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MULTIGRID METHODS FOR PARAMETER DEPENDENT PROBLEMS (*)

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Abstract. — *Multigrid methods for parameter dependent problems are discussed. The contraction numbers of the algorithms are proved within a unifying framework to be bounded away from one, independent of the parameter and the mesth levels. Examples include the pure displacement and pure traction boundary value problems in planar linear elasticity, the Timoshenko beam problem, and the Reissner-Mindlin plate problem.*

Résumé. — *On discute des méthodes multigrilles pour les problèmes dépendant de paramètres. On prouve que la diminution du nombre d'itérations des algorithmes est bornée indépendamment du paramètre et du niveau de maillages, et ce dans un cadre général. On donne des exemples d'élasticité linéaire plane avec des conditions au bord de déplacement ou de traction, du problème de poutres de Timoshenko et du problème de plaques de Reissner-Mindlin.*

1. INTRODUCTION

In recent years finite element multigrid methods have been applied to problems in solid mechanics. The straight-forward approach using low order conforming finite elements in a displacement formulation runs into trouble with the phenomenon of locking, which occurs when a certain parameter (e.g., the thickness of a plate or a beam, the Poisson ratio of an elastic material) approaches a limit. The performance of the method deteriorates dramatically, as occurs for example when the Poisson ratio approaches 0.5 in the problem of linear elasticity (*cf.* [28], [30]).

It is well-known that locking can be avoided by using nonconforming finite elements or reduced integration, and the variational problems can usually be posed in either a displacement formulation or an equivalent mixed formulation with a penalty term. It turns out that even though the formulations are equivalent, the mixed formulation is superior for multigrid algorithms, as was observed in [6].

In this paper we present a unifying framework for some recent results ([6], [12], [13], [27]) of robust multigrid methods for parameter dependent prob-

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lems using mixed formulations. The problems treated in these papers are stated here as examples of our theory. The notation is more or less standard. (The definitions are given in the appendix).

Example 1.1. (Pure displacement boundary value problem in linear elasticity) :

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$. The continuous problem is :

Find $(\underset{\sim}{u}, p) \in \underset{\sim}{\mathring{H}}^1(\Omega) \times \hat{L}^2(\Omega)$ such that

$$(1.1) \quad \int_\Omega \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{u} : \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{v}\, dx + \int_\Omega p \operatorname{div} \underset{\sim}{v}\, dx = \frac{1}{\mu} \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx \quad \forall \underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1(\Omega)$$

$$\int_\Omega q \operatorname{div} \underset{\sim}{u}\, dx - \frac{\mu}{\mu + \lambda} \int_\Omega pq\, dx = 0 \quad \forall q \in L^2(\Omega).$$

Here $\underset{\sim}{u}$ is the displacement, $\underset{\sim}{f}$ is the body force, μ and λ are the Lamé constants, and $p = \frac{\mu + \lambda}{\mu} \operatorname{div} \underset{\sim}{u}$. The ranges for the Lamé constants are $0 < \mu_0 \leqslant \mu \leqslant \mu_1 < \infty$ and $0 < \lambda < \infty$. As λ approaches ∞, the material becomes nearly incompressible.

The discrete problem is obtained by replacing $\underset{\sim}{\mathring{H}}^1(\Omega) \times \hat{L}^2(\Omega)$ by the nonconforming finite element space (*cf.* [12], [15]) $\underset{\sim}{\mathring{M}}{}^1_*(\mathcal{T}_h) \times \hat{M}^0_{-1}(\mathcal{T}_h)$, $\underset{\approx}{\mathrm{grad}}$ by $\underset{\approx}{\mathrm{grad}}_h$ and div by div_h.

