M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 30, nº 4 (1996), p. 385-400
<http://www.numdam.org/item?id=M2AN_1996__30_4_385_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 30, n° 4, 1996, p. 385 à 400)

A POSTERIORI ERROR ESTIMATORS FOR NONCONFORMING FINITE ELEMENT METHODS

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Abstract. — *We introduce two* a posteriori *error estimators for piecewise linear nonconforming finite element approximation of second order elliptic problems. We prove that these estimators are equivalent to the energy norm of the error.*
Finally, we present several numerical experiments showing the good behavior of the estimators when they are used as local error indicators for adaptive refinement.

Résumé. — *On introduit deux « estimateurs » d'erreur* a posteriori *pour une approximation par une méthode d'éléments finis non conforme d'un problème elliptique du second ordre. On montre que ces « estimateurs » sont équivalents à la norme d'énergie de l'erreur. Finalement, on présente diverses expériences numériques montrant le bon comportement de ces estimateurs, lorsqu'on les utilise comme indicateurs locaux d'erreurs pour un raffinement adaptatif.*

1. INTRODUCTION

This paper deals with a posteriori error estimators and adaptivity for nonconforming finite element methods.

There have been several motivations for introducing nonconforming methods. For example, to avoid the necessity of smooth elements in fourth order problems or to treat constrained minimization problems such as the Stokes equations (see [8] for a review of this kind of methods). Also, they have been shown to be related with mixed methods (see [1], [9]). We also refer to [2] for more recent applications in elasticity.

As a first step we consider here the case of nonconforming piecewise linear approximations of second order elliptic problems [7]. However,

(*) Manuscript received February 26, 1993.

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similar techniques can be applied in other situations. In particular, in a forthcoming paper we obtain similar results for the Stokes equations.

In the conforming case several approaches have been introduced to define error estimators for different problems by using the residual equation (see for example [4], [5], [6], [12]). In order to extend these techniques to nonconforming approximations, the main difficulty is the treatment of the consistency terms arising in the error equation in this case. These terms depend on the exact solution and cannot be neglected. Our technique is based on the use of a Helmholtz decomposition together with some orthogonality relations for the error. In this way we are able to extend to the nonconforming case the ideas developed in [12].

We define two error estimators and prove that they are equivalent to the energy norm of the error.

The rest of the paper is organized as follows. In Section 2 we introduce the model problem and recall the finite element approximation. Section 3 deals with the error estimators and their equivalence with the error and finally in Section 4 we present several numerical results.

2. MODEL PROBLEM AND FINITE ELEMENT APPROXIMATION

Let $\Omega \subset \mathbf{R}^2$ be a simply connected polygon. We consider the model problem

$$\begin{cases} - \operatorname{div}\,(a\,\nabla u) = f\,, & \text{in} \quad \Omega \\ u = g_1\,, & \text{on} \quad \Gamma_1 \\ a\,\dfrac{\partial u}{\partial n} = g_2\,, & \text{on} \quad \Gamma_2 \end{cases} \tag{2.1}$$

where Γ_1 and Γ_2 are disjoint sets such that $\Gamma_1 \neq \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and the coefficient $a = a(x)$ is bounded by above and below by positive constants.

We use standard notation for Sobolev spaces, norms and seminorms and for $\Gamma \subset \partial\Omega$ we set

$$H^1_\Gamma = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}\,.$$

Then, the solution of problem (2.1) satisfies

$$\int_\Omega a\,\nabla u\,.\,\nabla v = \int_\Omega f v + \int_{\Gamma_2} g_2\,v\,, \quad \forall v \in H^1_{\Gamma_1}\,. \tag{2.2}$$

Assume that we have a family $\{\mathcal{T}_j\}$ of triangulations of Ω such that any two triangles in $\mathcal{T}_j$ share at most a vertex or an edge and any $\mathcal{T}_j$ is consistent with the boundary data, i.e., a boundary side is contained in either Γ_1 or Γ_2.

For any $\mathcal{T}_j$ we introduce the nonconforming finite element space

$$V^j = \left\{v \in L^2(\Omega) : v\,|_T \in \mathscr{P}_1, \quad \forall T \in \mathcal{T}_j \text{ and } v \text{ is continuous at midside points}\right\}$$

($\mathscr{P}_1$ denotes the space of linear polynomials) and for $\Gamma \subset \partial\Omega$ let,

$$V^j_\Gamma = \{v \in V^j : v = 0 \text{ at midside points contained in } \Gamma\} .$$

In our analysis we will also use the standard conforming piecewise linear finite element spaces,

$$M^j = \left\{v \in H^1(\Omega) : v\,|_T \in \mathscr{P}_1, \quad \forall T \in \mathcal{T}_j\right\}$$

and

$$M^j_\Gamma = M^j \cap H^1_\Gamma .$$

Let M_ℓ be the midpoint of a side ℓ. Then the nonconforming finite element approximation to the solution of problem (3.1) is defined by $u_j \in V^j$ and,

$$\sum_{T \in \mathcal{T}_j} \int_T a\, \nabla u_j \,.\, \nabla v = \int_\Omega fv + \int_{\Gamma_2} g_2\, v\,, \quad \forall v \in V^j_{\Gamma_1} \tag{2.3}$$

and

$$u_j(M_\ell) = g_1(M_\ell)\,, \quad \forall \ell \subset \Gamma_1 .$$

We end this section with some notation that we will need in the definition and analysis of the error estimators.

