q$, then $\mathbf{v} = \nabla q \in X$ because $\mathbf{v} \times \mathbf{n} = 0$ on Γ by using $q|_{\Gamma_i} = c_i$, for $0 \leqslant i \leqslant m$ Thus $b(\mathbf{v}, q) = \|\mathbf{v}\|_X \|q\|_Q$, which shows the inf sup condition with $\beta = 1$

$p = 0$ follows immediately by taking $\mathbf{v} = \nabla p$ in (6 7) (recall Remark 6 2) Thus by Lemma 2 1, the solution $(\mathbf{E}, p)$ to (6 7)-(6 8) exists uniquely On the other hand, the previous derivations of (6 7)-(6 8) indicate that the solution of (5 1) is also the solution of (6 7)-(6 8) Now the conclusion of Theorem 6 1 follows by the uniqueness of $(\mathbf{E}, p)$ and Lemma 5 1 □

6.2. The Neumann problem

To derive the variational formulation of (5 2), multiplying the first equation of (5 2) by any $\mathbf{v} \in H(\mathbf{curl}, \Omega)$ and using (6 1), $\nabla \cdot \mathbf{B} = 0$ and Neumann boundary condition, we come to

$$\int_\Omega (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{v})\, dx = \int_\Omega \mathbf{f} \cdot (\nabla \times \mathbf{v})\, dx \tag{6 11}$$

Now multiplying $\nabla \cdot \mathbf{B} = 0$ by any $q \in H^1(\Omega)$ and integrating over Ω by parts, using $\mathbf{B} \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n}$ on Γ and $\nabla \cdot \mathbf{B}_0 = 0$ in Ω yields

$$\int_\Omega \mathbf{B} \cdot \nabla q\, dx = \int_\Omega \mathbf{B}_0 \cdot \nabla q\, dx \tag{6 12}$$

From the above derivations, we obtain the following variational formulation for the Neumann problem (5 2) find $\mathbf{B} \in H(\mathbf{curl}, \Omega)$ such that

$$\int_\Omega (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{v})\, dx = \int_\Omega \mathbf{f} \cdot (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in H(\mathbf{curl}, \Omega) \tag{6 13}$$

$$\int_\Omega \mathbf{B} \cdot \nabla q\, dx = \int_\Omega \mathbf{B}_0 \cdot \nabla q\, dx, \quad \forall q \in H^1(\Omega)/\mathbb{R}, \tag{6 14}$$

which is equivalent to the problem :
find $(\mathbf{B}, p) \in H(\mathbf{curl}\, ; \Omega) \times H^1(\Omega)/\mathbb{R}$ such that

$$(6.15) \quad \int_\Omega (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{v})\, dx + \int_\Omega \mathbf{v} \cdot \nabla p\, dx = \int_\Omega \mathbf{f} \cdot (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in H(\mathbf{curl}\, ; \Omega)\,,$$

$$(6.16) \quad \int_\Omega \mathbf{B} \cdot \nabla q\, dx = \int_\Omega \mathbf{B}_0 \cdot \nabla q\, dx, \quad \forall q \in H^1(\Omega)/\mathbb{R}\,.$$

The equivalency is easy to see by taking $\mathbf{v} = \nabla p$ in (6.15). We have the following theorem for the above system :

THEOREM 6.2 : *There exists a unique solution $(\mathbf{B}, p)$ to (6.15)-(6.16) with $p = 0$ in Ω up to a constant. Moreover,* $\mathbf{B}$ *is also the solution of (5.2) and therefore* $\mathbf{B} \in H(\mathbf{curl}, div\, ; \Omega)$.

Proof : We first apply Theorem 2.1 for the existence of a unique solution $(\mathbf{B}, p)$ to (6.15)-(6.16). We define the spaces and linear functionals used in Theorem 2.1 as follows :

$$X = H(\mathbf{curl}\, ; \Omega), \quad Q = H^1(\Omega)/\mathbb{R}\,,$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})\, dx, \quad f(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot (\nabla \times \mathbf{v})\, dx\,,$$

$$b(\mathbf{v}, q) = \int_\Omega \mathbf{v} \cdot \nabla q\, dx, \quad g(q) = \int_\Omega \mathbf{B}_0 \cdot \nabla q\, dx\,,$$

for any $\mathbf{u}, \mathbf{v} \in X$ and $q \in Q$. Then the closed subspace V of X is

$$V = \{ v \in X\, ; \quad b(\mathbf{v}, q) = 0, \quad \forall q \in Q \}\,,$$

which, by Green's formula (2.1), may be written as

$$V = \{ v \in X\, ; \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}\,.$$

We claim next that there exists a constant $C > 0$ such that

$$(6.17) \quad \| \mathbf{v} \|_0 \le C \| \nabla \times \mathbf{v} \|_0, \quad \forall \mathbf{v} \in V\,.$$

Otherwise we have a sequence $\{\mathbf{v}_n\} \in H(\mathbf{curl}\,,\Omega)$ such that $\nabla \cdot \mathbf{v}_n = 0$ in Ω, $\mathbf{v}_n \cdot \mathbf{n} = 0$ on Γ, and

$$\|\nabla \times \mathbf{v}_n\|_0 < 1/n, \quad \|\mathbf{v}_n\|_0 = 1 \tag{6 18}$$

Then by the compact imbedding theorem (*cf* Weber [20]), $\{\mathbf{v}_n\}$ has a convergent subsequence (still denoted by $\{\mathbf{v}_n\}$) in $(L^2(\Omega))^3$ This combining with (6 18) implies that $\{\mathbf{v}_n\}$ converges to some $\mathbf{v}$ in $H(\mathbf{curl}\,,\Omega)$ Then one can readily see that $\|\mathbf{v}\|_0 = 1$, $\nabla \cdot \mathbf{v} = 0$ in Ω, $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ and $\nabla \times \mathbf{v} = 0$ in Ω This last equation ensures the existence of a function $q \in H^1(\Omega)$ such that $\mathbf{v} = \nabla q$ Now using $\mathbf{v} \in V$, and hence $b(\mathbf{v}, q) = 0$, we have $\|\mathbf{v}\|_0 = 0$, this contradicts with $\|\mathbf{v}\|_0 = 1$ Therefore (6 17) holds, and so does the V-ellipticity of $a(\,\cdot\,,\cdot\,)$

For the inf-sup condition take $\mathbf{v} = \nabla q \in X$ for any given $q \in Q$ Then $b(\mathbf{v}, q) = \|\mathbf{v}\|_X \|q\|_Q$, so the inf-sup condition holds for $\beta = 1$

Finally, using $\mathbf{v} = \nabla p$ in (6 15) yields $p = 0$ up to a constant in Ω Thus Lemma 2 1 shows the existence of a unique solution $(\mathbf{B}, p)$ to (6 15)-(6 16) On the other hand, we already know that the solution of (5 2) is also the solution of (6 15)-(6 16) Then the conclusion of Theorem 6 2 follows by the uniqueness of the solution $(\mathbf{B}, p)$ and Lemma 5 1 □

6.3. Finite element solution for the Dirichlet problem

We will make use of Nedelec's mixed finite elements [15] to approximate the variational problem (6 7)-(6 8)

Let $\mathcal{T}^h = \cup K$ be a **shape regular** triangulation of the domain Ω made of tetrahedra For any element K, let $\mathcal{P}_1(K)$ be the space of linear functions in K and $\mathcal{R}_1(K)$ be defined as

$$\mathcal{R}_1(K) = \{\mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{x} \text{ on } K\,, \quad \mathbf{a} \in \mathbb{R}^3, \quad \mathbf{b} \in \mathbb{R}^3\}$$

Remark 6 3 In order to construct finite dimensional subspaces of $H^1(\Omega)$ and $H(\mathbf{curl}\,,\Omega)$, we briefly outline some fundamental properties of finite elements derived from $\mathcal{P}_1(K)$ and $\mathcal{R}_1(K)$ A function of $\mathcal{P}_1(K)$ is uniquely determined by its values at the vertices of K Moreover, as stated in Theorem 4 2 1 of [7], a function whose restriction on any tetrahedron K belongs to $\mathcal{P}_1(K)$ is in $H^1(\Omega)$ if and only if it is continuous in Ω Clearly, such a function is completely defined by its values at the vertices of $\mathcal{T}^h$ For $\mathcal{R}_1(K)$, Nedelec [15] proved that a function $\mathbf{v}$ satisfying $\mathbf{v}|_K \in \mathcal{R}_1(K)$

belongs to $H(\mathbf{curl}\,;\Omega)$ if and only if $\mathbf{v}\times\mathbf{n}$ is continuous across the faces. Moreover, such a $\mathbf{v}$ is uniquely defined by its moments $\left\{\int_e \mathbf{v}\,.\,\tau_e\,dl\right\}_{e\in E}$, where E is the set of edges of the triangulation $\mathcal{T}^h$ and τ_e a unit vector parallel to e, and $\mathbf{v}$ is always locally divergence free.