Example 1.2 (Pure traction boundary value problem in linear elasticity) :

Let Ω be a convex polygonal domain in $\mathbb{R}^2$, $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$ and $\underset{\sim}{g} \in \underset{\sim}{H}^{1/2}(\partial\Omega)$. The continuous problem is :

Find $(\underset{\sim}{u}, p) \in \underset{\sim}{H}^1_\perp(\Omega) \times L^2(\Omega)$ such that

$$(1.2) \quad \int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\epsilon}(\underset{\sim}{v})\, dx + \int_\Omega p \operatorname{div} \underset{\sim}{v}\, dx$$

$$= \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds\right] \quad \forall \underset{\sim}{v} \in \underset{\sim}{H}^1_\perp(\Omega)$$

$$\int_\Omega q \operatorname{div} \underset{\sim}{u}\, dx - \frac{2\mu}{\lambda} \int_\Omega pq\, dx = 0 \quad \forall q \in L^2(\Omega).$$

Here $\underset{\sim}{u}$ is the displacement, $\underset{\sim}{f}$ is the body force, $\underset{\sim}{g}$ is the traction on the boundary, $p = \frac{\lambda}{2\mu} \operatorname{div} \underset{\sim}{u}$ and $(\underset{\sim}{f}, \underset{\sim}{g})$ satisfy the compatibility condition

$$\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds = 0 \quad \forall \underset{\sim}{v} \in \underset{\sim}{\mathrm{RM}}.$$

The ranges for the Lamé constants μ and λ are $0 < \mu_0 \leqslant \mu \leqslant \mu_1 < \infty$ and $0 < \lambda < \infty$. The subscript « $\perp$ » indicates the L^2-orthogonality to $\underset{\sim}{RM}$.

The discrete problem is obtained by using the nonconforming finite element space $\underset{\sim}{M}^1_{*,\perp}(\mathcal{T}_h) \times M^0_{-1}(\mathcal{T}_{2h})$. The operator div is replaced by div_h, and the strain tensor $\underset{\approx}{\epsilon}$ is replaced by $\overset{*}{\underset{\approx}{\epsilon}}_h$, where (*cf.* [13], [23]) $\overset{*}{\underset{\approx}{\epsilon}}_h(\underset{\sim}{v}) := \underset{\approx}{\mathrm{grad}}_h \underset{\sim}{v} - \frac{1}{2}(P_{2h} \mathrm{rot}_h \underset{\sim}{v}) \underset{\approx}{\chi}$. ($P_{2h}$ is the L^2-orthogonal projection onto the space $M^0_{-1}(\mathcal{T}_{2h})$ of piecewise constant functions on the coarser grid.)

Example 1.3 (Timoshenko beam problem) :

Let Ω be a finite interval in $\mathbb{R}$ and $f \in L^2(\Omega)$. The continuous problem is :

Find $(\phi, \omega, \zeta) \in \mathring{H}^1(\Omega) \times \mathring{H}^1(\Omega) \times L^2(\Omega)$ such that

$$(1.3) \qquad \int_\Omega \phi' \psi' dx + \int_\Omega \zeta(\psi - v')\, dx = \int_\Omega fv\, dx \quad \forall (\psi, v) \in \mathring{H}^1(\Omega) \times \mathring{H}^1(\Omega)$$

$$\int_\Omega (\phi - \omega')\, \eta\, dx - d^2 \int_\Omega \zeta\eta\, dx = 0 \quad \forall \eta \in L^2(\Omega)\,.$$

Here ϕ is the rotation of the vertical fiber, ω is the vertical displacement, $\zeta = d^{-2}(\phi - \omega')$, d is the thickness of the beam, and $d^2 f$ is the vertical body force.

The discrete problem is obtained by replacing $\mathring{H}^1(\Omega) \times \mathring{H}^1(\Omega) \times L^2(\Omega)$ with $\mathring{M}^1_0(\mathcal{T}_h) \times \mathring{M}^1_0(\mathcal{T}_h) \times M^0_{-1}(\mathcal{T}_h)$ (*cf.* [1]).

Example 1.4 (Reissner-Mindlin plate problem) :

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$. The continuous problem is :

Find $(\underset{\sim}{\phi}, p) \in \mathring{\underset{\sim}{H}}^1(\Omega) \times \hat{H}^1(\Omega)$ such that

$$(1.4) \qquad \frac{E}{12(1-v^2)} \int_\Omega [(1-v)\, \underset{\approx}{\epsilon}(\underset{\sim}{\phi}) : \underset{\approx}{\epsilon}(\underset{\sim}{\psi}) + v \,\mathrm{div}\, \underset{\sim}{\phi}\, \mathrm{div}\, \underset{\sim}{\psi}]\, dx$$

$$+ \int_\Omega p \,\mathrm{rot}\, \underset{\sim}{\psi}\, dx = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{\psi}\, dx \quad \forall \underset{\sim}{\psi} \in \mathring{\underset{\sim}{H}}^1(\Omega)$$

$$\int_\Omega q \,\mathrm{rot}\, \underset{\sim}{\phi}\, dx - t^2 \int_\Omega \underset{\sim}{\mathrm{curl}}\, p\, \underset{\sim}{\mathrm{curl}}\, q\, dx = 0 \quad \forall q \in \hat{H}^1(\Omega)\,.$$

