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CHEBYSHEV PSEUDOSPECTRAL-HYBRID FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL VORTICITY EQUATION (*)

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Abstract. — *Chebyshev pseudospectral-hybrid finite element schemes are proposed for two-dimensional vorticity equation. Some approximation results in non-isotropic Sobolev spaces are presented. The generalized stability and the convergence are proved. The hybrid finite element approximation provides the optimal convergence rate. The numerical results show the advantages of the approach. The technique in this paper is also applicable to other nonlinear problems in computational fluid dynamics.*

Key words : Two-dimensional vorticity equation, Chebyshev pseudospectral-finite element approximation.
Subject classification. AMS(MOS) : 65N30, 76D99.

Résumé. — *Des schémas pseudospectraux-hybrides de Chebyshev sont proposés pour l'équation de vorticité bi-dimensionnelle. On présente des résultats d'approximation dans des espaces de Sobolev non isotropes. La stabilité généralisée et la convergence sont établies. L'approximation par éléments finis hybrides conduit à un ordre de convergence optimal. Les résultats numériques montrent les avantages de cette approche. La méthode de cet article est également utilisable pour d'autres problèmes non linéaires de la dynamique des fluides numérique.*

1. INTRODUCTION

There is much literature concerning numerical solutions of partial differential equations describing fluid flows. The early work mainly concerned finite-difference method, e.g., see [1,2]. Since the seventies, finite element method has also been given much attention in computational fluid dynamics, see [3, 4]. As we know, the accuracy of both the above methods is limited by the given schemes. But the precision of spectral method increases as the smoothness of genuine solution increases. Thus spectral method has been used successfully in this field, see [5]. For instance Guo Ben-yu and Ma He-ping developed Fourier spectral and Fourier pseudospectral methods to solve periodic problems of two-dimensional vorticity equation, see [6, 7]. For semi-periodic problems, Fourier spectral (or pseudospectral)-finite difference (or finite element) methods have been developed, see [8-14]. On the other

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hand, some authors provided mixed Fourier-Chebyshev spectral (or pseudospectral) approximation to keep higher accuracy, see [15, 16]. But in fluid dynamics, most of practical problems are neither periodic nor semi-periodic. The domains are of complex geometry. Whereas the sections of domains might be rectangular in certain directions. For example, the fluid flow in a cylindrical container. To solve this kind of problems, it is natural to use Chebyshev spectral-finite element method, see [17]. But the calculation is quite complicated. The purpose of this paper is to develop a new mixed method suitable for non-periodic problems. By taking two-dimensional vorticity equation as an example, we propose Chebyshev pseudospectral-finite element schemes, including fully implicit and semi-implicit schemes, and Chebyshev pseudospectral-hybrid finite element approximation or usual mixed approximation. They are much easier to be performed than Chebyshev spectral-finite element schemes. In particular, it is easy to deal with nonlinear terms and saves much time in calculation. Of course, we can also use full Chebyshev approximation to solve this two-dimensional problem. But it is easy to generalize this new approach to three-dimensional problems with complex geometry. Besides the theoretical analysis in this paper sets up a framework for such mixed approximation, which is very useful for other nonlinear problems in computational fluid dynamics. The outline of this paper is as follows. We construct the schemes in Section 2, and present the numerical results in Section 3, which show the advantages of this method. Then we list some lemmas in Section 4. Finally we prove the generalized stability and convergence in the last two sections. The theoretical analysis shows that the hybrid finite element approximation raises the accuracy and provides the optimal convergence rate. It is concordant with the numerical experiments.

2. THE SCHEME

Let $I_x = (-1, 1)$, $I_y = (0, 1)$ and $\Omega = I_x \times I_y$. The boundary of Ω is denoted by $\partial\Omega$. We denote the vorticity, stream function and kinetic viscosity by $\xi(x, y, t)$, $\psi(x, y, t)$ and $\nu > 0$ respectively. $f(x, y, t)$ and $\xi_0(x, y)$ are given functions. Let $\partial_z = \frac{\partial}{\partial z}$, $z = x, y, t$. We consider the following two-dimensional vorticity equation

$$\begin{cases} \partial_t \xi + J(\xi, \psi) - \nu \nabla^2 \xi = f, & (x, y) \in \Omega, t \in (0, T], \\ -\nabla^2 \psi = \xi, & (x, y) \in \Omega, t \in (0, T], \\ \xi(x, y, 0) = \xi_0(x, y), & (x, y) \in \Omega \cup \partial\Omega, \end{cases} \tag{2.1}$$

where $J(\xi, \psi) = \partial_x \xi \, \partial_y \psi - \partial_y \xi \, \partial_x \psi$. Assume that the boundary is a fixed non-slip wall, and so $\psi = 0$ on $\partial\Omega$. For simplicity, we follow [18] to let $\xi = 0$ on $\partial\Omega$ also. The existance, uniqueness and regularity of global solution of (2.1) is discussed in Theorem 6.10 of [18].

Let N be a positive integer. Denote by $\mathscr{P}_N$ the set of all algebraic polynomials of degree less or equal N, defined on I_x. Let

$$V_N(I_x) = \{ u(x) \in \mathscr{P}_N(I_x) / u(-1) = u(1) = 0 \} .$$

Let $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$ and

$$(u, v)_{\omega, I_x} = \int_{-1}^{1} \omega u v \, dx , \quad \| u \|^2_{\omega, I_x} = (u, u)_{\omega, I_x} ,$$

$$L^2_\omega(I_x) = \{ u(x) / u \text{ is measurable on } I_x \text{ and } \| u \|_{\omega, I_x} < + \infty \} .$$

Also define

$$(u, v)_\omega = \iint_\Omega \omega u v \, dx \, dy , \quad \| u \|^2_\omega = (u, u)_\omega ,$$

$$L^2_\omega(\Omega) = \{ u(x, y) / u \text{ is measurable on } \Omega \text{ and } \| u \|_\omega < + \infty \} .$$

Let τ_h be a class of regular decomposition of I_y with the grid points y_l and subintervals $I_l = (y_{l-1}, y_l)$, $1 \leqslant l \leqslant M$, where

$$0 = y_0 < \dots < y_M = 1 .$$

Suppose that τ_h satisfies the inverse assumption (see [19]). Let

$$h = \max_{1 \leqslant l \leqslant M} |y_l - y_{l-1}| \quad \text{and} \quad \bar{h} = \min_{1 \leqslant l \leqslant M} |y_l - y_{l-1}| .$$

Then $\frac{h}{\bar{h}}$ is bounded above by a positive constant independent of h. Let

$$\tilde{S}^k_h(I_y) = \{ u(y) / u|_{I_l} \in \mathscr{P}_k(I_l), 1 \leqslant l \leqslant M \} , \quad S^k_h(I_y) = \tilde{S}^k_h(I_y) \cap H^1_0(I_y) .$$

$$\tilde{W}^k_{N,h}(\Omega) = V_N(I_x) \otimes \tilde{S}^k_h(I_y) , \quad W^k_{N,h}(\Omega) = V_N(I_x) \otimes S^k_h(I_y) .$$

Now let $x^{(j)}$ and $\omega^{(j)}$ be the nodes and weights of Gauss-Lobatto integration, i.e., $x^{(j)} = \cos \frac{j\pi}{N}$ for $0 \leqslant j \leqslant N$, $\omega^{(0)} = \omega^{(N)} = \frac{\pi}{2N}$, and $\omega^{(j)} = \frac{\pi}{N}$ for $1 \leqslant j \leqslant N-1$. It is well known that for any $g \in \mathscr{P}_{2N-1}(I_x)$,

$$\int_{-1}^{1} \omega(x) \, g(x) \, dx = \sum_{j=0}^{N} \omega^{(j)} \, g(x^{(j)}) . \tag{2.2}$$

The discrete inner products and norms are defined as

$$(u, v)_{N, \omega} = \sum_{j=0}^{N} \omega^{(j)} u(x^{(j)}) v(x^{(j)}), \quad \|u\|_{N, \omega}^2 = (u, u)_{N, \omega},$$

$$(u, v)_{N, h, \omega} = \sum_{j=0}^{N} \int_0^1 \omega^{(j)} u(x^{(j)}, y) v(x^{(j)}, y) \, dy, \quad \|u\|_{N, h, \omega}^2 = (u, u)_{N, h, \omega}.$$

Let $P_c : C(\bar{I}_x) \to \mathscr{P}_N(\bar{I}_x)$ be the interpolation operator, i.e., $P_c u(x^{(j)}) = u(x^{(j)})$ for $0 \leq j \leq N$. Let $\Pi_h^k : C(\bar{I}_y) \to \tilde{S}_h^k(I_y) \cap H^1(I_y)$ be the piecewise Lagrange interpolation operator of degree k over each I_l. For function $u(x, y)$, we still use the notations $P_c u$ and $\Pi_h^k u$.

Let τ be the step size of time t, and

$$\dot{R}_\tau = \left\{ t = l\tau/1 \leq l \leq \left[\frac{T}{\tau}\right] \right\}, \quad R_\tau = \dot{R}_\tau \cup \{0\}.$$

For simplicity, we denote $u(x, y, t)$ by $u(t)$ or u usually, and let

$$\hat{u}(t) = \frac{1}{2}(u(t+\tau) + u(t-\tau)), \quad u_{\hat{t}}(t) = \frac{1}{2\tau}(u(t+\tau) - u(t-\tau)).$$

Now let η and ϕ be the approximations to ξ and ψ respectively. For approximating the non-linear convection term in (2.1), we introduce

$$J_c(u, v) = \partial_x(P_c(u \, \partial_y v)) - \partial_y(P_c(u \, \partial_x v)).$$

Also let

$$a_{N, h, \omega}(u, v) = -(\partial_x^2 u, v)_{N, h, \omega} + (\partial_y u, \partial_y v)_{N, h, \omega}.$$

Let $k \geq 1$ and $\lambda = 0$ or 1. The Chebyshev pseudospectral-finite element scheme for solving (2.1) is to find $\eta(t) \in W_{N,h}^k(\Omega)$ and $\phi(t) \in W_{N,h}^{k+\lambda}(\Omega)$ for all $t \in R_\tau$ such that

$$\begin{cases} (\eta_{\hat{t}} + J_c(\hat{\eta}, \phi), v)_{N, h, \omega} + \nu a_{N, h, \omega}(\hat{\eta}, v) = (f, v)_{N, h, \omega}, & \forall v \in W_{N,h}^k(\Omega), t \in \dot{R}_\tau, \\ a_{N, h, \omega}(\phi, w) = (\eta, w)_{N, h, \omega}, & \forall w \in W_{N,h}^{k+\lambda}(\Omega), t \in R_\tau, \\ \eta(\tau) = P_{N,h}(\xi_0 + \tau \, \partial_t \xi(0)), & \\ \eta(0) = P_{N,h} \xi_0, & \end{cases} \tag{2.3}$$

where $P_{N,h} : L^2_\omega(\Omega) \to W^k_{N,h}(\Omega)$ is the $L^2_\omega(\Omega)$-orthogonal projection and

$$\partial_t \xi(0) = -J(\xi_0, \psi_0) + \nu \nabla^2 \xi_0 + f(0).$$

Scheme (2.3) is a fully implicit scheme, but it is much easier to be performed than the corresponding Chebyshev spectral-finite element scheme. Another scheme, which deals with the nonlinear term explicitly, is to find $\eta(t) \in W^k_{N,h}(\Omega)$ and $\phi(t) \in W^{k+\lambda}_{N,h}(\Omega)$ for all $t \in R_\tau$ such that

$$\begin{cases} (\eta_{\hat{t}} + J_c(\eta, \phi), v)_{N,h,\omega} + \nu a_{N,h,\omega}(\hat{\eta}, v) = (f, v)_{N,h,\omega}, & \forall v \in W^k_{N,h}(\Omega), t \in \dot{R}_\tau, \\ a_{N,h,\omega}(\phi, w) = (\eta, w)_{N,h,\omega}, & \forall w \in W^{k+\lambda}_{N,h}(\Omega), t \in R_\tau, \\ \eta(\tau) = P_{N,h}(\xi_0 + \tau\, \partial_t \xi(0)), & \\ \eta(0) = P_{N,h}\, \xi_0 . & \end{cases} \tag{2.4}$$

We shall analyze the generalized stability and the convergence for schemes (2.3) and (2.4) in the last two sections.