Let E_I be the set of all interior edges and E_T the set of edges of T. For each interior edge ℓ we choose an arbitrary normal direction n and denote the two triangles sharing this edge T_{in} and T_{out} where n points outwards T_{in}. For a boundary side ℓ we take n as the outward normal. If $n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ we define the tangent on ℓ by $t = \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix}$ and we set,

$$\left[\!\left[a \frac{\partial u_j}{\partial n} \right]\!\right]_\ell (x) = a\,\big|_{T_{\text{out}}}(x)\, \nabla (u_j\,|_{T_{\text{out}}}) \cdot n - a\,\big|_{T_{\text{in}}}(x)\, \nabla (u_j\,|_{T_{\text{in}}}) \cdot n\,, \quad \forall x \in \ell$$

and

$$\left[\!\left[\frac{\partial u_j}{\partial t} \right]\!\right]_\ell = \nabla (u_j\,|_{T_{\text{out}}}) \cdot t - \nabla (u_j\,|_{T_{\text{in}}}) \cdot t .$$

Note, that these values are independent of the choice of n.

We define the jump on ℓ for $v \in V^j$ by

$$[\![v]\!]_\ell(x) = v\,|_{T_{\text{out}}}(x) - v\,|_{T_{\text{in}}}(x)\,, \quad \forall x \in \ell\,.$$

Finally for $\varphi \in H^1(\Omega)$ we set,

$$\mathbf{curl}\ \varphi = \begin{pmatrix} -\dfrac{\partial \varphi}{\partial x_2} \\ \dfrac{\partial \varphi}{\partial x_1} \end{pmatrix}.$$

3. ERROR ESTIMATORS

In this section we introduce two error estimators and prove that they are equivalent to the energy norm of the error.

For the sake of simplicity we assume that,

$$a \quad \text{is piecewise constant}\,, \tag{3.1a}$$

$$g_1 \quad \text{is piecewise linear}\,, \tag{3.1b}$$

$$g_2 \quad \text{is piecewise constant}\,, \tag{3.1c}$$

and

$$f \quad \text{is piecewise constant} \tag{3.1d}$$

and denote with f_T the value of f on the element T.

Assumptions (3.1) are not very restrictive, in fact, in the general case we may replace the data by appropriate interpolations and it is not difficult to see that our theorems can be generalized assuming local regularity of the data (we refer to [3], [12] for details).

For a side ℓ define $J_{\ell,n}$ and $J_{\ell,t}$ by

$$J_{\ell,n} = \begin{cases} \left[\!\!\left[a \dfrac{\partial u_j}{\partial n} \right]\!\!\right]_\ell, & \text{if } \ell \in E_I \\ 2\left(g_2 - a \dfrac{\partial u_j}{\partial n} \right)\Big|_\ell, & \text{if } \ell \subset \Gamma_2 \\ 0\,, & \text{if } \ell \subset \Gamma_1 \end{cases}$$

and

$$J_{\ell,t} = \begin{cases} \left[\!\!\left[\dfrac{\partial u_j}{\partial t} \right]\!\!\right]_\ell, & \text{if } \ell \in E_I \\ 2\left(\dfrac{\partial g_1}{\partial t} - \dfrac{\partial u_j}{\partial t} \right)\Big|_\ell, & \text{if } \ell \subset \Gamma_1 \\ 0\,, & \text{if } \ell \subset \Gamma_2\,. \end{cases}$$

The following lemma gives an error equation which is one of the key points in our error analysis.

LEMMA 3.1 : *Under the assumptions (3.1) and for $v \in H^1_{\Gamma_1}$ and $\varphi \in H^1_{\Gamma_2}$ the error $e = u - u_j$ satisfies,*

$$\sum_{T \in \mathcal{T}_j} \int_T a \nabla e \cdot (\nabla v + a^{-1} \, \mathbf{curl} \, \varphi) =$$

$$= \sum_{T \in \mathcal{T}_j} \left\{ \int_T f v + \frac{1}{2} \sum_{\ell \in E_T} \int_\ell (J_{\ell, n} v + J_{\ell, t} \varphi) \right\} . \quad (3.2)$$

