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## ON A NON-HOMOGENEOUS SHALLOW-WATER PROBLEM (\*)

by François-Joseph CHATELON <sup>(1)</sup> and Pierre ORENGA <sup>(1)</sup>

*Abstract* — We present an existence theorem for a shallow-water problem with a depth-mean velocity formulation and non homogeneous boundary conditions expressing water entering. A result has been already shown in the case of homogeneous boundary conditions. If we prescribe a non-zero velocity (or normal velocity) on the boundary, we must also prescribe the water elevation on the part of the boundary where the flow enters. With these boundary conditions, we obtain a priori estimates that show the problem has a solution. We build a sequence of approximated solutions that preserves energy and to pass to the limit we use a trace theorem for the space of  $L^1$ -functions with  $L^1$ -divergence.

*Résumé* — Nous présentons un théorème d'existence de solutions d'un problème de shallow-water, en formulation hauteur-vitesse avec conditions aux limites non homogènes exprimant les entrées d'eau. Un résultat a déjà été montré dans le cas de conditions homogènes. Si on fixe la vitesse (ou la vitesse normale) non nulle sur le bord, on doit fixer également la hauteur d'eau sur la partie de la frontière où le flux est strictement entrant. Avec ce type de conditions, on obtient des majorations de type énergie qui servent à montrer que le problème a une solution. Nous construisons une suite de solutions approchées respectant les majorations de type énergie établies et pour passer à la limite, nous utilisons un théorème de trace sur l'espace des fonctions intégrables dont la divergence est intégrable.

### 1. INTRODUCTION

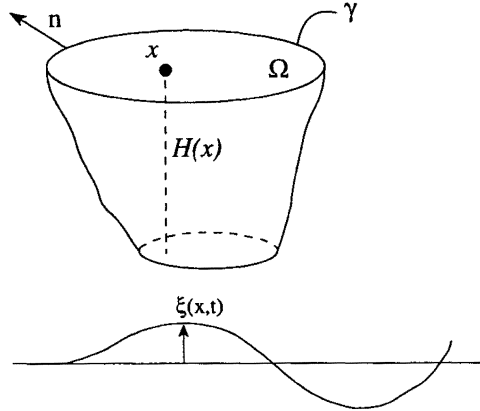
#### 1.1. Notations

Let  $\Omega$  be a fixed bounded smooth open domain of  $\mathbb{R}^2$  with boundary  $\gamma$ . Let  $n$  be the exterior unit normal to  $\Omega$  on  $\gamma$ . Physically,  $\Omega$  is the domain corresponding to the surface of the sea assumed to be horizontal. We denote by  $H(x)$  the depth of the sea at the point  $x(x_1, x_2)$  of  $\Omega$ . Let  $Q$  be the cylinder  $]0, T[ \times \Omega$  with boundary  $\partial Q$ . We denote by  $\gamma^-$  (resp  $\gamma^+$ ) the part of the boundary where the flow enters (resp. is outgoing or is zero).

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Then, we set :

$$\Sigma = ]0, T[ \times \gamma$$

$$\Sigma^- = ]0, T[ \times \gamma^-$$

$$\Sigma^+ = ]0, T[ \times \gamma^+ .$$

We will also denote by  $(\cdot, \cdot)$  and  $\|\cdot\|$  the scalar product and the associated norm of  $L^2(\Omega)$  and  $L^2(\Omega)^2$ . Moreover, if  $u = (u_1, u_2)$  is a vector function from  $\Omega$  into  $\mathbb{R}^2$  and  $q$  a scalar function from  $\Omega$  into  $\mathbb{R}$ , we define the following operators  $\alpha$ , curl, Curl and  $\nabla$  as follows :

$$\alpha(u) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \quad \text{Curl } q = \begin{pmatrix} \frac{\partial q}{\partial x_2} \\ -\frac{\partial q}{\partial x_1} \end{pmatrix} \quad \text{curl } u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{pmatrix} .$$

### 1.2. Presentation of the model

The three-dimensional hydrodynamic equations are the followings ([LTW]) :

$$v_t + \nabla \cdot (v \otimes v) + 2 \mathcal{A} \wedge v - \frac{\partial}{\partial x_3} \left( v \frac{\partial v}{\partial x_3} \right) - \nabla(\tilde{v} \nabla v) = -\nabla p + \rho g \quad (1.1a)$$

$$\text{div } v = 0 \quad (1.1b)$$

$$T_t + \nabla \cdot (Tv) = \Delta T \quad (1.1c)$$

$$S_t + \nabla \cdot (Sv) = \Delta S \quad (1.1d)$$

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)) . \quad (1.1e)$$

Here,  $v$  is the velocity vector field,  $T$  the temperature,  $S$  the salinity,  $\Lambda$  the earth rotation vector,  $p$  the pressure,  $\rho$  the specific mass of sea water,  $x_3$  the vertical coordinate,  $\nu$  the vertical eddy viscosity,  $\bar{\nu}$  the horizontal eddy viscosity,  $\rho_0$ ,  $T_0$ ,  $S_0$  reference values of the density, the temperature and the salinity respectively and  $\beta_T$ ,  $\beta_S$  are expansion coefficients (given constants).

Then, we make some particular assumptions on this model. The first consists in neglecting the horizontal turbulent diffusion compared to the vertical diffusion because horizontal length scales are much larger than vertical length scales and that, turbulent fluxes are proportionnal to the gradients of the mean quantities. Moreover, considering  $H$  a characteristic vertical length and  $L$  a characteristic horizontal length, we have  $\frac{H}{L} \ll 1$ . Then a scale analysis for the vertical component of the momentum equation allow us to write the so-called hydrostatic approximation :

$$\frac{\partial p}{\partial x_3} - \rho g = 0.$$

That means all the terms of the  $x_3$ -momentum equation can be neglected with respect to the acceleration of the gravity and the vertical component of the gradient of pressure. That also means we no longer have an evolution equation. We write  $v = u + v_3 \vec{e}_3$  where  $u$  denote the horizontal velocity.

Moreover, if we want to work with cartesian coordinates, we must do another approximation. We change the earth rotation vector by its component on the local vertical axis. Then, denoting the mean latitude by  $\lambda$ , we can write  $2 \Lambda \wedge v = \omega \wedge v$  with  $\omega = (2 \Lambda \sin \lambda) \vec{e}_3$ .

If we still denote the operator  $\left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$  by  $\nabla$ , we write the equations of the three dimensional model :

$$u_t + \nabla \cdot (u \otimes u) + \omega \wedge u + \frac{\partial}{\partial x_3} (u v_3) - \frac{\partial}{\partial x_3} \left( \nu \frac{\partial u}{\partial x_3} \right) = - \nabla p + \rho g \quad (1.2a)$$

$$\operatorname{div} u + \frac{\partial v_3}{\partial x_3} = 0 \quad (1.2b)$$

$$T_t + \nabla \cdot (T u) + \frac{\partial}{\partial x_3} (T v_3) = \Delta T \quad (1.2c)$$

$$S_t + \nabla \cdot (S u) + \frac{\partial}{\partial x_3} (S v_3) = \Delta S \quad (1.2d)$$

$$\rho = \rho_0 (1 - \beta_T (T - T_0) + \beta_S (S - S_0)). \quad (1.2e)$$

The depth averaged motion is described in terms of the mean velocity denoted by  $\bar{u}(x)$  (where  $x = (x_1, x_2)$  is a point of the surface  $\Omega$ ) defined by :

$$\bar{u} = \frac{1}{h} \int_{-H}^{\xi} u \, dx_3$$

where  $h$  is the total depth i.e.  $h(t, x) = H(x) + \xi(t, x)$ . Let  $u'$  be the deviations from the vertical mean, with  $\int_{-H}^{\xi} u' \, dx_3 = 0$ . To obtain a shallow water model, we consider that there is no stratification effect, i.e. the sea is well mixed ( $\rho$  is constant). Then the equations (1.2c) and (1.2d) are not necessary any longer. The shallow water equations are obtained by integrating the momentum equation (1.2a) over depth, we obtain :

$$\bar{u}_t + \operatorname{div} \mathcal{R} + \frac{1}{2} \nabla \bar{u}^2 + \operatorname{curl} \bar{u} \alpha(\bar{u}) + g \nabla \xi + F(\bar{u}) = f.$$

Here  $\mathcal{R}$  denotes the Reynolds stress tensor that results from the non linear interactions of the fluctuations products and  $g$  denotes the acceleration of gravity.  $F(\bar{u})$  and  $f$  come from the integration of the dissipation term  $\frac{\partial}{\partial x_3} \left( \nu \frac{\partial u}{\partial x_3} \right)$ . Indeed,  $f$  represents the wind effect at the surface and  $F(\bar{u})$  is the shear effect at the bottom. Usually, one can write  $F(\bar{u}) = D\bar{u}|_{\mathbb{R}^2}$ .