Let us now introduce two finite element subspaces

$$X_h=\left\{\mathbf{v}_h\in H_0(\mathbf{curl}\,;\Omega)\,;\quad \mathbf{v}_h|_K\in \mathcal{R}_1(K),\quad \forall K\in\mathcal{T}^h\right\},$$

$$Q_h=\left\{q_h\in H_c^1(\Omega)\,;\quad q_h|_K\in\mathcal{P}_1(K),\quad \forall K\in\mathcal{T}^h\right\},$$

corresponding to the continuous spaces

$$X=H_0(\mathbf{curl}\,;\Omega)\,;\quad Q=H_c^1(\Omega)\,.$$

For any $\mathbf{u}\in(H^1(K))^3$, let $I_K\,\mathbf{u}$ be the unique element in $\mathcal{R}_1(K)$ which has the same moments as $\mathbf{u}$ on the tetrahedron K. And let $\Pi_h\,\mathbf{u}$ be the interpolant of $\mathbf{u}$, for any $\mathbf{u}\in(H^1(\Omega))^3$, defined on X_h by :

$$\Pi_h\,\mathbf{u}=I_K\,\mathbf{u}\text{ on }K,\quad \forall K\in\mathcal{T}^h\,. \tag{6.19}$$

We can verify that

$$\Pi_h\,\mathbf{u}\in X_h,\quad \text{if }\mathbf{u}\in H_0(\mathbf{curl}\,;\Omega)\,.$$

The finite element approximation to the problem (6.7)-(6.8) is now formulated as : find $(\mathbf{E}_h,p_h)\in X_h\times Q_h$ such that

$$a(\mathbf{E}_h,\mathbf{v})+b(\mathbf{v},p_h)=\int_\Omega \mathbf{B}_1\,.\,(\nabla\times\mathbf{v})\,dx,\quad \forall\mathbf{v}\in X_h\,, \tag{6.20}$$

$$b(\mathbf{E}_h,q)=0,\quad \forall q\in Q_h\,, \tag{6.21}$$

where

$$a(\mathbf{E},\mathbf{v})=\int_\Omega(\nabla\times\mathbf{E})\,.\,(\nabla\times\mathbf{v})\,dx,$$

$$b(\mathbf{E},q)=\int_\Omega \mathbf{E}\,.\,\nabla q\,dx,\quad \forall\mathbf{E},\mathbf{v}\in X,\,q\in Q\,.$$

We can show the following convergence for the above finite element approximation :

THEOREM 6.3 : *Suppose that Ω is a polyhedral domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary, then there exists a unique solution $(\mathbf{E}_h, p_h)$ to (6.20)-(6.21). Moreover, $p_h = 0$ and the following error estimates hold*

$$\|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_0 \leqslant Ch|\mathbf{E}|_2, \quad \text{if } \mathbf{E} \in X \cap (H^2(\Omega))^3, \tag{6.22}$$

$$\lim_{h \to 0} \|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_0 = 0, \quad \text{if } \mathbf{E} \in X, \tag{6.23}$$

where $(\mathbf{E}, 0)$ is the solution to (6.7)-(6.8). If in addition the domain is convex, then

$$\|\mathbf{E} - \mathbf{E}_h\|_{0,\,curl} \leqslant Ch|\mathbf{E}|_2, \quad \text{if } \mathbf{E} \in X \cap (H^2(\Omega))^3, \tag{6.24}$$

$$\lim_{h \to 0} \|\mathbf{E} - \mathbf{E}_h\|_{0,\,curl} = 0, \quad \text{if } \mathbf{E} \in X. \tag{6.25}$$

Proof: We apply Theorem 2.2 for the proof. Let $\|\cdot\|_a = (a(\cdot,\cdot))^{1/2}$. This is a norm on X_h. To see this, let $\mathbf{v}_h \in V_h$ and $a(\mathbf{v}_h, \mathbf{v}_h) = 0$, then $\nabla \times \mathbf{v}_h = 0$ in Ω. Thus we have a function $q_h \in H^1(\Omega)/\mathbb{R}$ such that $\mathbf{v}_h = \nabla q_h$.

As $\mathbf{v}_h|_K = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}$ for any tetrahedra of the triangulation $\mathcal{T}^h$, $\nabla \times \mathbf{v}_h = 0$ implies $\mathbf{v}_h|_K = \mathbf{a}_K$. Thus $q_h|_K \in \mathcal{P}_1(K)$. Using the boundary condition $\mathbf{v}_h \times \mathbf{n} = 0$ on Γ, we know $\nabla q_h \times \mathbf{n} = 0$ on Γ, which indicates that q_h is a constant on each Γ_i. As q_h is unique up to a constant, we can choose a q_h such that $q_h = 0$ on Γ_0 and is constant on the remaining components Γ_i for $i \neq 0$. Therefore $q_h \in Q_h$ as q_h belongs to $H^1(\Omega)$ by definition. Combining with $b(\mathbf{v}_h, q_h) = 0$ gives $\mathbf{v}_h = 0$. Thus $\|\cdot\|_a$ is indeed a norm on X_h.

As $\|\cdot\|_a$ is naturally also a norm on the finite dimensional subspace

$$V_h(0) = \{\mathbf{v}_h \in X_h \,;\, b(\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in Q_h\},$$

there exists a constant $\underline{\alpha}(h)$ depending on h such that

$$a(\mathbf{v}_h, \mathbf{v}_h) \geqslant \underline{\alpha}(h)\, \|\mathbf{v}_h\|_0^2, \quad \mathbf{v}_h \in V_h(0), \tag{6.26}$$

which means the $V_h(0)$-ellipticity of $a(\cdot,\cdot)$.

To get the discrete inf-sup condition, for any q_h in Q_h, we define $\mathbf{v} = \nabla q_h$. Clearly, $\mathbf{v} \in X_h$ and $b(\mathbf{v}, q_h) = \|\mathbf{v}\|_X \|q_h\|_Q$, i.e. the discrete inf-sup condition holds for $\underline{\beta} = 1$. Then the existence and uniqueness of the solution $(\mathbf{E}_h, p_h)$ follows from Theorem 2.2.

$p_h = 0$ follows immediately by taking $\mathbf{v} = \nabla p_h$ in (6.20).

We next derive the error estimates. Using Theorem 2.2 (b) and (c) and noting $\beta = b_0 = a_0 = 1$ and $\inf_{q_h \in Q_h} \|p - q_h\|_Q = 0$ in our case, we derive that

$$\|\mathbf{E} - \mathbf{E}_h\|_a \leqslant 4 \inf_{\mathbf{v}_h \in X_h} \|\mathbf{E} - \mathbf{v}_h\|_X . \tag{6.27}$$

If $\mathbf{E} \in X \cap H^2(\Omega)^3$, then by Theorem 2 in Nedelec [15], we have

$$\inf_{\mathbf{v}_h \in X_h} \|\mathbf{E} - \mathbf{v}_h\|_X \leqslant \|\mathbf{E} - \Pi_h \mathbf{E}\|_X \leqslant Ch\,|\mathbf{E}|_2 , \tag{6.28}$$

this implies (6.22).

