M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 30, n° 5 (1996), p. 575-605
<http://www.numdam.org/item?id=M2AN_1996__30_5_575_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 30, n° 5, 1996, p. 575 à 605)

STATIONARY VOLTAGE CURRENT CHARACTERISTICS OF A PLASMA (*)

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Abstract. — *In this paper a one dimensional, relativistic model of a two component mono energetic plasma of electrons and ions is investigated. Given an applied electric potential* Φ_A *the electron and ion currents* j_e *and* j_i *are calculated. The analysis is based on the associated Lagrange functional. The model equation is a nonlinear ODE of order two in terms of the electric potential* ϕ. *Electron and ion current are unknown parameters. This ODE is subject to four boundary conditions. It is shown that infinitely many solutions* (ϕ, j_e, j_i) *exist but only one distinguished solution minimizes the Lagrange functional. The asymptotic voltage-current-characteristics in non relativistic, relativistic and ultra relativistic settings are computed and rigorously justified.*

Key words : relativistic particles in plasma, (non) (ultra) relativistic Lagrange function of motion in an electric field, nonlinear ODE of order two, two point boundary value problem, asymptotics of nonlinear ODE of order two
AMS(MOS) subject classification. 34B10, 34B15, 34E10, 70H35, 70H40.

Résumé. — *Un modèle relativiste unidimensionnel pour un plasma monoénergétique, composé d'ions et d'électrons, est étudié. Étant donné une différence de potentiel* Φ_A, *les courants électronique* j_e *et ionique* j_i *sont déterminés. L'analyse repose sur la minimisation de la fonctionnelle de Lagrange associée au système. Les équations d'Euler associées à cette fonctionnelle consistent en une équation différentielle semi-linéaire du second ordre, avec quatre conditions aux limites. On montre l'existence d'une infinité de solutions et l'unicité d'une solution minimisant la fonctionnelle. L'étude asymptotique de la caractéristique courant-tension est faite dans le cadre non relativiste et ultra relativiste.*

1. INTRODUCTION

The aim of this paper is to analyze a one-dimensional, mono energetic model of a two component, collisionless plasma of electrons and ions.

(*) Manuscript received February 2, 1995 ; revised January 4, 1996.
Part of this work was done while the first author was member of the Centre de mathématiques Appliquées of the Ecole Polytechnique and the second Author acknowledges support from EC-network, contract # ERBCHRXCT 930413, and from the project entitled « Mathematische Analysis und Numerik von Quantenhydodynamischen Modellen in der Halleiterphysik » funded by the Deutsche Forschungsgemeinschaft.
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We assume the plasma confined to the interval $(0, L)$ with $L > 0$ is in a stationary state.

The main objective of subsequent investigations is the calculation of voltage current characteristics. Given an applied voltage Φ_A we wish to determine corresponding electron and ion currents $j_e[\Phi_A]$, $j_i[\Phi_A]$.

The plasma under consideration can be described by the phase space densities $f_e(x, p)$, $f_i(x, p)$ of electrons and ions by the stationary relativistic Vlasov-Poisson system [8, 9]

$$(1) \qquad \begin{cases} \dfrac{cp}{\sqrt{m_e^2 c^2 + p^2}} \dfrac{\partial f_e}{\partial x} + \dfrac{e}{m_e} \dfrac{d\phi}{dx} \dfrac{\partial f_e}{\partial p} = 0 \,, \\ \dfrac{cp}{\sqrt{m_i^2 c^2 + p^2}} \dfrac{\partial f_i}{\partial x} - \dfrac{Ze}{m_i} \dfrac{d\phi}{dx} \dfrac{\partial f_e}{\partial p} = 0 \,, \\ \dfrac{d^2 \phi}{dx^2} = \dfrac{e}{\epsilon_0} (\rho_e - Z\rho_i) \,. \end{cases}$$

Here $x \in (0, L)$ is the spatial variable and $v \in (-\infty, +\infty)$ is the velocity variable. $c \approx 2.998 \,.\, 10^8 \,\mathrm{ms}^{-1}$ is the speed of light. The respective rest masses m_e and m_i of electrons and ions are approximatively $m_e \approx 9.109 \,.\, 10^{-31}$ kg and $m_i \approx N m_p$ where N is an integer and $m_p \approx 1.673 \,.\, 10^{-27}$ kg is the rest mass of a proton. $e \approx 1.602 \,.\, 10^{-19}$ C is the elementary charge and Ze with integer Z is the positive charge of an ion. The dielectricity constant of vacuum ϵ_0 is approximatively $8.854 \,.\, 10^{-12}\, \mathrm{C}^2\, \mathrm{kg}^{-1}\, \mathrm{m}^{-3}\, \mathrm{s}^2$. In (1) the electric potential is denoted by $\phi(x)$. The densities of electrons and ions, ρ_e, ρ_i are given by

$$\rho_e(x) = \int_{\mathbb{R}} f_e(x, p)\, dp \,, \quad \rho_i(x) = \int_{\mathbb{R}} f_i(x, p)\, dp \,.$$

Stationary and transient (non-) relativistic Vlasov-Poisson (Vlasov-Maxwell) systems have been extensively analyzed within the last fifteen years, see e.g. [17, 16, 9, 1] and the references therein. Vlasov-Poisson or Vlasov-Maxwell systems have found applicable in many areas of mathematical physics such as stellar dynamics ([8, 2]), plasma physics ([10, 11]) and semiconductor physics ([12, 6, 5]).

We are here interested in space charge limited regimes where the flow of particles is monoenergetic and the electric field vanishes at both electrodes. The hypotheses are as listed in [13] :

a) Electrons and ions are assumingly mono energetic. Hence the phase space densities are of the form

$$f_e(x, p) = \rho_e(x)\, \delta(p - g_e(j_e, \phi(x))) \,,$$

$$f_i(x, p) = \rho_i(x, p) = \rho_i(x)\, \delta(p - g_i(j_i, \phi(x))) \,.$$

Electron and ion current are divergence free which gives in one space dimension

$$j_e = e\int v_e(p)f_e(p)\,dp\,,\quad j_i = Ze\int v_i(p)f_i(p)\,dp \text{ constant}$$

where

$$v_e(p) = \frac{cp}{\sqrt{m_e^2c^2+p^2}}\,,\quad v_i(p) = \frac{cp}{\sqrt{m_i^2c^2+p^2}}$$

are the velocities of electrons and ions respectively. Furthermore, particles are assumed to enter plasma with vanishing velocities $p_e(0) = p_i(L) = 0$ (this is the Child-langmuir regime). This leads formally to boundary data

$$(2)\qquad ev_e(p)f_e(0,p) = j_e\delta(p)\,,\quad Zev_i(p)f_i(L,p) = j_i\delta(p)\,.$$

b) Electrons leave plasma at $x=0$, ions leave plasma at $x=L$. Hence $f_e(L,p)=0$ for $p<0$ and $f_i(0,p)=0$ for $p>0$.

c) The electric potential ϕ is assumed to be subject to four (!) boundary conditions

$$\phi(0)=0\,,\quad \phi(L)=\Phi_A\,,$$

$$\phi'(0)=\phi'(L)=0\,.$$

Actually this condition, called in the physical litterature the Space Charge Limited regime condition, determines the value of the currents, and should be obtained via the Child-Langmuir asymptotics of the full Vlasov-Poisson system in the spirit of [5, 6, 3, 7, 4]. The reason why we do not investigate this limit in this paper is that the results we have are preliminary, the difficulty being the lack of information on the shape of the potential (monotonicity). This kind of problems was already noticed in [3] where some the ions are described in a simpler manner (they have a fixed constant density).

Due to the mono energy assumption the respective velocities of electrons and ions assume in dependence on the spatial variable x a single value

$v_e(x)$, $v_i(x)$. Therefore a relativistic non statistical model in terms of particle densities and velocities should be applicable. The model analyzed in the sequel is settled on the Lagrange functional (see [14], p. 55)

$$\mathscr{L}(\rho_e, v_e, \rho_i, v_i, \phi) \equiv \mathscr{L}_{\text{rel}}(\rho_e, v_e) + \mathscr{L}_{\text{rel}}(\rho_i, v_i) + \mathscr{L}_{\text{elstat}}(\rho_e, \rho_i, \phi)$$

$$= \left(\int_0^L - m_e \rho_e c^2 \sqrt{1 - (v_e/c)^2} + m_e \rho_e c^2 \right)$$

$$+ \left(\int_0^L - m_i \rho_i c^2 \sqrt{1 - (v_i/c)2} + m_i \rho_i c^2 \right)$$

$$+ \left(\int_0^L e\rho_e \phi + Ze\rho_i(\Phi_A - \phi) + \frac{\epsilon_0}{2} \int_0^L (\phi')^2 \right).$$

The model equations are constituted by the requirement to minimize this Lagrange functional taking into account boundary conditions a), b), c). As we shall see in Section 2 one gets an autonomous ODE of order two.

$$(3) \qquad \begin{cases} \rho_e = \dfrac{|j_e|}{ec} \dfrac{m_e c^2 + e\phi}{\sqrt{(m_e c^2 + e\phi)^2 - m_e^2 c^4}}, \\ \rho_i = \dfrac{|j_i|}{Zec} \dfrac{m_i c^2 + Ze(\Phi_A - \phi)}{\sqrt{(m_i c^2 + Ze(\Phi_A - \phi))^2 - m_i^2 c^4}} \\ \phi'' = \dfrac{e}{\epsilon_0}(\rho_e - Z\rho_i) \end{cases}$$

subject to the boundary conditions

$$(4) \qquad \begin{cases} \phi(0) = 0, \phi(L) = \Phi_A \\ \phi'(0) = \phi'(L) = 0 \end{cases}$$

with unknown constants j_e and j_i. In Section 3. existence and uniqueness of solutions ϕ in dependence on j_e, j_i will be discussed.

