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ON THE CHEBYSHEV PENALTY METHOD FOR PARABOLIC AND HYPERBOLIC EQUATIONS (*)

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Abstract. — *We propose, in this article, several versions of the penalty method for the pseudospectral Chebyshev discretization of hyperbolic and parabolic equations. We demonstrate the stability for a range of the penalty parameter. For the hyperbolic equation we find the optimal estimate for the parameter, whereas the estimate for the parabolic case is not optimal.*

Résumé. — *Plusieurs versions de la méthode de pénalisation sont proposées pour une discrétisation pseudospectrale Chebyshev d'équations hyperboliques ou paraboliques. On montre la stabilité lorsque le paramètre de pénalisation est dans un intervalle. Nous estimons la valeur optimale du paramètre dans le cas hyperbolique, tandis que notre estimation n'est pas optimale dans le cas parabolique.*

1. INTRODUCTION

Boundary conditions play a crucial role in spectral and pseudospectral methods. One way of applying a collocation method to a partial differential equation is to satisfy the equation at the interior points and impose boundary conditions at the boundary points. An alternative way is to collocate the differential equation at all the grid points and introduce the boundary conditions as a penalty term.

The main difference between the two methods is that, in the latter, the boundary conditions are not satisfied exactly but only at the limit. The penalty method, in the context of a Legendre collocation approximation of a linear hyperbolic systems, has been first inroduced by Funaro and Gottlieb in [7]. In [6] the case of a Chebyshev penalty method for a scalar hyperbolic equation has been studied.

In this paper we present first a Chebyshev penalty method for a linear hyperbolic equation. The method differs from the one considered in [6] in the functional form of the penalty term. In our method the penalty term is non zero also at the interior points ; which introduces a correction at all the interpolation

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points. We prove stability of the semidiscrete scheme in the appropriate norm and we give an optimal estimate of the penalty parameter. We present also a new Chebyshev penalty method for a linear parabolic equation and we prove the stability of the scheme in L^2_ω corresponding to the Chebyshev weight function $\omega(x) = (1 - x^2)^{-1/2}$.

The paper is organized as follows. In Section 1 we present some standard results on Chebyshev approximation. Section 2 is devoted to a new version of the Chebyshev penalty method for linear hyperbolic equations. Stability of the scheme is proven. In Section 3 we consider a Chebyshev penalty method for a linear parabolic equation. Stability of the scheme in the usual L^2_ω norm is proven. In Section 4 we present some numerical results for the Maxwell equation.

2. PRELIMINARIES

In this section we summarize the notation and some general results regarding the Chebyshev polynomials. For an overview of those results the reader is referred to [1, 4] and [2]. In the article we will work on the following space :

$$\mathbf{P}_N = \{\text{polynomials of degree} \leqslant N\} . \tag{1}$$

We denote by T_N the Chebyshev polynomial of degree N,

$$T_N(x) = \cos\left(N \cos^{-1} x\right) .$$

Throughout the article we will make use of the following formulas :

$$2\, T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k+1}}{k-1} , \tag{3}$$

$$xT_k = \frac{1}{2}\left(T_{k+1} + T_{k-1}\right) ; \tag{4}$$

and the orthogonality property :

$$\int_{-1}^{1} T_N\, T_M \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2}\, \delta_{NM} . \tag{5}$$

We will use the Gauss Lobatto set of collocation points :

$$x_j^N = \cos\left(\frac{\pi j}{N}\right), \quad 0 \leqslant j \leqslant N . \tag{6}$$

These points are the zeros of $(1 - x^2)\, T'_N(x)$. Associated to this set of collocation points we define the discrete scalar product $(\cdot, \cdot)_N$:

$$(f, g)_N = \sum_{j=0}^{N} f(x_j^N)\, g(x_j^N)\, \omega_j\,, \tag{7}$$

where

$$\begin{cases} \omega_j = \dfrac{\pi}{N}\,, & 1 \leq j \leq N\,, \\ \omega_0 = \omega_N = \dfrac{\pi}{2N}\,. \end{cases} \tag{8}$$

If $fg \in \mathbf{P}_{2N-1}$ we have:

$$\int_{-1}^{1} f(x)\, g(x)\, \omega(x)\, dx = \sum_{j=0}^{N} f(x_j^N)\, g(x_j^N)\, \omega_j\,, \tag{9}$$

where $\omega(x) = (1 - x^2)^{-1/2}$.