This is part of the Brezzi-Fortin formulation (*cf.* [17]) of the Reissner-Mindlin plate problem where $\underset{\sim}{\phi}$ is the rotation of the vertical fiber, E is the Young's modulus, ν is the Poisson ratio, t is the thickness of the plate, and the Lamé constant λ is taken to be 1.

The discrete problem is obtained by replacing $\underset{\sim}{\mathring{H}}{}^1(\Omega) \times \mathring{H}^1(\Omega)$ with $\underset{\sim}{\mathring{N}}{}^1_0(\mathcal{T}_h) \times \mathring{M}^1_0(\mathcal{T}_h)$ where $\underset{\sim}{\mathring{N}}{}^1_0(\mathcal{T}_h) = \mathring{M}^1_0(\mathcal{T}_h) \oplus \underset{\sim}{B}{}^3(\mathcal{T}_h)$ (*cf.* [2], [3]).

For general background on multigrid methods we recommend the books [24], [29] and [8], and for mixed methods the book [18]. Earlier work on multigrid methods for mixed formulations (without parameters) can be found in [25], [34], and [35]. Other ways of applying multigrid methods to parameter dependent problems are discussed in [31] and [33].

The rest of this paper is organized as follows. In Section 2 we set up the framework, followed by the multigrid algorithm in Section 3. Three lemmas crucial to the convergence analysis are discussed in Section 4 and the convergence analysis is carried out in Section 5. Section 6 contains some concluding remarks and the appendix contains the definitions of the notation.

2. THE FRAMEWORK

The Continuous Problem

Ω is a convex polygonal domain in $\mathbb{R}^n$, $n = 1$ or 2. Let $\underset{\sim}{V}$ be a closed subspace of $\underset{\sim}{H}{}^1(\Omega)$ and W be a closed subspace of $H^\alpha(\Omega)$ where $\alpha = 0$ or 1. For each parameter $t \in (0, T]$ $(0 < T < \infty)$ the continuous problem is described by a symmetric bilinear form $\mathfrak{B}^t(\cdot\,,\cdot\,)$ on $\underset{\sim}{V} \times W$ which satisfies

$$\text{(A.1)}\quad |\mathfrak{B}^t((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))| \leq C(\|\underset{\sim}{v}_1\|_{\underset{\sim}{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)} + t^\alpha \|q_1\|_{H^\alpha(\Omega)}) \times (\|\underset{\sim}{v}_2\|_{\underset{\sim}{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)} + t^\alpha \|q_2\|_{H^\alpha(\Omega)})\,.$$

(Throughout this paper, C denotes a generic constant independent of both the mesh level and the parameter t.)

We assume the following stability estimate which, together with (A.1), implies the unique solvability of the continuous problem.

$$\text{(A.2)}\qquad \sup_{(\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W} \frac{|\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q))|}{\|\underset{\sim}{v}\|_{\underset{\sim}{H}^1(\Omega)} + \|q\|_{L^2(\Omega)} + t^\alpha \|q\|_{H^\alpha(\Omega)}} \geq C(\|\underset{\sim}{w}\|_{\underset{\sim}{H}^1(\Omega)} + \|r\|_{L^2(\Omega)} + t^\alpha \|r\|_{H^\alpha(\Omega)})\,.$$

For elliptic regularity we require that

$$\text{(A.3}a) \qquad \| \underset{\sim}{w} \|_{H^2(\Omega)} + \| r \|_{H^1(\Omega)} + t^{\alpha} \| r \|_{H^{1+\alpha}(\Omega)} \leqslant C \| \underset{\sim}{f} \|_{L^2(\Omega)}$$

for $(\underset{\sim}{w}, r, \underset{\sim}{f}) \in \underset{\sim}{V} \times W \times \underset{\sim}{L}^2(\Omega)$ satisfying