Aside from these general investigations it is important to know asymptotics of $j_e(\Phi_A)$, $j_i(\Phi_A)$ corresponding to different orders of magnitude of Φ_A :

A) Non relativistic case : The rest energy of electrons and ions is large compared to their maximal kinetic energy :

$$\frac{m_e c^2}{e\Phi_A} \equiv r \to \infty, \quad \mu r \equiv \frac{Nm_p}{Zm_e} r = \frac{Nm_p c^2}{Ze\Phi_A} \to \infty .$$

B) Relativistic electrons and non relativistic ions : Increasing the applied potential Φ_A amounts to the necessity of taking into account relativistic effects for electrons. For Φ_A not « too large » the maximal kinetic energy of an electron is of the order of magnitude of its rest energy :

$$r \approx 1 , \quad \mu r \to \infty ,$$

i.e. $\mu \to \infty$ for fixed r. (The ratio $\mu = \frac{N m_p}{Z m_e} \approx 5.10^3$ is rather large.)

C) Ultra relativistic electrons and non relativistic ions : The maximal kinetic energy of an electron exceeds considerably its rest energy while relativistic effects are negligeable for ions. This amounts to

$$r \ll 1 , \quad \mu r \to \infty .$$

A discussion of ultra relativistic electrons and non relativistic ions for $N = Z = 1$ can be found in [13] where the following expressions for j_e, j_i are given :

$$j_e = c \epsilon_0 \frac{\pi^2}{2} \frac{\Phi_A}{L^2} , \quad j_i = \sqrt{\frac{2e}{m_i}} \epsilon_0 \frac{\pi^2}{4} \frac{\Phi_A^{3/2}}{L^2} . \tag{5}$$

Most interesting the current j_i is almost identical with the current $J(\Phi_A)$ of an $N^+ - N - N^+$ diode in case of hot electrons, see [5] :

$$J(\Phi_A) = \sqrt{\frac{2e}{m_e}} \epsilon_0 \frac{4}{9} \frac{\Phi_A^{3/2}}{L^2} .$$

D) Ultra relativistic electrons and relativistic ions : The ions will become relativistic when Φ_A increases. If the maximal kinetic energy of an ion is approximately equal to its rest energy then

$$r \ll 1 , \quad \mu r \ll 1 .$$

The paper is organized as follows.

In Section 2 a derivation of the model equations (3), (4) based on the associated Lagrange functional is given. Section 3 is concerned with ODE's of the form

$$\phi'' = \alpha f_1(\phi) - \beta f_2(\phi)$$

$$\phi(0) = 0 , \quad \phi(1) = 1$$

$$\phi'(0) = \phi'(1) = 0$$

with α, β *a priori* unknown. Existence results are established and it is shown that a distinguished solution (the « minimizing solution ») minimizes the associated Lagrange functional

$$\mathscr{L}_h \equiv \alpha \int_0^1 F_1(\phi) + \beta \int_0^1 F_2(\phi) + \frac{1}{2} \int_0^1 (\phi')^2$$

with $(F_1)' = f_1$, $(F_2)' = f_2$. The limiting behaviour of the minimizing solutions are investigated for $f_1 \to f_{10}$, $f_2 \to f_{20}$ in $L^1(0,1)$. In Section 4 the existence results of Section 3 are applied to a scaled version of (3), (4). The discussion of limiting problems A), B), C), D) based on the respective results of Section 3 is carried out and (5) is recovered.

2. DERIVATION OF MODEL EQUATIONS

Recall that the Lagrange functional associated with the investigated plasma is given by

$$\begin{aligned}\mathscr{L}(\rho_e, v_e, \rho_i, v_i, \phi) &\equiv \mathscr{L}_{\text{rel}}(\rho_e, v_e) + \mathscr{L}_{\text{rel}}(\rho_i, v_i) + \mathscr{L}_{\text{elstat}}(\rho_e, \rho_i, \phi) \\ &= \left(\int_0^L - m_e \rho_e c^2 \sqrt{1 - (v_e/c)^2} + m_e \rho_e c^2 \right) \\ &\quad + \left(\int_0^L - m_i \rho_i c^2 \sqrt{1 - (v_i/c)2} + m_i \rho_i c^2 \right) \\ &\quad + \left(\int_0^L e\rho_e \phi + Ze\rho_i(\Phi_A - \phi) + \frac{\epsilon_0}{2} \int_0^L (\phi')^2 \right).\end{aligned}$$

Since

$$j_e = e\rho_e v_e = \text{constant}, \quad j_i = Ze\rho_i v_i = \text{constant}$$

we can express v_e and v_i in terms of j_e and j_i to get the modified Lagrange functional

$$\begin{aligned}\mathscr{L}^*(\rho_e, \rho_i, \phi; j_e, j_i) &\equiv \mathscr{L}^*_{\text{rel}}(\rho_e; j_e) + \mathscr{L}^*_{\text{rel}}(\rho_i; j_i) + \mathscr{L}^*_{\text{elstat}}(\rho_e, \rho_i, \phi) \\ &= \int_0^L \left[- m_e e^{-1} \sqrt{c^2 e^2 \rho_e^2 - j_e^2} + m_e \rho_e c^2 \right] \\ &\quad + \int_0^L \left[- m_i (Ze)^{-1} \sqrt{Z^2 c^2 e^2 \rho_i^2 - j_i^2} + m_i \rho_i c^2 \right] \\ &\quad + \int_0^L [e\rho_e \phi + Ze\rho_i(\Phi_A - \phi)] + \frac{\epsilon_0}{2} \int_0^L (\phi')^2 .\end{aligned}$$

Now the model equations are constituted by $\frac{\partial L^*}{\partial \rho_e} = \frac{\partial L^*}{\partial \rho_i} = \frac{\partial L^*}{\partial \phi} = 0$, i.e.

$$(6)\quad \begin{cases} \rho_e = \dfrac{|j_e|}{ec} \dfrac{m_e c^2 + e\phi}{\sqrt{(m_e c^2 + e\phi)^2 - m_e^2 c^4}}, \\ \rho_i = \dfrac{|j_i|}{Zec} \dfrac{m_i c^2 + Ze(\Phi_A - \phi)}{\sqrt{(m_i c^2 + Ze(\Phi_A - \phi))^2 - m_i^2 c^4}} \\ \phi'' = \dfrac{e}{\epsilon_0} (\rho_e - Z\rho_i) \end{cases}$$

subject to the boundary conditions

$$(7)\quad \begin{cases} \phi(0) = 0, \phi(L) = \Phi_A \\ \phi'(0) = \phi'(L) = 0 . \end{cases}$$

In (6), (7) the constants j_e and j_i and the function ϕ are unknown. Keeping j_e and j_i fixed, problem (6)-(7) is in general not solvable (too many boundary conditions). On the other hand the demand for solvability shall fix j_e and j_i, but this is not exactly the case as we will see in subsequent Section 3. To distinguish the (unique) experimentally observable solution we have to take into account the functional $\mathcal{L}_h$ of action obtained by replacing ρ_e, ρ_i via (6) by ϕ :

$$\mathcal{L}_h(\phi ; j_e, j_i) \equiv \frac{|j_e|}{ec} \int_0^L \left[\sqrt{m_e c^2 + e\phi} - m_e c^2 \right]$$
$$+ \frac{|j_e|}{Zec} \left[\int_0^L \sqrt{m_i c^2 + Ze(\Phi_A - \phi)} - m_i c^2 \right]$$
$$+ \frac{\epsilon_0}{2} \int_0^L (\phi')^2 .$$

The requirement

$$\mathcal{L}_h \to \mathcal{MIN} \, !$$

allows to distinguish *exactly one* solution (the « minimizing solution ») of (6), (7) (see Section 3).

3. ON AN AUTONOMOUS ODE OF SECOND ORDER

We wish to analyze the following two-point boundary value problem :

(1) $$\Phi'' = \alpha f_1(\Phi) - \beta f_2(\Phi) \quad \text{on} \quad (0, 1)$$

(2) $$\begin{cases} \Phi(0) = 0, \Phi(1) = 1 \\ \Phi'(0) = \Phi'(1) = 0 . \end{cases}$$

Here we assume

(A1) $f_1, f_2 \in L^1(0, 1), f_1, f_2 \geq 0, f_1 \not\equiv 0 \not\equiv f_2$.

(A2) Set

$$F_i(x) = \int_0^x f_i(s)\, ds \quad \text{and set} \quad h(x) = F_2(1) f_1(x) - F_1(1) f_2(x).$$

Define now

$$H(x) = F_2(1)\, F_1(x) - F_1(1)\, F_2(x) = \int_0^x h(s)\, ds .$$

We assume that $H(x) > 0$ on (0,1).

In (1), (2) the parameters $\alpha, \beta \in \mathbb{R}$ and the function Φ are unknown.