3. HYPERBOLIC EQUATIONS

In this section we present a Chebyshev penalty method for the linear hyperbolic equation.

$$\begin{cases} v_t = v_x, & x \in (-1, 1), \quad t > 0\,, \\ v(1, t) = g^+(t), & t > 0\,, \\ v(x, 0) = v_0, & x \in (-1, 1)\,. \end{cases} \tag{10}$$

We seek an approximate solution u in the space $\mathbf{P}_{N-1}$ that satisfies:

$$u_t(x) = u_x - \tau(u(1, t) - g^+(t))\, \frac{T'_N(x)}{N^2}\,, \tag{11}$$

at the points x_j^N, $0 \leq j \leq N$. Note that the penalty term $T'_N(x)$ is zero at the interior points but it is non zero at *both* boundary points. A different Chebyshev penalty method for equation (10) has been studied in [7], where the penalty term was $\dfrac{(1 + x)\, T'_N(x)}{2N^2}$.

Remark 3.1 : In a similar way we can define a penalty method for the equation

$$\begin{cases} v_t = -v_x, & x \in (-1, 1), \quad t > 0, \\ v(-1, t) = g^-(t), \quad t > 0. \end{cases} \tag{12}$$

In this case the penalty term would be :

$$-\tau(u(-1, t) - g^-(t)) \frac{T_N'(x)}{N^2}. \tag{13}$$

The extension to this case of all the results is straightforward.

As shown in [1], problem (10) is not well posed in the usual Chebyshev norm, corresponding to the weight function $\omega(x) = (1 - x^2)^{-1/2}$. Instead we prove stability in the norm corresponding to the weight function

$$\tilde{\omega}(x) = (1 + x)\omega(x) = (1 + x)^{1/2}(1 - x)^{-1/2}. \tag{14}$$

THEOREM 3.1 : *(Coercivity) Let $u \in \mathbf{P}_{N-1}$ and $\omega_0 = \pi/2N$ be the Chebyshev weight defined in (8). Then*

$$\int_{-1}^{1} u_x(x)\, u(x)\, \tilde{\omega}(x)\, dx \leqslant \tau\omega_0\, u^2(1), \tag{15}$$

provided τ satisfies

$$\tau \geqslant \frac{N^2}{2}. \tag{16}$$

Remark 3.2 : Note that in the penalty method (11) the approximate solution u does not satisfy the boundary condition exactly but only as N tends to ∞.

Remark 3.3 : Let $f(x)$ be a function such that

$$\lim_{x \to 1} \tilde{\omega}(x) f(x) = 0, \tag{17}$$

then we can prove the following coercivity property

$$\int_{-1}^{1} f(x) f_x(x)\, \tilde{\omega}(x)\, dx \leqslant 0. \tag{18}$$

In our case the function $u(x)$ does not satisfy zero boundary conditions and (18) does not hold. However, as shown in Theorem (3.1), coercivity can be proved, if we restrict the set of functions to polynomial and we impose a condition on the penalty term. This is sufficient for our purpose since we will use Theorem (3.1) for the solution of (11), $u(x)$, which is a polynomial.