3. NUMERICAL RESULTS

In this section, we examine the numerical performances of scheme (2.3) and (2.4). We take the following test functions

$$\psi(x, y, t) = A e^{-Bt}(x^2 - 1)(x^2 - 5) \sin \pi y,$$

$$\xi(x, y, t) = -\nabla^2 \psi(x, y, t).$$

We use Chebyshev pseudospectral-finite element schemes (CPSFE). The interval I_y is uniformly partitioned with the mesh size $h_y = 1/M$. For comparison, we also consider the bilinear finite element schemes (FEM). In this case, the domain is divided uniformly into rectangular subdomains with the length $h_x = 2/N^*$ in x-direction and $h_y = 1/M$ in y-direction. For describing the errors of numerical solutions, let

$$\hat{I}_x = \left\{ x_j / x_j = \cos \frac{j\pi}{N},\ 1 \leqslant j \leqslant N - 1 \right\}, \qquad \text{for CPSFE,}$$

$$\hat{I}_x = \left\{ x_j / x_j = -1 + j h_x,\ 1 \leqslant j \leqslant N^* - 1 \right\}, \quad \text{for FEM,}$$

$$\hat{I}_y = \left\{ y_j / y_j = j h_y,\ 1 \leqslant j \leqslant M - 1 \right\}, \qquad \text{for CPSFE and FEM}$$

and

$$E(u(t)) = \left(\frac{\sum_{x \in \hat{I}_x} \sum_{y \in \hat{I}_y} |u(x, y, t) - v(x, y, t)|^2}{\sum_{x \in \hat{I}_x} \sum_{y \in \hat{I}_y} |u(x, y, t)|^2} \right)^{1/2}$$

where $u = \xi$ and $v = \eta$, or $u = \psi$ and $v = \phi$.

In calculations, we take $A = 0.5$ and $B = 0.4$. For scheme CPSFE, we take $k = 1$, $M = 10$ and $N = 4$. For scheme FEM, we take $N^* = 10$. Besides, $\tau = 0.005$. Firstly, we use semi-implicit scheme (2.4) with $\lambda = 0$. The numerical results comparied with scheme FEM are shown in Table I and Table II. We find that the scheme CPSFE gives much better numerical results than the scheme FEM. The high accuracy is obtained even for relatively small

Table I. — The errors of scheme (2.4) and FEM, $\nu = 0.001$.

Scheme (2.4), $\lambda = 0$			Scheme FEM	
t	$E(\xi(t))$	$E(\psi(t))$	$E(\xi(t))$	$E(\psi(t))$
0.5	0.2220E-03	0.6736E-02	0.3836E-02	0.1180E-01
1.0	0.3886E-03	0.6932E-02	0.8192E-02	0.1619E-01
1.5	0.6387E-03	0.7174E-02	0.1352E-01	0.2154E-01
2.0	0.9341E-03	0.7485E-02	0.2007E-01	0.2805E-01
2.5	0.1295E-02	0.7838E-02	0.2806E-01	0.3597E-01

Table II. — The errors of scheme (2.4) and FEM, $\nu = 0.0001$.

Scheme (2.4), $\lambda = 0$			Scheme FEM	
t	$E(\xi(t))$	$E(\psi(t))$	$E(\xi(t))$	$E(\psi(t))$
0.5	0.1862E-03	0.6684E-02	0.3925E-02	0.1186E-01
1.0	0.2923E-03	0.6824E-02	0.8380E-02	0.1635E-01
1.5	0.4843E-03	0.7000E-02	0.1390E-01	0.2184E-01
2.0	0.7001E-03	0.7230E-02	0.2064E-01	0.2856E-01
2.5	0.9625E-03	0.7484E-02	0.2895E-01	0.3678E-01

N. Table III and Table IV list the numerical results of scheme (2.4) with $\lambda = 1$. They tell us that the Chebyshev pseudospectral approximation with hybrid finite element approximation provides better numerical results than usual mixed Chebyshev pseudospectral-finite element method. Table V and Table VI show the numerical results of implicit scheme (2.3) with $\lambda = 1$. The numerical results is nearly the same as those of scheme (2.4) with $\lambda = 1$. The numerical experiments are concordant with the theoritical analysis in the last two sections.

Table III. — The errors of scheme (2.4), $\nu = 0.001$, $\lambda = 1$.

t	$E(\xi(t))$	$E(\psi(t))$
0.5	0.5933E-04	0.5821E-02
1.0	0.1054E-03	0.5858E-02
1.5	0.1631E-03	0.5902E-02
2.0	0.2393E-03	0.5956E-02
2.5	0.3343E-03	0.6012E-02

Table IV. — The errors of scheme (2.4), $\nu = 0.0001$, $\lambda = 1$.

t	$E(\xi(t))$	$E(\psi(t))$
0.5	0.4092E-04	0.5794E-02
1.0	0.5033E-04	0.5799E-02
1.5	0.5506E-04	0.5804E-02
2.0	0.6132E-04	0.5812E-02
2.5	0.1030E-03	0.5818E-02

Table V. — The errors of scheme (2.3), $\nu = 0.001$, $\lambda = 1$.

t	$E(\xi(t))$	$E(\psi(t))$
0.5	0.5842E-04	0.5820E-02
1.0	0.1060E-03	0.5858E-02
1.5	0.1629E-03	0.5902E-02
2.0	0.2393E-03	0.5956E-02
2.5	0.3340E-03	0.6021E-02

Table VI. — The errors of scheme (2.3), $\nu = 0.0001$, $\lambda = 1$.

t	$E(\xi(t))$	$E(\psi(t))$
0.5	0.4098E-04	0.5794E-02
1.0	0.4949E-04	0.5799E-02
1.5	0.5369E-04	0.5803E-02
2.0	0.6038E-04	0.5812E-02
2.5	0.1010E-03	0.5818E-02

4. SOME LEMMAS

For error estimations, we need some notations and lemmas. Firstly we introduce some Sobolev spaces of functions defined on I_x, with the weight $\omega(x)$. For any integer $r \geqslant 0$, set

$$|u|_{r,\omega,I_x} = \|\partial_x^r u\|_{\omega,I_x}, \quad \|u\|_{r,\omega,I_x} = \left(\sum_{m=0}^{r} |u|_{m,\omega,I_x}^2\right)^{1/2},$$

$$H_\omega^r(I_x) = \{u(x) / \|u\|_{r,\omega,I_x} < \infty\}.$$

For any real $r > 0$, the space $H_\omega^r(I_x)$ is defined by the complex interpolation between the spaces $H_\omega^{[r]}(I_x)$ and $H_\omega^{[r]+1}(I_x)$. Furthermore,

$$C_0^\infty(I_x) = \{u(x) / u \text{ is infinitely differentiable, and has a compact support in } I_x\}.$$

Denote by $H_{0,\omega}^r(I_x)$ the closure of $C_0^\infty(I_x)$ in $H_\omega^r(I_x)$. Besides $L^\infty(I_x)$ is the space of essentially bounded functions with the norm $\|\cdot\|_{\infty,I_x}$.

Next, let B be a Banach space with the norm $\|\cdot\|_B$ and I be an interval in $\mathcal{R}$. Define

$$L^2(I,B) = \{u(z) : I \to B / u \text{ is strongly measurable and } \|u\|_{L^2(I,B)} < \infty\},$$

$$C(I,B) = \{u(z) : I \to B / u \text{ is strongly continuous and } \|u\|_{C(I,B)} < \infty\}$$

where

$$\|u\|_{L^2(I,B)} = \left(\int_I \|u(z)\|_B^2\, dz\right)^{1/2}, \quad \|u\|_{C(I,B)} = \max_{z \in I} \|u(z)\|_B.$$

For any integer $s \geqslant 0$, define

$$H^s(I, B) = \{u(z) / \|u\|_{H^s(I,B)} < \infty\}$$

equipped with the semi-norm and norm

$$|u|_{H^s(I,B)} = \|\partial_z^s u\|_{L^2(I,B)}, \quad \|u\|_{H^s(I,B)} = \left(\sum_{m=0}^{s} |u|^2_{H^m(I,B)}\right)^{1/2}.$$

For real $s \geqslant 0$, the space $H^s(I, B)$ is defined by the complex interpolation as before.

We now introduce the non-isotropic space

$$H_\omega^{r,s}(\Omega) = L^2(I_y, H_\omega^r(I_x)) \cap H^s(I_y, L_\omega^2(I_x)), \quad r, s \geqslant 0$$

with the norm

$$\|u\|^2_{H_\omega^{r,s}(\Omega)} = \|u\|^2_{L^2(I_y, H_\omega^r(I_x))} + \|u\|^2_{H^s(I_y, L_\omega^2(I_x))}.$$

Also let

$$M_\omega^{r,s}(\Omega) = H_\omega^{r,s}(\Omega) \cap H^{s-1}(I_y, H_\omega^1(I_x)) \cap H^1(I_y, H_\omega^{r-1}(I_x)), \quad r, s \geqslant 1,$$

$$X_\omega^{r,s}(\Omega) = H^1(I_y, H_\omega^1(I_x)) \cap H^s(I_y, H_\omega^{r+1}(I_x))$$
$$\cap H^{s+1}(I_y, H_\omega^r(I_x)), \quad r, s \geqslant 0,$$

$$Y_\omega^{r,s,\delta}(\Omega) = H_\omega^{r,s}(\Omega) \cap H^{1/2+\delta}(I_y, H_\omega^{[r/2]+1/2+\delta}(I_x))$$
$$\cap H^{[s/2]+1/2+\delta}(I_y, H_\omega^{1/2+\delta}(I_x)), \quad r, s, \delta \geqslant 0.$$

Their norms are defined in the way similar to $\|\cdot\|_{H_\omega^{r,s}(\Omega)}$. Furthermore let $H_{0,\omega}^{r,s}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H_\omega^{r,s}(\Omega)$. If $r = s$, then $H_\omega^{r,r}(\Omega) = H_\omega^r(\Omega)$, $H_{0,\omega}^{r,r}(\Omega) = H_{0,\omega}^r(\Omega)$ and denote their semi-norm and norm by $|\cdot|_{r,\omega}$ and $\|\cdot\|_{r,\omega}$ respectively, etc. Moreover we denote by $L_\omega^p(\Omega)$ and $W_\omega^{m,p}(\Omega)$ the weighted Sobolev spaces. In addition we denote by $L^\infty(\Omega)$ and $W^{q,\infty}(\Omega)$ the usual Sobolev spaces of essentially bounded function with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{q,\infty}$, etc.