Instead of considering this problem, we are going to replace the elevation of the free surface  $\xi$  by the water elevation  $h$  that leads to the appearance of a new term  $g \nabla H$  (non-dependent on time) that we put in the term of the right-hand side of the equation.

Finally, we can simply write a good approximation of the Reynolds stress tensor (see in particular [N]) :

$$\operatorname{div} \mathcal{R} = -A \Delta \bar{u}$$

where  $A$  is the eddy viscosity.

In the same way, with the integration of the continuity equation over depth, we obtain :

$$h_t + \operatorname{div} (\bar{u}h) = 0.$$

To simplify notations, we are going to denote the mean velocity  $\bar{u}$  by  $u$  and we take  $g = 1$ .

Let us recall the momentum and continuity equation :

$$u_t - A \Delta u + \frac{1}{2} \nabla u^2 + \operatorname{curl} u \alpha(u) + \nabla h + Du|u| = f \quad (1.3)$$

$$h_t + \operatorname{div} (uh) = 0.$$

A set of natural conditions for this problem to be well-posed consists in the boundary conditions and the following Cauchy data at time :

$$\begin{aligned} u(t=0, x) &= u_0(x) \quad \text{in } \Omega \\ h(t=0, x) &= h_0(x) \quad \text{in } \Omega. \end{aligned} \tag{1.4}$$

We present an existence theorem for a shallow water problem with a depth-mean velocity formulation and non-homogeneous boundary conditions expressing water entering. If we have to prescribe the velocity on the boundary, we prescribe the water elevation on the part of the boundary where the flow enters (as it is shown in the third paragraph). In this case, we get *a priori* estimates that show the problem has a solution. Particularly, about the water elevation, we show that  $h$  and  $h \log h$  are bounded into  $L^1(Q)$  which prove that  $h$  is in a subspace of equi-integrable functions of  $L^1(Q)$ . To verify the boundary condition, we have shown a trace theorem on the space of integrable functions whose divergence is integrable. We know how to prove the existence of solutions when we consider two cases of boundary conditions.

- $u \cdot n$  prescribed and  $\text{curl } u = 0$  on the boundary.
- $u$  prescribed on the boundary.

In the first case, if we want to obtain a global solution on  $[0, \infty[$ , we have to take into account the dissipation term at the bottom  $Du|u|_{\mathbb{R}^2}$  if the domain is not simply connected. Sometimes, models neglect the shear effect at the bottom and in this case the theorem remains true if the domain is simply connected. Otherwise, we always obtain a solution on  $(0, T)$  where  $T$  is « small ».

In the second case, we prove the existence of solutions on  $(0, +\infty)$  for all domain  $\Omega$ . In the literature (for example [AAPQS]) other boundary conditions on  $h$  are proposed ; in this case, we do not know if the existence result is still true (we do not find *a priori* estimates).

After acting the theorem, we explain in the third paragraph, how we obtain *a priori* estimates for the problem. And then we present in three lemma how we can pass to the limit with the approximated solutions whose construction is developed in a sixth part.

We would like to thank P. L. Lions for his kind help.

## 2. AN EXISTENCE THEOREM

We give our existence theorem in the case where  $u \cdot n = G$  and  $\text{curl } u = 0$  on the boundary. Let  $f \in L^2(0, T; H^{-1}(\Omega)^2)$ ,  $G, \frac{\partial G}{\partial t} \in L^2(0, T; H^{1/2}(\gamma))$ ,  $h = \mu \in L^1(0, T; L^1(\gamma^-))$  and  $u_0 \in H^1(\Omega)^2$ . We also need the compatibility condition  $u_0 \cdot n = G(0)$ .

The problem we consider is the following :

$$(P) \begin{cases} u_t - A \Delta u + \frac{1}{2} \nabla u^2 + \operatorname{curl} u \alpha(u) + Du|u| + \nabla h = f & \text{in } Q \\ u \cdot n = G & \text{on } \Sigma \\ \operatorname{curl} u = 0 & \text{on } \Sigma \\ u(t=0) = u_0 & \text{in } \Omega \\ h_t + \operatorname{div}(uh) = 0 & \text{in } Q \\ h = \mu \geq 0 & \text{on } \Sigma^- \\ h(t=0) = h_0 \geq 0 & \text{in } \Omega. \end{cases}$$

Let us observe that we have not included the Coriolis term. Indeed, if this term is important from a numerical point of view, mathematically, this term does not make any difference in the theoretical analysis.

### 2.1. Weak formulation

We solve the above problem (P) using a weak formulation. We are going to transform equations in order to obtain an homogeneous problem.

Since  $G \in L^2(0, T; H^{1/2}(\gamma))$  and  $\frac{\partial G}{\partial t} \in L^2(0, T; H^{1/2}(\gamma))$ , we make sense to  $G(t)$  for each  $t$  and then we can solve for each  $t$  the following scalar problem (S) :

$$(S) \begin{cases} -\Delta p(t) = f_1(t) \in L^\infty(\Omega) \\ \frac{\partial p(t)}{\partial n} = G(t) \in H^{\frac{1}{2}}(\gamma) \end{cases}$$

where  $f_1$  is chosen in such a way that  $\int_{\Omega} f_1 + \int_{\gamma} G = 0$  and  $f_1 \in H^1(0, T; L^\infty(\Omega))$  in order to have a solution. Then the function  $w(t) = \nabla p(t)$  satisfies :

$$w \in H^1(0, T; H^1(\Omega)^2) \quad (2.1a)$$

$$\operatorname{div} w \in H^1(0, T; L^\infty(\Omega)) \quad (2.1b)$$

$$\operatorname{curl} w = 0 \quad (2.1c)$$

$$w \cdot n = G \quad (2.1d)$$

$$\int_{\Omega} f_1 + \int_{\gamma} G = 0. \quad (2.1e)$$

Setting  $u = v + w$ , we get the problem  $(P')$  :

$$(P') \begin{cases} v_t + w_t - A \Delta(v + w) + \frac{1}{2} \nabla(v + w)^2 \\ \quad + \operatorname{curl}(v + w) \alpha(v + w) + \nabla h + D(v + w) |v + w| = f \quad \text{in } Q \\ v \cdot n = 0 \quad \text{on } \Sigma \\ \operatorname{curl} v = 0 \quad \text{on } \Sigma \\ v(t = 0) = u_0 - w(t = 0) \quad \text{in } \Omega \\ h_t + \operatorname{div}(vh) + \operatorname{div}(wh) = 0 \quad \text{in } Q \\ h = \mu \geq 0 \quad \text{on } \Sigma^- \\ h(t = 0) = h_0 \geq 0 \quad \text{in } \Omega. \end{cases}$$

Classically, we obtain the weak formulation (denoted by  $(\mathcal{V})$ ) associated to the problem  $(P')$  :

$$\begin{aligned} & (v_r, \varphi) + Aa(v, \varphi) - \frac{1}{2} (v^2, \operatorname{div} \varphi) - (vw, \operatorname{div} \varphi) + (\operatorname{curl} v \alpha(v), \varphi) \\ & + (\operatorname{curl} v \alpha(w), \varphi) - (h, \operatorname{div} \varphi) + (D(v + w) |v + w|, \varphi) = (f, \varphi) \\ & + \frac{1}{2} (w^2, \operatorname{div} \varphi) - Aa(w, \varphi) - (w_r, \varphi) \quad \forall \varphi \in V \cap H^3(\Omega). \end{aligned} \quad (2.2)$$

Here and below, we denote by  $a(u, \varphi)$  the following bilinear form :

$$a(u, \varphi) = \int_{\Omega} \operatorname{div} u \operatorname{div} \varphi + \int_{\Omega} \operatorname{curl} u \operatorname{curl} \varphi.$$

The space  $V$  in which we work is the following :

$$V = \{ \varphi \in L^2(\Omega)^2, \operatorname{div} \varphi \in L^2(\Omega), \operatorname{curl} \varphi \in L^2(\Omega), \varphi \cdot n = 0 \text{ on } \gamma \}.$$

$V$  is equipped with the norm  $\|\varphi\|_V^2 = \|\varphi\|_{L^2}^2 + \|\operatorname{div} \varphi\|_{L^2}^2 + \|\operatorname{curl} \varphi\|_{L^2}^2$ . Then, with this norm, the bilinear form  $a$  is elliptic if the domain  $\Omega$  is simply connected. Note that if  $\Omega$  is sufficiently smooth, this space  $V$  is algebraically and topologically equal to the space  $\{ \varphi \in H^1(\Omega), \varphi \cdot n = 0 \}$ .