DEFINITION 1 : *A triple* $(\alpha, \beta, \Phi) \in \mathbb{R} \times \mathbb{R} \times W^{2,1}_{loc}(0, 1)$ *is called a « weak solution of (1), (2) » if*

(i) $\Phi(x) \in (0, 1)$ *for almost all* $x \in (0, 1)$,

(ii) Φ *satisfies (1) in the sense of distributions, i.e.*

$$\forall \vartheta \in C_0^\infty((0, 1)) : \int \phi'' \vartheta \equiv - \int \phi' \vartheta' = \int (\alpha f_1(\phi) - \beta f_2(\phi))\, \vartheta ,$$

(iii)

(3) $$\begin{cases} \lim\limits_{x \to 0^+} \Phi(x) = 0 , & \lim\limits_{x \to 1^-} \Phi(x) = 1 \\ \lim\limits_{x \to 0^+} \dfrac{\Phi(x)}{x} = 0 , & \lim\limits_{x \to 1^-} \dfrac{1 - \Phi(x)}{1 - x} = 0 . \end{cases}$$

Remark 1 :

a) $W^{2,1}_{loc}(0, 1) \subset C^1((0, 1)) \subset C((0, 1))$ implies that $\Phi(x)$ is pointwise well defined for all $x \in (0, 1)$ and so are all terms of (3).

b) If (α, β, Φ) is a weak solution of (1), (2) then $\Phi'' = \alpha f_1(\Phi) - \beta f_2(\Phi)$ a.e. on (0, 1).

Before entering the analysis of (1), (2) we shall collect some simple facts whose proofs can be left to the reader.

Remark 2 :

a) Thanks to (A1) we have $f_1, f_2 \in L^1(0,1)$ and therefore $F_1, F_2 \in AC([0,1])$ where $AC([0,1])$ is the set of all absolute continuous, real valued functions defined on [0, 1]. Clearly $F_1' = f_1$ and $F_2' = f_2$ (where « ′ » denotes differentiation in the sense of distributions) whereas (A1) implies that $F_1(1), F_2(1) > 0$.

b) We have $H \in AC([0,1])$ and $H(0) = H(1) = 0$. If $H^{-1/2} \in L^1(0,1)$ does hold set

$$\gamma = \frac{1}{\sqrt{2}} \int_0^1 \frac{ds}{\sqrt{H(s)}} \tag{4}$$

and put

$$\alpha_0 \equiv \gamma^2 F_2(1), \quad \beta_0 \equiv \gamma^2 F_1(1). \tag{5}$$

We check easily that the function $\Phi_0 : [0,1] \to \mathbb{R}$ implicitly defined via

$$\int_0^{\Phi_0(x)} \frac{ds}{\sqrt{H(s)}} = \sqrt{2}\, \gamma x \tag{6}$$

belongs to $C^1([0,1])$ and satisfies the following differential equation

$$\Phi_0' = \gamma \sqrt{2} \sqrt{H(\Phi_0)} > 0 \quad \text{on} \quad (0,1)$$

with boundary conditions (2).

3.1. Existence of solutions

The main result of this Section is :

THEOREM 1 : *Assumed that (A1), (A2) do hold. Then :*

(A) *All weak solutions of (1), (2) belong to* $C^1([0,1])$.

(B) *(1), (2) has a weak solution iff* $H^{-1/2} \in L^1(0,1)$.

(C) *Assume that* $H^{-1/2} \in L^1(0,1)$. *Then*

(i) $(\alpha_0, \beta_0, \Phi_0)$ *is a weak solution of (1), (2).*

(ii) $(\alpha_0, \beta_0, \Phi_0)$ *is the unique weak solution of (1), (2) iff* $\Phi_0 \notin W^{2,1}(0,1)$.

(iii) *If* $\Phi_0 \in W^{2,1}(0,1)$ *then (1), (2) has infinitely many weak solutions* $(\alpha_l, \beta_l, \Phi_l)$, $l \in \mathbb{N} \cup \{0\}$:

$$\alpha_l = (2l+1)^2 \alpha_0, \quad \beta_l = (2l+1)^2 \beta_0$$

$$(7) \qquad \Phi_l(x) = \begin{cases} \Phi_0((2l+1)x) \cdot \mathbf{1}_{[x_0, x_1]}(x) \\ + \sum_{\nu=1}^{l} \Phi_0(2\nu - (2l+1)x) \cdot \mathbf{1}_{[x_{2\nu-1}, x_{2\nu}]}(x) \\ + \sum_{\nu=1}^{l} \Phi_0((2l+1)x - 2\nu) \cdot \mathbf{1}_{[x_{2\nu}, x_{2\nu+1}]}(x) \end{cases}$$

with $x_\nu \equiv \dfrac{\nu}{2l+1}$ *for* $\nu = 0, \dots, 2l+1$.

Proof:

• *Proof of* **(A)**

Assume that (α, β, Φ) is a weak solution of (1), (2). Then

$$(8) \qquad \Phi \in C([0,1]) \cap W^{2,1}_{\mathrm{loc}}(0,1) \subset C([0,1]) \cap C^1((0,1)).$$

Let $\Omega \subset\subset (0,1)$. Then there exists an $M_\Omega >$ such that $|\Phi'| \leqslant M_\Omega$ on Ω. Let now define the function g

$$g(x) = \begin{cases} x^2, & \text{if } |x| \leqslant 2M_\Omega \\ 4M_\Omega^2, & \text{if } |x| \leqslant 2M_\Omega. \end{cases}$$

The function g belongs to $C^{0,1}(\mathbb{R})$ and since $\Phi' \in W^{1,1}(\Omega)$ we can make use of the chain rule (see e.g. Ziemer [18], Theorem 2.1.11., p. 48) and get

$$(9) \qquad \tfrac{1}{2}[(\Phi')^2]' = \tfrac{1}{2}[g(\Phi')]' = \tfrac{1}{2} g'(\Phi')\Phi'' = \Phi'\Phi'' \text{ a.e. on } \Omega$$

and $\Phi'\Phi'' \in L^1(\Omega)$. Since $\Omega \subset\subset (0,1)$ is arbitrary and $\Phi'' = \alpha f_1(\Phi) - \beta f_2(\Phi)$ a.e. on (0,1) formula (9) establishes

$$(10) \qquad \tfrac{1}{2}[(\Phi')^2]' = (\alpha f_1(\Phi) - \beta f_2(\Phi))\Phi'$$

a.e. on (0, 1). On the other hand

$$\alpha F_1 - \beta F_2 \in AC([0,1]), \quad F_1' = f_1, \quad F_2' = f_2, \quad \Phi \in W^{1,1}_{loc}((0,1))$$

(see (8)) implies (see Marcus and Mizel [15], Theorem 4.3, pp. 315-316)

$$[\alpha F_1(\Phi) - \beta F_2(\Phi)]' = (\alpha f_1(\Phi) - \beta f_2(\Phi))\,\Phi'$$

which, in connection with (10), implies the existence of $c^* \in \mathbb{R}$ such that

(11) $$(\Phi')^2(x) = 2(\alpha F_1(\Phi) - \beta F_2(\Phi))(x) + c^* \in C([0,1]).$$

Letting tend x in the right-hand side of (11) to 0 and 1, respectively, yields

$$\lim_{x \to 0^+} (\Phi')^2(x) = c^*,$$

$$\lim_{x \to 1^-} (\Phi')^2(x) = 2(\alpha F_1(1) - \beta F_2(1)) + c^*.$$

This gives together with (3) and the Mean Value Theorem of elementary calculus

$$\Phi \in C^1([0,1])$$

with

$$\Phi'(0) = \Phi'(1) = c^* = 0.$$

This proves **(A)**.

Finally, there exists $\lambda \in \mathbb{R}$ such that

(12) $$\alpha = \lambda F_2(1), \quad \beta = \lambda F_1(1).$$

We deduce from this and from (10) that

(13) $$\Phi''\Phi' = \lambda h(\Phi).$$

• *Proof of* **(B)**

Assume that Φ is a weak solution of (1), (2). Using (12) we re-write (11) as

$$(\Phi')^2 = 2\lambda H(\Phi).$$

We note that Φ' cannot vanish identically on (0, 1) because $\Phi(0) = 0 < 1 = \Phi(1)$. Hence $\lambda > 0$.

Since the function H vanishes only at 0 and 1, then

$$\Phi' = 0 \quad \text{iff} \quad \Phi = 0 \text{ or } \Phi = 1 . \tag{14}$$

Besides, the open set $\{x \in (0,1) : \Phi'(x) \neq 0\}$ is an at most countable union of disjoint open intervals

$$\{x \in (0,1) : \Phi'(x) \neq 0\} = \bigcup_{k \in \mathcal{N}} (a_k, b_k)$$

where $\Phi \neq \mathcal{N} \subset \mathbb{N}$ and $(a_k, b_k) \cap (a_l, b_l) = \Phi$ for $k \neq l$. Moreover, since $\Phi'(a_k) = \Phi'(b_k) = 0$, then, thanks to (14), one of the four following possibilities occurs,

i) $\Phi(a_k) = 0, \Phi(b_k) = 0$ or ii) $\Phi(a_k) = 1, \Phi(b_k) = 1$ or
iii) $\Phi(a_k) = 0, \Phi(b_k) = 1$ or iv) $\Phi(a_k) = 1, \Phi(b_k) = 0$.

But $(\Phi')^2 > 0$ on (a_k, b_k) for all $k \in \mathcal{N}$ and $\Phi' \in C([0,1])$ implies that the function Φ is either strictly monotone increasing or strictly monotone decreasing on $[a_k, b_k]$. Hence for given $k \in \mathcal{N}$ either (iii) or (iv) must hold.

Let $k \in \mathcal{N}$ and assume that (iii) is valid. Then

$$\frac{\Phi'}{\sqrt{H(\Phi)}} = \sqrt{2\lambda} > 0 \quad \text{a.e. on} \quad (a_k, b_k) .$$

This leads after integration to

$$\int_x^y \frac{\Phi'(u)}{\sqrt{H(\Phi(u))}}\, du = \int_{\Phi(x)}^{\Phi(y)} \frac{ds}{\sqrt{H(s)}} = \sqrt{2\lambda}\,(y - x) , \quad a_k < x < y < b_k .$$

Letting x and y tend respectively to a_k and b_k implies that $H^{-1/2} \in L^1(0,1)$, $\int_0^1 H^{-1/2}(s)\, ds = \sqrt{2\lambda}\,(b_k - a_k)$ and

$$\int_0^{\Phi(x)} \frac{ds}{\sqrt{H(s)}} = \sqrt{2\lambda}\,(x - a_k) \quad \text{for all} \quad x \in [a_k, b_k] .$$

Hence $b_k - a_k = \gamma/\sqrt{\lambda}$ where γ is given in (4).