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_{\Omega} \underset{\sim}{f} \cdot \underset{\sim}{v} \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W .$$

Moreover, in the case where $\alpha = 1$ we also require that

$$\text{(A.3}b) \qquad \| \underset{\sim}{w} \|_{H^2(\Omega)} + \| r \|_{H^1(\Omega)} + t \| r \|_{H^2(\Omega)} \leqslant C t^{-1} \| g \|_{L^2(\Omega)}$$

for $(\underset{\sim}{w}, r, g) \in \underset{\sim}{V} \times W \times L^2(\Omega)$ satisfying

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_{\Omega} g q \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W .$$

Examples : Pure displacement problem In Example 1.1, we take $\underset{\sim}{V} = \underset{\sim}{\mathring{H}}^1(\Omega)$, $W = \hat{L}^2(\Omega)$, and $t^2 = \frac{\mu}{\mu + \lambda}$. The bilinear form $\mathfrak{B}^t(\cdot, \cdot)$ is defined by

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_{\Omega} \{ \underset{\approx}{\operatorname{grad}} \, \underset{\sim}{w} : \underset{\approx}{\operatorname{grad}} \, \underset{\sim}{v} + r \operatorname{div} \underset{\sim}{v} + q \operatorname{div} \underset{\sim}{w} - t^2 r q \} \, dx .$$

The continuous problem is :

Find $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ such that

$$\mathfrak{B}^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \frac{1}{\mu} \int_{\Omega} \underset{\sim}{f} \cdot \underset{\sim}{v} \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W .$$

The stability and regularity estimates for this example can be found in [12] and [15].

Pure traction problem In Example 1.2 we take $\underset{\sim}{V} = \underset{\sim}{H}^1_{\perp}(\Omega)$, $W = L^2(\Omega)$, and $t^2 = \frac{2\mu}{\lambda}$. The bilinear form $\mathfrak{B}^t(\cdot, \cdot)$ is defined by

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_{\Omega} \{ \underset{\approx}{\epsilon}(\underset{\sim}{w}) : \underset{\approx}{\epsilon}(\underset{\sim}{v}) + r \operatorname{div} \underset{\sim}{v} + q \operatorname{div} \underset{\sim}{w} - t^2 r q \} \, dx .$$

The continuous problem is :
Find $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ such that

$$\mathfrak{B}^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds\right] \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W .$$

The stability and regularity estimates for this example can be found in [13] and [15].

Timoshenko beam problem In Example 1.3 we take $\underset{\sim}{V} = \overset{\circ}{\underset{\sim}{H}}{}^1(\Omega)$, $W = L^2(\Omega)$, and $t^2 = d^2$. The bilinear form $\mathfrak{B}^t(\cdot, \cdot)$ is defined by

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_\Omega \{w_1' v_1' + r(v_1 - v_2') + (w_1 - w_2')\, q - t^2\, rq\}\, dx .$$

The continuous problem is :
Find $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ such that

$$\mathfrak{B}^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \int_\Omega f v_2\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W .$$

The stability and regularity estimates for this example can be found in [1] and [6].

Reissner-Mindlin plate problem In Example 1.4 we take $\underset{\sim}{V} = \overset{\circ}{\underset{\sim}{H}}{}^1(\Omega)$ and $W = \hat{H}^1(\Omega)$. The bilinear form $\mathfrak{B}^t(\cdot, \cdot)$ is defined by

$$\begin{aligned}\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) &= \int_\Omega \left\{ \frac{E}{12(1 - \nu^2)} [(1 - \nu)\, \underset{\approx}{\epsilon}(\underset{\sim}{w}) : \underset{\approx}{\epsilon}(\underset{\sim}{v}) + \nu \operatorname{div} \underset{\sim}{w} \operatorname{div} \underset{\sim}{v}] \right. \\ &\quad \left. + r \operatorname{rot} \underset{\sim}{v} + q \operatorname{rot} \underset{\sim}{w} - t^2\, \underset{\sim}{\operatorname{curl}}\, r\, \underset{\sim}{\operatorname{curl}}\, q \right\} dx .\end{aligned}$$

The continuous problem is :
Find $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ such that

$$\mathfrak{B}^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx .$$

The stability and regularity estimates for this example can be found in [3] and [27].

