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GLOBAL SUPERCONVERGENCE APPROXIMATIONS OF THE MIXED FINITE ELEMENT METHOD FOR THE STOKES PROBLEM AND THE LINEAR ELASTICITY EQUATION (*)

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Abstract. — *A three fields formulation for solving the Stokes problem and the equation of linear elasticity proposed by Baranger and Sandri is developed in this paper, using finite element subspaces based upon a modified pair of « $Q_1 - Q_0$ » element. It is proved that global superconvergence approximations can be obtained for all three fields, not only extra stress tensor, velocity but also pressure.*

Key words : Superconvergence, mixed finite element, Stokes and linear elasticity problem.

Résumé. — *On développe une formulation à trois champs pour résoudre le problème de Stokes et l'équation d'élasticité linéaire proposée par Baranger et Sandri. On utilise les sous-espaces éléments finis basés sur l'élément modifié $Q_1 - Q_0$. On prouve que les approximations de superconvergence peuvent être obtenues pour tous les champs, non seulement pour le tenseur des extracontraintes et la vitesse, mais aussi la pression.*

1. INTRODUCTION

Let Ω be a domain consisting of connected rectangles and

$$\Sigma = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ij}, \tau_{ij} \in L^2(\Omega), i, j = 1, 2\},$$

$$\mathbf{V} = \{\mathbf{v} = (v_1, v_2) : v_i \in H_0^1(\Omega), i = 1, 2\},$$

$$L = L_0^2(\Omega) \equiv \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}.$$

It is known that both the Stokes problem and the linear elasticity equation can be written in Oldroyd's formulation (see Baranger and Sandri [1992]) :

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Find $(\sigma, \mathbf{u}, p) \in \Sigma \times \mathbf{V} \times L$ such that

$$(1) \begin{cases} (\sigma, \tau) - 2\,\alpha(d(\mathbf{u}), \tau) & = 0, \quad \forall \tau \in \Sigma\,, \\ (\sigma, d(\mathbf{u})) + 2(1-\alpha)\,(d(\mathbf{u}), d(\mathbf{v})) - (p, \nabla \cdot \mathbf{v}) & = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (\nabla \cdot \mathbf{u}, q) + \beta(p, q) & = 0, \quad \forall q \in L\,, \end{cases}$$

where $\alpha \in [0, 1]$ and $\beta \geqslant 0$ are constants.

In the context of the Stokes problem, σ denotes extra stress tensor, $\mathbf{u}$ velocity and p pressure ; $d(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$ rate of strain tensor and $\nabla \mathbf{u}$ gradient velocity tensor. While for the linear elasticity problem, $\mathbf{u}$ denotes displacement, $\sigma = 2\,\alpha\varepsilon(\mathbf{u})$ a scaled « non Newtonian » extra stress tensor and $\varepsilon(\mathbf{u})$ the strain tensor and $\beta = (1 - 2\,\nu)/2\,\nu$ with the Poisson ratio ν.

Mixed finite element schemes for this version have been recently proposed by Fortin and Pierre [1989] and Franca and Stenberg [1991] on the Maxwell model corresponding to $\alpha = 1$ and by Baranger and Sandri [1992] for $\alpha \in [0,1)$. For the approximation accuracy, only have the usual estimates been obtained in literature. Regarding the version of Oldroyd's model, whether superconvergence approximations can be obtained is an interesting problem and to solve it is certainly valuful for both the theory and practice.

In this paper, we shall pay attention to superconvergence aspect in this version. It is proved that global superconvergence approximations can be obtained for all three variables, not only extra stress tensor, velocity but also pressure if finite element spaces based upon a modified pair of « $Q_1 - Q_0$ » elements are chosen. Our theoretical bases are the identity techniques and interpolation arguments developed by Lin, Yan and Zhou [19], see a survey by Zhou, Li and Yan [26]. It has been shown that these techniques and arguments are powerful for convergence analysis especially for improving accuracy analysis in mixed finite element methods for solving the Stokes equations provided certain preprocesses on the finite element mesh and imposed a kind of postprocess on the finite element approximation. We mention here related work [18, 25, 26] in this direction.