Assume now that (iv) holds on an interval (a_m, b_m) with $m \in \mathcal{N}$. We proceed in analogy and get $H^{-1/2} \in L^1(0,1)$,

$$\int_{\Phi(x)}^1 \frac{ds}{\sqrt{H(s)}} = \sqrt{2\lambda}\,(x - a_m) \quad \text{for} \quad x \in [a_m, b_m]$$

and $\quad b_m - a_m = \gamma/\sqrt{\lambda}$.

We conclude from this case-distinction that $H^{-1/2} \in L^1(0, 1)$ and $1 \geq b_k - a_k \equiv \gamma/\sqrt{\lambda}$ independently of $k \in \mathcal{N}$. Therefore $\mathcal{N}$ must be finite and we deduce from (13) that $\Phi'' = \lambda h(\Phi)$ on (a_k, b_k). But his implies that

$$\alpha = \frac{\lambda \alpha_0}{\gamma^2}, \quad \beta = \frac{\lambda \beta_0}{\gamma^2},$$

where α_0 and β_0 are defined in (5). For later reference the following observation is useful. Since $\Phi(0) = 0$ and $\Phi(1) = 1$ one has $|\mathcal{N}| = 2l + 1$ with $l \in \mathbb{N} \cup \{0\} L$:

$$\begin{cases} \Phi(a_1) = \Phi'(a_1) = 0, & \Phi(b_1) = 1, \quad \Phi'(b_1) = 0, \\ \Phi(a_{2l+1}) = \Phi'(a_{2l+1}) = 0, & \Phi(b_{2l+1}) = 1, \quad \Phi'(b_{2l+1}) = 0. \end{cases}$$

Since $H^{-1/2} \in L^1(0, 1)$ we can define Φ_0 as in (6) to write Φ as

$$\Phi(x) = \begin{cases} \Phi_0\left(\frac{\sqrt{\lambda}(x - a_1)}{\gamma} \cdot \mathbf{1}_{[a_l, b_l]}(x)\right) \\ + \sum_{\nu=1}^{l} \Phi_0\left(1 - \frac{\sqrt{\lambda}(x - a_{2\nu})}{\gamma}\right) \cdot \mathbf{1}_{(a_{2\nu}, b_{2\nu}]}(x) \\ + \sum_{\nu=1}^{l} \Phi_0\left(\frac{\sqrt{\lambda}(x - a_{2\nu+1})}{\gamma}\right) \cdot \mathbf{1}_{(a_{2\nu+l}, b_{2\nu+l}]}(x) \\ + \sum_{\nu=1}^{l} \mathbf{1}_{(b_{2\nu-l}, a_{2\nu}]}(x) + \mathbf{1}_{(b_{2l+l}, l]}(x). \end{cases}$$

We observe that

$$\{x \in (0, 1) : \Phi(x) = 0\}$$
$$= (0, a_1] \cup [b_2, a_3] \cup [b_4, a_5] \cup \ldots \cup [b_{2l}, a_{2l+1}],$$

$$\{x \in (0, 1) : \Phi(x) = 1\}$$
$$= [b_1, a_2] \cup [b_3, a_4] \cup [b_3, a_4] \cup \ldots \cup [b_{2l-1}, a_{2l}] \cup [b_{2l+1}, 1).$$

Since (α, β, Φ) is a weak solution of (1), (2) it follows from Definition 1(i) that both sets must have measure zero. This proves that

$$0 = a_1, b_1 = a_2, b_2 = a_3, \dots, b_{2l-1} = a_{2l}, b_{2l+1} = 1 .$$

Hence

$$b_k - a_k = \gamma/\sqrt{\lambda} \equiv \frac{1}{2l+1} \quad \text{and} \quad \alpha = \alpha_l = (2l+1)^2 \alpha_0, \beta = \beta_l = (2l+1)^2 \beta_0$$

for an $l \in \mathbb{N} \cup \{0\}$.

Let us now assume conversely that $H^{-1/2} \in L^1(0,1)$ and prove that Φ_0 is a solution of (1), (2). Recall first that Φ_0 is defined via

$$\int_0^{\Phi_0(x)} \frac{ds}{\sqrt{H(s)}} = \sqrt{2}\,\gamma x .$$

As mentioned in Remark 2 the function $\Phi_0 : [0,1] \to [0,1]$ is well-defined, strictly monotone increasing and belongs to $C^1([0,1])$ with $\Phi_0(0) = 0$, $\Phi_0(1) = 1$, $\Phi_0'(0) = 0$, $\Phi_0'(1) = 0$. This establishes $0 < \Phi_0 < 1$ on (0,1) and therefore $\Phi_0(x) \in (0,1)$ for almost all $x \in (0,1)$. Furthermore,

$$\Phi_0'(x) = \sqrt{2}\,\gamma \sqrt{H(\Phi_0)} > 0$$

on (0,1). Employing these properties of Φ_0 it is an easy task to show that $(\alpha_0, \beta_0, \Phi_0)$ is actually a weak solution of (1), (2). This finishes the proof of **(B)** and establishes **(C)**(i).

• *Proof of* **(C)**

We can turn our attention to **(C)**(ii) and **(C)**(iii).

First of all assume that $\Phi_0 \in W^{2,1}(0,1)$. Referring to the proof of **(B)**, all weak solutions of (1), (2) must be as in (7). A change of variable argument immediately settles $\Phi_l \in W^{2,1}(0,1)$, i.e. $h(\Phi_l) \in L^1(0,1)$ for all $l \in \mathbb{N} \cup \{0\}$. We already know that $H^{-1/2} \in L^1(0,1)$ implies that $(\alpha_0, \beta_0, \Phi_0)$ is a weak solution of (1), (2) no matter if $\Phi_0 \in W^{2,1}(0,1)$ or not. We just have to prove that Φ_l is a solution of (1), (2) for any $l \in \mathbb{N}$. But this is obvious since

$$\Phi_l'' = \gamma_l^2\, h(\Phi_l)$$

on $(x_\nu, x_\nu + 1)$ where $\gamma_l = (2l+1)\gamma$. The equality is then true almost everywhere and then in the sence of distributions. Now it remain to prove that Φ_l is in $W^{2,1}_{\text{loc}}(0,1)$. For this aim, we deduce from $\Phi_0 \in W^{2,1}(0,1)$, that $h(\Phi_l)$ is in $L^1(x_\nu, x_\nu + 1)$ which leads to $h(\Phi_l) \in L^1(0,1)$.

Now suppose that (1), (2) is not uniquely solvable. Then it follows from the previous discussion that there has to exist an $l \in \mathbb{N}$ such that $(\alpha_l, \beta_l, \Phi_l)$ is a weak solution of (1), (2). Then $\Phi_l'' \in L^1_{\text{loc}}(0, 1)$ and a change of variable immediately gives $\Phi_0'' \in L^1(0, 1)$ and finishes the proof of **(C)**(ii), **(C)**(iii). □

For the sake of completeness we shall discuss in short which properties of h are sufficient for $\Phi_0 \in W^{2,1}(0, 1)$. To answer this the following definition is useful :

DEFINITION 2 : *For* $x_0 \in [0, 1]$ *the function* $h \in L^1(0, 1)$ *is called « locally semi-bounded at* x_0 *» if there exists an* $\epsilon > 0$ *such that*

$$h^+|_{(x_0-\epsilon, x_0+\epsilon)\cap(0,1)} \in L^\infty(0, 1)$$

or

$$h^-|_{(x_0-\epsilon, x_0+\epsilon)\cap(0,1)} \in L^\infty(0, 1) .$$

The point of this Definition is

LEMMA 1 : *Assume that (A1), (A2) and* $H^{-1/2} \in L^1(0, 1)$ *do hold. Assume that* h *is locally semibounded at* 0 *and* 1. *Then* $\Phi_0 \in W^{2,1}(0, 1)$.