For simplicity of statements, we denote throughout this paper by c a positive constant independent of h, N, τ and any functions, which may be different in different cases. Let $P_N^0 : L_\omega^2(I_x) \to V_N(I_x)$ be the orthogonal projection. Then for any $u \in L_\omega^2(I_x)$,

$$(P_N^0 u - u, v)_{\omega, I_x} = 0, \quad \forall v \in V_N(I_x).$$

Besides, let P_N be the truncated Chebyshev projection. It is shown in [5] that for any $u \in C(\bar{I}_x)$ and $v \in \mathscr{P}_N(I_x)$,

$$\|v\|_{\omega, I_x} \leqslant \|v\|_{N, \omega} \leqslant \sqrt{2}\, \|v\|_{\omega, I_x}, \tag{4.1}$$

$$|(u, v)_{N, \omega} - (u, v)_{\omega, I_x}| \leqslant c(\|u - P_{N-1} u\|_{\omega, I_x} + \|u - P_c u\|_{\omega, I_x})\|v\|_{\omega, I_x}. \tag{4.2}$$

If in addition $u \in H_\omega^r(I_x)$ and $r > \frac{1}{2}$, then

$$|(u, v)_{N, \omega} - (u, v)_{\omega, I_x}| \leqslant cN^{-r}\|u\|_{r, \omega, I_x}\|v\|_{\omega, I_x}. \tag{4.3}$$

Moreover for any $u \in L_\omega^2(I_x)$ and $v \in H_{0, \omega}^1(I_x)$ (see [20, 21]),

$$|(u, \partial_x(\omega v))_{L^2(I_x)}| \leqslant 2\|u\|_{\omega, I_x}|v|_{1, \omega, I_x}, \tag{4.4}$$

$$\|\omega^2 v\|_{\omega, I_x} \leqslant c|v|_{1, \omega, I_x}, \tag{4.5}$$

$$(\partial_x v, \partial_x(\omega v))_{L^2(I_x)} \geqslant \frac{1}{4}\|v\|_{1, \omega, I_x}^2. \tag{4.6}$$

If $u, v \in \mathscr{P}_N(I_x) \times L^2(I_y)$ and $uv \in \mathscr{P}_{2N-1}(I_x)$ for $y \in I_y$, then by (2.2),

$$(u, v)_{N, h, \omega} = (u, v)_\omega. \tag{4.7}$$

Let $P_h^0 : L^2(I_y) \to S_h^k(I_y)$ be the $L^2(I_y)$-projection, then for any $u \in L^2(I_y)$,

$$(P_h^0 u - u, v)_{L^2(I_y)} = 0, \quad \forall v \in S_h^k(I_y).$$

Let $P_{N,h} = P_N^0 \cdot P_h^0 = P_h^0 \cdot P_N^0$. Then for any $u \in L_\omega^2(\Omega)$,

$$(P_{N,h} u - u, v)_\omega = 0, \quad \forall v \in W_{N,h}^k(\Omega).$$

We now turn to list some lemmas. Hereafter let $\bar{s} = \min(s, k+1)$. In some lemmas, we require that there exist positive constant c_1 and c_2 such that

$$c_1 N \leqslant \frac{1}{h} \leqslant c_2 N. \tag{4.8}$$

LEMMA 1 : *If* $u \in C(\bar{I}_x) \times L^2(I_y)$ *and* $v \in \mathscr{P}_N(I_x) \times L^2(I_y)$, *then*

$$\| v \|_{\omega} \leqslant \| v \|_{N, h, \omega} \leqslant \sqrt{2} \, \| v \|_{\omega} ,$$

$$|(u, v)_{N, h, \omega} - (u, v)_{\omega}| \leqslant c(\| u - P_{N-1} u \|_{\omega} + \| u - P_c u \|_{\omega}) \, \| v \|_{\omega} .$$

If in addition $u \in H_{\omega}^{r, 0}(\Omega)$ *and* $r > \frac{1}{2}$, *then*

$$|(u, v)_{N, h, \omega} - (u, v)_{\omega}| \leqslant c N^{-r} \| u \|_{H_{\omega}^{r, 0}(\Omega)} \| v \|_{\omega} .$$

This lemma comes from (4.1)-(4.3).

LEMMA 2 : *For any* $u \in W_{N, h}^{k}(\Omega)$ *with* $k \geqslant 1$, *we have* $a_{N, h, \omega}(u, u) \geqslant \frac{1}{4} \| u \|_{1, \omega}^{2}$.

Proof: We have from (4.7) that

$$a_{N, h, \omega}(u, u) = - (\partial_x^2 u, u)_{\omega} + \| \partial_y u \|_{N, h, \omega}^{2} .$$

Then by integrating by parts and (4.6),

$$a_{N, h, \omega}(u, u) = (\partial_x u, \partial_x(\omega u))_{L^2(\Omega)} + \| \partial_y u \|_{N, h, \omega}^{2} \geqslant \frac{1}{4} \| u \|_{1, \omega}^{2} .$$

LEMMA 3 : (Lemma 5 of [17]). *For any* $u \in \tilde{W}_{N, h}^{k}(\Omega)$ *with* $k \geqslant 1$, *we have* $\| u \|_{\infty} \leqslant c \left(\frac{N}{h} \right)^{1/2} \| u \|_{\omega}$. *Moreover for any* $u \in H_{\omega}^{1}(\Omega)$, *we have* $\| P_N u \|_{\infty} \leqslant c (\ln N)^{\frac{1}{2}} \| u \|_{1, \omega}$.

LEMMA 4 : (Lemma 2 of [17]). *Let* $u \in H_{0, \omega}^{r, 1}(\Omega) \cap H_{\omega}^{r, s}(\Omega)$ *with* $0 \leqslant r \leqslant 1$, $s \geqslant 0$, *or* $u \in H_{0, \omega}^{1}(\Omega) \cap H_{\omega}^{r, s}(\Omega)$ *with* $r > 1$, $s \geqslant 0$. *Then*

$$\| u - P_{N, h} u \|_{\omega} \leqslant c(N^{-r} + h^{\bar{s}}) \, \| u \|_{H_{\omega}^{r, \bar{s}}(\Omega)} .$$

LEMMA 5 : *If* $u \in H^{\beta}(I_y, H_{\omega}^{r}(I_x))$ *with* $r > \frac{1}{2}$, $0 \leqslant \alpha \leqslant r$ *and* $\beta \geqslant 0$, *then*

$$\| u - P_c u \|_{H^{\beta}(I_y, H_{\omega}^{\alpha}(I_x))} \leqslant c N^{2 \alpha - r} \| u \|_{H^{\beta}(I_y, H_{\omega}^{r}(I_x))} .$$

Proof: We have from (9.5.20) of [5] that if $v \in H_{\omega}^{r}(I_x)$, $r > \frac{1}{2}$ and $0 \leqslant \alpha \leqslant r$, then

$$\| v - P_c v \|_{\alpha, \omega, I_x} \leqslant c N^{2 \alpha - r} \| v \|_{r, \omega, I_x} . \tag{4.9}$$

Thus the conclusion follows.

Remark 1 : If in addition $r' > \frac{1}{2}$, then

$$\| u - P_c u \|_{H_\omega^{\alpha,\beta}(\Omega)} \leqslant cN^{2\alpha - r} \| u \|_{L^2(I_y, H_\omega^r(I_x))} + cN^{-r'} \| u \|_{H^\beta(I_y, H_\omega^{r'}(I_x))} .$$

We also have

$$\| u - P_c u \|_{H_\omega^{\alpha,\beta}(\Omega)} \leqslant cN^{2\alpha - r} \| u \|_{H^\beta(I_y, H_\omega^r(I_x))} .$$

LEMMA 6 : *If* $u \in H_\omega^{r,s}(\Omega)$, $\frac{1}{r} + \frac{1}{s} < 2$ *and* $k \geqslant 1$, *then*

$$\| u - \Pi_h^k P_c u \|_\omega \leqslant c(N^{-r} + h^{\bar{s}}) \| u \|_{H_\omega^{r,\bar{s}}(\Omega)} .$$

Proof : We follow the method in Section 9.5.3 of [5]. Set $I_\theta = (0, 2\pi)$ and

$$\tilde{V}_N = \left\{ v : I_\theta \to C / v(\theta) = \sum_{n=-N}^{N} \hat{v}_n e^{in\theta}, \hat{v}_N = \hat{v}_{-N} \right\} .$$

Let $\tilde{P}_c : C(\bar{I}_\theta) \to \tilde{V}_N$ be interpolation operator such that

$$\tilde{P}_c v(\theta_j) = v(\theta_j) , \quad \theta_j = \frac{j\pi}{N} , \quad j = 0, 1, \ldots, 2N .$$

We make use of the transformation $v(x) \to \tilde{v}(\theta) = v(\cos\theta)$. Then $\widetilde{P_c v} = \tilde{P}_c \tilde{v}$. Furthermore by using the mapping $\tilde{u}(\theta, y) = u(\cos\theta, y)$ for $(\theta, y) \in \bar{I}_\theta \times \bar{I}_y$, we have from [5] and the error estimation of Fourier pseudospectral-finite element approximation (see [22]),

$$\begin{aligned} \| u - \Pi_h^k P_c u \|_\omega &= \frac{1}{\sqrt{2}} \| \tilde{u} - \Pi_h^k \tilde{P}_c \tilde{u} \|_{L^2(I_\theta \times I_y)} \\ &\leqslant c(N^{-r} + h^{\bar{s}}) \| \tilde{u} \|_{H^{r,\bar{s}}(I_\theta \times I_y)} \leqslant c(N^{-r} + h^{\bar{s}}) \| u \|_{H_\omega^{r,\bar{s}}(\Omega)} . \end{aligned}$$

Let $a_\omega(u, v) = (\nabla u, \nabla(\omega v))_{L^2(\Omega)}$. In order to obtain optimal error estimation, we introduce the projection $P_{N,h}^* : H_{0,\omega}^1(\Omega) \to W_{N,h}^k(\Omega)$ such that for any $u \in H_{0,\omega}^1(\Omega)$,

$$a_\omega(P_{N,h}^* u, v) = a_\omega(u, v) , \quad \forall v \in W_{N,h}^k(\Omega) ,$$

and the projection $P_{N,h}^c : H_{0,\omega}^1(\Omega) \to W_{N,h}^k(\Omega)$ such that for any $u \in H_{0,\omega}^1(\Omega)$,

$$a_{N,h,\omega}(P_{N,h}^c u, v) = a_\omega(u, v) , \quad \forall v \in W_{N,h}^k(\Omega) .$$

LEMMA 7 : *Let* (4.8) *hold and* $k \geqslant 1$. *If* $u \in H^1_{0,\omega}(\Omega) \cap M^{r,s}_{\omega}(\Omega)$ *with* r, $s \geqslant 1$, *then*

$$\| u - P^*_{N,h} u \|_{\mu,\omega} \leqslant c(N^{\mu - r} + h^{\bar{s} - \mu}) \, \| u \|_{M^{r,\bar{s}}_{\omega}(\Omega)} , \quad \mu = 0, 1 , \tag{4.10}$$

$$\| u - P^c_{N,h} u \|_{\mu,\omega} \leqslant c(N^{\mu - r} + h^{\bar{s} - \mu}) \, \| u \|_{M^{r,\bar{s}}_{\omega}(\Omega)} , \quad \mu = 0, 1 . \tag{4.11}$$

Proof : The first conclusion is Lemma 3 of [17]. We now prove the second one. Firstly, we have from Lemma 2, (4.7) and the definitions of $P^*_{N,h}$ and $P^c_{N,h}$ that

$$\begin{aligned} \| P^c_{N,h} u - P^*_{N,h} u \|^2_{1,\omega} &\leqslant 4 \, a_{N,h,\omega}(P^c_{N,h} u - P^*_{N,h} u, P^c_{N,h} u - P^*_{N,h} u) \\ &= 4 \, a_{\omega}(P^*_{N,h} u, P^c_{N,h} u - P^*_{N,h} u) \\ &\quad - 4 \, a_{N,h,\omega}(P^*_{N,h} u, P^c_{N,h} u - P^*_{N,h} u) \\ &= 4(\partial_y P^*_{N,h} u, \partial_y(P^c_{N,h} u - P^*_{N,h} u))_{\omega} \\ &\quad - 4(\partial_y P^*_{N,h} u, \partial_y(P^c_{N,h} u - P^*_{N,h} u))_{N,h,\omega} . \end{aligned}$$

It is shown in [20] that for $z, v \in \mathscr{P}_N(I_x)$,

$$| (z, v)_{N,\omega} - (z, v)_{\omega, I_x} | = | z_N v_N |$$

where $T_k(x)$ is the Chebyshev polynomial of order k, $c_0 = 2$, $c_k = 1$ $(k \geqslant 1)$ and

$$z_k = \frac{2}{\pi c_k} \int_{-1}^{1} z(x) \, T_k(x) \, \omega(x) \, dx .$$

Let ϑ be the identity operator. Then

$$\begin{aligned} \| P^c_{N,h} u - P^*_{N,h} u \|^2_{1,\omega} &= 4 | ((\vartheta - P_{N-1}) \, \partial_y P^*_{N,h} u, \partial_y (P^c_{N,h} u - P^*_{N,h} u))_{\omega} | \\ &\leqslant c(\| (\vartheta - P_{N-1}) \, \partial_y (P^*_{N,h} u - u) \|_{\omega} \\ &\quad + \| (\vartheta - P_{N-1}) \, \partial_y u \|_{\omega}) \, | P^c_{N,h} u - P^*_{N,h} u |_{1,\omega} \\ &\leqslant c(| P^*_{N,h} u - u |_{1,\omega} \\ &\quad + \| (\vartheta - P_{N-1}) \, \partial_y u \|_{\omega}) \, | P^c_{N,h} u - P^*_{N,h} u |_{1,\omega} \\ &\leqslant c(N^{1-r} + h^{\bar{s}-1}) \, \| u \|_{M^{r,\bar{s}}_{\omega}(\Omega)} | P^c_{N,h} u - P^*_{N,h} u |_{1,\omega} . \end{aligned}$$