Proof (of the Coercivity Theorem) : Let us define

$$U(x) = u(x) - u(1)\, T_N(x)\,. \tag{19}$$

It is clear that $U(1) = 0$. We write :

$$\begin{aligned}\int_{-1}^{1} u(x)\, u_x(x)\, (1+x)\, \omega(x)\, dx &= \int_{-1}^{1} U(x)\, U_x(x)\, \tilde{\omega}(x)\, dx \\ &+ u(1) \int_{-1}^{1} u(x)\, T_N'(x)\, (1+x)\, \omega(x)\, dx \\ &+ u(1) \int_{-1}^{1} u_x(x)\, T_N(x)\, (1+x)\, \omega(x)\, dx \\ &+ u^2(1) \int_{-1}^{1} T_N(x)\, T_N'(x)\, (1+x)\, \omega(x)\, dx\,. \end{aligned} \tag{20}$$

Due to the orthogonality property (5) and the fact that $u_x(x) \in \mathbf{P}_{N-2}$ we have :

$$\int_{-1}^{1} u(x)\, T_N'(x)\, (1+x)\, \omega(x)\, dx = 0 = \int_{-1}^{1} T_N(x)\, T_N'(x)\, \omega\, dx\,. \tag{21}$$

Combining Remark (3.3), (4) and (9) we obtain :

$$\begin{aligned}\int_{-1}^{1} u(x)\, u_x(x)\, \tilde{\omega}(x)\, dx &\leqslant \pi N u^2(1) - \frac{u^2(1)}{2} \int_{-1}^{1} T_{N-1}\, T_N\, \omega(x)\, dx \\ &\leqslant \pi N u^2(1) - \frac{\pi}{2} N u^2(1) \leqslant \frac{\pi}{2} N u^2(1)\,. \end{aligned} \tag{22}$$

Therefore we obtain :

$$\int_{-1}^{1} u(x)\, u_x(x)\, \tilde{\omega}(x)\, dx - \tau \omega_0\, u^2(1) \leqslant \left(\frac{N}{2} - \frac{\tau}{N}\right) \pi u^2(1)\,. \tag{23}$$

We conclude that if (16) is satisfied (15) holds. □

Using the previous Lemma we can prove the following stability result for the scheme (11).

THEOREM 3.2 : *Let u be the solution of (10) with $g^+ = 0$. If τ satisfies (30) then the Chebyshev penalty method is stable in L^2_{ω}. More precisely,*

$$\frac{d}{dt}\sum_{j=0}^{N} u^2(x_j^N)\,(1 + x_j^N)\,\omega_j \leqslant 0\,. \tag{24}$$

Remark 3.4 : The condition on the penalty parameter is optimal in this case. This can easily be seen by taking, for example, $u(x) = T_{N-1}(x)$.

4. PARABOLIC EQUATION

Consider the equation :

$$\begin{cases} v_t = v_{xx}, & x \in (-1, 1), \quad t > 0\,, \\ v(-1, t) = g^-(t), v(1, t) = g^+(t), & t > 0\,, \\ v(x, 0) = v_0, & x \in (-1, 1)\,. \end{cases} \tag{25}$$

We present a penalty method in the following way : we seek an approximate solution u in the space $\mathbf{P}_N$ that satisfies :

$$u_t(x) = u_{xx}(x) - \tau_0(u(1, t) - g^+(t))\,\frac{(1 + x)\,T'_N(x)}{2\,N^2}\,,$$

$$-\,\tau_N(u(-1) - g^-(t))\,\frac{(1 - x)T'_N(x)}{2\,N^2}\,, \tag{26}$$

at the points $x_j^N, 0 \leqslant j \leqslant N$.

Remark 4.1 : Note that in the penalty method (26) the approximate solution u does not satisfy the boundary conditions exactly but only as N tends to ∞.