Proof : We assume for the moment that h has the property

$$h^-|_{(0,\epsilon)} \in L^\infty(0, 1) \quad \text{and} \quad h^-|_{(1-\epsilon,1)} \in L^\infty(0, 1) . \tag{15}$$

Since $h(\Phi_0) = \gamma^{-2}\Phi_0'' \in L^1_{\text{loc}}(0, 1)$ it sufficies to prove that there exists a $\delta > 0$ with

$$h_1(\Phi_0) \equiv h(\Phi_0)|_{(0,\delta)}, \quad h_2(\Phi_0) \equiv h(\Phi_0)|_{(1-\delta,1)} \in L^1(0, 1) .$$

As $\Phi_0 \in C([0, 1])$ is strictly monotone increasing and $\Phi_0(0) = 0$, $\Phi_0(1) = 1$ there is a $\delta > 0$ such that

$$\Phi_0((0, \delta)) \subset (0, \epsilon), \quad \Phi_0((1-\delta, 1)) \subset (1-\epsilon, 1) .$$

We fix this $\delta > 0$ and choose a sequence (δ_n), $n \in \mathbb{N}$ of test functions $\vartheta_n \in C_0^\infty((0, \delta))$ such that for all $n \in \mathbb{N}$

$$\text{supp}(\vartheta_n) = \left[\frac{1}{2^{n+2}}\delta, \left(1 - \frac{1}{2^{n+2}}\right)\delta\right],$$

$$\vartheta_n \geq 0, \vartheta_n = 1 \quad \text{on} \quad \left[\frac{1}{2^{n+1}}\delta, \left(1 - \frac{1}{2^{n+1}}\delta\right)\right],$$

$$\vartheta_n' \geq 0 \text{ on} \left[\frac{1}{2^{n+2}}\delta, \frac{1}{2^{n+1}}\delta\right], \quad \vartheta_n' \leq 0 \text{ on} \left[\left(1 - \frac{1}{2^{n+1}}\right)\delta, \left(1 - \frac{1}{2^{n+2}}\right)\delta\right].$$

Then $0 \leqslant \vartheta_n \leqslant \vartheta_{n+1} \leqslant 1$ and $\vartheta_n \to \mathbf{1}_{(0,\delta)}$ a.e. (0, 1) as $n \to \infty$. Since $0 \leqslant h^- \leqslant m \in \mathbb{R}$ a.e. $(0, \epsilon)$ we have

$$(h(\Phi_0) + m)\,\vartheta_n \to h_1(\Phi_0) + m \,.\, \mathbf{1}_{(0,\delta)} \geqslant 0$$

a.e. on (0, 1). By non-negativity of $h(\Phi_0) + m$ and thanks to $\vartheta_n \leqslant \vartheta_{n+1}$ the sequence $(h(\Phi_0) + m)\,\vartheta_n$ is monotone increasing a.e. on (0,1). But Φ_0 is also a weak solution of (1), (2), i.e.

$$-\int \Phi_0' \vartheta_n' + m \int \vartheta_n = \int (h(\Phi_0) + m)\,\vartheta_n$$

for all $n \in \mathbb{N}$. Since $\Phi_0' > 0$ on (0, 1) we have

$$\left|\int (h(\Phi_0) + m)\,\vartheta_n\right| \leqslant \int |\Phi_0'\vartheta_n'| + m\left|\int \vartheta_n\right|$$

$$\leqslant \|\Phi_0'\|_{L^\infty(0,1)}\left(\int_{\frac{\delta}{2^{n+2}}}^{\frac{\delta}{2^{n+1}}} \vartheta_n' - \int_{\left(1-\frac{1}{2^{n+1}}\right)\delta}^{\left(1-\frac{1}{2^{n+2}}\right)\delta} \vartheta_n'\right) + m\delta$$

$$= 2\,\|\Phi_0'\|_{L^\infty(0,1)} + m\delta\,.$$

Now the Monotone Convergence Theorem provides that

$$h_1(\Phi_0) + m \,.\, \mathbf{1}_{(0,\delta)} \in L^1(0,1)$$

and consequently

$$h_1(\Phi_0) \in L^1(0,1)\,.$$

We prove in analogy

$$h_2(\Phi_0) \in L^1(0,1)\,.$$

The remaining cases can be treated in analogy. □

3.2. Existence and Stability of a minimal solution

Referring to the subsequent discussion of Section 3, the function h of the model equations (6), (7) is locally semibounded on [0, 1]. Theorem 1 and

Lemma 1 imply that (6), (7) has infinitely many solutions. To isolate amongst these the physically relevant solution we introduce the Lagrange functional $\mathscr{L}_h(\Phi\,;\alpha,\beta)$ — where the subscript « h » refers to the nonlinear function h — associated with (1), (2)

$$\mathscr{L}_h(\Phi\,;\alpha,\beta) \equiv \alpha\int_0^L F_1(\Phi) + \beta\int_0^L F_2(\Phi) + \frac{1}{2}\int_0^L (\Phi')^2\,,$$

i.e. (1) is the Euler Lagrange equation $\dfrac{\partial \mathscr{L}_h}{\partial \Phi} = 0$.

We assume that h is locally semibounded at 0 and 1. According to Theorem 1 and Lemma 1 the class of all solutions of problem (1), (2) is given by $((2l+1)^2\alpha_0, (2l+1)^2\beta_0, \Phi_l)$. A straight-forward computation gives

$$\forall l \in \mathbb{N}\cup\{0\} : \mathscr{L}_h(\Phi_l\,;\alpha_l,\beta_l) = (2l+1)^2\,\mathscr{L}_h(\Phi_0\,;\alpha_0,\beta_0)$$

which means that $\mathscr{L}_h(\Phi_l\,;\alpha_l,\beta_l)$ is minimal for $l=0$. We formulate this result as

COROLLARY 1: *Assume that (A1), (A2) do hold and let* $H^{-1/2}\in L^1(0,1)$. *Assume that h is locally semibounded at* 0 *and* 1. *Then* $(\alpha_0,\beta_0,\Phi_0)$ *defined in (5), (6) is the unique minimizer of* $\mathscr{L}_h$ *in the set of all weak solutions of (1), (2).*

To stress the distinguished importance of $(\alpha_0,\beta_0,\Phi_0)$ we give

DEFINITION 3: *Assuming the validity of (A1), (A2) and requiring* $H^{-1/2}\in L^1(0,1)$ *the triple* $(\alpha_0,\beta_0,\Phi_0)$ *is called « minimizing solution (of (1), (2)) ».*

Our next aim is to analyze the limiting behavior for η tending to zero of the minimizing solutions $(\alpha_{0\eta},\beta_{0\eta},\Phi_{0\eta})$ of

(16) $$\Phi''_{0\eta} = \alpha_{0\eta} f_{1\eta}(\Phi_{0\eta}) - \beta_{0\eta} f_{2\eta}(\Phi_{0\eta}) \quad \text{on} \quad (0,1)$$

(17) $$\begin{cases} \Phi_{0\eta}(0) = 0,\ \Phi_{0\eta}(1) = 1 \\ \Phi'_{0\eta}(0) = \Phi'_{0\eta}(1) = 0 \end{cases}$$

where we assume that

(L1) $\forall \eta \geqslant 0 : f_{1\eta}, f_{2\eta} \in L^1(0,1), f_{1\eta}, f_{2\eta} \geqslant 0, f_{1\eta} \not\equiv 0 \not\equiv f_{2\eta}$.

$$\text{(L2)}\ \forall \eta \geqslant 0 : \begin{cases} H_\eta(x) & \equiv \int_0^1 f_{2\eta}(s)\,ds \int_0^x f_{1\eta}(s)\,ds - \int_0^1 f_{1\eta}(s)\,ds \int_0^x f_{2\eta}(s)\,ds \\ & \equiv F_{2\eta}(1)\,F_{1\eta}(x) - F_{1\eta}(1)\,F_{2\eta}(x) \\ & \equiv \int_0^x h_\eta(s)\,ds > 0 \text{ on } (0,1). \end{cases}$$

(L3) $\forall \eta \geqslant 0 : h_\eta$ is locally semibounded at 0 and 1.

(L4) $\forall \eta \geqslant 0 : H_\eta^{-1/2} \in L^1(0,1)$.

(L5) $f_{1\eta}$, $f_{2\eta}$, and $H_\eta^{-1/2}$ converge strongly in $L^1(0,1)$ respectively to f_{10}, f_{20} and $H_0^{-1/2}$.

Remark 3 : Thanks to (L1)-(L4) and Theorem 1, Lemma 1 the minimizing solution $(\alpha_{0\eta}, \beta_{0\eta}, \Phi_{0\eta})$ of (16), (17) is well-defined for all $\eta \geqslant 0$.

Concerning the limit $\eta \to 0$ we have

THEOREM 2 : *Assume that (L1)-(L5) do hold. Then*

(A) $\Phi_{0\eta} \to \Phi_{00}$ *in* $C^1([0,1])$ *and* $\alpha_{0\eta} \to \alpha_{00}$, $\beta_{0\eta} \to \beta_{00}$ *as* $\eta \to 0$.

(B) *If* $\forall \eta \geqslant 0 : h_\eta \in C((0,1))$ *and if* $h_\eta \to h_0$ *in* $C_{loc}((0,1))$ *as* $\eta \to 0$ *then* $\forall \eta \geqslant 0 : \Phi_{0\eta} \in C^2((0,1))$ *and* $\Phi_{0\eta} \to \Phi_{00}$ *in* $C^2_{loc}((0,1))$ *as* $\eta \to 0$.

Proof :

- *Proof of* **(A)**

Recall that $\forall \eta \geqslant 0 : F_{1\eta}(x) = \int_0^x f_{1\eta}(s)\,ds$ and $F_{2\eta}(x) = \int_0^x f_{2\eta}(s)\,ds$,

$$\alpha_{0\eta} = \frac{F_{2\eta}(1)}{2} \int_0^1 H_\eta^{-1/2}(s) = F_{2\eta(1)}\,\gamma_\eta^2,$$

$$\beta_{0\eta} = \frac{F_{1\eta}(1)}{2} \int_0^1 H_\eta^{-1/2}(s) = F_{1\eta}(1)\,\gamma_\eta^2.$$

Hence $\alpha_{0\eta} \to \alpha_{00}$, $\beta_{0\eta} \to \beta_{00}$ as $\eta \to 0$ follows from (L5). Now we shall prove that

$$\Phi_{0\eta} \to \Phi_{00} \text{ in } C([0,1]) \text{ as } \eta \to 0. \tag{18}$$

Recall that for $\eta \geqslant 0$ and $x \in [0, 1]$

$$\int_0^{\Phi_{0\eta}(x)} \frac{ds}{\sqrt{\gamma_\eta^2 H_\eta(s)}} = \sqrt{2}\, x . \tag{19}$$

Since $H_\eta^{-1/2}$ converges in L^1 to $H_0^{-1/2}$, the antiderivatives converge uniformly. As a consequence, we can pass to the limit pointwise in (19) and deduce that $\Phi_{0\eta}$ converges pointwize to Φ_{00}, and since the $\Phi_{0\eta}$ are monotone increasing functions, a result of elementary calculus implies that the convergence is uniform.