2. PRELIMINARIES

The space Σ of symmetric tensors with $L^2(\Omega)$ components is equiped with the scalar product $(\sigma, \tau) = \int_\Omega \sigma_{ij}\,\tau_{ij}$ with associated norm $|\tau|_0$; $\mathbf{V}$ is equiped with the scale product $(\mathbf{u}, \mathbf{v})_{\mathbf{V}} = (d(\mathbf{u}), d(\mathbf{v}))$ with associated norm $|\mathbf{v}|_1 = (d(\mathbf{u}), d(\mathbf{v}))^{1/2}$. Other notations see Baranger and Sandri [1992] and Girault and Raviart [1986].

The variational formulation of problem (1) can be written in the following form (see Baranger and Sandri [1992]) :

Find $(\sigma, \mathbf{u}, p) \in \mathbf{\Sigma} \times \mathbf{V} \times L$ such that

$$(2) \qquad B(\sigma, \mathbf{u}, p\,; \tau, \mathbf{v}, q) = -2\,\alpha(\mathbf{f}, \mathbf{v})\,, \quad \forall(\tau, \mathbf{v}, q) \in \mathbf{\Sigma} \times \mathbf{V} \times L\,,$$

where

$$\begin{aligned} B(\sigma, \mathbf{u}, p\,; \tau, \mathbf{v}, q) = (\sigma, \tau) &- 2\,\alpha(d(\mathbf{u}), \tau) - 2\,\alpha(d(\mathbf{v}), \sigma) \\ &- 4\,\alpha(1-\alpha)\,(d(\mathbf{u}), d(\mathbf{v})) \\ &+ 2\,\alpha(\nabla \cdot \mathbf{v}, p) + 2\,\alpha(\nabla \cdot \mathbf{u}, q) + 2\,\alpha\beta(p, q) \end{aligned}$$

with $\beta \in [0, \beta_0)$.

For approximating (2), we need introduce three finite element spaces. We start with a subdivision T_h of $\Omega \subset R^2$ into rectangles whose edges respectively parallel to $x-$ and $y-$ axis with size h. Subsequently we divide each rectangle into four smaller rectangles by joining the midpoints, thus creating another subdivision $T_{h/2}$ of Ω into rectangles. For a path $e \in T_h$ consisting of elements $k_{e,i} \in T_{h/2}(1 = 1, 2, 3, 4)$ numbered in the counterclockwise sense. Define a pressure space L_h through the following choice of basis. For each rectangle $e \in T_h$ we define three basis functions

$$\mu_{e,1} = \begin{cases} 1, & \text{on } k_{e,1} \cup k_{e,2}\,, \\ 0, & \text{others}\,, \end{cases}$$

$$\mu_{e,2} = \begin{cases} 1, & \text{on } k_{e,1} \cup k_{e,4}\,, \\ 0, & \text{others}\,, \end{cases}$$

$$\mu_{e,3} = \begin{cases} 1, & \text{on } k_{e,3} \cup k_{e,4}\,, \\ 0, & \text{others}\,. \end{cases}$$

Of course, outside the particular rectangle $e \in T_h$, the basis functions vanish as well. This pressure space L_h consists of three of the four possible piecewise constants associated with the four rectangles in $T_{h/2}$ contained within a single rectangle in T_h and that have zero mean over Ω, i.e.,

$$(3) \qquad L_h = \{q \in L_0^2(\Omega) : q|_e \in \operatorname{span}\{\mu_{e,i} : i = 1, 2, 3\}, e \in T_h\}\,.$$

The velocity space $\mathbf{V_h}$ and the extra stress tensor space $\boldsymbol{\Sigma}_\mathbf{h}$ are defined as

$$\mathbf{V_h} = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_k \in Q_1(k)^2, \forall k \in T_{h/2}\}, \tag{4}$$

$$\boldsymbol{\Sigma}_\mathbf{h} = \{\sigma \in \boldsymbol{\Sigma} : \sigma_{ij}|_e \in Q_0(e), 1 \leqslant i, j \leqslant 2, \forall e \in T_h\}. \tag{5}$$

It is known that the pair of spaces $(\mathbf{V_h}, L_h)$ satisfies the inf-sup condition (see Brezzi and Fortin [1991], Girault and Raviart [1986], or Gunzburger [1986]) :

$$\sup_{\mathbf{v} \in \mathbf{V_h}} \frac{(\nabla \cdot \mathbf{v}, q)}{|\mathbf{v}|_1} \geqslant c|q|_0, \quad \forall q \in L_h \tag{6}$$

for some constant $c > 0$.