The continuity equation will be solved into  $L^1(0, T; W^{-1,1}(\Omega))$  with its boundary conditions :

$$h_t + \operatorname{div}(vh) + \operatorname{div}(wh) = 0$$

$$h|_{\Sigma^-} = \mu.$$



We add the initial conditions

$$v(t=0, x) = u_0(x) - w(t=0, x) = v_0(x) \quad a \in \Omega$$

$$h(t=0, x) = h_0(x) \geq 0 \quad a \in \Omega$$

*Remark 1* If we have Dirichlet boundary conditions  $u = G$  on  $\gamma$ , we consider  $w$  (like in the problem (S)) solution of the Stokes problem with  $p \in L^2(\Omega)$  <sup>(2)</sup>

$$(D) \quad \begin{cases} -\Delta w(t) + \nabla p(t) = f_2(t) \in H^{-1}(\Omega)^2 \\ \operatorname{div} w(t) = \phi(t) \in L^\infty(\Omega) \\ w(t)|_\gamma = G(t) \in H^{\frac{1}{2}}(\gamma) \end{cases}$$

*Remark 2* One often neglect the shear effect at the bottom. In this case and if  $\Omega$  is not simply connected, we know the existence of solutions when  $T$  is small. If  $\Omega$  is simply connected, the bilinear form  $a$  is elliptic because the norm  $\|\operatorname{div} \varphi\|_{L^2} + \|\operatorname{curl} \varphi\|_{L^2}$  is equivalent to  $\|\varphi\|_V$ .

*Remark 3* If we consider the condition  $u = G$  on the boundary, the existence result is true for all smooth domain  $\Omega$  even if we neglect the viscous effect at the bottom.

## 2.2. Theorem

We need just a few notations. We first consider  $C_t$  and  $C_j$  some injection constants defined as follows

$$\|u\|_{L^4(\Omega)}^2 \leq C_t \|u\|_{H^1(\Omega)}^2 \quad (2.3a)$$

$$C_j \|u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^3(\Omega)}^2 \quad (2.3b)$$

We denote by  $C$  the best constant associated to Gagliardo Nirenberg's inequality

$$\|u\|_{L^4(\Omega)}^2 \leq C \|u\|_V \|u\| \quad (2.3c)$$

The eddy viscosity  $A$  and the coefficient of viscous effects at the bottom  $D$  are given constants. Let  $\theta \in ]0, 1[$  and  $\lambda \in \mathbb{R}^*$ . We denote by  $\bar{h}$  the averaged water elevation.

<sup>(2)</sup> See [T] for the existence of this function

We present a global existence result with controlled data and we assume the data are not too much larger and satisfy the following conditions :

$$\begin{aligned} & \frac{1}{\lambda} \|f\|_{L^2(0, T, H^{-1}(\Omega)^2)}^2 + 2\left(\bar{h} \text{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G\|_{L^1(\Sigma^+)}\right) \\ & \quad + 2 \|h_0 \log h_0\|_{L^1(\Omega)} \\ & - 2 \int_{\Sigma^-} G \cdot \mu \log \mu + \left(\frac{1}{2} + 2D\right) \|w\|_{L^4(0, T, L^4(\Omega)^2)}^4 + A \|\text{div } w\|_{L^2(\Omega)}^2 + \|v_0\|^2 \\ & + \|w_t\|_{L^2(0, T, L^4(\Omega)^2)}^2 + \int_0^1 \sup_{\Omega} (f_1) \left[ \|h_0\|_{L^1(\Omega)} - \int_{\Sigma^+} G_{\mu} \right] < \theta^2 \frac{K^2}{C^2} \end{aligned} \tag{2.3d}$$

and

$$\begin{aligned} K = & \left( 2 \inf(A, C, D) - \lambda - \frac{7}{2} - 2D - A \right) \\ & - 2 C_t (\|w\|_{L^4(\Omega)}^2 + 2D \|w\|_{L^2(\Omega)}) \end{aligned} \tag{2.3e}$$

where  $w$  is the function of the variable change previously found in (2.1). The number  $K$  which only depends on the data is assumed to be non-negative.

**THEOREM :** Assume that  $\Omega$  is a bounded smooth open domain of  $\mathbb{R}^2$  with boundary  $\gamma$ . Let  $v_0 \in H^1(\Omega)^2$ ,  $h_0 \in L^1(\Omega)$  and  $f$  satisfy the following conditions :

$$h_0 \log h_0 \in L^1(\Omega), f \in L^2(0, T; H^{-1}(\Omega)^2) \tag{2.4a}$$

$$\|v_0\| < \theta \frac{K}{C}. \tag{2.4b}$$

Then, for each  $\theta, \lambda, f, v_0$  et  $h_0$  satisfying the previous conditions, the weak problem ( $\mathcal{V}$ ) has a solution  $(v, h)$  such as :

$$(v, h) \in \{(L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))) \times L^\infty(0, T; L^1(\Omega))\}.$$

Moreover, the solution satisfies the following estimates :

$$\begin{aligned} \|v\|_{L^\infty(0, T, L^2(\Omega)^2)} & \leq \theta \frac{K}{C} \\ \|v\|_{L^2(0, T, V)}^2 & \leq \theta^2 \frac{K}{C^2(1-\theta)} \end{aligned}$$

$$\begin{aligned} & \text{Sup}_t \int_{\Omega} h \log h + \int_{\Sigma^+} G \cdot h \log h \\ & \leq \theta^2 \frac{K^2}{C^2} + 2\left(\bar{h} \text{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G\|_{L^1(\Sigma^+)}\right). \end{aligned}$$

The proof of this theorem lays on the following lemma that we will present and prove :

- *a priori* estimates lemma.
- Passage to the limit into the continuity equation.
- Passage to the limit into the momentum equation.
- Construction of approached solutions.

### 3. A PRIORI ESTIMATES

LEMMA : If  $(v, h)$  is a classical solution of the problem  $(\mathcal{V})$ , and if the relations (2.3) are satisfied, then we have :

$$\frac{d}{dt} \int_{\Omega} h = - \int_{\gamma} Gh \quad (3.1a)$$

$$\begin{aligned} & - 2 \left( \bar{h} \text{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G\|_{L^1(\Sigma^+)} \right) \\ & \leq \|v\|_{L^\infty(0,T, L^2(\Omega)^2)}^2 + 2 \sup_{\Gamma} \int_{\Omega} h \log h \\ & + \|v\|_{L^2(0,T, V)}^2 (K - C \|v\|_{L^\infty(0,T, L^2(\Omega)^2)}) + 2 \int_{\Sigma^+} G \cdot h \log h \leq C_0 \end{aligned} \quad (3.1b)$$

$$K - C \|v\|_{L^2(\Omega)^2} > 0 \quad (3.1c)$$

$$h \geq 0. \quad (3.1d)$$

*Proof* : The result (3.1d) is a classical result. Let us recall briefly the proof of this fact : we consider the path  $(\mathcal{T})$  defined by the equation  $\frac{dx}{dt} = u$  where the data is  $x(t = \tau) = z$  if  $\tau \neq 0$  or  $x(t = 0) = x_0$  if  $\tau = 0$ . The solution of this problem is denoted by  $x(t, \tau, z)$  and it satisfies :

$$x(\tau, \tau, z) = z \quad \text{if } \tau \neq 0$$

$$x(0, 0, z) = x_0 \quad \text{if } \tau = 0.$$

We solve the continuity equation along the curves  $x(t, \tau, z)$  with the condition  $h(x_0)$  fixed or  $h(z)$  fixed. These data are given by the initial conditions  $h_0(x)$  or the boundary condition  $h = \mu$  on  $\Sigma^-$ .

We obtain :

$$\frac{dh}{dt} = - h \operatorname{div} u , \text{ i.e.}$$

$$\frac{dh(t, x(t))}{dt} = - h(t, x(t)) (\operatorname{div} u) (t, x(t)) .$$

Then the solution of this problem ( $\mathcal{H}$ ) is

$$h(t, x(t, \tau, z)) = C_1 \cdot e^{-\int_{\tau}^t (\operatorname{div} u) (\xi, x(\xi, \tau, z)) d\xi}$$

where  $C_1$  is a constant obtained by :

$$h(\tau, x(\tau, \tau, z)) = \mu \quad \text{if } \tau \neq 0 \text{ and } z \in \gamma^-$$

$$h(0, x(0, 0, z)) = h_0 \quad \text{if } \tau = 0 \text{ and } z \in \Omega .$$

This proves that if  $h_0$  and  $\mu$  are non-negative functions,  $C_1$  is a non-negative constant and thus  $h$  is a non-negative function.