Remark 4.2 : Let $f(x)$ be a function such that

$$\lim_{x \to \pm 1} \omega(x) f(x) = 0\,, \tag{27}$$

we can prove the following coercivity property

$$\int_{-1}^{1} f(x) f_{xx}(x)\,\omega(x)\,dx \leqslant 0\,. \tag{28}$$

In our case the function $u(x)$ does not satisfy zero boundary conditions and (28) does not hold. However, a similar result holds, if we restrict the set of functions to polynomials and we impose a condition on the penalty terms τ_0 and τ_N. This is sufficient for our purpose since we want to apply the result to the solution of (26), $u(x)$, which is a polynomial.

The crucial step in proving the stability of the scheme (26) is given by the following Theorem.

THEOREM 4.1 : *(Coercivity) Consider a polynomial $u \in \mathbf{P}_N$ and let $\omega_0 = \omega_N = \pi/2N$ be the Chebyshev weights defined in (8), then*

$$\int_{-1}^{1} u(x)\, u_{xx}(x)\, \omega(x)\, dx \leqslant \tau_0\, \omega_0\, u^2(1) + \tau_N\, \omega_N\, u^2(-1)\,, \tag{29}$$

provided that τ_0 and τ_N satisfy

$$\tau_0 \geqslant O(N^5)\,, \quad \tau_N \geqslant O(N^5)\,. \tag{30}$$

Remark 4.3 : In the case where $u(\pm 1) = 0$ *(29) becomes (28) and the coercivity follows from standard arguments.*

Remark 4.4 : Numerical computations suggest that the condition on the coefficients τ_0 and τ_N is not optimal. As we will show later on (see (45)) there are cases where the choice of τ_0, $\tau_N = O(N^4)$ is sufficient to guarantee stability of the penalty method.

In order to prove the Coercivity property (29) we define two *boundary polynomials* $P^+(x)$ and $P^-(x)$,

$$P^+(x) = -\frac{p^2-1}{2}\left[\frac{T_{p+1}}{2(p+1)} - \frac{T_{p-1}}{2(p-1)}\right] - \frac{q^2-1}{2}\left[\frac{T_{q+1}}{2(q+1)} - \frac{T_{q-1}}{2(q-1)}\right],$$

$$P^-(x) = -\frac{p^2-1}{2}\left[\frac{T_{p+1}}{2(p+1)} - \frac{T_{p-1}}{2(p-1)}\right] - \frac{q^2-1}{2}\left[\frac{T_{q+1}}{2(q+1)} - \frac{T_{q-1}}{2(q-1)}\right]; \tag{31}$$

where we assume that p is even and q is odd.

The following Lemma collects all the important properties of the two boundary polynomials.

LEMMA 4.1 : *Let $P^+(x)$ and $P^-(x)$ be defined as in (31) ; then*

$$p^{\pm}_{xx}(x) = \mp \frac{p^2-1}{2} T'_p + \frac{q^2-1}{2} T'_q . \tag{32}$$

Moreover

$$P^+(1) = P^-(1) = 1 , \quad P^+(-1) = P^-(1) = 0 . \tag{33}$$

In the proof of the Coercivity Theorem (4.1) we will need that

$$P^{\pm} \quad \text{are } \omega - \text{orthogonal to } \mathbf{P}_{N-2} , \tag{34}$$

therefore we choose :

$$p = N , \quad q = N+1 . \tag{35}$$

Proof : (of the Coercivity Theorem) Let us define

$$U(x) = u(x) - u(1) P^+(x) - u(-1) P^-(x) ; \tag{36}$$

from (33) it is clear that $U(\pm 1) = 0$. Therefore, using (34), we get that

$$\int_{-1}^{1} u(x)\, u_{xx}(x)\, \omega(x)\, dx = \int_{-1}^{1} U(x)\, U_{xx}(x)\, \omega(x)\, dx$$

$$+ u(1) \int_{-1}^{1} uP^+_{xx}\, \omega(x)\, dx + u(-1) \int_{-1}^{1} uP^-_{xx}\, \omega(x)\, dx$$

$$- \int_{-1}^{1} (u(1) P^+ + u(-1) P^-)(u(1) P^+ + u(-1) P^-)\, \omega(x)\, dx . \tag{37}$$