Denote

$$j_h p = \tilde{j}_h p + (-1)^i \alpha_e(p), \quad \text{on } k_{e,i}, \quad i = 1, 2, 3, 4,$$

where

$$\tilde{j}_h p = \frac{1}{|k_{e,i}|} \int_{k_{e,i}} p, \quad \text{on } k_{e,i},$$

$$\alpha_e(p) = \frac{1}{4}\left(\frac{1}{|k_{e,1}|}\int_{k_{e,1}} p - \frac{1}{|k_{e,2}|}\int_{k_{e,2}} p + \frac{1}{|k_{e,3}|}\int_{k_{e,3}} p - \frac{1}{|k_{e,4}|}\int_{k_{e,4}} p\right).$$

One sees that $\alpha_e(p) = 0$ if p is linear on e, which yields, by Bramble-Hilbert lemma :

$$|\alpha_e(p)| \leqslant ch \, \|p\|_{2,e}, \tag{7}$$

$$\|j_h p - \tilde{j}_h p\|_0 \leqslant ch^2 \|p\|_2. \tag{8}$$

LEMMA :

If $(\sigma, \mathbf{u}, p) \in (H^2(\Omega))^4 \times (H_0^1(\Omega) \cap H^3(\Omega))^2 \times (L_0^2(\Omega) \cap H^2(\Omega))$,

then

$$B(\sigma - \mathbf{j_h}\sigma, \mathbf{u} - \mathbf{i_h}\mathbf{u}, p - j_h p; \tau, \mathbf{v}, q) \tag{9}$$
$$\leqslant ch^2(|\tau|_0 + |\mathbf{v}|_1 + \|q\|_0), \quad \forall(\tau, \mathbf{v}, q) \in \boldsymbol{\Sigma}_\mathbf{h} \times \mathbf{V_h} \times L_h,$$

where i_h *is the usual bilinear interpolation operator with respect to* $T_{h/2}$ and $\mathbf{j_h}\,\sigma = (j_h(\sigma)_{ij})_{1 \le i,j \le 2}$ for $\sigma = ((\sigma)_{ij})_{1 \le i,j \le 2}$.

Proof: For any $k \in T_{h/2}$, let (x_k, y_k) be the center of k, $h_{k,x}$, $h_{k,y}$ be its widths in the x- and y-direction, respectively, and denote $F_k = \frac{1}{2}((y - y_k)^2 - (h_{k,y}/2)^2)$ and $E_k = \frac{1}{2}((x - x_k)^2 - (h_{k,x}/2)^2)$.

By the identities

$$\int_k \partial_x(w - i_h w) = \int_k F_k\, \partial_y^2 \partial_x w\,, \quad k \in T_{h/2}\,, \tag{10}$$

$$\int_k \partial_y(w - i_h w) = \int_k E_k\, \partial_x^2 \partial_y w\,, \quad k \in T_{h/2}\,, \tag{11}$$

we obtain firstly

$$|(\nabla \cdot (\mathbf{u} - \mathbf{i_h}\,\mathbf{u}), q)| \leqslant ch^2 \|\mathbf{u}\|_3 \|q\|_0\,, \quad \forall q \in L_h\,, \tag{12}$$

$$|(d(\mathbf{u} - \mathbf{i_h}\,\mathbf{u}), \tau)| \leqslant ch^2 \|\mathbf{u}\|_3 |\tau|_0\,, \quad \forall \tau \in \mathbf{\Sigma}_h\,. \tag{13}$$

Next we turn to prove

$$|(d(\mathbf{u} - \mathbf{i_h}\,\mathbf{u}), d(\mathbf{v}))| \leqslant ch^2 \|\mathbf{u}\|_3 |\mathbf{v}|_1\,, \quad \forall \mathbf{v} \in \mathbf{V}_h\,. \tag{14}$$