The above inequality and (4.10) lead to the result (4.11) for $\mu = 1$. Next, we prove the result for $\mu = 0$ by a duality argument. Let $g = u - P_{N,h}^c u$ and $u_g \in H_{0,\omega}^1(\Omega)$ be the solution of auxilarly problem

$$a_\omega(v, u_g) = (g, v)_\omega, \quad \forall v \in H_{0,\omega}^1(\Omega). \tag{4.12}$$

It is easy to see that u_g satisfies

$$\begin{cases} -\nabla^2(u_g\,\omega) = g\omega, & (x, y) \in \Omega, \\ u_g\,\omega = 0, & (x, y) \in \partial\Omega. \end{cases} \tag{4.13}$$

Let $\zeta(x) = \omega^{-1}(x)$ and define the spaces $L_\zeta^2(\Omega)$, $H_\zeta^r(\Omega)$ and the norms $\|\cdot\|_\zeta$ and $\|\cdot\|_{r,\zeta}$ in the way similar to $H_\omega^r(\Omega)$ and $\|\cdot\|_{r,\omega}$, etc. We shall prove that

$$\|u_g\,\omega\|_{2,\zeta} \leqslant c\|g\omega\|_\zeta = c\|g\|_\omega. \tag{4.14}$$

To do this, let $\phi_{l,m} = \sin l\pi\left(\frac{1+x}{2}\right)\sin m\pi y$ and

$$g\omega = \sum_{l,m=1}^{\infty} g_{l,m}\,\phi_{l,m}, \quad g_N = \sum_{l,m=1}^{N} g_{l,m}\,\phi_{l,m}.$$

Then $\|g\omega - g_N\|_\zeta \leqslant \|g\omega - g_N\|_{L^2(\Omega)} \to 0$, as $N \to \infty$. Next, let u_N be the solution of problem

$$\begin{cases} -\nabla^2 u_N = g_N, & (x, y) \in \Omega, \\ u_N = 0, & (x, y) \in \partial\Omega. \end{cases} \tag{4.15}$$

By multiplying the above equation by $u_N\zeta$ and integrating by parts, we get

$$(\partial_x u_N, \partial_x(u_N\zeta))_{L^2(\Omega)} + \|\partial_y u_N\|_\zeta^2 \leqslant \|g_N\|_\zeta\|u_N\|_\zeta.$$

It can be verified that

$$(\partial_x u_N, \partial_x(u_N\zeta))_{L^2(\Omega)} = \|\partial_x u_N\|_\zeta^2 + \frac{1}{2}\int_\Omega \omega^3\,u_N^2\,dx\,dy$$

and so $\|u_N\|_{1,\zeta} \leqslant c\|g_N\|_\zeta$. Moreover by (4.15),

$$\|\partial_{xx}u_N\|_\zeta^2 + \|\partial_{yy}u_N\|_\zeta^2 + 2(\partial_x(\partial_y u_N), \partial_x(\partial_y u_N\zeta))_{L^2(\Omega)} = \|g_N\|_\zeta^2$$

and thus by an argument as before, we obtain

$$\| u_N \|_{2,\zeta} \leqslant c \| g_N \|_{\zeta} .$$

It is an easy matter to note that $\{u_N\}$ is a Cauchy sequence in $H^2_\zeta(\Omega)$, which tends to u^*. Therefore,

$$\| u^* \|_{2,\zeta} \leqslant c \| g\omega \|_{\zeta} .$$

Clearly, for any $v \in H^1_{0,\zeta}(\Omega)$,

$$(\nabla u^*, \nabla(v\zeta))_{L^2(\Omega)} = \lim_{N\to\infty} (\nabla u_N, \nabla(v\zeta))_{L^2(\Omega)} = \lim_{N\to\infty} (g_N, v)_\zeta = (g\omega, v)_\zeta .$$

Hence we know from (4.13) that $u^* = u_g\,\omega$. Thus (4.14) follows. Furthermore we have from (4.12) and the definition of $P^c_{N,h}$ that for any $v^* \in W^k_{N,h}(\Omega)$,

$$\begin{aligned} |(u - P^c_{N,h} u, g)_\omega| &= |a_\omega(u - P^c_{N,h} u, u_g)| \\ &= |a_\omega(u - P^c_{N,h} u, u_g - v^*) + a_{N,h,\omega}(P^c_{N,h} u, v^*) \\ &\quad - a_\omega(P^c_{N,h} u, v^*)| \leqslant cA_1 + cA_2 \end{aligned} \tag{4.16}$$

where

$$A_1 = \| u - P^c_{N,h} u \|_{1,\omega} |(u_g - v^*)\,\omega|_{1,\zeta} ,$$

$$A_2 = |((\vartheta - P_{N-1})\,\partial_y P^c_{N,h} u, \partial_y v^*)_\omega| .$$

It is a very technical argument to choose v^* suitably. To do this, we follow the method in [23]. Let

$$U_N(I_x) = \{ u/\zeta u \in \mathscr{P}_{N-1}(I_x) \} ,$$

and $d_N : L^2_\zeta(I_x) \to U_N(I_x)$ be such that

$$(u - d_N u, v)_{L^2_\zeta(I_x)} = 0 , \quad \forall v \in U_N(I_x) .$$

We have from Lemma 4.4 of [23] that

$$\| u - d_N u \|_{\zeta, I_x} \leqslant cN^{-1} \| u \|_{1,\zeta,I_x} , \quad \forall u \in H^1_\zeta(I_x) .$$

Also for $u \in H^1_{0,\zeta}(I_x)$, we define $d^1_N u$ by

$$d^1_N u = \int_{-1}^{x} d_N \frac{du(x')}{dx'}\, dx' .$$

As in Section 4 of [23], it can be shown that $\zeta d_N^1 u \in V_N(I_x)$. Let

$$G(x) = \int_{-1}^{x} (u(x') - d_N^1 u(x'))\, dx'.$$

Then

$$\| u - d_N^1 u \|_{\zeta, I_x}^2 = - \int_{-1}^{1} \left(\frac{du}{dx} - d_N \frac{du}{dx} \right) G\zeta \, dx - \int_{-1}^{1} (u - d_N^1 u)\, G \frac{d\zeta}{dx} \, dx$$

$$= - \int_{-1}^{1} \left(\frac{du}{dx} - d_N \frac{du}{dx} \right) (G - d_N G)\zeta \, dx + \frac{1}{2} \int_{-1}^{1} \frac{dG^2}{dx} x\omega \, dx .$$

Since the last term equals $-\frac{1}{2} \int_{-1}^{1} G^2 \omega^3 \, dx$,

$$\| u - d_N^1 u \|_{\zeta, I_x} \leqslant cN^{-1} \| u \|_{1, \zeta, I_x}, \quad \forall u \in H_{0,\zeta}^1(I_x) .$$

Next we define $d_h : L^2(I_y) \to \tilde{S}_h^{k-1}(I_y)$ such that

$$(u - d_h u, v)_{L^2(I_y)} = 0 , \quad \forall v \in \tilde{S}_h^{k-1}(I_y) .$$

We have

$$\| u - d_h u \|_{L^2(I_y)} \leqslant ch \| u \|_{1, I_y}, \quad \forall u \in H^1(I_y) .$$

Also for $u \in H_0^1(I_y)$, we define

$$d_h^1 u = \int_0^y d_h \frac{du(y')}{dy'} dy' .$$

It is easy to see that $d_h^1 u(1) = 0$ and $d_h^1 u \in S_h^k(I_y)$. Let $\tilde{g} = u - d_h^1 u$ and $\tilde{G} = \int_0^y \tilde{g}\,(y')\, dy'$. We have

$$\| u - d_h^1 u \|_{L^2(I_y)}^2 = \int_0^1 (u - d_h^1 u)\, \tilde{g} \; dy$$

$$= - \int_0^1 \left(\frac{du}{dy} - d_h \frac{du}{dy} \right) \tilde{G}(y)\, dy$$

$$\leqslant \left\| (\vartheta - d_h) \frac{du}{dy} \right\|_{L^2(I_y)} \| (\vartheta - d_h)\, G \|_{L^2(I_y)}$$

$$\leqslant ch \| u \|_{1, I_y} \| \tilde{g} \|_{L^2(I_y)} .$$

Thus for any $u \in H_0^1(I_y)$,

$$\| u - d_h^1 u \|_{L^2(I_y)} \leqslant ch \| u \|_{1, I_y} .$$

Now we choose

$$v^* \omega = d_N^1 d_h^1 (u_g \omega) = \int_{-1}^{x} \int_0^{y} d_N d_h \partial_x \partial_y (u_g \omega) \, dx \, dy .$$

Then $v^* \in W_{N,h}^k(\Omega)$ and

$$\partial_x (v^* \omega) = d_N d_h^1 \partial_x (u_g \omega) , \quad \partial_y (v^* \omega) = d_h d_N^1 \partial_y (u_g \omega) .$$

Therefore

$$\begin{aligned} |(u_g - v^*) \omega|_{1,\zeta} \leqslant & \; \| (\vartheta - d_N) \partial_x (u_g \omega) \|_\zeta + \| d_N (\vartheta - d_h^1) \partial_x (u_g \omega) \|_\zeta \\ & + \| (\vartheta - d_h) \partial_y (u_g \omega) \|_\zeta + \| d_h (\vartheta - d_N^1) \partial_y (u_g \omega) \|_\zeta \\ & \leqslant c(N^{-1} + h) \| u_g \omega \|_{2,\zeta} \leqslant c(N^{-1} + h) \| g \|_\omega . \end{aligned}$$

Thus we have

$$A_1 \leqslant c(N^{1-r} + h^{\bar{s}-1})(N^{-1} + h) \| u \|_{M_\omega^{r,\bar{s}}(\Omega)} \| g \|_\omega .$$

Furthermore,

$$\begin{aligned} A_2 & \leqslant |((\vartheta - P_{N-1}) \partial_y P_{N,h}^c u, \zeta(\partial_y(v^* \omega) - d_N d_h \partial_y (u_g \omega)))_\omega| \\ & \leqslant (\| \partial_y (P_{N,h}^c u - u) \|_\omega + \| (\vartheta - P_{N-1}) \partial_y u \|_\omega) \| d_h (d_N^1 - d_N) \partial_y (u_g \omega) \|_\zeta \\ & \leqslant c(N^{1-r} + h^{\bar{s}-1}) N^{-1} \| u \|_{M_\omega^{r,\bar{s}}(\Omega)} \| g \|_\omega . \end{aligned}$$

Recalling the definition of g and substituting the above two estimations into (4.16), we obtain the result (4.11) for $\mu = 0$.

LEMMA 8 : *Let* (4.8) *hold. If* $u \in H_{0,\omega}^1(\Omega) \cap H^s(I_y, H_\omega^r(I_x))$ *with* r, $s > 1/2$, *then*

$$\| P_{N,h}^c u \|_\infty \leqslant c \| u \|_{H_\omega^1(\Omega) \cap H^s(I_y, H_\omega^r(I_x))} .$$

If in addition $u \in X_\omega^{r,s}(\Omega)$ with r, $s > 1/2$, then

$$\| P_{N,h}^c u \|_{1,\infty} \leqslant c \| u \|_{X_\omega^{r,s}(\Omega)} .$$

Proof: The results can be got in the same way as in the proof of Lemma 6 of [17].