On the other hand, if $\mathbf{E} \in X$ only, we can use the density of $X \cap (H^2(\Omega))^3$ in X to find a function $\mathbf{v}_\varepsilon \in X \cap (H^2(\Omega))^3$ for any $\varepsilon > 0$ such that

$$\|\mathbf{E} - \mathbf{v}_\varepsilon\|_X \leqslant \varepsilon/2 .$$

But from the interpolation result (6.28), we know that there exists a h_ε such that

$$\|\mathbf{v}_\varepsilon - \Pi_h \mathbf{v}_\varepsilon\|_X \leqslant \varepsilon/2, \quad \forall h \leqslant h_\varepsilon .$$

Now (6.23) follows by taking $\mathbf{v}_h = \Pi_h \mathbf{v}_\varepsilon$ in (6.27) and the triangle inequality.

If the domain is convex, the following Lemma 6.4 with Remark 1 of Section 2 yields the improved result (6.24)-(6.25). □

LEMMA 6.4 : *Suppose here that Ω is* **convex.** *Then there exists a positive constant C independent of h such that*

$$\int_\Omega |\mathbf{curl}\, \mathbf{v}_h|^2\, dx \geqslant C \int_\Omega |\mathbf{v}_h|^2\, dx, \quad \forall \mathbf{v}_h \in V_h(0) .$$

Proof: This was proved by Girault-Raviart [9] (Proposition 5.1, Chap. 3). □

6.4. Finite element solution for the Neumann problem

We solve the Neumann problem (6.15)-(6.16) by means of Nedelec's finite elements. Let us first introduce two finite element subspaces

$$X_h = \{\mathbf{v}_h \in H(\mathbf{curl}\,; \Omega)\,; \quad \mathbf{v}_h|_K \in \mathcal{R}_1(K), \quad \forall K \in \mathcal{T}^h\} ,$$

$$Q_h = \{q_h \in H^1(\Omega)/\mathbb{R}\,; \quad q_h|_K \in \mathcal{P}_1(K), \quad \forall K \in \mathcal{T}^h\} ,$$

corresponding to the two continuous spaces

$$X = H(\mathbf{curl}\,;\Omega)\,; \quad Q = H^1(\Omega)/\mathbb{R}\,.$$

Then the finite element problem for solving the Neumann system (6.15)-(6.16) is formulated as follows : find $(\mathbf{B}_h, p_h) \in X_h \times Q_h$ such that

$$(6.29) \qquad a(\mathbf{B}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = \int_\Omega \mathbf{f} \,.\, (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in X_h,$$

$$(6.30) \qquad b(\mathbf{B}_h, q) = \int_\Omega \mathbf{B}_0 \,.\, \nabla q\, dx, \quad \forall q \in Q_h\,,$$

where

$$a(\mathbf{B}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{B}) \,.\, (\nabla \times \mathbf{v})\, dx,$$

$$b(\mathbf{B}, q) = \int_\Omega \mathbf{B} \,.\, \nabla q\, dx, \quad \forall \mathbf{B}, \mathbf{v} \in X, q \in Q\,.$$

For the convergence of the finite element approximation, we have the following theorem :

THEOREM 6.5 : *Suppose that* Ω *is any polyhedral domain in* $\mathbb{R}^3$ *with a Lipschitz continuous boundary. Then there exists a unique solution* $(\mathbf{B}_h, p_h)$ *to the finite element problem (6.29)-(6.30). Moreover,* $p_h = 0$ *up to a constant and the following error estimates hold*

$$(6.31) \qquad \|\nabla \times (\mathbf{B} - \mathbf{B}_h)\|_0 \leqslant Ch|\mathbf{B}|_2, \quad \text{if}\, \mathbf{B} \in X \times (H^2(\Omega))^3\,,$$

$$(6.32) \qquad \lim_{h \to 0} \|\nabla \times (\mathbf{B} - \mathbf{B}_h)\|_0 = 0, \quad \text{if}\, \mathbf{B} \in X\,,$$

where $(\mathbf{B}, 0)$ *is the solution to (6.15)-(6.16). If in addition the domain is convex, then*

$$(6.33) \qquad \|\mathbf{B} - \mathbf{B}_h\|_{0,\,curl} \leqslant Ch|\mathbf{B}|_2, \quad \text{if}\, \mathbf{B} \in X \times (H^2(\Omega))^3\,,$$

$$(6.34) \qquad \lim_{h \to 0} \|\mathbf{B} - \mathbf{B}_h\|_{0,\,curl} = 0, \quad \text{if}\, \mathbf{B} \in X\,.$$

Proof : The proof is almost the same as the one for Theorem 6.3. We can first prove that $a(\,.\,,\,.\,)$ is a norm on $V_h(0)$ and have the $V_h(0)$-ellipticity. The discrete inf-sup condition can be done similarly and we also obtain $\underline{\beta} = 1$. The only minor difference is that $\Pi_h\, \mathbf{u}$ belongs to X_h naturally here.

If the domain is convex, the following Lemma 6.6 with the previous proved results leads to the conclusion. □

LEMMA 6.6 : *Suppose here that Ω is* **convex.** *Then $V_h(0)$ is defined as*

$$V_h(0) = \left\{ \mathbf{w}_h \in X_h \; ; \int_\Omega \mathbf{w}_h \, . \, \nabla q_h \, dx = 0, \quad \forall q_h \in Q_h \right\},$$

and there exists a positive constant C independent of h such that

$$\int_\Omega |\mathbf{curl}\, \mathbf{v}_h|^2 \, dx \geqslant C \int_\Omega |\mathbf{v}_h|^2 \, dx, \quad \forall \mathbf{v}_h \in V_h(0) \, .$$

Proof: This corresponds to Theorem 1 (inequality (22)) in [16]. □

7. $H(\mathbf{curl}, \mathrm{div} \,; \Omega)$ FORMULATION

In this section, we consider the $H(\mathbf{curl}, \mathrm{div} \,; \Omega)$ formulations for the Dirichlet and Neumann problems (5.1) and (5.2). Different from the $H(\mathbf{curl} \,; \Omega)$ formulations and their finite element methods discussed in Section 6, the variational formulations of this section will enable us to use the standard $H^1(\Omega)$ conforming finite element methods for solving the systems (5.1) and (5.2). The finite element methods will be discussed in Section 8.

7.1. The Dirichlet problem

We will show that the solution $\mathbf{E}$ of the Dirichlet problem (5.1) satisfies also the following variational problem: find $(\mathbf{E}, p) \in H_{0c}(\Omega) \times Q$ such that

$$a(\mathbf{E}, \mathbf{v}) + b(\mathbf{v}, p) = \int_\Omega \mathbf{B}_1 \, . \, (\nabla \times \mathbf{v}) \, dx, \quad \forall \mathbf{v} \in H_{0c}(\Omega), \tag{7.1}$$

$$b(\mathbf{E}, q) = 0, \quad \forall q \in Q \,, \tag{7.2}$$

where

$$H_{0c}(\Omega) = \{ \mathbf{v} \in H(\mathbf{curl}, \mathrm{div} \,; \Omega) \,; \quad \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma \}, \quad Q = L^2(\Omega) \,,$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{u}) \, . \, (\nabla \times \mathbf{v}) \, dx$$

$$+ \int_\Omega (\nabla \, . \, \mathbf{u}) \, (\nabla \, . \, \mathbf{v}) \, dx + \sum_{i=0}^{m} \langle \mathbf{u} \, . \, \mathbf{n}, 1 \rangle_{\Gamma_i} \langle \mathbf{v} \, . \, \mathbf{n}, 1 \rangle_{\Gamma_i} \,,$$

$$b(\mathbf{v}, q) = \int_\Omega (\nabla \, . \, \mathbf{v}) \, q \, dx \, .$$

Before proving the existence and uniqueness of the solutions of the system (7.1)-(7.2), we first introduce two auxiliary spaces V and V_Γ :

$$V = \{ \mathbf{v} \in H_{0c}(\Omega) \, ; \nabla \, . \, \mathbf{v} = 0 \} ,$$

$$V_\Gamma = \{ \mathbf{v} \in V \, ; \langle \mathbf{v} \, . \, \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, 0 \leqslant i \leqslant m \} .$$