In fact, combining the identities (10), (11) and

$$\int_k \partial_x(w - i_h w)\,(y - y_k) = \frac{1}{3} \int_k F_k (y - y_k)\, \partial_y^2 \partial_x w\,,$$

$$\int_k \partial_x(w - i_h w)\,(x - x_k) = -\int_k E_k\, \partial_x^2 w\,,$$

we obtain (*cf.* Lin, Yan and Zhou [19], Lin and Zhu [20] or Zhou and Li [25]), for bilinear function φ on k,

$$\int_k \partial_x(w - i_h w)\, \partial_x\varphi = \int_k \partial_x(w - i_h w)\,(\partial_x\varphi(x, y_k) + (y - y_k)\,\partial_x\partial_y\varphi(x, y))$$

$$= \int_k F_k\, \partial_y^2\partial_x w\, \partial_x\varphi - \frac{2}{3}\int_k F_k(y - y_k)\,\partial_y^2\partial_x w\, \partial_y\partial_x\varphi(x, y)\,, \tag{15}$$

$$\int_k \partial_x(w - i_h w)\, \partial_y\varphi = \int_k \partial_x(w - i_h w)\,(\partial_y\varphi(x_k, y) + (x - x_k)\,\partial_x\partial_y\varphi(x, y))$$

$$= \int_k F_k(\partial_y\varphi - (x - x_k)\,\partial_x\partial_y\varphi(x, y))\,\partial_y^2\partial_x w$$

$$- E_k\,\partial_x\partial_y\varphi\,\partial_x^2 w$$

$$= -\left(\int_{s_{k,4}} - \int_{s_{k,3}}\right) E_k\,\partial_x\varphi\,\partial_x^2 w + \int_k E_k\,\partial_x\varphi\,\partial_x^2\partial_y w$$

$$+ \int_k F_k(\partial_y\varphi - (x - x_k)\,\partial_x\partial_y\varphi(x, y))\,\partial_y^2\partial_x w\,, \tag{16}$$

where $s_{k,i}(i = 3, 4)$ are the down and up edges of k. If $T_{h/2}$ is almost uniformly, i.e., for any pair of neighboured elements k and k', there holds

$$|h_{k,x} - h_{k',x}| + |h_{k,y} - h_{k',y}| \leqslant ch^2\,,$$

then by Abel's summation, (14) is obtained from (15), (16) and similar identities to

$$\int_k \partial_y(w - i_h w)\,\partial_y\varphi \quad \text{and} \quad \int_k \partial_y(w - i_h w)\,\partial_x\varphi\,. \tag{17}$$

Finally, by the similar arguments, for bilinear function φ on k, we have

$$\int_k (p - \tilde{j}_h p)\,\partial_x\varphi = \int_k (p - \tilde{j}_h p)\,(\partial_x\varphi(x, y_k) + (y - y_k)\,\partial_x\partial_y\varphi(x, y))$$

$$= \int_k (p - \tilde{j}_h p)\,(y - y_k)\,\partial_x\partial_y\varphi(x, y)$$

$$= -\int_k F_k\,\partial_y p\,\partial_x\partial_y\varphi(x, y)$$

$$= -\left(\int_{s_{k,2}} - \int_{s_{k,1}}\right) F_k\,\partial_y p\,\partial_y\varphi + \int_k F_k\,\partial_x\partial_y p\,\partial_y\varphi\,,$$

where $s_{k,i}(i = 1, 2)$ are the left and right edges of k. If $T_{h/2}$ is almost uniformly, we obtain, by Abel's summation

$$(18) \qquad |(\nabla \cdot \mathbf{v}, p - \tilde{j}_h p)| \leqslant ch^2 \|p\|_2 |\mathbf{v}|_1, \quad \forall \mathbf{v} \in \mathbf{V_h},$$

$$(19) \qquad |(d(\mathbf{v}), \sigma - \tilde{j}_{\mathbf{h}} \sigma)| \leqslant ch^2 |\sigma|_2 |\mathbf{v}|_1, \quad \forall \mathbf{v} \in \mathbf{V_h}.$$