LEMMA 9 : *If $u,\ v \in Y_\omega^{r,s,\delta}(\Omega)$ with $r,\ s \geqslant 0$ and $\delta > 0$, then*

$$\| uv \|_{H_\omega^{r,s}(\Omega)} \leqslant c \| u \|_{Y_\omega^{r,s,\delta}(\Omega)} \| v \|_{Y_\omega^{r,s,\delta}(\Omega)} .$$

Proof: We first assume that r and s are integers. Then

$$\partial_x^r(uv) = u\, \partial_x^r v + r\, \partial_x u \partial_x^{r-1} v + \cdots + v\, \partial_x^r u ,$$

and so by embedding theory,

$$\| \partial_x^r(uv) \|_\omega$$

$$\leqslant c \| u \|_{C(I_y, W^{\left[\frac{r}{2}\right], \infty}(I_x)) \cap L^2(I_y, H_\omega^r(I_x))} \| v \|_{C(I_y, W^{\left[\frac{r}{2}\right], \infty}(I_x)) \cap L^2(I_y, H_\omega^r(I_x))}$$

$$\leqslant c \| u \|_{H^{\frac{1}{2}+\delta}(I_y, H_\omega^{\left[\frac{r}{2}\right]+\frac{1}{2}+\delta}(I_x)) \cap L^2(I_y, H_\omega^r(I_x))} \| v \|_{H^{\frac{1}{2}+\delta}(I_y, H_\omega^{\left[\frac{r}{2}\right]+\frac{1}{2}+\delta}(I_x)) \cap L^2(I_y, H_\omega^r(I_x))} .$$

For real $r \geqslant 0$, we can get the same result by using the interpolation of spaces. Similarly

$$\| \partial_y^s(uv) \|_\omega \leqslant c \| u \|_{H^{\left[\frac{s}{2}\right]+\frac{1}{2}+\delta}(I_y, H_\omega^{\frac{1}{2}+\delta}(I_x)) \cap H^s(I_y, L_\omega^2(I_x))} \cdot$$

$$\| v \|_{H^{\left[\frac{s}{2}\right]+\frac{1}{2}+\delta}(I_y, H_\omega^{\frac{1}{2}+\delta}(I_x)) \cap H^s(I_y, L_\omega^2(I_x))} .$$

LEMMA 10 : *If $u \in L_\omega^2(\Omega)$ and $v \in H_{0,\omega}^1(\Omega)$, then*

$$|(u, \partial_x(\omega v))_{L^2(\Omega)}| \leqslant 2 \| u \|_\omega | v |_{1,\omega} .$$

Proof: We have from (4.4) that

$$|(u, \partial_x(\omega v))_{L^2(\Omega)}| \leqslant 2 \int_0^1 \| u \|_{\omega, I_x} | v |_{1,\omega,I_x}\, dy \leqslant 2 \| u \|_\omega | v |_{1,\omega} .$$

LEMMA 11 : *If $u,\ v \in H_{0,\omega}^1(\Omega)$ and $z \in W^{1,\infty}(\Omega)$, then*

$$|(J(u, z), v)_\omega| \leqslant 3 \| u \|_\omega \| z \|_{1,\infty} | v |_{1,\omega} .$$

Proof: We have

$$(J(u, z), v)_\omega = - (u\, \partial_y z, \partial_x(\omega v))_{L^2(\Omega)} + (u\, \partial_x z, \partial_y v)_\omega .$$

Then the conclusion follows from Lemma 10.

LEMMA 12 : *If* $u \in \mathscr{P}_{2N}(I_x)$, *then*

$$\|P_c u\|_{\omega, I_x} \leqslant \sqrt{2}\, \|u\|_{\omega, I_x}.$$

Proof: Let $T_k(x)$ be the Chebyshev polynomial of degree k, $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_n = 1$ for $1 \leqslant n \leqslant N-1$. By (2.4.19) of [5],

$$(T_n, T_m)_{N,\omega} = \begin{cases} \frac{\pi}{2}\bar{c}_n, & \text{if } m = 2lN \pm n, \quad l \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$u = \sum_{m=0}^{2N} \hat{u}_m T_m(x), \quad P_c u = \sum_{n=0}^{N} \tilde{u}_n T_n(x).$$

Then

$$\tilde{u}_n = \frac{2}{\pi \bar{c}_n}(u, T_n)_{N,\omega} = (\hat{u}_n + \hat{u}_{2N-n}), \quad \text{for } 0 \leqslant n \leqslant N-1$$

and $\tilde{u}_N = \hat{u}_N$. Denoting that $c_n = \bar{c}_n (n \neq N)$ and $c_N = 1$, we have

$$\begin{aligned}\|P_c u\|^2_{\omega, I_x} &= \frac{\pi}{2}\sum_{n=0}^{N} c_n \tilde{u}_n^2 \\ &\leqslant \pi \sum_{n=0}^{N-1} c_n(\hat{u}_n^2 + \hat{u}_{2N-n}^2) + \frac{\pi}{2}\hat{u}_N \\ &\leqslant \pi \sum_{n=0}^{2N} c_n \hat{u}_n^2 \leqslant 2\|u\|^2_{\omega, I_x}.\end{aligned}$$

LEMMA 13 : *For any* $u, v, z \in W^k_{N,h}(\Omega)$ *and* $k \geqslant 1$, *we have*

$$|(J_c(u,z), v)_{N,h,\omega}| \leqslant c\|u\|_\omega |z|_{1,\infty} |v|_{1,\omega}, \tag{4.17}$$

$$|(J_c(u,z), v)_{N,h,\omega}| \leqslant c\|u\|_{L^4_\omega(\Omega)} |z|_{W^{1,4}_\omega(\Omega)} |v|_{1,\omega}. \tag{4.18}$$

Proof: For any fixed y, $\partial_x P_c(u\, \partial_y z) \in \mathscr{P}_{N-1}(I_x)$. Hence by (4.7),

$$(J_c(u,z), v)_{N,h,\omega} = D_1 + D_2$$

where

$$D_1 = -(P_c(u\,\partial_y z), \partial_x(\omega v))_{L^2(\Omega)}, \quad D_2 = (P_c(u\,\partial_x z), \partial_y v)_{N,h,\omega}.$$

From Lemma 10 and Lemma 12,

$$|D_1| \leqslant \|P_c(u\,\partial_y z)\|_\omega |v|_{1,\omega} \leqslant \sqrt{2}\,\|u\,\partial_y z\|_\omega |v|_{1,\omega} \leqslant \sqrt{2}\,\|u\|_\omega |z|_{1,\infty}|v|_{1,\omega}\,.$$

Similarly, from Lemma 1,

$$|D_2| \leqslant 2\|P_c(u\,\partial_x z)\|_\omega \|\partial_y v\|_\omega \leqslant 2\sqrt{2}\,\|u\|_\omega |z|_{1,\infty}|v|_{1,\omega}\,.$$

Then the first conclusion follows. We can derive the second one similarly by using Cauchy inequality.

LEMMA 14 : *Let $g \in \mathscr{P}_N(I_x)\times L^2(I_y)$ and $u \in W^k_{N,h}(\Omega)$ be the solution of*

$$a_{N,h,\omega}(u,v) = (g,v)_{N,h,\omega}\,, \quad \forall v \in W^k_{N,h}(\Omega)\,. \tag{4.19}$$

If (4.8) holds, then

$$\|u\|_{1,\omega} \leqslant 8\|g\|_\omega\,,$$

$$|u|_{W^{1,p}_\omega(\Omega)} \leqslant c\|g\|_\omega\,, \quad 1 \leqslant p < \infty\,,$$

$$|u|_{1,\infty} \leqslant c|\ln N|^{1/2}\|g\|_\omega\,.$$

Proof: By putting $v = u$, we have from Lemma 1 and Lemma 2 that

$$\frac{1}{4}\|u\|^2_{1,\omega} \leqslant \|u\|_{N,h,\omega}\|g\|_{N,h,\omega} \leqslant 2\|u\|_\omega\|g\|_\omega \leqslant 2\|u\|_{1,\omega}\|g\|_\omega\,.$$

Now we prove the second conclusion. We define $\tilde{P}u \in W^k_{N,h}(\Omega)$ by

$$(\tilde{P}u, v)_\omega = (u,v)_{N,h,\omega}\,, \quad \forall v \in W^k_{N,h}(\Omega)\,. \tag{4.20}$$

Then we have from Lemma 1 that

$$\|\tilde{P}u\|^2_\omega = (\tilde{P}u, \tilde{P}u)_\omega = (u, \tilde{P}u)_{N,h,\omega} \leqslant 2\|u\|_\omega \|\tilde{P}u\|_\omega\,.$$

So that $\|\tilde{P}u\|_\omega \leqslant 2\|u\|_\omega$. Set $g^* = \tilde{P}P_h g$. Let $\tilde{u} \in W^k_{N,h}(\Omega)$ be the solution of

$$a_\omega(\tilde{u}, v) = (g^*, v)_\omega\,, \quad \forall v \in W^k_{N,h}(\Omega)\,, \tag{4.21}$$

and $u^* \in H^1_{0,\omega}(\Omega)$ be the solution of

$$a_\omega(u^*, v) = (g^*, v)_\omega\,, \quad \forall v \in H^1_{0,\omega}(\Omega)\,.$$

Obviously $\tilde{u} = P^*_{N,h} u^*$. Moreover by Lemma 9 of [17] (A similar result is given in [23]),

$$\| u^* \|_{2,\omega} \leqslant c \| g^* \|_{\omega} \leqslant c \| g \|_{\omega} .$$

Using Lemma 3, we have

$$|u|_{1,p,\omega} \leqslant |u - \tilde{u}|_{1,p,\omega} + |\tilde{u}|_{1,p,\omega}$$
$$\leqslant c \sqrt{\frac{N}{h}}\, |u - \tilde{u}|_{1,\omega} + |\tilde{u}|_{1,p,\omega} .$$

By (4.19)-(4.21), $a_{N,h,\omega}(u, v) = a_{\omega}(\tilde{u}, v)$ for all $v \in W^k_{N,h}(\Omega)$. So from Lemma 2 and [20],

$$\begin{aligned}
|u - \tilde{u}|^2_{1,\omega} &\leqslant c a_{N,h,\omega}(u - \tilde{u}, u - \tilde{u}) \\
&= c a_{\omega}(\tilde{u}, u - \tilde{u}) - c a_{N,h,\omega}(\tilde{u}, u - \tilde{u}) \\
&= c(\partial_y \tilde{u}, \partial_y (u - \tilde{u}))_{\omega} - c(\partial_y \tilde{u}, \partial_y (u - \tilde{u}))_{N,h,\omega} \\
&= c|((\vartheta - P_{N-1})\, \partial_y \tilde{u}, \partial_y (u - \tilde{u}))_{\omega}| \\
&\leqslant c(\|(\vartheta - P_{N-1})\, \partial_y (\tilde{u} - u^*)\|_{\omega} \\
&\quad + \|(\vartheta - P_{N-1})\, \partial_y u^* \|_{\omega})\, |u - \tilde{u}|_{1,\omega} .
\end{aligned}$$

Therefore,

$$|u - \tilde{u}|_{1,\omega} \leqslant c(|P^*_{N,h} u^* - u^*|_{1,\omega} + N^{-1} \| u^* \|_{2,\omega}) \leqslant c(N^{-1} + h)\, \| g \|_{\omega} .$$

Next we introduce $\tilde{P}_h : L^2(I_y) \to \tilde{S}^k_h(I_y) \cap H^1(I_y)$ defined by

$$(\tilde{P}_h u - u, v)_{L^2(I_y)} = 0 , \quad \forall v \in \tilde{S}^k_h(I_y) \cap H^1(I_y) .$$

Then for any $u \in H^s(I_y)$ (see [13]),

$$\| \tilde{P}_h u - u \|_{H^\mu(I_y)} \leqslant c h^{\bar{s} - \mu} \| u \|_{H^{\bar{s}}(I_y)} , \quad 0 \leqslant \mu \leqslant 1 , \quad \mu \leqslant \bar{s} .$$