Proof : Integrating by parts in each element we have

$$\sum_{T \in \mathcal{T}_j} \int_T a \nabla e \cdot (\nabla v + a^{-1} \, \mathbf{curl} \, \varphi) =$$

$$= \sum_{T \in \mathcal{T}_j} \left\{ \int_T f v + \int_{\partial T} a \frac{\partial e}{\partial n} v + \int_{\partial T} \frac{\partial e}{\partial t} \varphi \right\}$$

$$= \sum_{T \in \mathcal{T}_j} \left\{ \int_T f v - \sum_{\ell \in E_T \cap E_I} \left[\int_\ell a \frac{\partial u_j}{\partial n} v + \int_\ell \frac{\partial u_j}{\partial t} \varphi \right] \right\}$$

$$+ \sum_{\ell \subset \Gamma_2} \int_\ell \left(g_2 - a \frac{\partial u_j}{\partial n} \right) v + \sum_{\ell \subset \Gamma_1} \int_\ell \left(\frac{\partial g_1}{\partial t} - \frac{\partial u_j}{\partial t} \right) \varphi$$

which, in view of the definitions of $J_{\ell, n}$ and $J_{\ell, t}$, gives 3.2. □

The second tool in the analysis are the orthogonality relations for the error which are given in the next lemma.

LEMMA 3.2 : *The error satisfies*

$$\sum_{T \in \mathcal{T}_j} \int_T a \nabla e \, . \, \nabla v = 0 \, , \quad \forall v \in M^j_{\Gamma_1} \quad (3.3)$$

$$\sum_{T \in \mathcal{T}_j} \int_T \nabla e \, . \, \mathbf{curl} \, \varphi = 0 \, , \quad \forall \varphi \in M^j \text{ such that } \frac{\partial \varphi}{\partial t} = 0 \text{ on } \Gamma_2 \, . \quad (3.4)$$

Proof : Since $M^j_{\Gamma_1} \subset H^1_{\Gamma_1} \cap V^j_{\Gamma_1}$ we can use (2.2) and (2.3) for $v \in M^j_{\Gamma_1}$ and substracting (2.3) from (2.2) we obtain (3.3).

The orthogonality property (3.4) is known (see for example [2]) and easy to verify, in fact, integration by parts yields,

$$\sum_{T \in \mathcal{T}_j} \int_T \nabla e \, . \, \mathbf{curl} \, \varphi = \sum_{\ell \subset \Gamma_1} \int_\ell (g_1 - u_j) \frac{\partial \varphi}{\partial t} - \sum_{\ell \in E_I} \int_\ell [\![u_j]\!]_\ell \frac{\partial \varphi}{\partial t}$$

and the right hand side vanishes because $\left.\frac{\partial \varphi}{\partial t}\right|_{\ell}$ is constant, $[\![u_j]\!]_{\ell}$ is zero at the midside point of ℓ for every $\ell \in E_I$ and $g_1 - u_j$ is zero at the midside point of ℓ for $\ell \subset \Gamma_1$. $\square$

Let us now define the local error estimator η_T by,

$$\eta_T^2 = f_T^2 |T|^2 + \frac{1}{2} \sum_{\ell \in E_T} (J_{\ell, n}^2 + J_{\ell, t}^2) |\ell|^2$$

and the global one by,

$$\eta^2 = \sum_{T \in \mathcal{T}_j} \eta_T^2 .$$

When doing adaptive refinement in problems involving singularities the generated meshes are usually highly nonuniform (i.e. the elements of $\mathcal{T}_j$ may have very different sizes). However, if the refinement is done in a proper way the family $\{\mathcal{T}_j\}$ can be constructed such that the minimum angle of any $\mathcal{T}_j$ is not less than half of the minimum angle of the starting triangulation (see [10]). Therefore, it is natural to seek error estimators which are equivalent to the error with constants depending only on the minimum angle (and not on the elements size). We will show that the estimator η is of this kind. In what follows the letter C denotes a generic constant depending only on the minimum angle.

Let $\nabla_j e$ be the L^2-vector defined by

$$\nabla_j e |_T = \nabla (e|_T) .$$

THEOREM 3.1 : *Under the assumptions (3.1), there exist positive constants C_1 and C_2 depending only on the minimum angle of $\mathcal{T}_j$ and on the bounds of $a(x)$ such that,*

$$C_1 \eta \leqslant \|\nabla_j e\|_0 \leqslant C_2 \eta . \tag{3.5}$$

Proof : Decompose the vector $a \nabla_j e$ as follows,

$$a \nabla_j e = a \nabla w + \mathbf{curl}\ \psi \tag{3.6}$$

with $w \in H^1_{\Gamma_1}$ and $\psi \in H^1$ such that $\frac{\partial \psi}{\partial t} = 0$ on Γ_2. Note that when $a \equiv 1$ this is a Helmholtz decomposition of $\nabla_j e$ and, in fact, (3.6) can be obtained as in that case. Indeed, let w be the solution of

$$\begin{cases} \operatorname{div}(a \nabla w) = \operatorname{div}(a \nabla_j e), & \text{in } \Omega \\ w = 0, & \text{on } \Gamma_1 \\ a \dfrac{\partial w}{\partial n} = a \nabla_j e \cdot n, & \text{on } \Gamma_2 \end{cases}$$

then, $\operatorname{div}(a \nabla_j e - a \nabla w) = 0$ and therefore there exists $\psi \in H^1$ such that

$$a \nabla_j e - a \nabla w = \mathbf{curl}\ \psi$$

and in view of the boundary condition on Γ_2 we have $\frac{\partial \psi}{\partial t} = \mathbf{curl}\ \psi \,.\, n = 0$ on Γ_2, which means that ψ is constant on each connected component of Γ_2.