Now we are going to prove (3.1a) ; contrarily to the homogeneous case, we do not have the result  $\int_{\Omega} h = \int_{\Omega} h_0$ , but by integration of the continuity equation, we have the following result :

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} h &= - \int_{\gamma} Gh \\ \Leftrightarrow \int_{\Omega} h + \int_0^T \int_{\gamma^+} Gh &= \int_{\Omega} h_0 - \int_0^T \int_{\gamma^-} G\mu \end{aligned}$$

i.e.

$$\|h\|_{L^1(\Omega)} + \int_{\Sigma^+} Gh \leq C_2 . \tag{3.2}$$

To obtain (3.1b), the difficulty comes from the continuity equation ; we do not have  $\operatorname{div} u = 0$  as in the Stokes problem. The only estimate on  $h$  is (3.2)

and it does not allow to avoid the difficulties of the weak topology of  $L^1(\Omega)$ . Thus, we write the energy inequality in order to obtain another estimate on  $h$ ; then, changing  $\varphi$  by  $v$  in (2.2) yields :

$$\begin{aligned} 2(v_p, v) + 2Aa(v, v) - (v^2, \operatorname{div} v) - 2(h, \operatorname{div} v) \\ + (D(v+w) |(v+w)|, v) = 2(f, v) \\ + 2(vw, \operatorname{div} v) - 2(\operatorname{curl} v\alpha(w), v) - 2(w_p, v) - 2Aa(w, v) + (w^2, \operatorname{div} v). \end{aligned}$$

We have two terms that we need to estimate <sup>(3)</sup> :

$$(v^2, \operatorname{div} v) \quad \text{and} \quad (\nabla h, v).$$

The last term will be transformed using the continuity equation as follows. Formally, we can write :

$$(\nabla h, v) = \left( \frac{\nabla h}{h}, vh \right) = (\nabla \log h, vh) = -(\log h, \operatorname{div}(vh)).$$

Using the continuity equation it follows :

$$\begin{aligned} (\nabla h, v) &= (\log h, h_t) + (\log h, \operatorname{div}(wh)) \\ &= \frac{d}{dt} (h \log h - h, 1) - \int_{\Omega} \nabla h \cdot w + \int_{\gamma} G h \log h \\ &= \frac{d}{dt} (h \log h - h, 1) + \int_{\Omega} h \operatorname{div} w + \int_{\gamma} G (h \log h - h) \\ &= \frac{d}{dt} (h \log h, 1) + \int_{\gamma} G (h \log h) + \int_{\Omega} h \operatorname{div} w. \end{aligned}$$

Thanks to (2.1b) and (3.2), the term  $\int_{\Omega} h \operatorname{div} w$  can be easily bounded and then, it is put in the right-hand side of the equality.

In order to estimate the term  $(v^2, \operatorname{div} v)$ , we build a stability space as following :

First, we bound all the terms in the right-hand side of the equation like :

$$|(w^2, \operatorname{div} v)| \leq \|w\|_{L^4}^2 \|v\|_V \leq \frac{1}{2} \|w\|_{L^4}^4 + \frac{1}{2} \|v\|_V^2$$

---

<sup>(3)</sup> These terms are not equal to zero as in the Stokes problem

and

$$|(f, v)| \leq \frac{\lambda}{2} \|v\|_V^2 + \frac{1}{2\lambda} \|f\|_{H^{-1}}^2.$$

Thus, we obtain the energy inequality :

$$\begin{aligned} \frac{d}{dt} \|v\|^2 + K \|v\|_V^2 - (v^2, \operatorname{div} v) + 2 \frac{d}{dt} \int_{\Omega} h \log h + 2 \int_{\gamma} G \cdot h \log h \\ \leq \frac{1}{\lambda} \|f\|_{H^{-1}}^2 + \frac{1}{2} \|w\|_{L^4}^4 + \|w_t\|^2 + \|\operatorname{div} w\|^2 \\ + \sup_{\Omega} (f_1) \left[ \|h_0\|_{L^1(\Omega)} - \int_{\gamma^-} G \mu \right] \end{aligned}$$

i.e.

$$\frac{d}{dt} \|v\|^2 + K \|v\|_V^2 - (v^2, \operatorname{div} v) + 2 \frac{d}{dt} \int_{\Omega} h \log h + 2 \int_{\Omega} G \cdot h \log h \leq C_3.$$

If we choose data satisfying  $K \|v\|_V^2 - |(v^2, \operatorname{div} v)| \geq 0$ , then we automatically have  $K \|v\|_V^2 - (v^2, \operatorname{div} v) \geq 0$ .

Now, using Gagliardo Nirenberg's inequality, we bound the term  $(v^2, \operatorname{div} v)$  as follows :

$$|(v^2, \operatorname{div} v)| \leq \|v\|_{L^4}^2 \|v\|_V \leq C \|v\|_V^2 \|v\|.$$

Then, as we wish to have  $K \|v\|_V^2 - (v^2, \operatorname{div} v) \geq 0$ , we choose data as follows :

$$K - C \|v(t)\| \geq 0 \quad \forall t.$$

We are going to prove that we must have  $\|v(t)\| < \frac{K}{C}$ . We assume the solution is continuous from  $[0, T[$  to  $L^2(\Omega)^2$ . Since  $\|v_0\| < \frac{K}{C}$ , there exists a time  $t_1$  such that  $\|v(t)\| < \frac{K}{C}$  on  $[0, t_1[$ . Suppose that  $\|v(t_1)\| = \frac{K}{C}$ , we can write :

$$\begin{aligned} \left(\frac{K}{C}\right)^2 - 2 \left( \bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G\|_{L^1(\Sigma^+)} \right) \leq \frac{1}{\lambda} \|f\|_{L^2(0,T,H^{-1}(\Omega)^2)}^2 + \|v_0\|^2 \\ - 2 \int_{\Sigma^+} G \cdot \mu \log \mu + A \| \operatorname{div} w \|_{L^2(\Omega)}^2 + 2 \|h_0 \log h_0\|_{L^1(\Omega)} + \|w_t\|_{L^2(0,T,L^2(\Omega)^2)}^2 \\ + \left(\frac{1}{2} + 2D\right) \|w\|_{L^4(0,T,L^4(\Omega)^2)}^4 + \int_0^T \sup_{\Omega} (f_1) \left[ \|h_0\|_{L^1(\Omega)} - \int_{\Sigma^+} G \mu \right] \end{aligned}$$

which contradicts the condition (2 3d)

Finally, we obtain (3 1b) by integrating the energy inequality

$$\begin{aligned}
& - 2 \left( \bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G\|_{L^1(\Sigma^+)} \right) \\
\leq & \|v\|_{L^2(0,T;L^2(\Omega)^2)}^2 + 2 \sup_t \int_{\Omega} h \log h \\
& + \|v\|_{L^2(0,T;V)(K-C(v))_{L^2(0,T;L^2(\Omega)^2)}}^2 + 2 \int_{\Sigma^+} G \cdot h \log h \\
\leq & \frac{1}{\lambda} \|f\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 + \|v_0\|^2 + 2 \|h_0 \log h_0\|_{L^1(\Omega)} \\
& + A \| \operatorname{div} w \|_{L^2(Q)}^2 + \left( \frac{1}{2} + 2D \right) \|w\|_{L^4(0,T;L^4(\Omega)^2)}^4 - 2 \int_{\Sigma^-} G \cdot \mu \log \mu \\
& + \|w_t\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \int_0^T \sup_t (f_1) \left[ \|h_0\|_{L^1(\Omega)} - \int_{\Sigma^+} G \mu \right]
\end{aligned}$$

The left-hand side of the inequality is obtained with these two relations

$$\int_{\Omega} h(t) \log h(t) \geq -\bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} \quad (3 3a)$$

$$\int_{\gamma^+} h(t) \log h(t) \cdot G \geq -\frac{1}{e} \|G\|_{L^1(\gamma^+)} \quad (3 3b)$$

It is easy to check the estimate (3 3b) by studying the function  $x \mapsto x \log x$ . The estimate (3 3a) is obtained using the following convexity inequality

$$h \log h \geq \bar{h} \log \bar{h} + (\log \bar{h} + 1) (h - \bar{h})$$

hence, we obtain

$$\begin{aligned}
& \int_{\Omega} h \log \left( \frac{h}{\bar{h}} \right) \geq \int_{\Omega} (h - \bar{h}) = 0 \\
\Leftrightarrow & \int_{\Omega} h \log \left( \frac{h}{\bar{h}} \right) \geq \int_{\Omega} h \log h - \bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} \geq 0 \\
\Leftrightarrow & \int_{\Omega} h \log h \geq -\bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} \quad \blacksquare
\end{aligned}$$

*Remark 4 :* With Dirichlet boundary conditions, the estimate is the same ; the values of the constants are slightly different.

**4. A TRACE THEOREM**

We denote by :

$$L^1_{\text{div}}(\Omega) = \{u \in L^1(\Omega)^n, \text{div } u \in L^1(\Omega)\}$$

equipped with the graph-norm  $\|u\|_{L^1_{\text{div}}}$  :

$$\|u\|_{L^1_{\text{div}}} = \|u\|_{L^1(\Omega)^n} + \|\text{div } u\|_{L^1(\Omega)}.$$

We also denote by  $L^1_{0,\text{div}}(\Omega)$ , the closure of  $\mathcal{D}(\Omega)^n$  in  $L^1_{\text{div}}(\Omega)$ .