In the space V_Γ, we have

LEMMA 7.1 : *There exists a constant $C > 0$ such that*

$$\| \phi \|_0 \leqslant C \| \nabla \times \phi \|_0, \quad \forall \phi \in V_\Gamma .$$

Proof: We know that, *cf.* Girault-Raviart [9] (Theorem 3.6, Chap. 1),

$$\forall \mathbf{u} \in H_0(\operatorname{div} 0 \, ; \Omega), \quad \exists ! \phi \in V_\Gamma \text{ such that } \mathbf{u} = \nabla \times \phi .$$

Thus **curl** is a one-to-one, continuous linear mapping from V_Γ to $H_0(\operatorname{div}0 \, ; \Omega)$. As in addition, V_Γ and $H_0(\operatorname{div}0 \, ; \Omega)$ are Banach spaces, **curl** is an isomorphism from V_Γ to $H_0(\operatorname{div}0 \, ; \Omega)$. Also, $\| \nabla \times \phi \|_0$ is a norm on V_Γ equivalent to the standard norm $\| \, . \, \|_{0, \text{curl}, \text{div}}$ of $H_{0c}(\Omega)$ (as a subspace of $H(\mathbf{curl}, \operatorname{div} \, ; \Omega)$). □

Let us now define another norm in $H_{0c}(\Omega)$ by

$$\| \mathbf{v} \|_{H_{0c}} = (a(\mathbf{v}, \mathbf{v}))^{1/2} , \tag{7.3}$$

for which we have

LEMMA 7.2 : *In the space $H_{0c}(\Omega)$, the norm $\| \, . \, \|_{H_{0c}}$ defined in (7.3) is equivalent to the classical norm $\| \, . \, \|_{0, curl, div}$.*

Proof: For any $\mathbf{v}$ in $H_{0c}(\Omega)$, we define $q \in H^1_c(\Omega)$ to be the solution to

$$\Delta q = \nabla \, . \, \mathbf{v} \text{ in } \Omega \, ; \quad C\mathbf{q} = \int_\Omega \mathbf{v} \, . \, \nabla \chi \, dx .$$

Here $C = (c_{ij})_{0 \leqslant i, j \leqslant m}$ is the capacitance matrix introduced in Theorem 4.1 and $\mathbf{q} = (q|_{\Gamma_j})_{0 \leqslant j \leqslant m}$. q is uniquely defined. Then let $\phi = \mathbf{v} - \nabla q$, it is easy to verify that $\phi \in V_\Gamma$. Thus by Lemma 7.1,

$$\| \phi \|_0 \leqslant C \| \nabla \times \phi \|_0 = C \| \nabla \times \mathbf{v} \|_0 . \tag{7.4}$$

On the other hand,

$$\int_{\Omega} |\nabla q|^2 \, dx = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, dx + \langle \mathbf{v} \cdot \mathbf{n}, q \rangle_{\Gamma}$$

$$= - \int_{\Omega} q \nabla \cdot \mathbf{v} \, dx + \sum_j q_{|\Gamma_j} \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j},$$

that implies

$$(7.5) \qquad \|\nabla q\|_0^2 \leqslant \|\nabla \cdot \mathbf{v}\|_0 \, \|q\|_0 + m^{1/2} \left(\max_j |q_{|\Gamma_j}| \right) \left(\sum_j \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j}^2 \right)^{1/2}.$$

Note that

$$\|q\|_{0, \Gamma} \leqslant \|q\|_{1/2, \Gamma} \leqslant \|q\|_1, \qquad \|q\|_{0, \Gamma}^2 = \sum_j meas\,(\Gamma_j)\, |q_{|\Gamma_j}|^2,$$

which gives $\max_j |q_{|\Gamma_j}| \leqslant C \|q\|_1$. Combining with (7.5) yields

$$(7.6) \qquad \|\nabla q\|_0 \leqslant C \left(\|\nabla \cdot \mathbf{v}\|_0 + \left(\sum_j \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j}^2 \right)^{1/2} \right),$$

for a constant C which depends only on Ω.

Finally, as $\|\mathbf{v}\|_0 \leqslant \|\phi\|_0 + \|\nabla q\|_0$, using (7.4) and (7.6), we obtain

$$\|\mathbf{v}\|_0 \leqslant C \left(\|\nabla \times \mathbf{v}\|_0 + \|\nabla \cdot \mathbf{v}\|_0 + \left(\sum_j \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j}^2 \right)^{1/2} \right).$$

This proves the existence of a constant $C > 0$ depending only on Ω such that

$$\|\mathbf{v}\|_{0, \mathrm{curl}, \mathrm{div}} \leqslant C \|\mathbf{v}\|_{H_{0c}}, \quad \forall \mathbf{v} \in H_{0c}(\Omega).$$

To conclude the proof, one simply sees by Green's formula (2.1) that

$$|\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_j}| = |\langle \mathbf{v} \cdot \mathbf{n}, \chi_j \rangle_{\Gamma}|$$

$$= \left| \int_{\Omega} (\nabla \cdot \mathbf{v}) \chi_j \, dx + \int_{\Omega} \mathbf{v} \cdot \nabla \chi_j \, dx \right|$$

$$\leqslant C \{ \|\mathbf{v}\|_0 + \|\nabla \cdot \mathbf{v}\|_0 \},$$

which implies there exists a constant C depending only on Ω such that,

$$\|\mathbf{v}\|_{H_{0c}} \leqslant C\|\mathbf{v}\|_{0,\,\mathrm{curl},\,\mathrm{div}}, \quad \forall \mathbf{v} \in H_{0c}.$$

□

THEOREM 7.3 : *The pair* $(\mathbf{E}, 0)$, *where* $\mathbf{E}$ *is the solution of the Dirichlet problem (5.1), may be characterized as the unique solution of (7.1)-(7.2).*

Proof: We first apply Theorem 2.1 to show that (7.1)-(7.2) has a unique solution $(\mathbf{E}, p) \in H_{0c}(\Omega) \times L^2(\Omega)$. Obviously, Lemma 7.2 indicates the V-ellipticity of $a(\,.\,,\,.\,)$.

To check the inf-sup condition, for any $q \in L^2(\Omega)$, we take $\mathbf{v} = \nabla\phi$ with ϕ satisfying

$$\Delta\phi = q \text{ in } \Omega\,; \quad \phi = 0 \text{ on } \Gamma\,.$$

Immediately we know $\mathbf{v} \times \mathbf{n} = 0$ on Γ. Hence $\mathbf{v} \in H_{0c}(\Omega)$. By multiplying $\Delta\phi = q$ by ϕ and using integration by parts, we come to

$$\|\nabla\phi\|_0 \leqslant C\|q\|_0\,.$$

Thus we see

$$\|\mathbf{v}\|^2_{H_{0c}} = \|\nabla\,.\,\mathbf{v}\|^2_0 + \|\nabla \times \mathbf{v}\|^2_0 + \sum_{i=0}^{m}\langle \mathbf{v}\,.\,\mathbf{n}, 1\rangle^2_{\Gamma_i}$$

$$= \|q\|^2_0 + \sum_{i=0}^{m}\left\langle \frac{\partial\phi}{\partial n}, 1\right\rangle^2_{\Gamma_i}\,.$$

By using Green's formula and the functions χ_i introduced in Theorem 4.1,

$$\sum_{i=0}^{m}\left\langle \frac{\partial\phi}{\partial n}, 1\right\rangle^2_{\Gamma_i} = \sum_{i=0}^{m}\left\langle \frac{\partial\phi}{\partial n}, \chi_i\right\rangle^2_{\Gamma}$$

$$= \sum_{i=0}^{m}\left(\int_\Omega \Delta\phi\,\chi_i\,dx + \int_\Omega \nabla\phi\,.\,\nabla\chi_i\,dx\right)^2$$

$$\leqslant \sum_{i=0}^{m}(\|q\|_0\,\|\chi_i\|_0 + \|\nabla\phi\|_0\,\|\nabla\chi_i\|_0)^2$$

$$\leqslant C\,\|q\|^2_0\,.$$

From above, we obtain that

$$\|\mathbf{v}\|_{H_{0c}} \leqslant C \|q\|_0 ,$$

which implies

$$\sup_{\mathbf{w} \in H_{0c}} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{H_{0c}}} \geqslant \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H_{0c}}} = \frac{\|q\|_0^2}{\|\mathbf{v}\|_{H_{0c}}} \geqslant C \|q\|_0 .$$

Then the existence of a unique solution $(\mathbf{E}, p)$ of (7.1)-(7.2) follows from Lemma 2.1.