Let us evaluate the second and third integral in the RHS of (37). Using (32) and (9) we can rewrite the integrals as discrete scalar products based on $p+1$ and $q+1$ points respectively and get :

$$u(1) \int_{-1}^{1} uP^+_{xx}\, \omega\, dx + u(-1) \int_{-1}^{1} uP^-_{xx}\, \omega\, dx$$

$$= -\frac{\pi}{4} p(p^2-1) [u^2(1) - 2\, u(1)\, u(-1) + u^2(-1)]$$

$$- \frac{\pi}{4} q(q^2-1) [u^2(1) + 2\, u(1)\, u(-1) + u^2(-1)]$$

$$\leqslant 0 . \tag{38}$$

Consider now the last integral in the RHS of (37). We start by evaluating $\int_{-1}^{1} P^+ P^+_{xx}\, \omega(x)\, dx$. Using (31), (5) and switching to the discrete scalar products, we obtain

$$\int_{-1}^{1} P^+ P^+_{xx}\, \omega(x)\, dx = -\frac{(p^2-1)^2}{8(p-1)} \int_{-1}^{1} T'_p T_{p-1}\, \omega\, dx$$
$$-\frac{(q^2-1)^2}{8(q-1)} \int_{-1}^{1} T'_q T_{q-1}\, \omega\, dx$$
$$-\frac{\pi}{8} p(p+1)(p^2-1) - \frac{\pi}{8} q(q+1)(q^2-1)\,. \tag{39}$$

Note that, due to the fact that p is even and q is odd, the « mixed » integrals are zero. For example :

$$\int_{-1}^{1} T'_p \left[\frac{T_{q-1}}{2(q+1)} - \frac{T_{q-1}}{2(q-1)} \right] \omega(x)\, dx = 0\,. \tag{40}$$

It is easy to see that :

$$\int_{-1}^{1} P^- P^-_{xx}\, \omega(x)\, dx = \int_{-1}^{1} P^+ P^+_{xx}\, \omega(x)\, dx\,. \tag{41}$$

In a similar way we evaluate $\int_{-1}^{1} P^+ P^-_{xx}\, \omega(x)\, dx$ and $\int_{-1}^{1} P^- P^+_{xx}\, \omega(x)\, dx$ to get :

$$-\int_{-1}^{1} (u(1) P^+ + u(-1) P^-)(u(1) P^+ + u(-1) P^-)_{xx}\, \omega(x)\, dx =$$
$$= u^2(1) \frac{\pi}{8} [p(p+1)(p^2-1) + q(q+1)(q^2-1)]$$
$$- u(1) u(-1) \frac{\pi}{8} [p(p+1)(p^2-1) + q(q+1)(q^2-1)]$$
$$+ u^2(-1) \frac{\pi}{8} [p(p+1)(p^2-1) + q(q+1)(q^2-1)]\,. \tag{42}$$

Therefore, choosing $p = N$ and $q = N+1$ we conclude that if (30) is satisfied (29) holds. □

Theorem (4.1) yields the stability of the scheme (26) in the norm corresponding to the weight function $\omega(x) = (1-x^2)^{-1/2}$.

THEOREM 4.2 : *(Stability of the penalty method) Let u be the solution of (26), with $g^+ = g^- = 0$. If τ_0 and τ_N satisfy (30), then the Chebyshev penalty method is stable in the L^2_ω norm. More precisely,*

$$\frac{d}{dt}\sum_{j=0}^{N} u^2(x_j^N)\,\omega_j \leqslant 0\,. \tag{43}$$

Proof : Taking the discrete scalar product of (26) with $u(x)$ we obtain :

$$\frac{d}{dt}\sum_{j=0}^{N} u^2(x_j^N)\,\omega_j = (u_{xx}, u)_N - \tau_0\,\omega_0\,u^2(1) - \tau_N\,\omega_N\,u^2(-1)\,. \tag{44}$$