In the next paragraph, we will need the following result :

**THEOREM :** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ , ( $n \in \mathbb{N}^*$ ) and  $\gamma$  its Lipschitz-continuous boundary. We denote by  $\vec{n}$  the normal extern unit to  $\Omega$  on  $\gamma$ .*

*Then we have :*

- i) The space  $\mathcal{D}(\overline{\Omega})^n$  is dense in  $L^1_{\text{div}}(\Omega)$ .*
- ii) The map  $\gamma_n : u \mapsto u \cdot \vec{n}|_{\gamma}$  defined on  $\mathcal{D}(\overline{\Omega})^n$  can be extended to a linear continuous map from  $L^1_{\text{div}}$  into  $[W^{1,\infty}(\gamma)]$ .*
- iii) The kernel of  $\gamma_n$  is the space  $L^1_{0,\text{div}}(\Omega)$ .*

Before proving this theorem, we give a lemma that characterizes the continuous linear forms onto  $L^1_{\text{div}}(\Omega)$ .

**LEMMA 2 :** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $F \in (L^1_{\text{div}}(\Omega))'$ . Then there exists  $f_0 \in L^\infty(\Omega)^n$  and  $f_1 \in L^\infty(\Omega)$  as :*

$$\langle F, v \rangle = \int_{\Omega} f_0 v + \int_{\Omega} f_1 \text{div } v \quad \forall v \in L^1_{\text{div}}(\Omega).$$

*Proof :* Let  $T$  be the map defined by :

$$T : L^1_{\text{div}}(\Omega) \rightarrow L^1(\Omega)^n \times L^1(\Omega)$$

$$v \mapsto (v, \text{div } v).$$



$T$  is an isometry from  $L^1_{\text{div}}(\Omega)$  into  $L^1(\Omega)^n \times L^1(\Omega)$ . We note  $T(L^1_{\text{div}}(\Omega)) = G$ .  $G$  is equipped with the induced norm of  $L^1(\Omega)^n \times L^1(\Omega)$  :

$$\|h\|_{L^1 \times L^1} = \|h_0\|_{L^1} + \|h_1\|_{L^1}.$$

We also denote by  $S$  the map that applies  $G$  on  $L^1_{\text{div}}(\Omega)$ . The map that for  $h \in G$  ( $h = (\varphi, \text{div } \varphi)$ ) associates  $\langle F, Sh \rangle$  is a linear continuous form on  $G$ .

Thanks to the Hahn-Banach theorem, it can be extended to a linear continuous form on  $L^1(\Omega)^n \times L^1(\Omega)$ , noted  $\phi$ .

Using the Riesz theorem, there exists  $f_0 \in L^\infty(\Omega)^n$  and  $f_1 \in L^\infty(\Omega)$  as :

$$\langle \phi, h \rangle = \int_{\Omega} f_0 h_0 + \int_{\Omega} f_1 h_1 \quad \forall h \in (L^1(\Omega)^n \times L^1(\Omega)).$$

Then

$$\begin{aligned} \langle \phi, h \rangle &= \int_{\Omega} f_0 h_0 + \int_{\Omega} f_1 h_1 \quad \forall h \in G \\ \langle \phi, h \rangle &= \int_{\Omega} f_0 h + \int_{\Omega} f_1 \text{div } h \quad \forall h \in G \end{aligned}$$

that achieves the proof. ■

*Proof of the theorem :*

i) To show the density of  $\mathcal{D}(\overline{\Omega})^n$  into  $L^1_{\text{div}}(\Omega)$ , we use the classical following result :

The proposition

$$\{\text{if } \exists f \in (L^1_{\text{div}}(\Omega))' / \langle F, v \rangle = 0, \forall v \in \mathcal{D}(\overline{\Omega})^n \text{ then } F = 0\}.$$

is equivalent to  $\mathcal{D}(\overline{\Omega})^n$  is dense in  $L^1_{\text{div}}(\Omega)$ .

Let  $F$  be a linear continuous form on  $L^1_{\text{div}}(\Omega)$  such as

$$\langle F, v \rangle = 0 \quad \forall v \in \mathcal{D}(\overline{\Omega})^n$$

by the lemma 2, we can write  $\langle F, v \rangle$  as :

$$\langle F, v \rangle = \int_{\Omega} f_0 v + \int_{\Omega} f_1 \text{div } v \quad \text{with } f_0 \in L^\infty(\Omega)^n, f_1 \in L^\infty(\Omega).$$

Then :

$$\begin{aligned} \int_{\Omega} f_0 v + \int_{\Omega} f_1 \operatorname{div} v &= 0 \quad \forall v \in \mathcal{D}(\Omega)^n \\ \langle f_0 - \nabla f_1, v \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} &= 0 \quad \forall v \in \mathcal{D}(\Omega)^n \\ \Rightarrow f_0 &= \nabla f_1 \quad \text{in the distribution sense .} \end{aligned}$$

As  $f_0$  belongs to  $L^\infty(\Omega)^n$ , and therefore to  $L^2(\Omega)^n$ , then we have  $f_1 \in H^1(\Omega)$ .

Consequently

$$\int_{\Omega} \nabla f_1 v + \int_{\Omega} f_1 \operatorname{div} v = 0 \quad \forall v \in \mathcal{D}(\overline{\Omega})^n .$$

By the Green's formula, we have :

$$\int_{\gamma} v \cdot \vec{n} f_1 = 0 \quad \forall v \in \mathcal{D}(\overline{\Omega})^n .$$

And by density of  $\mathcal{D}(\overline{\Omega})^n$  in  $H^1(\Omega)^n$  and by the continuity trace map,

$$\int_{\gamma} v \cdot \vec{n} f_1 = 0 \quad \forall v \in H^1(\Omega)^n$$

$$\Rightarrow \gamma_0 f_1 = 0 \quad \text{that implies that } f_1 \in H_0^1(\Omega) .$$

Then we take a sequence  $f_{1,k} \in \mathcal{D}(\Omega)^n$  that converges to  $f_1$  into  $H^1(\Omega)$  and for all  $v \in L^1_{\operatorname{div}}(\Omega)$ , we can get by the green's formula,

$$\int_{\Omega} \nabla f_{1,k} v + \int_{\Omega} f_{1,k} \operatorname{div} v = 0 \quad \forall v \in L^1_{\operatorname{div}}(\Omega)$$

$$\downarrow k \rightarrow +\infty$$

$$\langle F, v \rangle = \int_{\Omega} \nabla f_1 v + \int_{\Omega} f_1 \operatorname{div} v = 0 \quad \forall v \in L^1_{\operatorname{div}}(\Omega) .$$

Hence

$$F = 0$$

ii) We write the Green's formula :

$$(v, \nabla \varphi) + (\operatorname{div} v, \varphi) = \int_{\gamma} v \cdot \vec{n} \varphi \, d\gamma \quad \forall v \in \mathcal{D}(\overline{\Omega})^n, \forall \varphi \in W^{1,\infty}(\Omega).$$

Then we deduct :

$$\begin{aligned} \left| \int_{\gamma} v \cdot \vec{n} \varphi \, d\gamma \right| &\leq \|v\|_{L^1} \|\nabla \varphi\|_{W^{1,\infty}(\Omega)} + \|\operatorname{div} v\|_{L^1} \|\varphi\|_{W^{1,\infty}(\Omega)} \\ &\leq \|v\|_{L^1_{\operatorname{div}}(\Omega)} \|\varphi\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

The first term is only dependent on the trace of  $\varphi$  onto  $\gamma$ . Moreover,

$$\|\mu\|_{W^{1,\infty}(\gamma)} = \inf_{\varphi \in W^{1,\infty}(\Omega), \varphi|_{\gamma} = \mu} \|\varphi\|_{W^{1,\infty}(\Omega)}.$$

Then we have :

$$\left| \int_{\gamma} v \cdot \vec{n} \mu \, d\gamma \right| \leq \|v\|_{L^1_{\operatorname{div}}(\Omega)} \|\mu\|_{W^{1,\infty}(\gamma)}.$$

The map  $\gamma_n : v \mapsto v \cdot \vec{n}|_{\gamma} \in (W^{1,\infty}(\gamma))'$  defined on  $\mathcal{D}(\overline{\Omega})^n$  equipped with the norm of the space  $L^1_{\operatorname{div}}(\Omega)$  is continuous. It can be extended by continuity to the space  $L^1_{\operatorname{div}}(\Omega)$  by (i). ■

iii) We must show :

$$\ker \gamma_n = L^1_{0, \operatorname{div}}(\Omega).$$

First, it is easy to see that  $L^1_{0, \operatorname{div}}(\Omega) \subset \ker \gamma_n$ , by the continuity trace map. On an other side,  $\ker \gamma_n$  is closed, it is a banach space.