Remarks : In Examples 1.1-1.4, we have

$$\mathfrak{B}^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = a(\underset{\sim}{w}, \underset{\sim}{v}) + b(\underset{\sim}{v}, r) + b(\underset{\sim}{w}, q) - t^2 c(r, q) .$$

The stability estimate for $\mathfrak{B}^t$ follows from appropriate coercivity assumptions on $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$. We refer the readers to [1], [26], [6], [18] and [27].

In the case $\alpha = 0$ (Examples 1.1-1.3), the terms involving t can be absorbed into other terms and hence do not appear in (A.1), (A.2) and (A.3). In Example 1.4 where $\alpha = 1$, we have a singular perturbation of the Stokes-like problem and the parameter t cannot be eliminated from the estimates (*cf.* [3], [4]).

The Discrete Problem

Let $\mathcal{T}_k (k \geq 1)$ be a sequence of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of each of the triangles in $\mathcal{T}_k$. Let h_k be the mesh parameter. Then

$$h_{k-1} = 2\, h_k . \tag{2.1}$$

The corresponding finite element spaces are $\underset{\sim}{V}_k \times W_k$. We assume that $W_k \subseteq W_{k+1} \subseteq W \ \forall k$, but we assume neither $\underset{\sim}{V}_k \subseteq \underset{\sim}{V}_{k+1}$ nor $\underset{\sim}{V}_k \subseteq \underset{\sim}{V}$. Our theory thus allows for the possibility of using nonnested or nonconforming finite element spaces $\underset{\sim}{V}_k$ in the discrete problem. In any case, we assume that $\underset{\sim}{V}_k \subseteq \underset{\sim}{M}^{\ell}_{-1}(\mathcal{T}_k)$ for some ℓ (and hence $(\underset{\sim}{V}_k \subseteq \underset{\sim}{L}^2(\Omega))$.

The discrete problem is described by a symmetric bilinear form $\mathfrak{B}^t_k(\cdot, \cdot)$ on $\underset{\sim}{V}_k \times W_k$. We assume that

$$\begin{aligned} |\mathfrak{B}^t_k((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))| \leq\ & C(\|\underset{\sim}{v}_1\|_k + \|q_1\|_{L^2(\Omega)} + t^{\alpha}\|q_1\|_{H^{\alpha}(\Omega)}) \\ & \times (\|\underset{\sim}{v}_2\|_k + \|q_2\|_{L^2(\Omega)} + t^{\alpha}\|q_2\|_{H^{\alpha}(\Omega)}) \end{aligned} \tag{A.4}$$

where

$$\|\underset{\sim}{v}\|_k := \|\underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} + \|\underset{\approx}{\mathrm{grad}}_k\, \underset{\sim}{v}\|_{\underset{\approx}{L}^2(\Omega)} , \tag{2.2}$$

and the following stability estimate holds

$$\text{(A.5)}\qquad \sup_{(\underset{\sim}{v},q)\in \underset{\sim}{V}_k\times W_k} \frac{|\mathfrak{B}_k^t((\underset{\sim}{w},r),(\underset{\sim}{v},q))|}{\|\underset{\sim}{v}\|_k+\|q\|_{L^2(\Omega)}+t^\alpha\|q\|_{H^\alpha(\Omega)}} \geq C(\|\underset{\sim}{w}\|_k+\|r\|_{L^2(\Omega)}+t^\alpha\|r\|_{H^\alpha(\Omega)})\,.$$

In particular, the bilinear form $\mathfrak{B}_k^t(\cdot,\cdot)$ is nondegenerate.

The approximation property of $\underset{\sim}{V}_k\times W_k$ is described by an interpolation operator

$$\Pi_k : (\underset{\sim}{V}\times W)\cap(\underset{\sim}{H}^2(\Omega)\times H^{1+\alpha}(\Omega))\to \underset{\sim}{V}_k\times W_k$$

with the property that

$$\text{(A.6)}\qquad \|(\underset{\sim}{w},r)-\Pi_k(\underset{\sim}{w},r)\|_{L^2(\Omega)} \leq C(h_k^2|\underset{\sim}{w}|_{\underset{\sim}{H}^2(\Omega)}+\min(h_k|r|_{H^1(\Omega)},h_k^{1+\alpha}|r|_{H^{1+\alpha}(\Omega)}))\,.$$

We shall denote by Π_k^v and Π_k^s the vector and scalar parts of Π_k.