To establish the C^1 convergence of $\Phi_{0\eta}$ to Φ_{00}, it remains to prove the uniform convergence of the derivatives but this is straightforward since

$$\Phi'_{0\eta} = \sqrt{2}\, \gamma_\eta\, H_\eta(\Phi_{0\eta})$$

and H_η and $\Phi_{0\eta}$ converge uniformly.

• *Proof of* **(B)**

Thanks to $h_\eta \in C((0, 1))$ and $\Phi_{0\eta} \in C^1([0, 1])$ for all $\eta \geqslant 0$ we have $\Phi''_{0\eta} = \gamma_\eta^2 h_\eta(\Phi_{0\eta}) \in C((0, 1))$ and therefore $\Phi_{0\eta} \in C^2((0, 1))$ for all $\eta \geqslant 0$. The limit $\Phi_{0\eta} \to \Phi_{00}$ in $C^2_{\text{loc}}((0, 1))$ follows at once from $h_\eta \to h_0$ in $C_{\text{loc}}((0, 1))$. This proves **(B)**. □

4. VOLTAGE CURRENT CHARACTERISTICS

In this Section the model equations (6), (7) are rescaled and the framework of Section 2 is applied to discuss the non relativistic, relativistic and ultra relativistic settings of A)-D) of Section 1.

Remark 4 : The notation of a weak solution of Section 2 *a priori* excludes that the potential Φ equals to zero or one on a set of nonzero measure. However we do not ignore any physically relevant solution with this concept : If $\{\Phi = 0\}$ or $\{\Phi = 1\}$ had nonzero measure then $\int \rho_e = \infty$ or $\int \rho_i = \infty$ would follow. These infinite masses clearly make no sense from a physical point of view.

Naturally the subsequent discussion is restricted to minimizing solutions.

4.1. The general case

We rescale (6), (7) via

$$x \equiv L x_s\,, \quad \Phi \equiv \Phi_A\,\Phi_s\,, \quad j_e \equiv \frac{c\epsilon_0\,\Phi_A}{L^2}\,j_{e,s}\,, \quad j_i \equiv \frac{c\epsilon_0\,\Phi_A}{L^2}\,j_{i,s}\,.$$

and get by omitting the index « s » henceforth

$$\Phi'' = |j_e|\,\frac{\Phi + r}{\sqrt{\Phi^2 + 2\,r\Phi}} - |j_i|\,\frac{(1-\Phi) + \mu r}{\sqrt{(1-\Phi)^2 + 2\,\mu r(1-\Phi)}} \tag{1}$$

$$\begin{cases} \Phi(0) = 0,\ \Phi(1) = 1 \\ \Phi'(0) = \Phi'(1) = 0 \end{cases} \tag{2}$$

where we recall that $r = m_e\,c^2/(e\Phi_A)$ and

$$\mu = m_i/(Zm_e) = Nm_p/(Zm_e).$$

Employing the notations of Section 2 we have $\alpha = |j_e|$, $\beta = |j_i|$,

$$f_1(x) = \frac{x + r}{\sqrt{x^2 + 2\,rx}}\,, \quad f_2(x) = \frac{(1-x) + \mu r}{\sqrt{(1-x)^2 + 2\,\mu r(1-x)}}\,,$$

$$F_1(x) = \sqrt{x^2 + 2\,rx}\,, \quad F_2(x) = \sqrt{1 + 2\,\mu r} - \sqrt{(1-x)^2 + 2\,\mu r(1-x)}\,.$$

Obviously $f_1, f_2 \in L^1(0,1)$, $f_1, f_2 \geqslant 0$ and $f_1 \not\equiv 0 \not\equiv f_2$. Hence (A1) does hold. The validity of (A2) can be checked by

LEMMA 2 : *Assume that* $g_1, g_2 \in L^1(0,1)$ *and that*

$$h^*(x) \equiv g_1(x)\int_0^1 g_2(s)\,ds - g_2(x)\int_0^1 g_1(s)$$

is monotone decreasing with $\operatorname{meas}(\{x \in (0,1) : h^*(x) = 0\}) = 0$. *Then*

$$H^*(x) \equiv \int_0^x h^*(s)\,ds > 0 \text{ on } (0,1)\,.$$

Proof: We note that $H^* \in C([0,1])$ and $H(0)=H(1)=0$. Assume that there is an $x_1 \in (0,1)$ with $H^*(x_1)=\int_0^{x_1} h^*(s)\,ds \leq 0$. Since meas $(\{x \in (0,1): h^*(x)=0\})=0$ there must be an $\epsilon > 0$ and a $z^* \in (0, x_1)$ such that $h^*(z^*) \leq -\epsilon$. As h^* is monotone decreasing we have $h^*(x_1) \leq h^*(z^*) \leq -\epsilon$ for all $x \in [z^*, 1)$. This implies

$$H^*(1)=\int_0^{x_1} h^*(s)\,ds+\int_{x_1}^1 h^*(s)\,ds \leq 0-\epsilon<0$$

which contradicts $H^*(1)=0$. □

Since $h(x)=F_2(1) f_1(x)-F_1(1) f_2(x)$ where $F_1(1)$, $F_2(1)>0$ and f_1, f_2 are respectively strictly monotone decreasing and h is *strictly* monotone decreasing. Hence meas $(\{x \in (0,1): h(x)=0\})=0$ and Lemma (2) applies. This proves (A2).

To verify $H^{-1/2} \in L^1(0,1)$ recall that

$$H(x) \equiv F_2(1) F_1(x)-F_1(1) F_2(x)$$

is continuous on [0,1] and $H>0$ on (0, 1). Hence it sufficies to check the behaviour of $H^{-1/2}(x)$ at 0 and 1. We get

$$\lim_{x \rightarrow 0^+} \frac{H(x)}{\sqrt{x}}=F_2(1) \sqrt{2 r}, \quad \lim_{x \rightarrow 1^-} \frac{H(x)}{\sqrt{1-x}}=F_1(1) \sqrt{2 \mu Z r}$$

and consequently

$$H^{-1/2}(x)=O\left(\frac{1}{\sqrt[4]{x}}\right) \text{ as } x \rightarrow 0^+, \quad H^{-1/2}(x)=O\left(\frac{1}{\sqrt[4]{1-x}}\right) \text{ as } x \rightarrow 1^-$$

which establishes $H^{-1/2} \in L^1(0,1)$.

Furthermore we note that $h(0^+)=+\infty$ and $h(1^-)=-\infty$ implies that h is locally semibounded at 0 and 1. The function h also belongs to $C((0,1))$.

Having thus checked all assumptions of Theorem 1, Lemma 1 and Theorem 2 we deduce.

COROLLARY 2: *Problem (1), (2) has a unique minimizing solution* $(|j_e|, |j_i|, \Phi_0)$. *The potential* $\Phi_0 \in C^2((0,1)) \cap C^1([0,1])$ *defined via*

$$\int_0^{\Phi_0(x)} \frac{ds}{\sqrt{H(s)}}=\sqrt{2}\, \gamma x$$

with $\gamma = \int_0^1 H^{-1/2}(s)\,ds$ *is strictly monotone increasing. The currents are given by*

$$|j_e| = \gamma^2 F_2(1) = \gamma^2 \sqrt{1 + 2\mu r}\,, \quad |j_i| = \gamma^2 F_1(1) = \gamma^2 \sqrt{1 + 2r}\,.$$

Remark 5 : Since Φ_0 is strictly monotone increasing the negatively charged electrons movein the positive x-direction while the negatively charged ions move in the negative x-direction. Hence $|j_e| = j_e$, $|j_i| = j_i$.

4.2. Non relativistic voltage current characteristics

If the moduli of the velocities of electrons and ions are small compared with the speed of light c we expect that the kinetic terms $\mathscr{L}_{\text{rel}}$ of the Lagrange functional $\mathscr{L}$ defined in Section 1 can be replaced with good accuracy by

$$\mathscr{L}_{\text{nonrel}}(\rho_e, v_e) = \frac{1}{2}\int_0^L m_e\,\rho_e\,v_e^2\,, \quad \mathscr{L}_{\text{nonrel}}(\rho_i, v_i) = \frac{1}{2}\int_0^L m_i\,\rho_i\,v_i^2$$

where the respective model equations can be derived in analogy to Section 1. Instead of doing this formally we carry out the limit $r \to \infty$.