Let  $F \in (L^1_{\operatorname{div}}(\Omega))' / \langle F, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)^n$ .

We must show  $\langle F, \varphi \rangle = 0 \quad \forall \varphi \in \ker \gamma_n$ .

We have, by the lemma 2,

$$\int_{\Omega} f_0 \varphi + \int_{\Omega} f_1 \operatorname{div} \varphi = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)^n.$$

We have shown in (i)  $f_0 = \nabla f_1$  in the sense of distribution and that  $f_1 \in H_0^1(\Omega)$ . We can apply generalized Green formula :

$$\begin{aligned} \int_{\Omega} f_0 \varphi + \int_{\Omega} f_1 \operatorname{div} \varphi &= \langle \gamma_n \varphi, f_{1|_Y} \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)^n \\ &= 0 \quad \forall \varphi \in \mathcal{D}(\Omega)^n \cap \ker \gamma_n . \end{aligned}$$

But  $\mathcal{D}(\Omega)^n \cap \ker \gamma_n$  is dense into  $\ker \gamma_n$ . From which the result : ■

$$F = 0 .$$

## 5. PASSAGE TO THE LIMIT

### 5.1. Continuity Equation

We present two lemma that allow us to pass to the limit in the continuity equation. The first lemma is proved in [O2].

Let  $h_n, v_n$  and  $w_n$  be three sequences satisfying the following conditions :

$$v_n \in L^2(0, T; H^3(\Omega)^2) \text{ and } v_n \rightharpoonup v \in L^2(0, T; H^1(\Omega)^2) \text{ weakly} \quad (5.1a)$$

$$w_n \in L^2(0, T; H^3(\Omega)^2) \text{ and } w_n \rightarrow w \in L^2(0, T; H^1(\Omega)^2) \text{ strongly} \quad (5.1b)$$

$$h_n \text{ and } h_n \log h_n \text{ bounded in } L^\infty(0, T; L^1(\Omega)) \quad (5.1c)$$

$$h_{n,t} + \operatorname{div}(v_n h_n) + \operatorname{div}(w_n h_n) = 0 . \quad (5.1d)$$

Then, we have these following results :

LEMMA 3 : *We can extract from  $v_n$  and  $h_n$  subsequences such that :*

$$\int_Q h_n \Theta \, dx \, dt \rightarrow \int_Q h \Theta \, dx \, dt \quad \text{for all } \Theta \in L^1(0, T; L^\infty(\Omega)) \quad (5.2a)$$

$$v_n h_n \text{ bounded in } L^2(0, T; L^1(\Omega)^2) \quad (5.2b)$$

$$v_n h_n \rightharpoonup \kappa_1 \quad \text{in } L^1(Q) \text{ weakly} \quad (5.2c)$$

$$\kappa_1 = v h . \quad (5.2d)$$

LEMMA 4 : Let  $h_n$  and  $v_n$  be two sequences satisfying the conditions of lemma 3, let  $G_n$  be the trace of  $u_n = v_n + w_n$  satisfying :

$$G_n \rightarrow G \text{ in } H^1(0, T; H^2(\gamma)) \text{ strongly} \quad (5.3a)$$

$$\mu_n \rightarrow \mu \text{ in } L^1(0, T; L^1(\gamma^-)) \text{ strongly; } h_n = \mu_n \text{ on } \Sigma^- \quad (5.3b)$$

$$h_{n,t} + \operatorname{div}(u_n h_n) = 0. \quad (5.3c)$$

Then

$$h = \mu \text{ on } \Sigma^-. \quad (5.3d)$$

*Proof* : We consider the vectors  $\Theta_n$  and  $\Theta$  such as :

$$\Theta_n = (h_n, u_{1,n} h_n, u_{2,n} h_n) \quad \text{and} \quad \Theta = (h, u_1 h, u_2 h).$$

We have  $\Theta \in L^1(Q)$  and :

$$\operatorname{div}_{t,x} \Theta = \frac{\partial h}{\partial t} + \operatorname{div}(uh) = 0 \in L^1(Q).$$

Then we have  $\Theta \in L^1_{\operatorname{div}}(Q)$  and  $\Theta_n \in L^1_{\operatorname{div}}(Q)$ . By the previous lemma, we have  $\Theta_n \rightharpoonup \Theta$  in the sense of  $L^1_{\operatorname{div}}(\Omega)$ . By the continuity of the trace map, we obtain :

$$\gamma \Theta_n \rightharpoonup \gamma \Theta \quad \text{in} \quad (W^{1,\infty}(\partial Q))'.$$

Then, denoting the external normal unit on  $\partial Q$  by  $\vec{N}$ ,

$$\gamma \Theta_n = \Theta_n \cdot \vec{N} = \begin{pmatrix} h_n \\ u_{1,n} h_n \\ u_{2,n} h_n \end{pmatrix} \cdot \begin{pmatrix} N \\ n_1 \\ n_2 \end{pmatrix} = h_n \cdot N + u_n \cdot \vec{n} h_n = h_n \cdot N + G_n h_n.$$

We obtain :

$$\langle h_n \cdot N + G_n h_n, \varphi \rangle \rightarrow \langle h \cdot N + Gh, \varphi \rangle \quad \forall \varphi \in W^{1,\infty}(\partial Q).$$

Choosing  $\varphi \in \mathcal{D}(\Sigma^-)$ , we get :

$$\langle G_n h_n, \varphi \rangle \rightarrow \langle Gh, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Sigma^-).$$

By (5.3b), we have  $G_n h_n = G_n \mu_n$  onto  $\Sigma^-$ . As  $G_n \mu_n \rightarrow G_\mu$  in  $L^1(\Sigma^-)$ , by the uniqueness of the limit, we obtain :

$$Gh = G\mu \quad \text{onto } \Sigma^- \quad \blacksquare$$

### 5.2. Momentum equation

We are going to give a lemma that allows us to pass to the limit in the Momentum equation.

Let  $\{\varphi_1, \dots, \varphi_n, \dots\}$  be a basis of the space  $V$ ,  $\varphi_n$  belongs to  $H^3(\Omega)^2$  and let  $V_n$  be the set of linear combinations of the  $n$  first elements of the basis. Assume that  $v_n$  and  $h_n$  satisfy the weak problem  $(\mathcal{V})$ ,  $\forall \varphi \in V_n$ ; assume that  $v_n(t=0) = v_{0,n} \in V_n$  with  $v_{0,n} \rightarrow v_0$  in  $V$  and  $h_n$  satisfies the condition (5.1d).

Finally, assume that  $v_n$  satisfies the following estimate :

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + \|v_n\|_{L^2(0,T;V)}^2 \leq C_4.$$

Then we have the result :

LEMMA 5 : *If  $f \in L^2(0, T; H^{-1}(\Omega))$ , then, we can extract a sequence from  $v_n$ , still denoted by  $v_n$ , such as :*

$$v_n \text{ weakly converge to } v \text{ in } L^2(0, T; V) \tag{5.4a}$$

$$v_n \text{ weakly - * converge to } v \text{ in } L^\infty(0, T; L^2(\Omega)^2) \tag{5.4b}$$

$$\text{curl } v_n \alpha(v_n) \text{ weakly converge to } \text{curl } v \alpha(v) \text{ in } L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega)^2) \tag{5.4c}$$

$$\nabla v_n^2 \text{ weakly converge to } \nabla v^2 \text{ in } L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega)^2) \tag{5.4d}$$

$$v_{n,t} \text{ is bounded in } L^{\frac{4}{3}}(0, T; H^{-3}(\Omega)^2) \text{ and } v(t=0) = v_0 \tag{5.4e}$$

$$v \text{ satisfies the weak problem } (\mathcal{V}), \forall \varphi \in V_n. \tag{5.4f}$$

*Proof* : The passage to the limit in the momentum equation is easier than the continuity equation. For the proof of this lemma, we refer to [O2].  $\blacksquare$

### 6. APPROXIMATED SOLUTIONS

We introduce a basis of  $V$  denoted by  $\{v_1, \dots, v_n, \dots\}$ ,  $v_n$  belongs to  $H^3(\Omega)^2$ , and  $V_n$  the set of the linear combination of the  $n$  first elements of the basis. We are looking for  $(v_n, h_n)$ , where  $v_n$  is of the form  $v_n = \sum_{i=1}^n a_i(t) v_i(x)$ , solution of the following weak problem  $(\mathcal{V}_n)$  :