We prove now that if $\mathbf{E}$ is the solution of (5.1), then $(\mathbf{E}, 0)$ is the solution of (7.1)-(7.2). Using (6.1) and $\nabla . \mathbf{E} = 0$, multiplying by $\mathbf{v} \in H_{0c}(\Omega)$ in both sides of the first equation of (5.1), integrating on Ω and using the Green's formula, we come to

$$\int_\Omega (\nabla \times \mathbf{E}) . (\nabla \times \mathbf{v})\, dx = \int_\Omega (\nabla \times \mathbf{v}) . \mathbf{B}_1\, dx ,$$

which implies (7.1). But (7.2) is obvious.

Now the conclusion of Theorem 7.3 follows from the uniqueness of $(\mathbf{E}, p)$. □

7.2. The Neumann problem

Now we consider the $H(\mathbf{curl}, \mathrm{div}\ ; \Omega)$ formulation for the Neumann problem (5.2). For the ease of exposition, we transform the problem (5.2) into a problem with homogeneous boundary condition. Let $\overline{\mathbf{B}} = \mathbf{B} - \mathbf{B}_0$ and $\overline{\mathbf{f}} = \mathbf{f} - \nabla \times \mathbf{B}_0$. Then by Lemma 5.1, $\overline{\mathbf{B}} \in H(\mathbf{curl}, \mathrm{div}\ ; \Omega)$ is the unique solution of the following problem :

$$\text{(7.7)} \qquad \begin{cases} -\Delta \overline{\mathbf{B}} = \nabla \times \overline{\mathbf{f}} & \text{in } \Omega , \\ \nabla . \overline{\mathbf{B}} = 0 & \text{in } \Omega , \\ \overline{\mathbf{B}} . \mathbf{n} = 0 & \text{on } \Gamma , \\ (\nabla \times \overline{\mathbf{B}}) \times \mathbf{n} = \overline{\mathbf{f}} \times \mathbf{n} & \text{on } \Gamma . \end{cases}$$

In the following we will prove that the above problem (7.7) may be characterized as the variational problem : find $(\overline{\mathbf{B}}, p) \in H_{0d}(\Omega) \times L_0^2(\Omega)$ such that

$$\text{(7.8)} \qquad a(\overline{\mathbf{B}}, \mathbf{v}) + b(\mathbf{v}, p) = \int_\Omega \overline{\mathbf{f}} . (\nabla \times \mathbf{v})\, dx, \quad \forall \mathbf{v} \in H_{0d}(\Omega) ,$$

$$\text{(7.9)} \qquad b(\overline{\mathbf{B}}, q) = 0, \quad \forall q \in L_0^2(\Omega) ,$$

where

$$H_{0d}(\Omega) = \{\mathbf{v} \in H(\mathbf{curl}, \mathrm{div}, \Omega), \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q\, dx = 0 \right\},$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v})\, dx + \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\, dx,$$

$$b(\mathbf{v}, q) = \int_\Omega (\nabla \cdot \mathbf{v})\, q\, dx$$

Before we prove our major result of this section, we give an auxiliary lemma

LEMMA 7 4 *In the Hilbert space* $H_{0d}(\Omega)$, $\| \cdot \|_a = (a(\cdot, \cdot))^{1/2}$ *defines a norm which is equivalent to* $\| \cdot \|_{0\ curl\ div}$ *Therefore there exists a constant* $\alpha > 0$ *such that* $a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_{0\ curl\ div}$, $\forall \mathbf{v} \in H_{0d}(\Omega)$

Proof This is exactly (3 38) of Lemma 3 6, Chap 1 in Girault-Raviart [9] □

We are now in a position to show our major result of the section

THEOREM 7 5 *The pair* $(\overline{\mathbf{B}}, 0)$ *where* $\overline{\mathbf{B}}$ *is the solution of (7 7) may be characterized as the unique solution of the variational problem (7 8)-(7 9)*

Proof We first apply Theorem 2 1 to show that the system (7 8)-(7 9) has a unique solution $(\overline{\mathbf{B}}, p) \in H_{0d}(\Omega) \times L_0^2(\Omega)$ Let

$$V = \{\mathbf{v} \in H_{0d}(\Omega), b(\mathbf{v}, q) = 0, \quad \forall q \in L_0^2(\Omega)\}$$

It suffices to verify the inf-sup condition and the V-ellipticity of $a(\cdot, \cdot)$

For the inf-sup condition for any $q \in L_0^2(\Omega)$, we take $\mathbf{v} = \nabla\phi \in H_{0d}(\Omega)$ with ϕ satisfying

$$\Delta\phi = q \quad \text{in } \Omega, \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma$$

We have the following *a priori* estimate for ϕ

$$\|\nabla\phi\|_0 \leq C\|q\|_0, \tag{7 10}$$

where C is a constant depending only on the domain Ω In fact, multiplying $\Delta\phi = q$ by ϕ and then integrating over Ω, we easily come to

$$\| \nabla\phi \|_0^2 \leqslant \| q \|_0 \, \| \phi \|_0 \tag{7 11}$$

But note that ϕ is only unique up to a constant, so the desired result (7 10) comes from (7 11) and Friedrichs' inequality

Thus using (7 10) we obtain

$$\| \mathbf{v} \|_{0\ curl\ div}^2 = \| \nabla . \mathbf{v} \|_0^2 + \| \mathbf{v} \|_0^2 = \| q \|_0^2 + \| \nabla\phi \|_0^2 \leqslant C \| q \|_0^2 ,$$

and by the definition of $b(. , .)$,

$$b(\mathbf{v}, q) = \int_\Omega (\nabla . \mathbf{v}) \, q \, dx = \int_\Omega q^2 \, dx = \| q \|_0^2$$

The last two relations imply that

$$\sup_{\mathbf{w} \in H_{0d}^1(\Omega)} \frac{b(\mathbf{w}, q)}{\| \mathbf{w} \|_{0\ curl\ div}} \geqslant \frac{b(\mathbf{v}, q)}{\| \mathbf{v} \|_{0\ curl\ div}} \geqslant C \, \| q \|_0 , \quad \forall q \in L_0^2(\Omega) ,$$

thus the inf-sup condition holds But the V-ellipticity stems from Lemma 7 4

Now the existence of a unique solution $(\bar{B}, p)$ of (7 8)-(7 9) is the consequence of Theorem 2 1

Finally it is straightforward to verify that the solution $\bar{\mathbf{B}}$ of the problem (7 7) together with $p = 0$ is also the solution of (7 8)-(7 9) □

Remark 7 1 We have used the space $L_0^2(\Omega)$ for functions q instead of $L^2(\Omega)$ To have the inf-sup condition, we defined ϕ as the solution of the Neumann problem $\Delta\phi = q$ in Ω and $\partial\phi/\partial n = 0$ on Γ For the Neumann problem, the compatibility condition is given by $\int_\Omega q \, dx = 0$

8. $H^1(\Omega)$ CONFORMING FINITE ELEMENT METHOD

This section will focus on the $H^1(\Omega)$ conforming finite element method for the solution of the $H($**curl**, div , $\Omega)$ formulated variational problems proposed in Section 7, with both Dirichlet and Neumann boundary conditions

We shall use the Hood-Taylor finite element [13] (*cf* Bercovier-Pironneau [4], Verfurth [19] and Girault-Raviart [9]) Let $\mathcal{T}^h$ be a triangulation of the domain Ω, with each element $K \in \mathcal{T}^h$ being a tetrahedron of diameter diam $(K) \leqslant h$. Then we refine the triangulation $\mathcal{T}^h$ by dividing each tetrahedron into eight sub-tetrahedra, each with a diameter $\leqslant h/2$. The resultant triangulation is denoted by $\mathcal{T}^{h/2}$.