And (8) and (18) implies

$$(20) \qquad |(\nabla \cdot \mathbf{v}, p - j_h p)| \leqslant ch^2 \|p\|_2 |\mathbf{v}|_1, \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

which together with (12)-(14), (19), (20) and

$$(21) \qquad (\sigma - \tilde{j}_{\mathbf{h}} \sigma, \tau) + 2\,\alpha\beta(p - \tilde{j}_h p, q) = 0, \quad \forall (\tau, q) \in \boldsymbol{\Sigma}_h \times L_h$$

leads to

$$(22) \qquad \begin{aligned} &B(\sigma - \tilde{j}_{\mathbf{h}} \sigma, \mathbf{u} - \mathbf{i_h} \mathbf{u}, p - j_h p\,; \tau, \mathbf{v}, q) \\ &\leqslant ch^2 (|\tau|_0 + |\mathbf{v}|_1 + \|q\|_0), \quad \forall (\tau, \mathbf{v}, q) \in \boldsymbol{\Sigma}_{\mathbf{h}} \times \mathbf{V_h} \times L_h, \end{aligned}$$

where $\tilde{j}_h \sigma = (\tilde{j}_h(\sigma)_{ij})_{1 \leqslant i,j \leqslant 2}$ for $\sigma = ((\sigma)_{ij})_{1 \leqslant i,j \leqslant 2}$. From (8), we complete the proof.

To obtain superconvergence approximations, we need to introduce two operators. One is the usual biquadratic nodal interpolation operator I_h on T_h, another is an interpolation operator J_h with respect to the partition T_h, which is defined by the follows : $J_h p|_e \in Q_1(e)$, for any $p \in L^2(\Omega)$ and $e \in T_h$ and satisfying (*cf.* Lin, Yan and Zhou [19] and Zhou and Li [25]) :

$$(23) \qquad \int_{k_{e,i}} J_h q = \int_{k_{e,i}} q, \quad \forall q \in L^2(\Omega), \quad \text{on } k_{e,i} \subset e.$$

It is easy to see that the operators I_h and J_h possess the following properties :

$$(24) \qquad I_h i_h = I_h, \quad J_h \tilde{j}_h = J_h,$$

$$(25) \qquad \|\mathbf{I_h} \mathbf{w} - \mathbf{w}\|_1 \leqslant ch^2 \|\mathbf{w}\|_3, \quad \|J_h q - q\|_0 \leqslant ch^2 \|q\|_2,$$

where $\mathbf{I_h} \mathbf{w} = (I_h w_1, I_h w_2)$ for $\mathbf{w} = (w_1, w_2)$.

Following Baranger and Sandri [1992], we present here the mixed finite element approximation to the problem (2) :

Find $(\sigma_{\mathbf{h}}, \mathbf{u_h}, p_h) \in \boldsymbol{\Sigma}_{\mathbf{h}} \times \mathbf{V_h} \times L_h$ such that

$$(26) \qquad B(\sigma_{\mathbf{h}}, \mathbf{u_h}, p_h) = -\,2\,\alpha(\mathbf{f}, \mathbf{v}), \forall (\tau, \mathbf{v}, q) \in \boldsymbol{\Sigma}_{\mathbf{h}} \times \mathbf{V_h} \times L_h.$$

For $\alpha \in [0, 1)$ and $\beta \in [0, \beta_0)$ with some constant $\beta_0 > 0$, the finite element solution $(\sigma_{\mathbf{h}}, \mathbf{u}_{\mathbf{h}}, p_h)$ exists, and the following optimal error estimate is valid (see Baranger and Sandri [1992]) :

$$(27) \quad \|\sigma_{\mathbf{h}} - \sigma\|_0 + \|\mathbf{u}_{\mathbf{h}} - \mathbf{u}\|_1 + \|p_h - p\|_0 \leqslant ch(\|\sigma\|_1 + |\mathbf{u}|_2 + |p|_1) .$$

3. SUPERCONVERGENCE

We shall prove that all the three approximations are superconvergent globally. First of all, we have an error equation

$$(28) \quad B(\sigma - \sigma_{\mathbf{h}}, \mathbf{u} - \mathbf{u}_{\mathbf{h}}, p - p_h ; \tau, \mathbf{v}, q) = 0 , \quad \forall (\tau, \mathbf{v}, q) \in \mathbf{\Sigma}_h \times \mathbf{V}_h \times L_h .$$

By Preliminaries in the previous Section, we obtain the following theorem :