Now, let w^I and ψ^I be continuous piecewise linear approximations of w and ψ respectively such that,

$$\| w - w^I \|_{0,T} \leqslant C\, |T|^{1/2} |w|_{1,\tilde{T}} \tag{3.7}$$

(and analogously for ψ) where $\tilde{T}$ is the union of all the triangles sharing a vertex with T. Moreover, we assume that the interpolation preserves boundary conditions, that is,

$$w^I \in M^j_{\Gamma_1} .$$

and

$$\psi - \psi^I \in H^1_{\Gamma_2} .$$

An interpolation satisfying all these conditions was constructed in [11].

Using the decomposition (3.6) and Lemma 3.2 we have,

$$\begin{aligned}\int_\Omega a |\nabla_j e|^2 &= \int_\Omega a \nabla_j e \cdot (\nabla w + a^{-1}\, \mathbf{curl}\ \psi) \\ &= \int_\Omega a \nabla_j e \cdot [\nabla(w - w^I) + a^{-1}\, \mathbf{curl}\ (\psi - \psi^I)]\end{aligned}$$

hence, from Lemma 3.1 with $v = w - w^I$ and $\varphi = \psi - \psi^I$ we obtain

$$\int_\Omega a |\nabla_j e|^2 = \sum_{T \in \mathcal{T}_j} \left\{ \int_T f(w - w^I) + \right. \\ \left. + \frac{1}{2} \sum_{\ell \in E_T} \int_\ell [J_{\ell,n}(w - w^I) + J_{\ell,t}(\psi - \psi^I)] \right\}$$

and using Schwartz inequality and (3.7) we get

$$\int_\Omega a |\nabla_j e|^2 \leqslant C \eta \left(|w|_1 + |\psi|_1 \right) \leqslant C \eta\, \|a \nabla_j e\|_0$$

(the last inequality is a consequence of the orthogonality of $a \nabla w$ and $\mathbf{curl}\ \psi$ in a weighted L^2-norm), and this proves the right inequality of (3.5).

To prove the other inequality we follow the ideas developed by Verfürth [12] for the conforming case (see also [3]). We use the error equation (3.2) with a particular choice of $v \in H^1_{\Gamma_1}$ and $\varphi \in H^1_{\Gamma_2}$ satisfying,

$$\int_T v = f_T |T|^2, \qquad \forall T \in \mathcal{T}_j \tag{3.8}$$

$$\int_\ell v = J_{\ell,n} |\ell|^2, \qquad \forall \ell \tag{3.9}$$

$$\int_\ell \varphi = J_{\ell,t} |\ell|^2, \qquad \forall \ell \tag{3.10}$$

and

$$|v|_{1,T} + |\varphi|_{1,T} \leq C \eta_T . \tag{3.11}$$

It is not difficult to see that those v and φ exist. Indeed, φ can be taken as a continuous piecewise quadratic polynomial vanishing at every vertex of the triangulation and v as a continuous piecewise polynomial of degree three (in fact, quadratic augmented with local bubbles). We refer to [12] for the details.

Now, (3.2) together with (3.8), (3.9), (3.10) and (3.11) yields,

$$\eta^2 = \sum_{T \in \mathcal{T}_j} \int_T a \nabla e \cdot (\nabla v + a^{-1} \mathbf{curl}\, \varphi) \leq C \|\nabla_j e\|_0 \, \eta$$

which concludes the theorem. □

A simpler equivalent error estimator can be defined by observing that the term corresponding to the jump of the flux is dominated by that corresponding to f. Indeed, given an interior ℓ let $v_\ell \in V^j_{\Gamma_1}$ be the basis function which is equal to one at M_ℓ and vanish at any other node. Then, from (2.3) we have,

$$\int_{T_{\text{in}}} a \nabla u_j \,.\, \nabla v_\ell + \int_{T_{\text{out}}} a \nabla u_j \,.\, \nabla v_\ell = \int_{T_1 \cup T_2} f v_\ell$$

and integrating by parts we obtain

$$-\int_\ell J_{\ell,n} v_\ell = \int_{T_1 \cup T_2} f v_\ell$$

thus

$$-J_{\ell,n} |\ell| = f_{T_1} \frac{|T|}{3} + f_{T_2} \frac{|T|}{3} \tag{3.12}$$

(the relation (3.12) was observed in [9] in a different context).