8.1. The Neumann problem (7.8)-(7.9)

Based on two triangulations $\mathcal{T}^h$ and $\mathcal{T}^{h/2}$, if we let $P_1(K) = (\mathcal{P}_1(K))^3$, we define the finite element spaces X_h and Q_h by

$$X_h = \{\mathbf{v}_h \in C^0(\overline{\Omega})\,;\, \mathbf{v}_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}^{h/2}, \mathbf{v}_h\,.\,\mathbf{n} = 0 \text{ on } \Gamma\},$$

$$Q_h = \{q_h \in C^0(\overline{\Omega})\,;\, q_h|_K \in \mathcal{P}_1(K), \quad \forall K \in \mathcal{T}^h\},$$

and their subspaces

$$W_{0h} = \{\mathbf{v}_h \in X_h\,;\, \mathbf{v}_h = 0 \text{ on } \Gamma\},$$

$$Q_{0h} = \left\{q_h \in Q_h\,;\, \int_\Omega q_h\,dx = 0\right\}.$$

Here X_h and Q_h are the subspaces of two continuous spaces X and Q defined by

$$X = H_{0d}(\Omega), \quad Q = L_0^2(\Omega).$$

We remark that the restriction $\mathbf{v}_h\,.\,\mathbf{n} = 0$ on Γ appearing in the definition of X_h is imposed only on the boundary faces of all boundary elements in $\mathcal{T}^{h/2}$. Thus the space X_h is well-defined.

One can prove (*cf.* Verfurth [19] and Raviart [18])

$$\sup_{\mathbf{v}_h \in W_{0h}} \frac{(\nabla\,.\,\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geqslant C\,\|q_h\|_0, \quad q_h \in Q_{0h}. \tag{8.1}$$

Now the finite element method for the Neumann problem (7.8)-(7.9) can be formulated as follows : find $(\overline{\mathbf{B}}_h, p_h) \in X_h \times Q_{0h}$ such that

$$a(\overline{\mathbf{B}}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \int_\Omega \bar{\mathbf{f}}\,.\,(\nabla \times \mathbf{v}_h)\,dx, \quad \forall \mathbf{v}_h \in X_h, \tag{8.2}$$

$$b(\overline{\mathbf{B}}_h, q_h) = 0, \quad \forall q \in Q_{0h}. \tag{8.3}$$

We have the following convergence results :

THEOREM 8.1 : *Let Ω be any polyhedral domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary. There exists a unique solution $(\overline{\mathbf{B}}_h, p_h)$ to (8.2)-(8.3). And we have the following error estimates*

$$\|\overline{\mathbf{B}} - \overline{\mathbf{B}}_h\|_{0, curl, div} \leq Ch \|\overline{\mathbf{B}}\|_2, \quad \text{if } \overline{\mathbf{B}} \in X \cap (H^2(\Omega))^3 ; \tag{8.4}$$

$$\lim_{h \to 0} \|\overline{\mathbf{B}} - \overline{\mathbf{B}}_h\|_{0, curl, div} = 0, \quad \text{if } \overline{\mathbf{B}} \in X . \tag{8.5}$$

where $(\overline{\mathbf{B}}, 0) \in X \times L_0^2(\Omega)$ is the solution to (7.8)-(7.9). If in addition the domain is convex, then

$$\|\overline{\mathbf{B}} - \overline{\mathbf{B}}_h\|_1 \leq Ch \|\overline{\mathbf{B}}\|_2, \quad \text{if } \overline{\mathbf{B}} \in X \cap (H^2(\Omega))^3 ; \tag{8.6}$$

$$\lim_{h \to 0} \|\overline{\mathbf{B}} - \overline{\mathbf{B}}_h\|_1 = 0, \quad \text{if } \overline{\mathbf{B}} \in X . \tag{8.7}$$

Proof : The V_h-ellipticity of $a(\cdot, \cdot)$ immediately stems from Lemma 7.4 (with a constant independent of h).

Since there exists a constant $C > 0$ such that $\|\mathbf{v}\|_{0, curl, div} \leq C \|\mathbf{v}\|_1$, $\forall \mathbf{v} \in (H^1(\Omega))^3$, from (8.1) and $W_{0h} \subset X_h$ we immediately have the following inf-sup condition :

$$\sup_{\mathbf{v}_h \in X_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{0, curl, div}} \geq \beta_0 \|q_h\|_0, \quad \forall q_h \in Q_{0h} . \tag{8.8}$$

Then the existence and uniqueness of the solutions to (8.2)-(8.3) follows from Theorem 2.2.

Next we show the convergence of the finite element method. Recall Theorem 2.2, we easily see that here $b_0 = b_1 = 1$, $a_0 = 1$ and $V_h(g) = V_h(0)$. Thus from Theorem 2.2(b) and (c), we obtain

$$\begin{aligned}\|\overline{\mathbf{B}} - \overline{\mathbf{B}}_h\|_a &\leq C \|\overline{\mathbf{B}} - \Pi_h \overline{\mathbf{B}}\|_a + \|p - \pi_h p\|_Q \\ &\leq C \|\overline{\mathbf{B}} - \Pi_h \overline{\mathbf{B}}\|_1 \leq Ch \|\overline{\mathbf{B}}\|_2 ,\end{aligned}$$

using the standard interpolation result and the fact that p is a constant. Here Π_h(resp. π_h) is the interpolation operator defined on X_h(resp. Q_h).

The rest of the result in the general case can be proved similarly to the proof of Theorem 6.3 by using the density of the subspace $X \cap (H^2(\Omega))^3$ in the space X and Lemma 7.4. If the domain is convex, the desired results follow from the following Lemma 8.2. □

LEMMA 8.2 : *If Ω is a convex polyhedron, the space X is continuously imbedded in $(H^1(\Omega))^3$ and $\|\mathbf{v}\|_1 \leqslant C \|\mathbf{v}\|_{0, curl, div}$, $\forall \mathbf{v} \in X$. Here C is a positive constant.*

Proof : This is Theorem 3.9, Chap. 1 in Girault-Raviart [9]. □

8.2. The Dirichlet problem (7.1)-(7.2)

Based on the previously described triangulations $\mathcal{T}^h$, we first define the finite element space Q_h associated with the pressure by

$$Q_h = \{ q_h \in C^0(\overline{\Omega}) ; \quad q_h|_K \in \mathcal{P}_1(K), \quad \forall K \in \mathcal{T}^h \} .$$

Then one possibility is to construct a finite element space approximating $H_{0c}(\Omega)$ by using the way described in Raviart [18]. The main idea is to impose the boundary condition $\mathbf{v} \times \mathbf{n} = 0$ on Γ in a weak form, that is

$$\int_\Gamma (\mathbf{v} \times \mathbf{n}) \cdot \mu_h \, d\sigma = 0, \quad \forall \mu_h \in (V_h(\Gamma))^3 , \tag{8.9}$$

where $V_h(\Gamma)$ is the standard piecewise linear finite element space defined on $\mathcal{T}^{h/2}(\Gamma) = \Gamma \cap \mathcal{T}^{h/2}$. To be more practically efficient, we further approximate (8.9) by a quadrature. For a triangle $T \in \mathcal{T}^{h/2}(\Gamma)$ with three vertices a_i, $i = 1, 2, 3$, using the following quadrature formula

$$\int_T \phi \, d\sigma \approx \frac{meas\,(T)}{3} \sum_{i=1}^{3} \phi(a_i) ,$$

we can approximate (8.9) by

$$\sum_{i \in I} \sum_{T \in A(i)} \frac{meas\,(T)}{3} (\mathbf{v}(a_i) \times \mathbf{n}_T) \cdot \mu_h(a_i) = 0, \quad \forall \mu_h \in (V_h(\Gamma))^3 ,$$

where $\{a_i ; i \in I\}$ denotes the set of all vertices of $\mathcal{T}^{h/2}(\Gamma)$, $A(i)$ the set of all triangles $T \in \mathcal{T}^{h/2}(\Gamma)$ which have a_i as a vertex and $\mathbf{n}_T$ the unit normal to T. Let $\mathbf{n}(a_i)$ be an approximate unit normal to Γ at the vertex $a_i \in \mathcal{T}^{h/2}(\Gamma)$ defined by

$$\mathbf{n}(a_i) = \Bigg(\sum_{T \in A(i)} meas\,(T)\, \mathbf{n}_T \Bigg) \Big/ \Bigg(\sum_{T \in A(i)} meas\,(T) \Bigg) ,$$

then finally (8.9) can be approximated by

$$X_h = \{\mathbf{v}_h \in C^0(\overline{\Omega})\,;\, \mathbf{v}_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}^{h/2}\,;$$
$$\mathbf{v}_h(a_i) \times \mathbf{n}(a_i) = 0, \quad \forall i \in I\}\,.$$