THEOREM : *Let*

$$(\sigma, \mathbf{u}, p) \in (H^2(\Omega))^4 \times (H_0^1(\Omega) \cap H^3(\Omega))^2 \times (L_0^2(\Omega) \cap H^2(\Omega))$$

be the exact solution of (2) and $(\sigma_{\mathbf{h}}, \mathbf{u}_{\mathbf{h}}, p_h) \in \mathbf{\Sigma}_{\mathbf{h}} \times \mathbf{V}_{\mathbf{h}} \times L_h$ *be the finite element solution of (26). Then*

$$(29) \quad \|\mathbf{J}_{\mathbf{h}} \sigma_{\mathbf{h}} - \sigma\|_0 + \|\mathbf{I}_{\mathbf{h}} \mathbf{u}_{\mathbf{h}} - \mathbf{u}\|_1 + \|J_h p_h - p\|_0 \leqslant ch^2 ,$$

where $\mathbf{J}_{\mathbf{h}} \sigma_{\mathbf{h}} = (J_h(\sigma_{\mathbf{h}})_{i,j})_{1 \leqslant i,j \leqslant 2}$ *for* $\sigma_{\mathbf{h}} = ((\sigma_{\mathbf{h}})_{i,j})_{1 \leqslant i,j \leqslant 2}$.

Proof: One sees that (28) implies

$$B(\sigma_{\mathbf{h}} - \mathbf{j}_{\mathbf{h}} \sigma, \mathbf{u}_{\mathbf{h}} - \mathbf{i}_{\mathbf{h}} \mathbf{u}, p_h - j_h p ; \tau, \mathbf{v}, q)$$

$$(30) \quad = B(\sigma - \mathbf{j}_{\mathbf{h}} \sigma, \mathbf{u} - \mathbf{i}_{\mathbf{h}} \mathbf{u}, p - j_h p ; \tau, \mathbf{v}, q) , \forall (\tau, \mathbf{v}, q) \in \mathbf{\Sigma}_h \times \mathbf{V}_h \times L_h .$$

Note that j_h maps $L_0^2(\Omega)$ into L_h, we have, by the BB stability (6), the lemma in the previous section and the theorems 4.1 and 4.2 in [2],

$$\|\sigma_{\mathbf{h}} - \mathbf{j}_{\mathbf{h}} \sigma\|_0 + \|\mathbf{u}_{\mathbf{h}} - \mathbf{i}_{\mathbf{h}} \mathbf{u}\|_1 + \|p_h - j_h p\|_0 \leqslant ch^2 ,$$

or

$$(31) \quad \|\sigma_{\mathbf{h}} - \tilde{j}_{\mathbf{h}} \sigma\|_0 + \|\mathbf{u}_{\mathbf{h}} - \mathbf{i}_{\mathbf{h}} \mathbf{u}\|_1 + \|p_h - \tilde{j}_h p\|_0 \leqslant ch^2 .$$

On the other hand, one sees that

$$\|\mathbf{J}_h \tau\|_0 \leq c\|\tau\|_0, \quad \forall \tau \in \Sigma_h, \tag{32}$$

$$\|\mathbf{I}_h \mathbf{v}\|_1 \leq c\|\mathbf{v}\|_1, \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{33}$$

$$\|J_h q\|_0 \leq c\|q\|_0, \quad \forall q \in L_h. \tag{34}$$

Thus, combining (8), (24), (25), (31)-(34) and the identities

$$\mathbf{J}_h \sigma_h - \sigma = \mathbf{J}_h(\sigma_h - \tilde{j}_h \sigma_h) + \mathbf{J}_h \tilde{j}_h \sigma - \sigma, \tag{35}$$

$$\mathbf{I}_h \mathbf{u}_h - \mathbf{u} = \mathbf{I}_h(\mathbf{u}_h - \mathbf{i}_h \mathbf{u}) + \mathbf{I}_h \mathbf{i}_h \mathbf{u} - \mathbf{u}, \tag{36}$$

$$J_h p_h - p = J_h(p_h - \tilde{j}_h p) + J_h \tilde{j}_h p - p, \tag{37}$$

we finish the proof.

In this paper, only are superconvergence approximations for $\alpha \in [0, 1)$ and « $Q_1 - Q_0$ » element presented. The forthcoming papers will deal with the case $\alpha = 1$ and other mixed elements.

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