Analogously, when $\ell \subset \Gamma_2$ is a side of T we can see that,

$$- J_{\ell, n} |\ell| = f_T \frac{|T|}{3} . \tag{3.13}$$

In view of all above, we define the error estimator

$$\tilde{\eta}_T^2 = f_T^2 |T|^2 + \frac{1}{2} \sum_{\ell \in E_T} J_{\ell, t}^2 |\ell|^2$$

and

$$\tilde{\eta}^2 = \sum_{T \in \mathcal{T}_j} \tilde{\eta}_T^2$$

and we have,

THEOREM 3.2 : *There exist positive constants C_3 and C_4 depending only on the minimum angle and on the bounds of $a(x)$ such that,*

$$C_3 \tilde{\eta} \leq \| \nabla_j e \|_0 \leq C_4 \tilde{\eta} .$$

Proof : Obviously $\tilde{\eta}_T \leq \eta_T$ and the theorem follows from (3.12), (3.13) and Theorem 3.1. □

4. NUMERICAL RESULTS

In this section we present the results of some numerical computations. We generate the meshes $\{\mathcal{T}_j\}$ in an adaptive way by using η_T as an error indicator at the element T. Starting with a uniform triangulation $\mathcal{T}_0$, $\mathcal{T}_{j+1}$ is obtained from $\mathcal{T}_j$ by refining the elements $T \in \mathcal{T}_j$ such that

$$\eta_T \geq \gamma \eta_{\max}$$

where $\eta_{\max} = \max_{T \in \mathcal{T}_j} \eta_T$ and $\gamma = 0.5$ in the first example and $\gamma = 0.7$ in the others. The refinement is propagated using the method described in [10] which guarantees that for every j the minimum angle of $\mathcal{T}_j$ is not less than half the minimum angle of $\mathcal{T}_0$.

As a first example consider the Dirichlet problem,

$$\begin{cases} \Delta u = 0 , & \text{in } \Omega \\ u = g , & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where Ω is the L-shaped domain $(-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ (see *fig.* 1) and g is the smooth function such that the solution of (4.1) in polar coordinates is,

$$u(r, \theta) = r^{2/3} \sin \left(\frac{2}{3} \theta \right) .$$

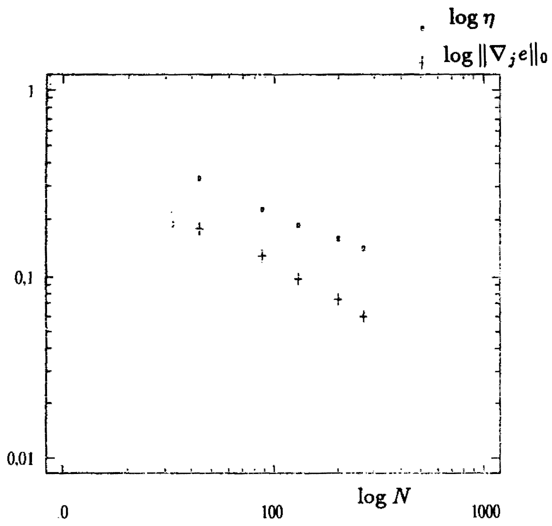

Figure 1.

Table 1 shows the results obtained in this example in four steps of the refinement procedure. We denote by N the number of nodes and by θ the ratio $\dfrac{\eta}{\|\nabla_j e\|_0}$.

Table 1

j	N	$\|\nabla_j e\|_0$	η	θ
0	44	0.18	0.329	1.83
1	87	0.126	0.232	1.84
2	130	0.096	0.193	2.01
3	200	0.0745	0.157	2.11
4				

The results in the table show in particular that the automatic refinement procedure allows us to obtain the optimal order of convergence with respect to the number of nodes for this singular solution, that is, the same order than in the regular case. Indeed, it is known that when the solution is regular and the meshes are refined uniformly,

$$\|\nabla_j e\|_0 = O(N^{-1/2}) \tag{4.2}$$

(4.2) also holds in this case as can be seen in Figure 1.