Since $uu_{xx} \in \mathbf{P}_{2N-1}$ we can apply (9). The result then follows from the Coercivity Theorem. □

As pointed out in Remark (4.4), we believe that the optimal value of the parameters τ_0 and τ_N is of order N^4. We are unable to prove this in the general case at this time but we can show that it is true in the case of a particular choice of the penalty term. This result will be true only in the case of a constant coefficient linear parabolic equation. Let us consider the problem (25). We seek an approximate solution u in the space $\mathbf{P}_N$ that satisfies :

$$u_t(x) = u_{xx} - \tau_0(u(1,t) - g^+)\,\frac{T''_{N+2}(x) + T''_{N+1}(x)}{2\,N^4}$$

$$- \tau_N(u(-1,t) - g^-)\,\frac{T''_{N+2}(x) - T''_{N+1}(x)}{2(-1)^N\,N^4} \tag{45}$$

at the points x_j^N, $0 \leqslant j \leqslant N$.

The stability of the scheme relies on the following Lemma.

LEMMA 4.2 : *Consider a polynomial $u \in \mathbf{P}_N$, then*

$$\int_{-1}^{1} u_{xx}(x)\,u(x)\,\omega(x)\,dx \leqslant u(1)\int_{-1}^{1} u(x)\,\frac{T''_{N+2}(x) + T''_{N+1}(x)}{2}\,\omega(x)\,dx \tag{46}$$

$$+\, u(-1)\int_{-1}^{1} u(x)\,\frac{T''_{N+2}(x) - T''_{N+1}(x)}{2(-1)^N}\,\omega(x)\,dx\,. \tag{47}$$

Proof : Let us define

$$U(x) = u(x) - u(1)\frac{T_{N+2}(x) + T_{N+1}(x)}{2} - u(-1)\frac{T_{N+2}(x) - T_{N+1}(x)}{2(-1)^N}. \tag{48}$$

It is clear that

$$U(\pm 1) = 0. \tag{49}$$

Note that $u_{xx}(x)$ is orthogonal to $\{T_{N+2}(x), T_{N+1}(x)\}$ and $\{T_{N+2}(x), T_{N+1}(x)\}$ is orthogonal to $\{T''_{N+2}(x), T''_{N+1}(x)\}$. Therefore we obtain :

$$\begin{aligned}\int_{-1}^{1} u(x)\, u_{xx}(x)\, \omega(x)\, dx = \int_{-1}^{1} & U(x)\, U_{xx}(x)\, \omega(x)\, dx \\ & + u(1)\int_{-1}^{1} u(x)\frac{T''_{N+2}(x) + T''_{N+1}(x)}{2}\,\omega(x)\, dx \\ & + u(-1)\int_{-1}^{1} u(x)\frac{T''_{N+2}(x) - T''_{N+1}(x)}{2(-1)^N}\,\omega(x)\, dx.\end{aligned} \tag{50}$$

The result now follows from Remark (4.2). □

Lemma (4.2) yields the stability of the scheme (45) in the norm corresponding to the weight function $\omega(x) = (1 - x^2)^{-1/2}$.

THEOREM 4.3 : *(Stability of the penalty method) Let u be the solution of (45) with $g^+ = g^- = 0$. If τ_0 and τ_N satisfy*

$$\tau_0 = N^4, \quad \tau_N = N^4, \tag{51}$$

then the Chebyshev penalty method is stable in L^2_ω. More precisely,

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2(x)\, \omega(x)\, dx \leqslant 0. \tag{52}$$

Proof: Multiply both sides of (45) by $u(x)\,\omega(x)$ and integrating we get :