Thus we have

$$|\tilde{u}|_{W^{1,p}_\omega(\Omega)} = |P^*_{N,h}\, u^*|_{W^{1,p}_\omega(\Omega)}$$

$$\leqslant c\sqrt{\frac{N}{h}}\,(\,|P^*_{N,h}\, u^* - u^*|_{1,\omega} + \|(\vartheta - P_N \tilde{P}_h)\,\partial_y u^*\|_\omega$$

$$+ \|(\vartheta - P_N \tilde{P}_h)\,\partial_y u^*\|_\omega\,) + \|P_N \tilde{P}_h\,\partial_x u^*\|_{L^p_\omega(\Omega)}$$

$$+ \|P_N \tilde{P}_h\,\partial_y u^*\|_{L^p_\omega(\Omega)}\,.$$

By Lemma 7, (9.5.7) of [5] and $H^1_\omega(\Omega) \hookrightarrow L^p_\omega(\Omega)$ for $p < \infty$, we get

$$|\tilde{u}|_{W^{1,p}_\omega(\Omega)} \leqslant c\|u^*\|_{2,\omega} + c(\,\|\tilde{P}_h\,\partial_x u^*\|_{1,\omega} + \|\tilde{P}_h\,\partial_y u^*\|_{1,\omega})$$

$$\leqslant c\|u^*\|_{2,\omega} \leqslant c\|g\|_\omega\,,$$

which gives the second conclusion. We can derive the third one in the same way. Indeed,

$$|u|_{1,\infty} \leqslant c\sqrt{\frac{N}{h}}\,|u - \tilde{u}|_{1,\omega} + |\tilde{u}|_{1,\infty}\,,$$

$$|\tilde{u}|_{1,\infty} \leqslant c\|u^*\|_{2,\omega} + c|\ln N|^{1/2}(\,\|\tilde{P}_h\,\partial_x u^*\|_{1,\omega} + \|\tilde{P}_h\,\partial_y u^*\|_{1,\omega})$$

$$\leqslant c|\ln N|^{1/2}\|g\|_\omega\,.$$

LEMMA 15 (Lemma 4.16 of [2]) : *Suppose that*

(i) b_0, b_1, b_2 *and* ρ *are non-negative constants,*

(ii) $E(t)$ *is a non-negative function defined on* R_τ,

(iii) $H(z)$ *is a function such that if* $0 \leqslant z \leqslant b_2$, *then* $H(z) \leqslant z$,

(iv) $E(0) \leqslant \rho$ *and for* $t \in \dot{R}_\tau$,

$$E(t) \leqslant \rho + \tau \sum_{t'=0}^{t-\tau} (b_0 E(t') + b_1 E^2(t') + H(E(t')))\,,$$

(v) *for some* $t_1 \in R_\tau$, $\rho e^{2 b_0 t_1} \leqslant \min\left(\frac{b_0}{b_1}, b_2\right)$.

Then for all $t \in R_\tau$ *and* $t \leqslant t_1$, *we have* $E(t) \leqslant \rho e^{2 b_0 t_1}$.

5. THE GENERALIZED STABILITY

We now consider the generalized stability of scheme (2.3). Assume that the initial values $\eta(0)$ and $\eta(\tau)$ have the errors $\tilde{\eta}(0)$ and $\tilde{\eta}(\tau)$, while the right sides of the first and second formulas of (2.3) have the errors $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ respectively. They induce the errors of η and ϕ, denoted by $\tilde{\eta}$ and $\tilde{\phi}$. Then

$$\begin{cases} (\tilde{\eta}_{\hat{t}} + J_c(\hat{\eta}, \tilde{\phi}) + J_c(\hat{\tilde{\eta}}, \phi + \tilde{\phi}), v)_{N,h,\omega} & \\ + \nu a_{N,h,\omega}(\hat{\tilde{\eta}}, v) = (\tilde{f}_1, v)_{N,h,\omega}, & \forall v \in W^k_{N,h}(\Omega), t \in \dot{R}_\tau, \\ a_{N,h,\omega}(\tilde{\phi}, w) = (\tilde{\eta} + \tilde{f}_2, w)_{N,h,\omega}, & \forall w \in W^{k+\lambda}_{N,h}(\Omega), t \in R_\tau . \end{cases} \tag{5.1}$$

For describing the errors, let

$$E(\tilde{\eta}, t) = \| \tilde{\eta}(t) \|^2_\omega + \frac{\nu\tau}{2} \sum_{t'=\tau}^{t-\tau} \| \hat{\tilde{\eta}}(t') \|^2_{1,\omega},$$

$$\rho(t) = 5 \| \tilde{\eta}(0) \|^2_\omega + 4 \| \tilde{\eta}(\tau) \|^2_\omega + 4\tau \sum_{t'=\tau}^{t-\tau} G_1(t')$$

with

$$G_1(t) = c \| \tilde{f}_1(t) \|^2_\omega + \frac{c}{\nu} \| \eta(t) \|^2_\infty \| \tilde{f}_2(t) \|^2_\omega .$$

Besides, let $|||z|||_{q,\infty} = \max_{t \in R_\tau} \| z(t) \|_{q,\infty}$, and $|||z|||_\infty = |||z|||_{0,\infty}$, etc.

THEOREM 1 : *We consider scheme* (2.3), *and let τ be suitably small. Also let $\tau |||\phi|||_{1,\infty} \leqslant c_0 \nu$ and $\| \tilde{f}_2(t) \|^2_\omega \leqslant c_1$, c_0 and c_1 being small positive constants. Then there are positive constants M_1 and M_2 depending only on $|||\eta|||_\infty$, $|||\phi|||_{1,\infty}$ and ν such that if $\rho(t_1)\, e^{M_1 t_1} \leqslant M_2$ for some $t_1 \in R_\tau$, then for all $t \in R_\tau$ and $t \leqslant t_1$, we have $E(\tilde{\eta}, t) \leqslant \rho e^{M_1 t}$.*

Proof: By taking $v = 2\hat{\tilde{\eta}}$ in the first formula of (5.1), we have from Lemma 2 that

$$(\| \tilde{\eta} \|^2_{N,h,\omega})_{\hat{t}} + \frac{\nu}{2} \| \hat{\tilde{\eta}} \|^2_{1,\omega} + \sum_{l=1}^{3} F_l \leqslant \frac{1}{4} \| \hat{\tilde{\eta}} \|^2_{N,h,\omega} + 4 \| \tilde{f}_1 \|^2_{N,h,\omega} \tag{5.2}$$

where $F_l = F_l(t)$ and

$$F_1 = 2(J_c(\hat{\eta}, \tilde{\phi}), \hat{\tilde{\eta}})_{N,h,\omega}, \quad F_2 = 2(J_c(\hat{\tilde{\eta}}, \phi), \hat{\tilde{\eta}})_{N,h,\omega},$$

$$F_3 = 2(J_c(\hat{\tilde{\eta}}, \tilde{\phi}), \hat{\tilde{\eta}})_{N,h,\omega} .$$

On the other hand, we have from the second formula of (5.1) and Lemma 14 that

$$\|\tilde{\phi}\|_{1,\omega}^2 \leq c(\|\tilde{\eta}\|_\omega^2 + \|\tilde{f}_2\|_\omega^2), \quad |\tilde{\phi}|_{W_\omega^{1,4}(\Omega)}^2 \leq c(\|\tilde{\eta}\|_\omega^2 + \|\tilde{f}_2\|_\omega^2). \tag{5.3}$$

Next, we estimate $|F_l|$ ($l = 1, 2, 3$). By Lemma 12 and (5.3), we obtain that

$$|F_1| \leq c\|\hat{\eta}\|_\infty \|\tilde{\phi}\|_{1,\omega}^2 \|\hat{\tilde{\eta}}\|_{1,\omega} \leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|_{1,\omega}^2 + \frac{c}{\nu}\|\hat{\eta}\|_\infty^2(\|\tilde{\eta}\|_\omega^2 + \|\tilde{f}_2\|_\omega^2),$$

$$|F_2| \leq c\|\phi\|_{1,\infty}\|\hat{\tilde{\eta}}\|_\omega \|\hat{\tilde{\eta}}\|_{1,\omega} \leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|_{1,\omega}^2 + \frac{c}{\nu}\|\phi\|_{1,\infty}^2\|\hat{\tilde{\eta}}\|_\omega^2.$$

Moreover, Lemma 13 and (5.3) lead to

$$\begin{aligned} |F_3| &\leq c\|\hat{\tilde{\eta}}\|_{L_\omega^4(\Omega)} |\tilde{\phi}|_{W_\omega^{1,4}(\Omega)} \|\hat{\tilde{\eta}}\|_{1,\omega} \\ &\leq c\|\hat{\tilde{\eta}}\|_{1,\omega} |\tilde{\phi}|_{W_\omega^{1,4}(\Omega)} \|\hat{\tilde{\eta}}\|_{1,\omega} \\ &\leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|_{1,\omega}^2 + \frac{c}{\nu}(\|\tilde{\eta}\|_\omega^2 + \|\tilde{f}_2\|_\omega^2)\|\hat{\tilde{\eta}}\|_{1,\omega}^2. \end{aligned}$$

By substituting the above three inequalities into (5.2) and using Lemma 1 again, we get

$$(\|\tilde{\eta}\|_{N,h,\omega}^2)_{\hat{t}} + \frac{\nu}{8}\|\hat{\tilde{\eta}}\|_{1,\omega}^2 \leq d_0\|\hat{\tilde{\eta}}\|_\omega^2 + d_1\|\tilde{\eta}\|_\omega^2 + d_2(\tilde{\eta}, \tilde{f}_2)\|\hat{\tilde{\eta}}\|_{1,\omega}^2 + G_1 \tag{5.4}$$

with

$$d_0 = \frac{1}{2} + \frac{c}{\nu}\left|\left|\left|\phi\right|\right|\right|_{1,\infty}^2, \quad d_1 = \frac{c}{\nu}\left|\left|\left|\eta\right|\right|\right|_\infty^2,$$

$$d_2(\tilde{\eta}, \tilde{f}_2) = -\frac{3}{16}\nu + \frac{c}{\nu}(\|\tilde{\eta}(t)\|_\omega^2 + \|\tilde{f}_2(t)\|_\omega^2).$$

By summing (5.4) for $t \in \dot{R}_\tau$, we get that

$$\begin{aligned} &\|\tilde{\eta}(t)\|_{N,h,\omega}^2 + \|\tilde{\eta}(t-\tau)\|_{N,h,\omega}^2 + \frac{\nu\tau}{4}\sum_{t'=\tau}^{t-\tau}\|\hat{\tilde{\eta}}(t')\|_{1,\omega}^2 \\ &\leq \|\tilde{\eta}(0)\|_{N,h,\omega}^2 + \|\tilde{\eta}(\tau)\|_{N,h,\omega}^2 \\ &\quad + 2\tau\sum_{t'=\tau}^{t-\tau}(d_0\|\hat{\tilde{\eta}}(t')\|_\omega^2 + d_1\|\tilde{\eta}(t')\|_\omega^2 \\ &\quad + d_2(\tilde{\eta}(t'), \tilde{f}_2(t'))\,\|\hat{\tilde{\eta}}(t')\|_{1,\omega}^2 + G_1(t')). \end{aligned}$$

Let τ be suitably small. Then by Lemma 1 and the fact that

$$\|\hat{\tilde{\eta}}(t)\|_{\omega}^{2} \leq \frac{1}{2}\|\tilde{\eta}(t+\tau)\|_{\omega}^{2}+\frac{1}{2}\|\tilde{\eta}(t-\tau)\|_{\omega}^{2},$$

we obtain

$$E(\tilde{\eta}, t) \leq \rho(t)+4 \tau \sum_{t'=\tau}^{t-\tau}\left(\left(d_{0}+d_{1}\right) E(\tilde{\eta}, t')+d_{2}(\tilde{\eta}(t'), \tilde{f}_{2}(t'))\|\hat{\tilde{\eta}}(t')\|_{1, \omega}^{2}\right).$$

Finally we apply Lemma 15 to completing the proof.