Find  $v_n \in L^2(0, T; V_n) \cap L^\infty(0, T; L^2(\Omega)^2)$  and  $h_n \in \mathcal{C}^1(\overline{Q})$  such as :

$$\begin{aligned} & (v_{n,r}, v) + Aa(v_n, v) - \frac{1}{2}(v_n^2, \operatorname{div} v) - (v_n w_n, \operatorname{div} v) \\ & + (\operatorname{curl} v_n \alpha(v_n), v) + (\operatorname{curl} v_n \alpha(w_n), v) + D((v_n + w_n) |v_n + w_n|, v) \\ = & (h_n, \operatorname{div} v) + (f, v) - (w_{n,r}, v) + \frac{1}{2}(w_n^2, \operatorname{div} v) - Aa(w_n, v) \quad \forall v \in V_n \end{aligned}$$

$$h_{n,t} + \operatorname{div}(v_n h_n) + \operatorname{div}(w_n h_n) = 0$$

$$h_n = \mu_n \in \mathcal{C}_c^1(\Sigma^-)$$

$$v_n(t=0) = v_{0,n} \in V_n$$

$$h_n(t=0) = h_{0,n} \in \mathcal{C}_c^1(\Omega)$$

where the data and the constants satisfy the conditions of the theorem, and where  $w_n \in H^1(0, T; H^3(\Omega)^2)$  with  $w_n \cdot n = G_n \in H^1(0, T; H^{5/2}(\gamma))$ . To prove that  $h_n \in \mathcal{C}^1(\overline{Q})$ , we argue as in the lemma 1 and we obtain the solution  $h_n$ , function of the data on  $\Sigma^- \cup \Omega \times \{0\}$ . If we choose  $h_{0,n} \in \mathcal{C}_c^1(\Omega)$  and  $\mu_n \in \mathcal{C}_c^1(\Sigma^-)$ , then  $h_n$  is in  $\mathcal{C}^1(\overline{Q})$ .

LEMMA 6 : *The problem  $(\mathcal{V}_n)$  has a solution satisfying :*

$$(v_n, h_n) \in \{L^\infty(0, T; V_n) \times \mathcal{C}^1(\overline{Q})\}.$$

*And the following estimate :*

$$\begin{aligned} & \|v_n\|_{L^\infty(0, T, L^2(\Omega)^2)}^2 + \|v_n\|_{L^2(0, T, V)}^2 (K - C \|v_n\|_{L^\infty(0, T, L^2(\Omega)^2)}) + \\ & + 2 \sup_t \int_\Omega h_n \log h_n + 2 \int_{\Sigma^-} G_n \cdot h_n \log h_n \leq C_5. \end{aligned}$$

*Proof:* To solve this problem, we apply the Schauder fixed point theorem. This theorem specifies that if we consider  $E$  a banach space,  $K_0$  a convex compact subset in  $E$  and  $\Pi$  a continuous map from  $K_0$  into itself, then there exists  $x_0 \in K_0 / x_0 = \Pi(x_0)$  (see [GT]).

To apply this theorem, we fix a function  $v_n^*$  in  $L^2(0, T; V_n)$  and we solve the following problem :

$$(H) \quad \begin{cases} k_{n,t} + \operatorname{div}(v_n^* k_n) + \operatorname{div}(w_n k_n) = 0 \\ k_n = \mu_n \geq 0 \text{ on } \Sigma^- \\ k_n(t=0) = h_{0n} \geq 0. \end{cases}$$

The solution  $k_n$  of the problem (H) belongs to  $L^\infty(0, T; L^2(\Omega))$  and we define the following map :

$$\Pi_1 : L^2(0, T; V_n) \rightarrow L^\infty(0, T; L^2(\Omega))$$

$$v_n^* \mapsto \Pi_1(v_n^*) = k_n.$$

Then, we solve the following problem based on the weak formulation of the moment equation with  $h_n = k_n$  previously found :

$$(U) \quad \begin{cases} (v_{n,r}, v) + Aa(v_n, v) - \frac{1}{2}(v_n^2, \operatorname{div} v) - (v_n w_n, \operatorname{div} v) + (\operatorname{curl} v_n \alpha(v_n), v) \\ \quad + (\operatorname{curl} v_n \alpha(w_n), v) + D((v_n + w_n) |v_n + w_n|, v) - (k_n, \operatorname{div} v) \\ = (f, v) + \frac{1}{2}(w_n^2, \operatorname{div} v) - (w_{n,r}, v) - Aa(w_n, v) \quad \forall v \in V_n \\ v_n(t=0) = v_{0n}. \end{cases}$$

We define the map  $\Pi_2$  that associates to  $k_n \in L^\infty(0, T; L^2(\Omega))$  the solution of the problem (U),  $v_n \in L^2(0, T; V_n)$ . Then we consider the map  $\Pi = \Pi_2 \circ \Pi_1$ . We are going to prove that this map satisfies the conditions of the Schauder's theorem and then has a fixed point. This map  $\Pi$  must be continuous and must apply a compact convex into himself. We note that thanks to the regularity of the basis,  $L^2(0, T; V_n)$  and  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$  are algebraically and topologically equal. Then we can obtain conditions onto  $\Pi$  for the weak topology of  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$  that is metrisable into a finite dimensional space.

The problem (H) is solved with the method of the characteristics. As in the lemma 1, the explicit solution of the problem (H) is :

$$k_n(t, x(t, \tau, z)) = C_6 \cdot e^{-\int_\tau^t (\operatorname{div} v_n^* + \operatorname{div} w_n)(\xi, x(\xi, \tau, z)) d\xi}$$



where  $\mathbb{C}_6$  is a constant obtained by :

$$\begin{aligned} k_n(\tau, x(\tau, \tau, z)) &= \mu_n \geq 0 \quad \text{if } \tau \neq 0 \text{ and } z \in \gamma^- \\ k_n(0, x(0, 0, z)) &= h_{0n} \geq 0 \quad \text{if } \tau = 0 \text{ and } z \in \Omega. \end{aligned}$$

By standard considerations (see [DPL1]), we have  $k_n \in \mathcal{C}^0([0, T]; \mathcal{C}^1(\overline{\Omega}))$ . Then by the explicit solution, we have  $k_n = \mu_n$  onto  $\Sigma^-$  and the initial condition. Moreover,  $k_n$  satisfies the following estimate obtained by the Gronwall's lemma :

$$\|k_n\|_{L^\infty(0, T, L^2(\Omega))}^2 \leq \left[ \|h_{0n}\|_{L^2(\Omega)}^2 - \int_{\Sigma^-} G_n \mu_n^2 \right] e^{\int_{\tau'}^t \|\operatorname{div}(v_n^* + w_n)\|_{L^\infty(\Omega)^2}}. \quad (6.2)$$

This estimate will be necessary to prove the conditions of Schauder's theorem.

We now solve the problem (U) :

$$(U) \begin{cases} (v_{n,r}, v) + Aa(v_n, v) - \frac{1}{2}(v_n^2, \operatorname{div} v) - (v_n w_n, \operatorname{div} v) + (\operatorname{curl} v_n \alpha(v_n), v) \\ \quad + (\operatorname{curl} v_n \alpha(w_n), v) + D((v_n + w_n) |v_n + w_n|, v) - (k_n, \operatorname{div} v) \\ = (f, v) + \frac{1}{2}(w_n^2, \operatorname{div} v) - (w_{n,r}, v) - Aa(w_n, v) \quad \forall v \in V_n \\ v_n(t=0) = v_{0n}. \end{cases}$$

where  $k_n$  is the solution of the problem (H). This problem can be reduced to a simply differential system with the estimate :

$$|(k_n, \operatorname{div} v_n)| \leq \frac{1}{4\varepsilon^2} \|k_n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|v_n\|_V^2.$$

The solution of this differential system satisfies :

$$\begin{aligned} \|v_n\|_{L^2(0, T, V_n)}^2 &((K - \varepsilon^2) - C \|v_n\|_{L^\infty(0, T, L^2(\Omega)^2)}) + \\ &+ \|v_n\|_{L^\infty(0, T, L^2(\Omega)^2)}^2 < \frac{1}{4\varepsilon^2} \|k_n\|_{L^2(\Omega)}^2 + \mathbb{C}_7. \end{aligned}$$

As in the lemma 1, we show that the left-hand side of the inequality is non-negative if we choose  $\varepsilon^2 < K(1 - \theta)$ . Then  $v_n = \Pi_2(k_n)$  satisfies :

$$\|v_n\|_{L^2(0, T, V_n)}^2 \leq \mathbb{C}_8 \left(\frac{T}{2\varepsilon}\right)^2 \|k_n\|_{L^\infty(0, T, L^2(\Omega))}^2 + \mathbb{C}_9 \quad (6.3)$$

where  $\mathbb{C}_8$  is a positive constant. The problem (U) has a solution  $v_n$  in  $L^2(0, T; V_n)$ . The initial condition is easily satisfied.