With the above introduced spaces Q_h and X_h, the finite element approximation to the system (7.1)-(7.2) is formulated as follows: find $(\mathbf{E}_h, p_h) \in X_h \times Q_h$ such that

$$a(\mathbf{E}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \int_\Omega \mathbf{B}_1 \,.\, (\nabla \times \mathbf{v}_h)\, dx, \quad \forall \mathbf{v}_h \in X_h\,,$$

$$b(\mathbf{E}_h, q_h) = 0, \quad \forall q_h \in Q_h\,,$$

where $b(\mathbf{v}, q) = \int_\Omega (\nabla \,.\, \mathbf{v})\, q\, dx$ and

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{u}) \,.\, (\nabla \times \mathbf{v})\, dx$$
$$+ \int_\Omega (\nabla \,.\, \mathbf{u})\,(\nabla \,.\, \mathbf{v})\, dx + \sum_{i=0}^{m} \langle \mathbf{u} \,.\, \mathbf{n}, 1\rangle_{\Gamma_i} \langle \mathbf{v} \,.\, \mathbf{n}, 1\rangle_{\Gamma_i}\,.$$

Unfortunately, it is still an open question whether the space pair (X_h, Q_h) satisfies the discrete inf-sup condition, and no error estimates can be derived yet, though numerical computations show the validity of this pair of finite element spaces if the domain is convex.

In the next section, we will introduce another way to deal with the $H(\mathbf{curl}, \mathrm{div}\,; \Omega)$ formulation for the Dirichlet problem (5.1), for which we can prove the inf-sup condition and derive the error estimates for its finite element approximation.

8.3. Transforming the Dirichlet problem (5.1) into a Neumann problem

In this section, we introduce some techniques to transform the Dirichlet problem (5.1) into a new Neumann problem, analogous to (5.2). And then instead of the use of the $H(\mathbf{curl}, \mathrm{div}\,; \Omega)$ formulation (7.1)-(7.2) for the Dirichlet problem (5.1), as described in Section 7.1, we can adopt the Neumann $H(\mathbf{curl}, \mathrm{div}\,; \Omega)$ formulation for this new Neumann problem, to which the finite element methods described in Section 7.2 can be applied and error estimates can then also be achieved.

To this aim, we first show

THEOREM 8.3 : *The unique solution* $\mathbf{E} \in H(\mathbf{curl}, \operatorname{div} ; \Omega)$ *of the Dirichlet problem (5.1) can be expressed as* $\mathbf{E} = \nabla \times \mathscr{E}$, *where* $\mathscr{E} \in H(\mathbf{curl}, \operatorname{div} ; \Omega)$ *is the unique solution of the following Neumann problem :*

$$(8.10) \qquad \Delta \mathscr{E} = \mathbf{B}_1, \quad \nabla . \mathscr{E} = 0, \quad \text{in } \Omega ,$$

$$(8.11) \qquad \mathscr{E} . \mathbf{n} = 0, \quad (\nabla \times \mathscr{E}) \times \mathbf{n} = 0, \quad \text{on } \Gamma .$$

Proof: The existence of a unique solution $\mathscr{E} \in H(\mathbf{curl}, \operatorname{div} ; \Omega)$ to the system (8.10)-(8.11) follows routinely as we did for the Neumann problem (5.2) in Section 7.2.

We now prove that if $\mathbf{E} = \nabla \times \mathscr{E}$ and $\mathscr{E} \in H(\mathbf{curl}, \operatorname{div} ; \Omega)$ is a solution of (8.10)-(8.11), then $\mathbf{E}$ is the solution of (5.1). The conditions $\nabla . \mathbf{E} = 0$ in Ω and $\mathbf{E} \times \mathbf{n} = 0$ on Γ are obvious. To show $\langle \mathbf{E} . \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$ $(0 \leqslant i \leqslant m)$, for each i we define a function $\theta_j \in C_0^\infty(\mathbb{R}^3)$ satisfying $0 \leqslant \theta_j(x) \leqslant 1$ in $\mathbb{R}^3$, $\theta_j(x) = \delta_{ij}$ in a neighborhood of Γ_j. Let $\mathbf{E}_j = \nabla \times (\theta_j \mathscr{E})$ and so $\nabla . \mathbf{E}_j = 0$ in Ω. By Green's formula we have

$$\langle \mathbf{E} . \mathbf{n}, 1 \rangle_{\Gamma_i} = \langle \mathbf{E}_i . \mathbf{n}, 1 \rangle_{\Gamma} = \int_\Omega \nabla . \mathbf{E}_i \, dx = 0 ,$$

which is the desired boundary condition. The equation $\Delta \mathbf{E} = \nabla \times \mathbf{B}_1$ follows readily by noting $-\nabla \times (\nabla \times \mathscr{E}) = \mathbf{B}_1$, $\mathbf{E} = \nabla \times \mathscr{E}$ and $\Delta \mathbf{E} = -\nabla \times (\nabla \times \mathbf{E})$. By Lemma 5.1, it follows that $\mathbf{E}$ belongs to $H(\mathbf{curl}, \operatorname{div} ; \Omega)$.

Next we prove that if $\mathbf{E}$ is a solution of (5.1), then there exists $\mathscr{E}$ in $H(\mathbf{curl}, \operatorname{div} ; \Omega)$ satisfying (8.10)-(8.11) such that $\mathbf{E} = \nabla \times \mathscr{E}$.

First of all, it is easy to show that there exists a unique solution $(\mathscr{E}, p) \in H(\mathbf{curl} ; \Omega) \times H^1(\Omega)/\mathbb{R}$ to the system

$$(8.12) \qquad \int_\Omega (\nabla \times \mathscr{E}) . (\nabla \times \mathbf{v}) \, dx + \int_\Omega \mathbf{v} . \nabla p \, dx = \\ = \int_\Omega \mathbf{E} . (\nabla \times \mathbf{v}) \, dx, \quad \forall \mathbf{v} \in H(\mathbf{curl} ; \Omega) ,$$

$$(8.13) \qquad \int_\Omega \mathscr{E} . \nabla q \, dx = 0, \quad \forall q \in H^1(\Omega)/\mathbb{R} ,$$

by applying Theorem 2.1 (the proof is similar to that of Theorem 6.2). $\nabla p = 0$ follows by taking $\mathbf{v} = \nabla p$ in (8.12).

Let us now prove that $\mathcal{E}$ satisfies the required conditions. The conditions $\nabla \cdot \mathcal{E} = 0$ in Ω and $\mathcal{E} \cdot \mathbf{n} = 0$ on Γ come immediately from (8.13) and Green's formula. Furthermore, applying Green's formula to (8.12) gives the boundary condition $(\nabla \times \mathcal{E}) \times \mathbf{n} = 0$ and

$$\Delta \mathcal{E} = -\nabla \times \mathbf{E} \quad \text{in} \quad \Omega . \tag{8.14}$$

The above relation (8.14) will lead to $\Delta \mathcal{E} = \mathbf{B}_1$ if we can show that

$$\nabla \times \mathbf{E} = -\mathbf{B}_1 .$$

For the purpose, we observe that $\nabla \times (\nabla \times \mathbf{E} + \mathbf{B}_1) = 0$ from the first equation of (5.1). Therefore there exists a $\phi \in H^1(\Omega)/\mathbb{R}$ (*cf.* Degond-Raviart [8], Lemma 1.1) such that

$$\mathbf{B}_1 + \nabla \times \mathbf{E} = \nabla \phi .$$

Hence Green's formula yields for any ψ in $\mathcal{D}(\overline{\Omega})/\mathbb{R}$ that

$$\begin{aligned}\int_\Omega \nabla \phi \cdot \nabla \psi \, dx &= \int_\Omega (\mathbf{B}_1 + \nabla \times \mathbf{E}) \cdot \nabla \psi \, dx \\ &= -\int_\Omega (\nabla \cdot \mathbf{B}_1)\, \psi \, dx + \langle \mathbf{B}_1 \cdot \mathbf{n}, \psi \rangle_\Gamma + \langle \mathbf{n} \times \mathbf{E}, \nabla \psi \rangle_\Gamma \\ &= 0 ,\end{aligned}$$

by recalling the condition $\mathbf{B}_1 \cdot \mathbf{n} = 0$ on Γ and $\nabla \cdot \mathbf{B}_1 = 0$ in Ω. Therefore we have $\nabla \phi = 0$ by density, that is, $\nabla \times \mathbf{E} + \mathbf{B}_1 = \nabla \phi = 0$, or $\nabla \times \mathbf{E} = -\mathbf{B}_1$.