Next, we consider the generalized stability of scheme (2.4). Note that the difference between (2.3) and (2.4) only lies in the nonlinear term. Thus we have the same equation as (5.2), but with

$$F_{1}=2(J_{c}(\eta, \tilde{\phi}), \hat{\tilde{\eta}})_{N, h, \omega}, \quad F_{2}=2(J_{c}(\tilde{\eta}, \phi), \hat{\tilde{\eta}})_{N, h, \omega},$$

$$F_{3}=2(J_{c}(\tilde{\eta}, \tilde{\phi}), \hat{\tilde{\eta}})_{N, h, \omega}.$$

We estimate $|F_1|$ and $|F_2|$ as before. We have from Lemma 13 and Lemma 14 that

$$\begin{aligned}|F_{3}| &\leq c\|\tilde{\eta}\|_{\omega}|\tilde{\phi}|_{1, \infty}\|\hat{\tilde{\eta}}\|_{1, \omega} \\ &\leq \frac{\nu}{16}\|\hat{\tilde{\eta}}\|_{1, \omega}^{2}+\frac{c \ln N}{\nu}\left(\|\tilde{\eta}\|_{\omega}^{2}+\|\tilde{f}_{2}\|_{\omega}^{2}\right)\|\tilde{\eta}\|_{\omega}^{2}.\end{aligned}$$

Following the same line as in the proof of Theorem 1 and using the same notations, we get the following result.

THEOREM 2 : *We consider* (2.4), *and let* τ *be suitably small. Also let* $\|\tilde{f}_2(t)\|_{\omega}^2 \leq \frac{c_0}{\ln N}$, c_0 *being a small positive constant. Then there are positive constants* M_1 *and* M_2 *depending only on* $|||\eta|||_{\infty}$, $|||\phi|||_{1,\infty}$ *and* ν *such that if* $\rho(t_1)\, e^{M_1 t_1} \leq \frac{M_2}{\ln N}$ *for some* $t_1 \in R_\tau$, *then for all* $t \in R_\tau$ *and* $t \leq t_1$, *we have* $E(\tilde{\eta}, t) \leq \rho e^{M_1 t}$.

Remark 2 : By Theorem 1 and Theorem 2, we find that for the stability of the fully implicit scheme (2.3), we only require that $\rho(t_1)\, e^{M_1 t_1} \leq M_2$. But for the scheme (2.4), we require that $\rho(t_1)\, e^{M_1 t_1} \leq \frac{M_2}{\ln N}$. Therefore the fully implicit treatment increases the generalized stability.

6. THE CONVERGENCE

In this section, we first deal with the convergence of scheme (2.3). We define $\tilde{P}^c_{N,h} : H^1_{0,\omega}(\Omega) \to W^{k+\lambda}_{N,h}(\Omega)$ such that for any $u \in H^1_{0,\omega}(\Omega)$,

$$a_{N, h, \omega}(\tilde{P}_{N, h}^{c} u, w)=a_{\omega}(u, w), \quad \forall w \in W_{N, h}^{k+\lambda}(\Omega).$$

Let $\xi^* = P^c_{N,h}\,\xi$, $\psi^* = \tilde{P}^c_{N,h}\,\psi$, $\tilde{\xi} = \eta - \xi^*$ and $\tilde{\psi} = \phi - \psi^*$. Then we have from (2.1) and (2.3) that

$$\begin{cases} (\tilde{\xi}_{\hat{t}} + J_c(\hat{\xi}^*, \tilde{\psi}) + J_c(\hat{\tilde{\xi}}, \psi^* + \tilde{\psi}), v)_{N,h,\omega} \\ \quad + \nu a_{N,h,\omega}(\hat{\tilde{\xi}}, v) = \sum_{j=1}^{4} A_j, & \forall v \in W^k_{N,h}(\Omega), \\ a_{N,h,\omega}(\tilde{\psi}, w) = (\tilde{\xi}, w)_{N,h,\omega} + A_5, & \forall w \in W^{k+\lambda}_{N,h}(\Omega), \\ \tilde{\xi}(\tau) = P_{N,h}(\xi_0 + \tau\,\partial_t\xi(0)) - P^c_{N,h}\,\xi(\tau), \\ \tilde{\xi}(0) = P_{N,h}\,\xi_0 - P^c_{N,h}\,\xi_0, \end{cases} \tag{6.1}$$

where $A_j = A_j(t)$ and

$$A_1 = (\partial_t\xi, v)_\omega - (\xi^*_{\hat{t}}, v)_{N,h,\omega}, \qquad A_2 = (J(\xi, \psi), v)_\omega - (J_c(\hat{\xi}^*, \psi^*), v)_{N,h,\omega},$$
$$A_3 = -\nu(\nabla^2\xi, v)_\omega + \nu(\nabla^2\hat{\xi}, v)_\omega, \quad A_4 = -(f, v)_\omega + (f, v)_{N,h,\omega},$$
$$A_5 = -(\xi, w)_\omega + (\xi^*, w)_{N,h,\omega}.$$

Let $\tilde{s} = \bar{s}$ for $\lambda = 1$, and $\tilde{s} = \overline{s+1} - 1$ for $\lambda = 0$.

THEOREM 3 : *Let (ξ, ψ) and (η, ϕ) be the solutions of* (2.1) *and* (2.3) *respectively. Assume that*
(i) *for* $r,\ s \geq 1,\ \alpha,\ \beta > \frac{1}{2}$ *and* $\delta > 0$,

$$\xi \in C(0,T; M^{r,\bar{s}}_\omega(\Omega) \cap X^{\alpha,\beta}_\omega(\Omega) \cap Y^{r,0,\delta}_\omega(\Omega)) \cap H^1(0,T; M^{r,\bar{s}}_\omega(\Omega))$$
$$\cap\, H^2(0,T; H^1_\omega(\Omega)) \cap H^3(0,T; L^2_\omega(\Omega)),$$
$$\psi \in C(0,T; M^{r+1,\bar{s}+1}_\omega(\Omega) \cap X^{\alpha,\beta}_\omega(\Omega) \cap Y^{r+1,0,\delta}_\omega(\Omega) \cap Y^{r,1,\delta}_\omega(\Omega)),$$
$$f \in C(0,T; H^{r,0}_\omega(\Omega)),$$

(ii) τ, N^{-1} *are suitably small.*
Then for some $t_1 \leq T$ *and all* $t \leq t_1$,

$$\|\xi(t) - \eta(t)\|_\omega \leq M_3(\tau^2 + N^{-r} + h^{\tilde{s}}),$$

M_3 being a positive constant depending only on ν and the norms of ξ, ψ and f in the spaces mentioned above. If in addition τ, N^{-1} are small enough, then $t_1 = T$.

Proof: For the convergence, we have to estimate the right terms in (6.1). Firstly,

$$|A_1| \leqslant |(\xi_{\bar{t}}^*, v)_{N,h,\omega} - (\xi_{\bar{t}}^*, v)_\omega| + |(\xi_{\bar{t}}^* - \xi_{\bar{t}}, v)_\omega| + |(\xi_{\bar{t}} - \partial_t \xi, v)_\omega| .$$

Since

$$|(\xi_{\bar{t}}^*, v)_{N,h,\omega} - (\xi_{\bar{t}}^*, v)_\omega| \leqslant c(\|\xi_{\bar{t}}^* - \xi_{\bar{t}}\|_\omega + \|(\vartheta - P_{N-1})\, \xi_{\bar{t}}\|_\omega)\, \|v\|_\omega ,$$

we have from Lemma 7 that

$$\begin{aligned}|A_1| &\leqslant c\|v\|_\omega(\|\xi_{\bar{t}}^* - \xi_{\bar{t}}\|_\omega + \|(\vartheta - P_{N-1})\, \xi_{\bar{t}}\|_\omega + \tau^{3/2} \|\xi\|_{H^3(t-\tau, t+\tau\,;\, L^2_\omega(\Omega))}) \\ &\leqslant c\tau^{-1/2}(N^{-\tau} + h^{\bar{s}})\|v\|_\omega \|\xi\|_{H^1(t-\tau, t+\tau\,;\, M^{r,\bar{s}}_\omega(\Omega))} \\ &\quad + c\tau^{3/2}\|v\|_\omega \|\xi\|_{H^3(t-\tau, t+\tau\,;\, L^2_\omega(\Omega))} .\end{aligned}$$

It is complicated to estimate $|A_2|$. Indeed $A_2 = \sum_{j=1}^{5} B_j$ where

$$\begin{aligned}B_1 &= (J(\xi, \psi) - J(\hat{\xi}, \psi), v)_\omega , & B_2 &= (J(\hat{\xi}, \psi) - J_c(\hat{\xi}, \psi), v)_\omega , \\ B_3 &= (J_c(\hat{\xi}, \psi), v)_\omega - (J_c(\hat{\xi}, \psi), v)_{N,h,\omega} , & B_4 &= (J_c(\hat{\xi} - \hat{\xi}^*, \psi), v)_{N,h,\omega} , \\ B_5 &= (J_c(\hat{\xi}^*, \psi - \psi^*), v)_{N,h,\omega} .\end{aligned}$$

Obviously by Lemma 11,

$$\begin{aligned}|B_1| &\leqslant c\|v\|_{1,\omega}\|\psi\|_{1,\infty}\|\xi - \hat{\xi}\|_\omega \\ &\leqslant c\tau^{3/2}\|v\|_{1,\omega}\|\psi\|_{1,\infty}\|\xi\|_{H^2(t-\tau, t+\tau\,;\, L^2_\omega(\Omega))} .\end{aligned}$$

By Lemma 5, Lemma 9 and Lemma 10, we have that for $r > \frac{1}{2}$,

$$\begin{aligned}|B_2| &\leqslant c\|v\|_{1,\omega}(\|(\vartheta - P_c)\, (\hat{\xi}\, \partial_y \psi)\|_\omega + \|(\vartheta - P_c)\, (\hat{\xi}\, \partial_x \psi)\|_\omega) \\ &\leqslant cN^{-r}\|v\|_{1,\omega}(\|\hat{\xi}\, \partial_y \psi\|_{H^{r,0}_\omega(\Omega)} + \|\hat{\xi}\, \partial_x \psi\|_{H^{r,0}_\omega(\Omega)}) \\ &\leqslant cN^{-r}\|v\|_{1,\omega}\|\hat{\xi}\|_{Y^{r,0,\delta}_\omega(\Omega)}\|\psi\|_{Y^{r+1,0,\delta}_\omega(\Omega) \cap Y^{r,1,\delta}_\omega(\Omega)} .\end{aligned}$$

Next, for any y, $\partial_x P_c(\hat{\xi}\,\partial_y \psi) \in \mathscr{P}_{N-1}(I_x)$. Hence by (4.7),

$$B_3 = (\partial_x P_c(\hat{\xi}\,\partial_y \psi), v)_\omega - (\partial_y P_c(\hat{\xi}\,\partial_x \psi), v)_\omega - (\partial_x P_c(\hat{\xi}\,\partial_y \psi), v)_{N,h,\omega}$$
$$+ (\partial_y P_c(\hat{\xi}\,\partial_x \psi), v)_{N,h,\omega}$$
$$= (P_c(\hat{\xi}\,\partial_x \psi), \partial_y v)_\omega - (P_c(\hat{\xi}\,\partial_x \psi), \partial_y v)_{N,h,\omega}\,.$$

Then Lemma 1, Lemma 5 and Lemma 9 lead to that for $r > \frac{1}{2}$,

$$|B_3| \leqslant c \| v \|_{1,\omega} \| (\vartheta - P_{N-1})\, P_c(\hat{\xi}\,\partial_x \psi) \|_\omega$$
$$\leqslant c \| v \|_{1,\omega} (\| (\vartheta - P_c)(\hat{\xi}\,\partial_x \psi) \|_\omega + \| (\vartheta - P_{N-1})(\hat{\xi}\,\partial_x \psi) \|_\omega)$$
$$\leqslant c N^{-r} \| v \|_{1,\omega} \| \hat{\xi}\,\partial_x \psi \|_{H^{r,0}_\omega(\Omega)}$$
$$\leqslant c N^{-r} \| v \|_{1,\omega} \| \hat{\xi} \|_{Y^{r,0,\delta}_\omega(\Omega)} \| \psi \|_{Y^{r+1,0,\delta}_\omega(\Omega)}\,.$$