There is a last point that must be checked :  $\Pi$  satisfies the conditions of Schauder's theorem. With (6.2) and (6.3),  $v_n = \Pi(v_n^*)$  satisfies :

$$\begin{aligned} \|v_n\|_{L^\infty(0, T, v_n)}^2 &< C_{10} \left(\frac{T}{2\varepsilon}\right)^2 e^{\int_0^T \|\operatorname{div}(v_n^* + \operatorname{div} w_n)\|_{L^\infty(\Omega)}^2} + C_{11} \\ &< C_{12} \left(\frac{T}{2\varepsilon}\right)^2 e^{\int_0^T \|v_n^*\|_{V_n}} + C_{11}. \end{aligned}$$

It is clear that if  $\|v_n^*\|_{V_n} < R$ , we are going to obtain  $\|v_n\|_{L^2(0, T, V_n)} < R$ , if  $R$  is well chosen and if  $T$  is sufficiently small. Then, the last estimate proves there exists  $R$  such as :

$$\Pi(B(O, R)) \subset B(O, R)$$

$B(O, R)$  is a convex compact set for the weak topology of  $L^2(0, T; V_n)$ .

Finally, we must show that the map  $\Pi$  is continuous. Thanks to the regularity of the basis, we use the weak topology of the space  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$  that is metrisable when the dimension of the space is finite. We consider a sequence  $v_n^*$  that weakly converges to  $v^*$  in  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$ . We are going to show  $v_n = \Pi(v_n^*)$  weakly-\* converges to  $v = \Pi(v^*)$  in  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$ .

If  $k_n$  is the solution of the problem (H) with  $v^* = v_n^*$  and  $k_n(0) = h_0$ , then  $k_n$  satisfies the following estimate :

$$\|k_n\|_{L^\infty(0, T, L^2(\Omega))}^2 \leq \left[ \|h_0\|_{L^2(\Omega)}^2 - \int_{\Sigma^-} G\mu^2 \right] e^{\int_0^T \|\operatorname{div}(v_n^* + w)\|_{L^\infty(\Omega)}^2}$$

and weakly-\* converges to  $h$  in  $L^\infty(0, T; L^2(\Omega))$ , with  $h$  the solution of the problem  $h_t + \operatorname{div}(v^* h) + \operatorname{div}(wh) = 0$ . Then  $v_n$ , the solution of the problem (U) with  $h = k_n$  and  $v_n(0) = v_0$ , weakly converges to  $v = \Pi(v^*)$  in  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$  by the lemma 5.

We have proved all the conditions of Schauder's theorem. The problem  $v_n = \Pi(v_n)$  has a solution in  $L^2(0, T; W^{1,\infty}(\Omega)^2 \cap V_n)$ . Since  $v_n$  is a solution of (U) and  $h_n$  solution of (H), we have  $v_n(t=0) = v_{0n}$  and  $h_n(t=0) = h_{0n}$ .

We have obtained the existence of a solution for small times. It is possible to extend this result for all  $T$  thanks to (6.1). Indeed, the solution satisfies this estimate and then is bounded.

The solution of the problem is such that  $v_n \in H^1(0, T; V_n)$ . Since the elements of the basis belong to  $H^3(\Omega)^2$ , we have  $v_n \in H^1(0, T; H^3(\Omega)^2)$ , i.e.

$$v_n \in \mathcal{C}^0([0, T], \mathcal{C}^1(\overline{\Omega})^2).$$

The solution  $h_n$  satisfies :

$$\begin{cases} h_{n,t} + \operatorname{div}(v_n h_n) + \operatorname{div}(v_n h_n) \\ h_n = \mu_n \in \mathcal{C}_c^1(\Sigma^-) \\ h_n(t=0) = h_{0n} \in \mathcal{C}_c^1(\Omega). \end{cases}$$

And then (as  $(\mu_n \in \mathcal{C}_c^1(\Sigma^-))$ ):

$$h_n \in \mathcal{C}^1(\overline{Q}).$$

Since  $h_n$  satisfies the continuity equation, we have :

$$\int_{\Omega} h_n + \int_{\Sigma^+} G_n h_n = \int_{\Omega} h_{0n} - \int_{\Sigma^-} G_n \mu_n.$$

And since  $\mu_n, G_{n|_{\gamma^+}}$  and  $h_{0n}$  are positives and  $G_{n|_{\gamma^-}}$  is negative, we obtain :

$$h_n \geq 0. \quad \blacksquare$$

## 7. PROOF OF THE THEOREM

Let  $\mathcal{C}_c^1(\Omega) \ni h_{0,n} \rightarrow h_0$  in  $L^1(\Omega)$  and  $V_n \ni v_{0,n} \rightarrow v_0$  in  $V$ , where  $v_0$  and  $h_0$  are the initials conditions of the problem  $(P)$ .

Let  $w_n$  be defined as in the previous paragraph, let  $(v_n, h_n)$  be the solution of the problem  $\mathcal{V}_n$  seen in the lemma 6 with  $v_{0,n}$  and  $h_{0,n}$  as initial conditions. Let  $\mu_n \rightarrow \mu \in L^1(\Sigma^-)$  strongly. The functions  $v_n$  and  $h_n$  satisfy :

$$v_n \in L^2(0, T; V_n) \cap L^\infty(0, T; L^2(\Omega)^2), \quad h_n \in \mathcal{C}^1(\overline{Q}) :$$

and

$$\begin{aligned} & (v_{n,r}, v) + Aa(v_n, v) - \frac{1}{2} (v_n^2, \operatorname{div} v) - (v_n w_n, \operatorname{div} v) \\ & + (\operatorname{curl} v_n \alpha(v_n), v) + (\operatorname{curl} v_n \alpha(w_n), v) + D((v_n + w_n) |v_n + w_n|, v) \\ = & (h_n, \operatorname{div} v) + (f, v) - (w_{n,r}, v) + \frac{1}{2} (w_n^2, \operatorname{div} v) - Aa(w_n, v) \quad \forall v \in V_n \end{aligned}$$

$$h_{n,t} + \operatorname{div} (v_n h_n) + \operatorname{div} (w_n h_n) = 0$$

$$h_n = \mu_n \in \mathcal{C}_c^1(\Sigma^-)$$

$$v_n(t=0) = v_{0n} \in V_n$$

$$h_n(t=0) = h_{0n} \in \mathcal{C}_c^1(\Omega).$$

By the lemma 6, the solution  $(v_n, h_n)$  satisfy the estimate :

$$\begin{aligned} & -2\left(\bar{h} \operatorname{meas}(\Omega) \cdot \log \bar{h} + \frac{1}{e} \|G_n\|_{L^1(\Sigma^+)}\right) \\ \leq & \|v_n\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + 2 \sup_t \int_\Omega h_n \log h_n \\ & + \|v_n\|_{L^2(0,T;V)}^2 (K - C \|v_n\|_{L^\infty(0,T;L^2(\Omega)^2)}) + 2 \int_{\Sigma^+} G_n \cdot h_n \log h_n \leq \mathbb{C}_4. \end{aligned}$$

Then, by the lemma 5, we can extract to  $v_n$  a subsequence, still denoted by  $v_n$ , such as :

$$v_n \overset{*}{\rightharpoonup} v \quad \text{in } L^2(0, T; V_n) \cap L^\infty(0, T; L^2(\Omega)^2).$$

Moreover, this subsequence satisfies the following results :

$$\operatorname{curl} v_n \alpha(v_n) \text{ weakly converge to } \operatorname{curl} v \alpha(v) \text{ in } L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega)^2)$$

$$\nabla v_n^2 \text{ weakly converge to } \nabla v^2 \text{ in } L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Omega)^2)$$

$$v_{n,t} \text{ is bounded in } L^{\frac{4}{3}}(0, T; H^{-3}(\Omega)^2) \text{ and } v(t=0) = v_0$$

Thanks to its strong convergence, we have clearly the same results for the sequence  $w_n$ .

Now, we can extract to  $h_n$  a subsequence denoted by  $h_n$ , by the lemma 3, such that :

$$h_n \rightharpoonup h \text{ in } L^p(0, T; L^1(\Omega)) \text{ for } p < \infty.$$

By using the second result of the lemma 3, we have  $v_n h_n \rightharpoonup vh \in L^1(Q)$  weakly and we can deduce  $\operatorname{div}(vh)$  belongs to  $L^1(0, T; W^{-1,1}(\Omega))$  i.e. that  $h_t \in L^1(0, T; W^{-1,1}(\Omega))$  and then,  $h(t=0) = h_0$  and finally, thanks to the lemma 4,  $h = \mu$  on  $\Sigma^-$ .

This concludes the proof that  $(v, h)$  is a solution of the problem  $\mathcal{V}$ . ■

*Remark 5 :* If we consider Dirichlet boundary conditions  $u = G$  on  $\gamma$ , the proof of the theorem is slightly different ; the main ideas are the same.

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