We introduce $\eta \equiv 1/2\,r$, rescale j_e, j_i,

$$j_{e\eta} \equiv \frac{j_e}{\sqrt{2\,\eta}}\,, \quad j_{i\eta} \equiv \frac{j_i}{\sqrt{2\,\eta}}\,,$$

and set

$$f_{1\eta} \equiv \frac{1}{\sqrt{2\,x}}\,\frac{1 + 2\,\eta x}{\sqrt{1 + \eta x}}\,, \quad f_{2\eta} \equiv \frac{1}{\sqrt{2(1-x)}}\,\frac{1 + 2\,\eta(1-x)}{\sqrt{1 + \eta(1-x)}}\,.$$

Then $\Phi_{0\eta}$ is defined to be the minimizing solution of

$$\Phi_{0\eta}'' = \begin{cases} j_{e\eta}\,\dfrac{1}{\sqrt{2\,\Phi_{0\eta}}}\,\dfrac{1 + 2\,\eta\Phi_{0\eta}}{\sqrt{1 + \eta\Phi_{0\eta}}} \\[2ex] -\; j_{i\eta}\,\dfrac{1}{\sqrt{2(1 - \Phi_{0\eta})}}\,\dfrac{1 + 2\,\eta(1 - \Phi_{0\eta})}{\sqrt{1 + \eta(1 - \Phi_{0\eta})}} \end{cases} \tag{3}$$

$$\begin{cases} \Phi_{0\eta}(0) = 0,\ \Phi_{0\eta}(1) = 1 \\ \Phi_{0\eta}'(0) = \Phi_{0\eta}'(1) = 0\,. \end{cases} \tag{4}$$

The functions $F_{1\eta}$, $F_{2\eta}$ are given by

$$F_{1\eta}(x) = \sqrt{2x}\sqrt{1+\eta x}, \quad F_{2\eta}(x) = \sqrt{2\mu + 2\eta} - \sqrt{2\mu(1-x) + 2\eta(1-x)^2}$$

so H_η becomes

$$H_\eta(x) = \begin{cases} \sqrt{2\mu + 2\eta}\left(\sqrt{2x}\sqrt{1+\eta x} - \sqrt{2}\sqrt{1+\eta}\right) \\ + \sqrt{2}\sqrt{1+\eta}\left(\sqrt{2\mu(1-x) + 2\eta(1-x)^2}\right). \end{cases}$$

4.2.1. $\eta = 0$

We get the non relativistic model by putting $\eta = 0$ in (3), (4) :

$$\Phi''_{00} = \frac{j_{e0}}{\sqrt{2\,\Phi_{00}}} - \frac{j_{i0}}{\sqrt{2(1-\Phi_{00})}} \tag{5}$$

$$\begin{cases} \Phi_{00}(0) = 0, \Phi_{00}(1) = 1 \\ \Phi'_{00}(0) = \Phi'_{00}(1) = 0. \end{cases} \tag{6}$$

Hence

$$f_{10}(x) = \frac{1}{2x}, \quad f_{20}(x) = \frac{1}{\sqrt{2(1-x)}},$$

$$F_{10}(x) = \sqrt{2x}, \quad F_{20}(x) = \sqrt{2\mu}\left(1 - \sqrt{1-x}\right), \tag{7}$$

$$H_0(x) = \sqrt{2\mu}\sqrt{2}\left((\sqrt{x} - 1) + \sqrt{1-x}\right).$$

We can proceed in analogy to the derivation of Corollary 2 to get

COROLLARY 3 : *Problem (5), (6) has a unique minimizing solution* $(|j_{e0}|, |j_{i0}|, \Phi_{00})$. *The potential* $\Phi_{00} \in C^2((0,1)) \cap C^1([0,1])$ *defined via*

$$\int_0^{\Phi_{00}(x)} \frac{ds}{\sqrt{H_0(s)}} = \sqrt{2}\,\gamma_0\, x$$

with H_0 *as in (7) and*

$$\gamma_0 = \frac{1}{2}\frac{1}{\sqrt[4]{\mu}}\int_0^1 \frac{ds}{\sqrt{\sqrt{s} - 1 + \sqrt{1-s}}} \equiv \frac{I_a}{\sqrt[4]{\mu}} \approx \frac{0.9105}{\sqrt[4]{\mu}}$$

is strictly monotone increasing. The currents are given by

$$j_{e0} = \sqrt{2}\, I_a^2 \approx 1.172\,, \quad j_{i0} = \frac{j_{e0}}{\sqrt{\mu}} \approx \frac{1.172}{\sqrt{\mu}}\,.$$

Remark 6: a) Since $\mu \gg 1$ *we have* $j_{e0} \gg j_{i0}$

b) In SI units we have for $\eta = 0$

$$j_{e0}^{SI} = \sqrt{2}\, I_a^2\, \epsilon_0 \frac{e}{m_e} \frac{\Phi_A^{3/2}}{L^2} \approx 4.353\ 10^{-6} \frac{\Phi_A^{3/2}}{L^2}\,,$$

$$j_{i0}^{SI} = \sqrt{2}\, I_a^2\, \epsilon_0 \frac{e}{m_p} \sqrt{\frac{Z}{N}} \frac{\Phi_A^{3/2}}{L^2} \approx 1.016\ 10^{-7} \sqrt{\frac{Z}{N}} \frac{\Phi_A^{3/2}}{L^2}\,,$$

4.2.2 $\eta \to 0$

We are concerned with the analysis of the limit $(j_{e\eta}, j_{i\eta}, \Phi_{0\eta}) \to (j_{e0}, j_{i0}, \Phi_{00})$. It is our aim to apply Theorem 2. Hence we have to check (L1)-(L5).

(L1), (L2), (L3), (L4) are easily verified by means of the argumentation of Section 3.1.

(L5): The convergence in $L^1(0,1)$ of $f_{1\eta}$ to f_{10} and $f_{2\eta} f_{20}$ follows immediately from Lebesgue's Theorem. It remains to verify $\| H_\eta^{1/2} - H_0^{1/2} \|_{L^1(0,1)} \to 0$. A straight-forward computation gives $H_\eta(x) = H_0(x) + O(\eta)$. We introduce for $\eta \geq 0$ and $x \in (0,1)$

$$P_\eta(x) \equiv \frac{H_\eta(x)}{\sqrt{2x}}\,, \quad Q_\eta(x) \equiv \frac{H_\eta(x)}{\sqrt{2(1-x)}}\,.$$

We easily see that $\lim_{x \to 0^+} P_\eta(x) = F_{2\eta}(1)$, $\lim_{x \to 1^-} Q_\eta(x) = \sqrt{\mu Z} F_{1\eta}(1)$ and we get after some elementary manipulations

$$\exists M, \eta_0 > 0 \cdot \forall \eta \in [0, \eta_0] \cdot \begin{cases} \forall x \in (0, 1/2] : P_\eta(x) \geq M^2 \\ \forall x \in [1/2, 1)\ \ Q_\eta(x) \geq M^2\,. \end{cases}$$

But then

$$\|H_\eta^{-1/2} - H_0^{-1/2}\|_{L^1(0,1)} = \int_0^1 |H_\eta^{-1/2}(x) - H_0^{-1/2}(x)|\, dx$$

$$= \int_0^{1/2} \frac{1}{\sqrt[4]{2x}} \frac{1}{\sqrt{P_\eta(x)\, P_0(x)}} \frac{|P_\eta(x) - P_0(x)|}{\sqrt{P_\eta(x)} + \sqrt{P_0(x)}}\, dx$$

$$+ \int_{1/2}^1 \frac{1}{\sqrt[4]{2x}} \frac{1}{\sqrt{Q_\eta(x)\, Q_0(x)}} \frac{|Q_\eta(x) - Q_0(x)|}{\sqrt{Q_\eta(x)} + \sqrt{Q_0(x)}}\, dx$$

$$\leqslant \frac{1}{2M^3} \int_0^{1/2} \frac{1}{\sqrt[4]{2x}} \frac{1}{\sqrt{2x}} |H_\eta(x) - H_0(x)|\, dx$$

$$+ \frac{1}{2M^3} \int_{1/2}^1 \frac{1}{\sqrt[4]{2(1-x)}} \frac{|H_\eta(x) - H_0(x)|\; 1}{\sqrt{2(1-x)}}\, dx$$

$$= O(\eta)\,.$$

This proves (L5).

We furthermore have $h_\eta \to h_0$ in $C_{\mathrm{loc}}((0, 1))$ as $\eta \to 0$ so we deduce from Theorem 2 and from $\|H_\eta^{-1/2} - H_0^{-1/2}\|_{L^1(0,1)} = O(\eta)$.

COROLLARY 4: *As η tends to zero, $\Phi_{0\eta}$ converges to Φ_{00} in $C^1([0, 1]) \cap C^2_{loc}((0, 1))$, $j_{e\eta}$ converges to j_{e0} and $j_{\iota\eta}$ converges to $j_{\iota 0}$. Moreover the following expansion holds*

$$j_{e\eta} = j_{e0} + O(\eta)\,, \quad j_{\iota\eta} = j_{\iota 0} + O(\eta)\,.$$

4.3. Relativistic electrons and non relativistic ions

Electrons and ions can be treated as non relativistic particles as long as the applied potential Φ_A is small enough. If $\Phi_A \approx 10^5$ V then $r \approx 5$ while $\mu r \approx 2.5.10^4$. In this case the electrons are already in the relativistic regime while ions are still non relativistic particles. This amounts to $r \approx 1$, $\mu \to \infty$.