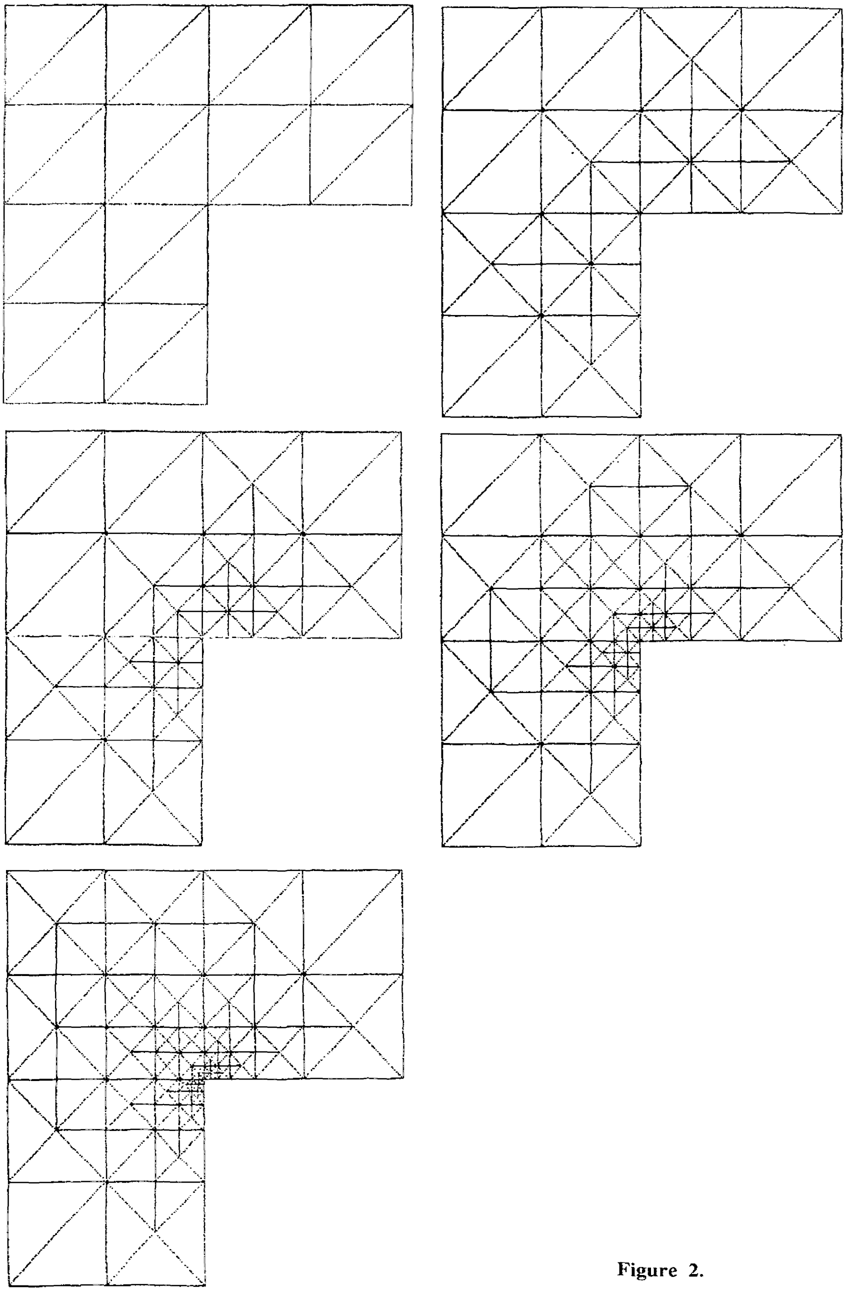

Figure 2.

Figure 2 shows the meshes $\mathcal{T}_0$ to $\mathcal{T}_4$, respectively.

In the next examples we consider again the Laplace equation but with mixed boundary conditions. For $k = 4,\ 6,\ 8$ we solve

$$\begin{cases} \Delta u = 0\,, & \text{in} \quad \Omega \\ u = 0\,, & \text{on} \quad \Gamma_1 \\ u = \sin\left(\dfrac{2\,\theta}{k}\right), & \text{on} \quad \Gamma_2 \\ \dfrac{\partial u}{\partial n} = 0\,, & \text{on} \quad \Gamma_3 \end{cases}$$

where

$$\begin{aligned} \Omega &= \left\{(r,\ \theta) : 0 < r < 1,\ 0 < \theta \leqslant \frac{k\pi}{4}\right\}, \\ \Gamma_1 &= \{(r,\ \theta) : 0 < r < 1,\ \theta = 0\}\,, \\ \Gamma_2 &= \left\{(r,\ \theta) : r = 1,\ 0 < \theta < \frac{k\pi}{4}\right\}, \end{aligned}$$

and

$$\Gamma_3 = \left\{(r,\ \theta) : 0 < r < 1,\ \theta = \frac{k\pi}{4}\right\}.$$

The solution of this problem is,

$$u(r,\ \theta) = r^{2/k} \sin\left(\frac{2\,\theta}{k}\right).$$

The meshes generated by the adaptive procedure are shown in figures 3, 4 and 5 for $k = 4$, 6 and 8 respectively.

In all these examples the same behavior of the error than in the first one was observed (i.e. : optimal order of convergence was obtained).

5. CONCLUSIONS

We have introduced a technique to construct and analyse a posteriori error estimators for nonconforming finite element methods.