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2(x)\,\omega(x)\,dx = \int_{-1}^{1} u_{xx}(x)\,u(x)\,\omega(x)\,dx$$

$$-\frac{\tau_0}{N^4}u(1)\int_{-1}^{1}\frac{T''_{N+2}(x)+T''_{N+1}(x)}{2}\,u(x)\,\omega(x)\,dx$$

$$-\frac{\tau_N}{N^4}u(-1)\int_{-1}^{1}\frac{T''_{N+2}(x)-T''_{N+1}(x)}{2(-1)^N}\,u(x)\,\omega(x)\,dx\,. \quad (53)$$

The results follows from Lemma (4.2) provided (51) is satisfied. □

5. NUMERICAL EXPERIMENTS

Consider the Maxwell equation

$$\begin{cases} \dfrac{\partial E}{\partial t} = -\,\epsilon c\,\dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial t} = -\,\mu c\,\dfrac{\partial E}{\partial x} \end{cases} \quad (54)$$

with the initial condition

$$E(x,0) = -\,e^{\sin(2\pi x)} \quad (55)$$

$$H(x,0) = \sqrt{\frac{\epsilon}{\mu}}\,e^{\sin(2\pi x)}\,.$$

We have the following exact solution

$$E(x,t) = -\,e^{\sin\left(2\pi\left(\sqrt{\epsilon\mu}ct+x\right)\right)} \quad (56)$$

$$H(x,t) = \sqrt{\frac{\epsilon}{\mu}}\,e^{\sin\left(2\pi\left(\sqrt{\epsilon\mu}ct+x\right)\right)}\,.$$

We will use the exact solution to impose the boundary condition along incoming characteristic. Thus we are indeed solving two scalar hyperbolic equations. One has positive eigenvalue $\sqrt{\varepsilon\mu}$ and the other has negative eigenvalue $-\sqrt{\epsilon\mu}$. We will use third order R-K scheme for time integration. In computation, take $\epsilon = 2$, $\mu = 1$, $c = 1$.

In numerical computation it is verified that the Method (11) is stable as long as $\tau \geqslant \frac{1}{2}$.

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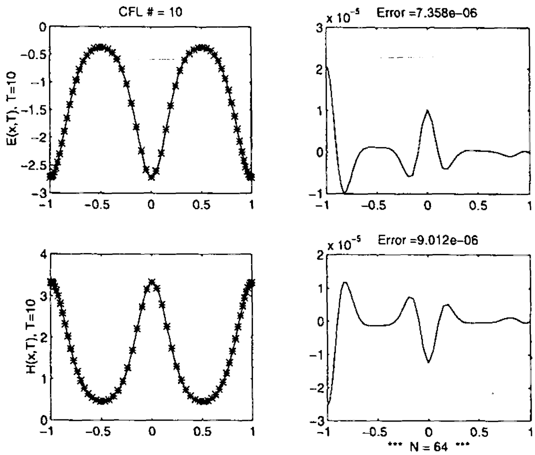

Figure 1. — **Numerical Solution : New Chebyshev Penalty Method ($\tau = 1/2$).**

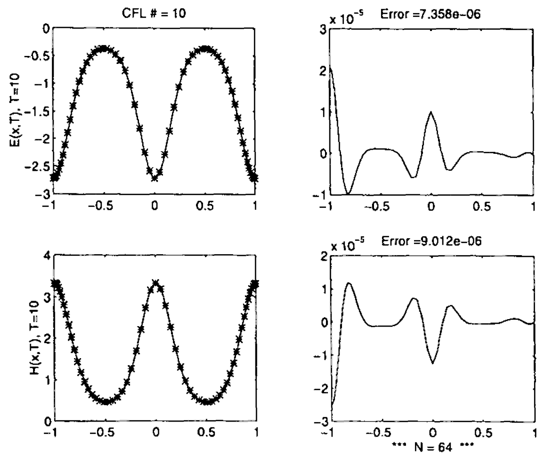

Figure 2. — **Numerical Solution : New Chebyshev Penalty Method ($\tau = 1/2$).**

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