We introduce $\eta \equiv 1/\mu$ and rescale the currents via

$$j_{e\eta} \equiv j_e\,, \quad j_{\iota\eta} \equiv \frac{j_\iota}{\sqrt{\eta}}\,.$$

Then we get or $\eta = 0$

$$f_{10}(x) = \frac{x+r}{\sqrt{x^2+rx}}, \quad f_{20}(x) = \frac{\sqrt{r}}{\sqrt{2(1-x)}},$$

(8) $$F_{10}(x) = \sqrt{x^2+2\,rx}, \quad F_{20}(x) = \sqrt{2\,r}\,(1-\sqrt{1-x}),$$

$$H_0(x) = \sqrt{2\,r}\left(\sqrt{x^2+2\,rx}+\sqrt{1+2\,r}\,(\sqrt{1-x}-1)\right)$$

and the model equations become

(9) $$\Phi_{00}'' = j_{e0}\frac{r+\Phi_{00}}{\sqrt{\Phi_{00}^2+r\Phi_{00}}} - j_{i\eta}\frac{\sqrt{r}}{\sqrt{2(1-\Phi_{00})}}$$

(10) $$\begin{cases} \Phi_{00}(0) = 0,\ \Phi_{00}(1) = 1 \\ \Phi_{00}'(0) = \Phi_{00}'(1) = 0 \end{cases}$$

and we proceed in analogy to Section 3.2 to get

COROLLARY 5 : *a)* Φ_{00} *is defined via*

$$\frac{1}{\sqrt[4]{2}}\int_0^{\Phi_{00}(x)} \frac{ds}{\left(\sqrt{s^2+2\,rs}+\sqrt{1+2\,r}\,(\sqrt{1-s}-1)\right)} = \sqrt{2}\,\gamma_0\,x$$

with

$$\gamma_0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt[4]{2}}\int_0^1 \frac{ds}{\left(\sqrt{s^2+2\,rs}+\sqrt{1+2\,r}\,(\sqrt{1-s}-1)\right)} \equiv \frac{1}{\sqrt{2}}\frac{1}{\sqrt[4]{2}}I_b(r)\,.$$

Φ_{00} *is strictly monotone increasing and the currents are given by*

$$j_{e0} = \frac{\sqrt{r}}{2}I_b^2(r)\,, \quad j_{i0} = \frac{\sqrt{r+1/2}}{2}I_b^2(r)$$

b) $j_{e\eta} = j_{e0} + O(\eta)$, $j_{i\eta} = j_{i0} + O(\eta)$.

c) In SI units we have

$$j_{e0}^{SI} = \frac{c^2 \epsilon_0}{2} \sqrt{\frac{m_e}{e}} I_b^2 \left(\frac{m_e c^2}{e} \frac{1}{\Phi_A} \right) \frac{\Phi_A^{1/2}}{L^2} \approx 0.9488\, I_b^2 \left(\frac{5.111 \cdot 10^5}{\Phi_A} \right) \frac{\Phi_A^{1/2}}{L^2},$$

$$j_{i0}^{SI} = \frac{c\epsilon_0}{2\sqrt{2}} \sqrt{\frac{m_e}{m_p}} \sqrt{\frac{Z}{N}} I_b^2 \left(\frac{m_e c^2}{e} \frac{1}{\Phi_A} \right) \sqrt{\frac{2\, m_e c^2}{e} + \Phi_A}\, \frac{\Phi_A^{1/2}}{L^2}$$

$$\approx 2.190 \cdot 10^{-5}\, I_b^2 \left(\frac{5.111 \cdot 10^5}{\Phi_A} \right) \sqrt{1.022 \cdot 10^6 + \Phi_A}\, \frac{\Phi_A^{1/2}}{L^2}.$$

4.4. Ultra relativistic electrons and non relativistic ions

Increasing the applied potential Φ_A up to the order of magnitude 10^7 we typically have $r \approx 0.05$, $\mu r \approx 250$. Hence the ions still behave as non relativistic particles while the electrons have already reached the « ultrarelativistic » (i.e. rest energy small compared with kinetic energy, see e.g. [14], p. 33) regime. As $\mu r \gg 1$ does hold we are obliged to consider the limit $r \to 0$, $\mu r \to \infty$ where typically $\mu \approx 5 \cdot 10^3 \approx r^{-3}$. Defining $\sigma > 0$ via $\mu(r) = r^{-1-\sigma}$ and setting $\eta \equiv r$ we put for $\eta > 0$

$$j_{e\eta} \equiv j_e, \quad j_{i\eta} \equiv \frac{j_i}{\sqrt{\eta^\sigma}}.$$

We get for $\eta = 0$

$$f_{10}(x) = 1, \quad f_{20}(x) = \frac{1}{\sqrt{2(1-x)}},$$

$$F_{10}(x) = x, \quad F_{20}(x) = \sqrt{2}(1 - \sqrt{1-x}), \tag{11}$$

$$H_0(x) = \sqrt{2}(x - 1 + \sqrt{1-x})$$

and the rescaled model equations read for $\eta = 0$

$$\Phi_{00}'' = j_{e0} - j_{i0} \frac{1}{\sqrt{2(1 - \Phi_{00})}} \tag{12}$$

$$\begin{cases} \Phi_{00}(0) = 0,\ \Phi_{00}(1) = 1 \\ \Phi_{00}'(0) = \Phi_{00}'(1) = 0. \end{cases} \tag{13}$$

We proceed in analogy to Section 3.2 to get

COROLLARY 6 : *a)* Φ_{00} *is defined via*

$$\frac{1}{\sqrt[4]{2}}\int_0^{\Phi_{00}(x)}\frac{ds}{\sqrt{s-1+\sqrt{1-s}}}=\sqrt{2}\,\gamma_0\,x$$

with

$$\gamma_0=\frac{1}{\sqrt{2}}\frac{1}{\sqrt[4]{2}}\int_0^1\frac{ds}{\sqrt{s-1+\sqrt{1-s}}}\equiv I_c=\frac{\pi}{\sqrt[4]{8}}\approx 1.868$$

is strictly monotone increasing and the currents are given by

$$j_{e0}=\sqrt{2}\,I_c^2=\frac{\pi^2}{2}\approx 4.935\,,\quad j_{i0}=I_c^2=\frac{\pi^2}{2\sqrt{2}}\approx 3.489\,.$$

b) $j_{e\eta}=j_{e0}+O(\eta^{\min\{1,\sigma\}})\,,\quad j_{i\eta}=j_{i0}+O(\eta^{\min\{1,\sigma\}})$.
c) For $\eta=0$ *the currents read in SI units*

$$(14)\qquad \begin{cases} j_{e0}^{SI}=\dfrac{c\epsilon_0\,\pi^2}{2}\dfrac{\Phi_A}{L^2}\approx 0.01310\dfrac{\Phi_A}{L^2}\,,\\[2ex] j_{i0}^{SI}=\sqrt{\dfrac{e}{m_p}}\,\epsilon_0\dfrac{\pi^2}{2\sqrt{2}}\sqrt{\dfrac{Z}{N}}\dfrac{\Phi_A^{3/2}}{L^2}\approx 3.023\cdot 10^{-7}\sqrt{\dfrac{Z}{N}}\dfrac{\Phi_A^{3/2}}{L^2}\,. \end{cases}$$

Remark 7: (14) recovers (5) for $Z=N=1$.

4.5. Ultra relativistic electrons and relativistic ions

The assumption that the ions are in the non relativistic regime ceases to be true as soon as $\mu r\approx 1$ which amounts to $\Phi_A\approx 10^9$ V and $r\leqslant 1$. Setting $\eta\equiv r$ we let

$$j_{e\eta}\equiv j_e\,,\quad j_{i\eta}\equiv j_i\,.$$

For $\eta=0$ we get

$$f_{10}(x)=1\,,\quad f_{20}(x)=\frac{(1-x)+\mu r}{\sqrt{(1-x)^2+2\,\mu r(1-x)}}\,,$$

$$(15)\qquad F_{10}(x)=x\,,\quad F_{20}(x)=\sqrt{1+2\,\mu r}-\sqrt{(1-x)^2+2\,\mu r(1-x)}\,,$$

$$H_0(x)=\sqrt{1+2\,\mu r}\,(x-1)+\sqrt{(1-x)^2+2\,\mu r(1-x)}$$

and the rescaled model equations read for $\eta = 0$

$$\Phi_{00}'' = j_{e0} - j_{i\eta} \frac{(1 - \Phi_{00}) + \mu r}{\sqrt{(1 - \Phi_{00})^2 + 2\,\mu r(1 - \Phi_{00})}} \tag{16}$$

$$\begin{cases} \Phi_{00}(0) = 0, \Phi_{00}(1) = 1 \\ \Phi_{00}'(0) = \Phi_{00}'(1) = 0 . \end{cases} \tag{17}$$

Again we easily deduce

COROLLARY 7 : *a)* Φ_{00} *is defined via*

$$\int_0^{\Phi_{00}(x)} \frac{ds}{\sqrt{\sqrt{1 + 2\,\mu r}\,(s - 1) + \sqrt{(1 - s)^2 + 2\,\mu r(1 - s)}}} = \sqrt{2}\,\gamma_0\, x$$

with

$$\gamma_0 = \frac{1}{\sqrt{2}\,\sqrt[4]{2}} \int_0^1 \frac{ds}{\sqrt{\sqrt{1 + 2\,\mu r}\,(s - 1) + \sqrt{(1 - s)^2 + 2\,\mu r(1 - s)}}} \equiv I_d(\mu r) .$$

Φ_{00} *is strictly monotone increasing and the currents are given by*

$$j_{e0} = \sqrt{1 + 2\,\mu r}\, I_d^2(\mu r) , \quad j_{i0} = I_d^2(\mu r)$$

b) $j_{e\eta} = j_{e0} + O(\eta)$, $\qquad j_{i\eta} = j_{i0} + O(\eta)$.
c) For $\eta = 0$ *the currents read in SI units*

$$j_{e0}^{SI} = c\epsilon_0\, I_d^2\left(\frac{m_p}{m_e}\frac{N}{Z}\right) \sqrt{\frac{2\, m_p\, c^2}{e}\frac{N}{Z} + \Phi_A}\; \frac{\Phi_A^{1/2}}{L^2}$$

$$\approx 2.654 \cdot 10^{-3}\, I_d^2\left(1\,837\,\frac{N}{Z}\right) \sqrt{1.887 \cdot 10^9\,\frac{N}{Z} + \Phi_A}\; \frac{\Phi_A^{1/2}}{L^2} ,$$

$$j_{i0}^{SI} = c\epsilon_0\, I_d^2\left(\frac{m_p}{m_e}\frac{N}{Z}\right) \frac{\Phi_A}{L^2} \approx 2.654 \cdot 10^{-3}\, I_d^2\left(1\,837\,\frac{N}{Z}\right) \frac{\Phi_A}{L^2} .$$

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