In the case of piecewise linear approximations of second order elliptic problems we have defined two error estimators which are equivalent to the discrete energy norm of the error. In particular, we have seen that the jumps across the elements of the tangential derivative of the approximate solution u_j (which can be written in terms of the jumps of u itself) play an important role. Instead, the jumps of the flux (which in the conforming case are

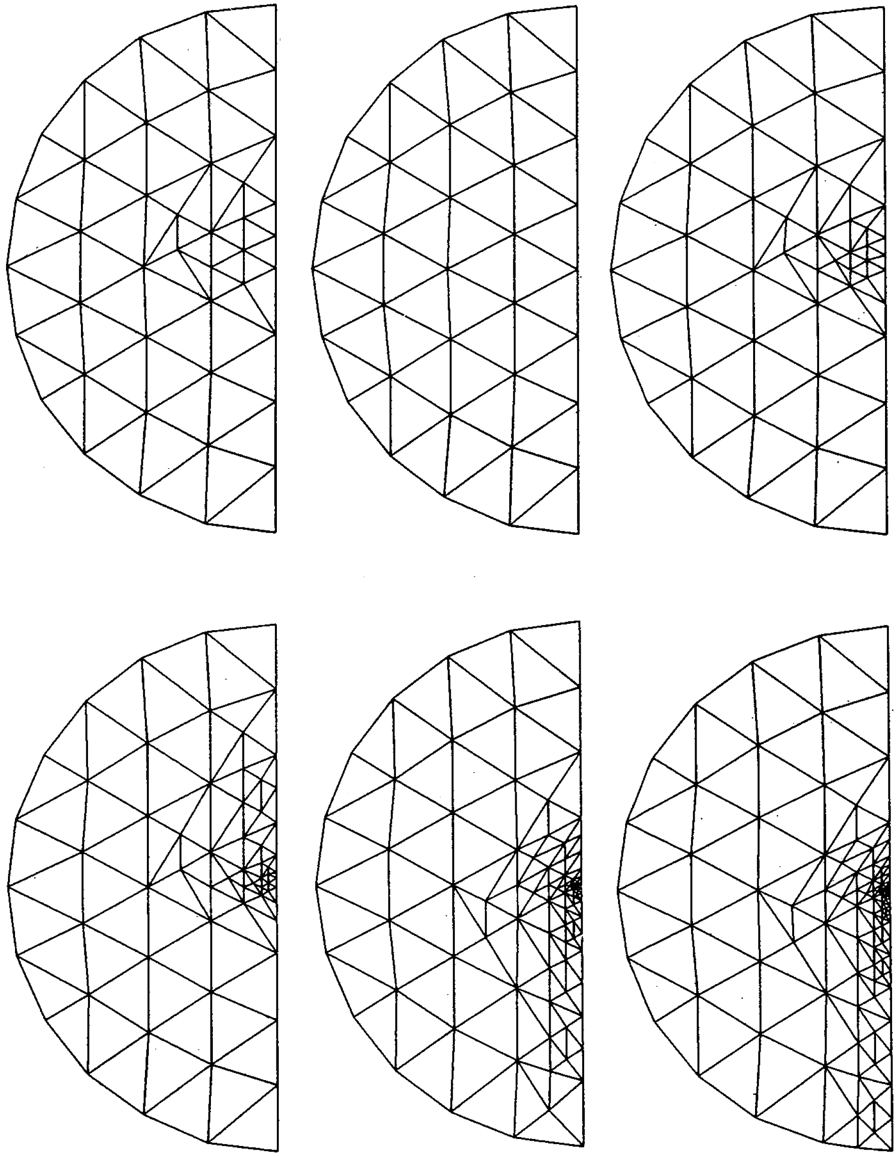

Figure 3.