So far we have proved that $\mathcal{E}$ is the solution to the system (8.10)-(8.11) and $\mathcal{E} \in H(\mathbf{curl}, \operatorname{div}; \Omega)$ as we know from Lemma 5.1.

Finally we show $\mathbf{E} = \nabla \times \mathcal{E}$. Let $\mathbf{g} = \nabla \times \mathcal{E} - \mathbf{E}$, the proved results indicate

$$\nabla \times \mathbf{g} = 0 \quad \text{in} \quad \Omega ; \quad \mathbf{g} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma .$$

Now, using Lemma 1.2 in Degond-Raviart [8] implies

$$\mathbf{g} = \nabla \phi$$

with $\phi \in H^1(\Omega)$ satisfying

$$\Delta \phi = \nabla \cdot \mathbf{g} = \nabla \cdot (\nabla \times \mathcal{E} - \mathbf{E}) = 0 \quad \text{in} \quad \Omega ,$$

$$\phi = \alpha_i \quad \text{on} \quad \Gamma_i, \quad 0 \leqslant i \leqslant m .$$

The α_i's are defined uniquely up to a constant as the solution of the linear system

$$\sum_{j=0}^{m} c_{ij} \alpha_j = \int_\Omega \mathbf{g} \cdot \nabla \chi_i \, dx, \quad 0 \leq i \leq m .$$

Here c_{ij} and χ_i are defined in Theorem 4.1 and $Ker(C) = span\{e\}$, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^{m+1}$. Now, for all $\tilde{\mathscr{E}} \in \mathscr{D}(\overline{\Omega})$, we get

$$\int_\Omega (\nabla \times \tilde{\mathscr{E}} - \mathbf{E}) \cdot \nabla \chi_i \, dx = \langle (\nabla \chi_i \times \mathbf{n}), \tilde{\mathscr{E}} \rangle_\Gamma - \langle \mathbf{E} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0 ,$$

as $\chi_i = \delta_{ij}$ on Γ_j. Thus we have $\int_\Omega \mathbf{g} \cdot \nabla \chi_i \, dx = 0$ by density. Therefore (α_j) belongs to $Ker(C)$, that means $\alpha_j = \alpha_0$ for $0 \leq i \leq m$, i.e. $\phi = \alpha_0$, or $\mathbf{g} = \nabla \phi = 0$. We proved that $\mathbf{E} = \nabla \times \mathscr{E}$ in Ω, which completes the proof of Theorem 8.3. □

8.3.1. *Piecewise linear finite element methods for the Neumann system (8.10)-(8.11)*

As the system (8.10)-(8.11) for the unknown $\mathscr{E}$ is a special case of the Neumann problem (7.7), therefore for solving $\mathscr{E}$ we can adopt the same finite element method based on piecewise linear spaces used in Section 8.1 for solving (7.7). All the results stated in Section 8.1 are valid for the present case. We omit the details.

8.3.2. *Piecewise quadratic finite element methods for the Neumann system (8.10)-(8.11)*

Our final aim is to calculate $\mathbf{E} = \nabla \times \mathscr{E}$, but by means of piecewise linear finite elements as described in Section 8.3.1, one can only have a piecewise constant approximation to the field $\mathbf{E}$. To achieve piecewise linear approximation for the field $\mathbf{E}$, we can make use of the piecewise quadratic finite elements for the solution of $\mathscr{E}$. Let $X = H_{0\,d}(\Omega)$. Adopting the same notation as in Section 8.1, we define

$$X_h = \{\mathbf{v}_h \in C^0(\overline{\Omega}) \, ; \mathbf{v}_h|_K \in P_2(K), \quad \forall K \in \mathcal{T}^{h/2}, \quad \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma\} ,$$

$$W_{1\,h} = \{\mathbf{v}_h \in X_h \, ; \quad \mathbf{v}_h = 0 \text{ on } \Gamma\} ,$$

$$Q_h = \{q_h \in C^0(\overline{\Omega}) \, ; q_h|_K \in \mathscr{P}_1(K), \quad \forall K \in \mathcal{T}^h\} ,$$

$$Q_{0\,h} = \left\{ q_h \in Q_h \, ; \quad \int_\Omega q_h \, dx = 0 \right\} .$$

Note that the pressure finite element space Q_{0h} is still piecewise linear which is enough for approximating the constant pressure p.

Obviously, $W_{0h} \subset W_{1h}$, where W_{0h} is the space defined in Section 8.1. This with (8.1) indicates that

$$\sup_{\mathbf{v}_h \in W_{1h}} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq C\|q_h\|_0, \quad \forall q_h \in Q_{0h}. \tag{8.15}$$

Analogous to (8.2)-(8.3), the finite element method for the Neumann problem (8.10)-(8.11) can be formulated as follows : find $(\mathscr{E}_h, p_h) \in X_h \times Q_{0h}$ such that

$$a(\mathscr{E}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \int_\Omega \mathbf{B}_1 \cdot \mathbf{v}_h \, dx, \quad \forall \mathbf{v}_h \in X_h, \tag{8.16}$$

$$b(\mathscr{E}_h, q_h) = 0, \quad \forall q_h \in Q_{0h}. \tag{8.17}$$

We have the following convergence results :

THEOREM 8.4 : *Let Ω be any polyhedral domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary. There exists a unique solution $(\mathscr{E}_h, p_h)$ to (8.16)-(8.17). And we have the following error estimates*

$$\|\mathscr{E} - \mathscr{E}_h\|_{0,\,curl,\,div} \leq Ch^{r-1}\|\mathscr{E}\|_r, \quad \text{if } \mathscr{E} \in X \cap (H^r(\Omega))^3 \text{ for some } 2 \leq r \leq 3\,; \tag{8.18}$$

$$\lim_{h \to 0} \|\mathscr{E} - \mathscr{E}_h\|_{0,\,curl,\,div} = 0, \quad \text{if } \mathscr{E} \in X, \tag{8.19}$$

where $\mathscr{E}$ is the solution to the system (8.12)-(8.13). If in addition the domain is convex, then

$$\|\mathscr{E} - \mathscr{E}_h\|_1 \leq Ch^{r-1}\|\mathscr{E}\|_r, \text{ if } \mathscr{E} \in X \cap (H^r(\Omega))^3 \text{ for some } 2 \leq r \leq 3\,; \tag{8.20}$$

$$\lim_{h \to 0} \|\mathscr{E} - \mathscr{E}_h\|_1 = 0, \quad \text{if } \mathscr{E} \in X. \tag{8.21}$$

Proof : The proof is similar to the one for Theorem 8.1, but one uses (8.15) here instead of (8.1). □

Authors' note

Recently, in a paper entitled "Vector Potentials in Three-Dimensional Nonsmooth Domains" (Technical Report, IRMAR #96-04, Rennes, France, 1996), Amrouche, Bernardi, Dauge and Girault extended Lemmas 6.4 and 6.6 to the case of non convex polyhedra (with a Lipschitz continuous boundary). Therefore, the conclusions of Theorems 6.3 and 6.5, i.e. estimates (6.24) and (6.33) and convergence properties (6.25) and (6.34), can be generalized to this class of domains.

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