Since $v \in W^k_{N,h}(\Omega)$, we have from (4.7), Lemma 1, Lemma 5 and Lemma 10 that

$$|B_4| = |(\partial_x P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_y \psi), v)_{N,h,\omega} - (\partial_y P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_x \psi), v)_{N,h,\omega}|$$
$$\leqslant |(P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_y \psi), \partial_x(\omega v))_{L^2(\Omega)}|$$
$$+ |(P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_x \psi), \partial_y v)_{N,h,\omega}|$$
$$\leqslant \| v \|_{1,\omega} (\| P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_y \psi) \|_{N,h,\omega} + \| P_c((\hat{\xi} - \hat{\xi}^*)\,\partial_x \psi) \|_{N,h,\omega})$$
$$\leqslant c \| v \|_{1,\omega} | \psi |_{1,\infty} \| P_c(\hat{\xi} - \hat{\xi}^*) \|_\omega$$
$$\leqslant c \| v \|_{1,\omega} | \psi |_{1,\infty} (\| P_c\,\hat{\xi} - \hat{\xi} \|_\omega + \| \hat{\xi} - \hat{\xi}^* \|_\omega)$$
$$\leqslant c(N^{-r} + h^{\bar{s}})\, \| v \|_{1,\omega} \| \psi \|_{1,\infty} \| \hat{\xi} \|_{M^{r,\bar{s}}_\omega(\Omega)}\,.$$

We now estimate $|B_5|$. By an argument as in the previous paragraph, we have

$$|B_5| \leqslant c \| v \|_{1,\omega} \| \hat{\xi}^* \|_\infty (\| P_c\,\partial_y \psi - \partial_y \psi^* \|_\omega + \| P_c\,\partial_x \psi - \partial_x \psi^* \|_\omega)$$
$$\leqslant c \| v \|_{1,\omega} \| \hat{\xi}^* \|_\infty (\| P_c\,\partial_y \psi - \partial_y \psi \|_\omega + \| \partial_y \psi - \partial_y \psi^* \|_\omega$$
$$+ \| P_c\,\partial_x \psi - \partial_x \psi \|_\omega + \| \partial_x \psi - \partial_x \psi^* \|_\omega)\,.$$

By Lemma 7, we have that for $\lambda = 1$,

$$\| \partial_x \psi - \partial_x \psi^* \|_\omega + \| \partial_y \psi - \partial_y \psi^* \|_\omega \leqslant c(N^{-r} + h^{\bar{s}}) \, \| \psi \|_{M_\omega^{r+1, \bar{s}+1}(\Omega)},$$

and for $\lambda = 0$,

$$\| \partial_x \psi - \partial_x \psi^* \|_\omega + \| \partial_y \psi - \partial_y \psi^* \|_\omega \leqslant c(N^{-r} + h^{\overline{s+1}-1}) \| \psi \|_{M_\omega^{r+1, \overline{s+1}}(\Omega)} .$$

Thus by Lemma 8,

$$|B_5| \leqslant c\lambda(N^{-r} + h^{\bar{s}}) \, \| v \|_{1,\omega} \| \hat{\xi} \|_{X_\omega^{\alpha,\beta}(\Omega)} \| \psi \|_{M_\omega^{r+1, \bar{s}+1}(\Omega)}$$

$$+ c(1 - \lambda)(N^{-r} + h^{\overline{s+1}-1}) \, \| v \|_{1,\omega} \| \hat{\xi} \|_{X_\omega^{\alpha,\beta}(\Omega)} \| \psi \|_{M_\omega^{r+1, \overline{s+1}}(\Omega)} .$$

Now we estimate $|A_3|$. We have

$$\begin{aligned} |A_3| &= \nu | (\partial_x(\xi - \hat{\xi}), \partial_x(\omega v))_{L_\omega^2(\Omega)} + (\partial_y(\xi - \hat{\xi}), \partial_y v)_\omega | \\ &\leqslant c \| v \|_{1,\omega} \| \xi - \hat{\xi} \|_{1,\omega} \\ &\leqslant c\tau^{3/2} \| v \|_{1,\omega} \| \xi \|_{H^2(t-\tau, t+\tau ; H_\omega^1(\Omega))} . \end{aligned}$$

We have from Lemma 1 that for $r > 1/2$,

$$|A_4| \leqslant cN^{-r} \| v \|_\omega \| f \|_{H_\omega^{r,0}(\Omega)} .$$

We have also that

$$\begin{aligned} |A_5| &\leqslant |(\xi - \xi^*, w)_\omega| + |(\xi^*, w)_\omega - (\xi^*, w)_{N,h,\omega}| \\ &\leqslant c \| w \|_\omega (\| \xi - \xi^* \|_\omega + \| (\vartheta - P_{N-1}) \xi^* \|_\omega) \\ &\leqslant c(N^{-r} + h^{\bar{s}}) \, \| w \|_\omega \| \xi \|_{M_\omega^{r,\bar{s}}(\Omega)} . \end{aligned}$$

Finally by Lemma 2 and Lemma 7,

$$\| \tilde{\xi}(0) \|_\omega \leqslant c(N^{-r} + h^{\bar{s}}) \| \xi_0 \|_{M_\omega^{r,\bar{s}}(\Omega)} ,$$

$$\begin{aligned} \| \tilde{\xi}(\tau) \|_\omega &\leqslant \| P_{N,h} \xi(\tau) - P_{N,h}^c \xi(\tau) \|_\omega + \| \xi(\tau) - \xi_0 - \tau \, \partial_t \xi(0) \|_\omega \\ &\leqslant c(N^{-r} + h^{\bar{s}}) \, \| \xi(\tau) \|_{M_\omega^{r,\bar{s}}(\Omega)} + c\tau^{3/2} \| \xi \|_{H^2(0,\tau ; L_\omega^2(\Omega))} . \end{aligned}$$

We now take $v = 2\hat{\hat{\xi}}$ in the first formula of (6.1). Then by an argument as in the proof of Theorem 1, we get that

$$E(\tilde{\xi}, t) \leqslant \tilde{\rho} + c\tau \sum_{t'=\tau}^{t-\tau} ((d_0^* + d_1^*) E(\tilde{\xi}, t') + d_2^*(\tilde{\xi}(t')) \|\tilde{\xi}(t')\|_{1,\omega}^2)$$

where

$$d_0^* = \frac{1}{2} + \frac{c}{\nu} |||\psi^*|||_{1,\infty}^2, \quad d_1^* = \frac{c}{\nu} |||\xi^*|||_{\infty}^2,$$

$$d_2^*(\tilde{\xi}) = -\frac{3}{16}\nu + \frac{c}{\nu}(\|\tilde{\xi}\|_{\omega}^2 + (N^{-r} + h^{\bar{s}}) \|\xi\|_{M_{\omega}^{r,\bar{s}}(\Omega)}^2),$$

$$E(\tilde{\xi}, t) = \|\tilde{\xi}(t)\|_{\omega}^2 + \frac{\nu\tau}{2} \sum_{t'=\tau}^{t-\tau} \|\hat{\tilde{\xi}}(t')\|_{1,\omega}^2,$$

$$\tilde{\rho} = c(N^{-r} + h^{\bar{s}})^2 \Big(\|\xi\|_{H^1(0,T;M_{\omega}^{r,\bar{s}}(\Omega))}^2 + |||\xi|||_{Y_{\omega}^{r,0,\delta}(\Omega)}^2 |||\psi|||_{Y_{\omega}^{r+1,0,\delta}(\Omega) \cap Y_{\omega}^{r,1,\delta}(\Omega)}^2$$

$$+ |||\xi|||_{M_{\omega}^{r,\bar{s}}(\Omega)}^2 \big(1 + |||\psi|||_{1,\infty}^2\big) + \lambda |||\xi|||_{X_{\omega}^{\alpha,\beta}(\Omega)}^2 |||\psi|||_{M_{\omega}^{r+1,\bar{s}+1}(\Omega)}^2$$

$$+ \|f\|_{C(0,T;H_{\omega}^{r,0}(\Omega))}^2 \Big)$$

$$+ c(1-\lambda)(N^{-r} + h^{\overline{s+1}-1})^2 |||\xi|||_{X_{\omega}^{\alpha,\beta}(\Omega)}^2 |||\psi|||_{M_{\omega}^{r+1,\overline{s+1}}(\Omega)}^2$$

$$+ c\tau^4 \Big(\|\xi\|_{H^2(0,T;H_{\omega}^1(\Omega))}^2 \big(1 + |||\psi|||_{1,\infty}^2\big) + \|\xi\|_{H^3(0,T;L_{\omega}^2(\Omega))}^2 \Big).$$

If $k \geqslant 1$ and $r, s \geqslant 1$, then $\bar{s} \geqslant 1$ and $\overline{s+1} - 1 \geqslant 1$. Thus by (4.8), $\tilde{\rho} \leqslant c(N^{-1} + \tau^4)^2$. Now let $E(t) = E(\tilde{\xi}, t)$ and $\rho = \tilde{\rho}$ in Lemma 15. If τ and N^{-1} are suitably small, then for certain $t_1 \leqslant T$ and all $t \leqslant t_1$, $\tilde{\rho}e^{\tilde{b}t} \leqslant \tilde{c}$, $\tilde{b}$ and $\tilde{c}$ being some constants. Finally the first conclusion follows from Lemma 15. If τ and N^{-1} are small enough, then for all $t \leqslant T$, $\tilde{\rho}e^{\tilde{b}t} \leqslant \tilde{c}$. Hence we can take $t_1 = T$.

Remark 3 : The hybrid finite element approximation ($\lambda = 1$) raises the accuracy and provides the optimal convergence rate. Indeed

$$\|\xi(t) - \eta(t)\|_{\omega} = \begin{cases} O(\tau^2 + N^{-r} + h^{\bar{s}}), & \text{for } \lambda = 1, \\ O(\tau^2 + N^{-r} + h^{\overline{s+1}-1}), & \text{for } \lambda = 0. \end{cases}$$

If $s \geq k+1$, then

$$\| \xi(t) - \eta(t) \|_{\omega} = \begin{cases} O(\tau^2 + N^{-r} + h^{k+1}), & \text{for } \lambda = 1, \\ O(\tau^2 + N^{-r} + h^k), & \text{for } \lambda = 0. \end{cases}$$

We next consider the convergence of scheme (2.4). We have the error equation similar to (6.1), but the first one becomes

$$(\tilde{\xi}_{\hat{t}} + J_c(\xi^*, \tilde{\psi}) + J_c(\tilde{\xi}, \psi^* + \tilde{\psi}), v)_{N,h,\omega} + \nu a_{N,h,\omega}(\hat{\tilde{\xi}}, v) = \sum_{j=1}^{4} A_j,$$

$$\forall v \in W^k_{N,h}(\Omega)$$

where A_1, A_3 and A_4 are the same as in (6.1), but A_2 should be modified as

$$A_2 = (J(\xi, \psi), v)_{\omega} - (J_c(\xi^*, \psi^*), v)_{N,h,\omega}.$$

By an argument as in the proof of Theorem 3, we can get the following result.

THEOREM 4 : *Let* (ξ, ψ) *and* (η, ϕ) *be the solutions of* (2.1) *and* (2.4) *respectively. Assume that condition* (i) *of Theorem 3 holds and* $\tau = O((\ln N)^{-1/4})$. *Then for some* $t_1 \leq T$ *and all* $t \leq t_1$,

$$\| \xi(t) - \eta(t) \|_{\omega} \leq M_3(\tau^2 + N^{-r} + h^{\bar{s}}),$$

M_3 *being a positive constant depending only on* ν *and the norms of* ξ, ψ *and* f *as in Theorem 3. If in addition* $\tau = o((\ln N)^{-1/4})$, *then* $t_1 = T$.

Remark 4 : We know from Theorem 3 and Theorem 4 that the fully implicit scheme (2.3) not only possesses better generalized stability than (2.4) (see Remark 2), but also weakens the restriction on τ for the convergence.

Remark 5 : If the term $(f, v)_{N,h,\omega}$ in (2.3) or (2.4) is replaced by $(\Pi^k_h f, v)_{N,h,\omega}$ and take

$$\eta(0) = \Pi^k_h P_c \xi_0, \quad \eta(\tau) = \Pi^k_h P_c(\xi_0 + \tau \partial_t \xi(0)),$$

then we can use Lemma 6 to get similar results as in Theorem 3 or Theorem 4.

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