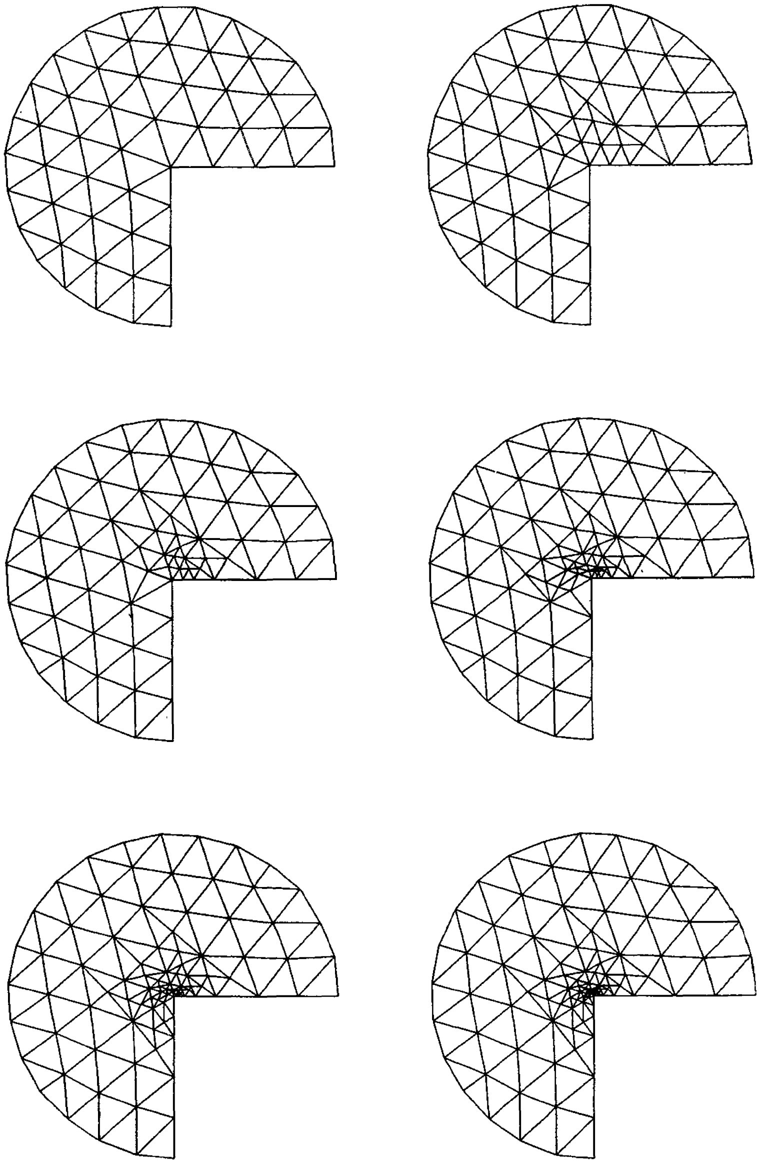

Figure 4.

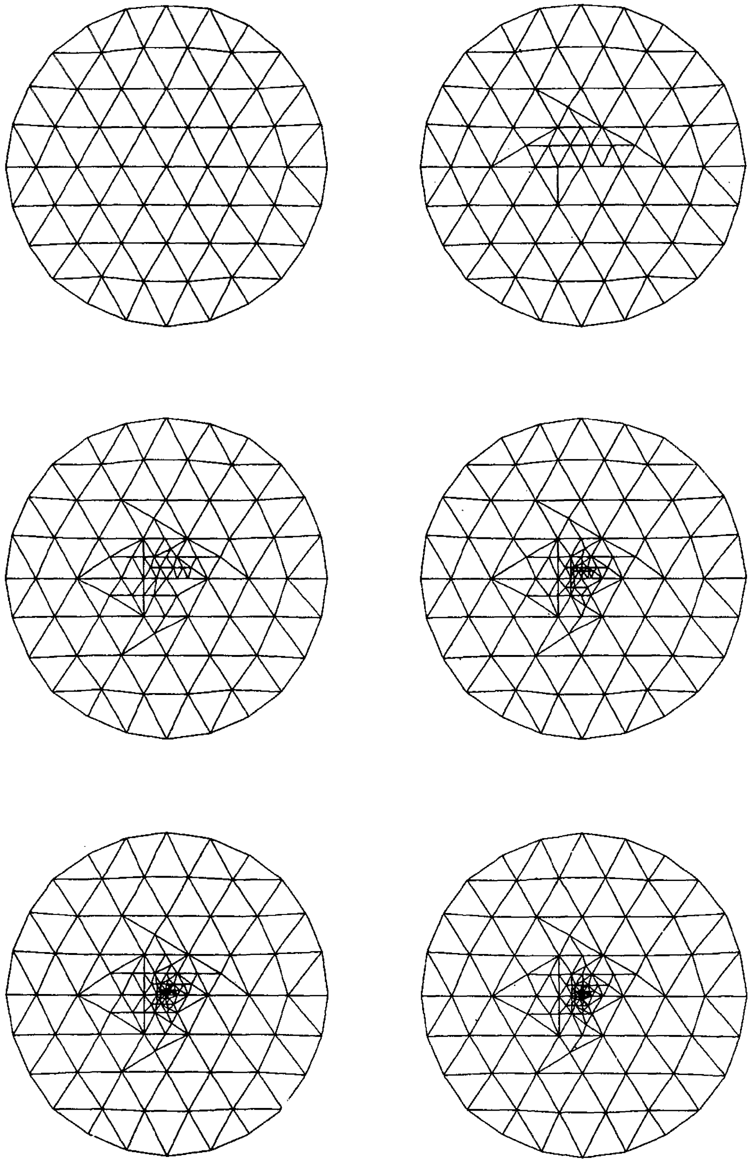

Figure 5.

essential) can be neglected since they are dominated by the right hand side of the equation (i.e. the local residual).

Our numerical computations show the good behavior of the estimator when used as an error indicator for adaptive refinement, one of the most important applications of a posteriori error estimates.

Clearly, the results can be extended to higher order nonconforming approximations. Also, similar ideas can be applied to other problems. In particular, in a forthcoming paper we introduce error estimators for nonconforming approximations of the Stokes problem.

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