Finally, we require the following discretization error estimate :

$$\text{(A.7}a\text{)}\qquad \|\underset{\sim}{w}-\underset{\sim}{w}_k\|_{\underset{\sim}{L}^2(\Omega)}+h_k(1+t^\alpha h_k^{-\alpha})\,\|r-r_k\|_{L^2(\Omega)}\leq Ch_k^2\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}$$

where $(\underset{\sim}{w},r)\in \underset{\sim}{V}\times W$ and $(\underset{\sim}{w}_k,r_k)\in \underset{\sim}{V}_k\times W_k$ satisfy

$$\mathfrak{B}^t((\underset{\sim}{w},r),(\underset{\sim}{v},q))=\int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx\quad \forall(\underset{\sim}{v},q)\in\underset{\sim}{V}\times W$$

and

$$\mathfrak{B}_k^t((\underset{\sim}{w}_k,r_k),(\underset{\sim}{v},q))=\int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx\quad \forall(\underset{\sim}{v},q)\in\underset{\sim}{V}_k\times W_k,$$

and in the case $\alpha=1$ we also require that

$$\text{(A.7}b\text{)}\qquad \|\underset{\sim}{w}-\underset{\sim}{w}_k\|_{\underset{\sim}{L}^2(\Omega)}+(h_k+t)\,\|r-r_k\|_{L^2(\Omega)}\leq Ct^{-1}h_k^2\|g\|_{L^2(\Omega)}$$

where $(\underset{\sim}{w},r)\in\underset{\sim}{V}\times W$ and $(\underset{\sim}{w}_k,r_k)\in\underset{\sim}{V}_k\times W_k$ satisfy

$$\mathfrak{B}^t((\underset{\sim}{w},r),(\underset{\sim}{v},q))=\int_\Omega gq\,dx\quad\forall(\underset{\sim}{v},q)\in\underset{\sim}{V}\times W$$

and

$$\mathfrak{B}_k^t((\underset{\sim}{w}_k, r_k), (\underset{\sim}{v}, q)) = \int_\Omega gq\,dx \quad \forall(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k.$$

Examples : Pure displacement problem For Example 1.1, we take $\underset{\sim}{V}_k = \overset{\circ}{\underset{\sim}{M}}{}^1_*(\mathcal{T}_k)$ and $W_k = \hat{M}^0_{-1}(\mathcal{T}_k)$. The bilinear form $\mathfrak{B}_k^t(\,\cdot\,,\cdot\,)$ is defined by

$$\mathfrak{B}_k^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_\Omega \{\underset{\approx}{\operatorname{grad}}_k \underset{\sim}{w} : \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{v} + r\operatorname{div}_k \underset{\sim}{v} + q \operatorname{div}_k \underset{\sim}{w} - t^2 rq\}\,dx\,,$$

and the discrete problem is :

Find $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times W_k$ such that

$$\mathfrak{B}_k^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \frac{1}{\mu}\int_\Omega \underset{\sim}{f}\cdot \underset{\sim}{v}\,dx \quad \forall(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k\,.$$

We use the Crouzeix-Raviart interpolation operator defined by $\Pi_k(\underset{\sim}{w}, r) = (\underset{\sim}{w}^*, r^*)$, where

$$\underset{\sim}{w}^*(m_e) = \frac{1}{|e|}\int_e \underset{\sim}{w}\,ds \tag{2.3}$$

at the midpoint m_e of the edge e in $\mathcal{T}_k$, and r^* is the L^2-orthogonal projection of r.

The stability, interpolation and error estimates in this case are found in [15] and [20].

Pure traction problem In Example 1.2 we take $\underset{\sim}{V}_k = \underset{\sim}{M}{}^1_{*,\perp}(\mathcal{T}_k)$ and $W_k = M^0_{-1}(\mathcal{T}_{k-1})$. The bliniear form $\mathfrak{B}_k^t(\,\cdot\,,\cdot\,)$ is defined by

$$\mathfrak{B}_k^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_\Omega \{\underset{\approx}{\epsilon}{}_k^*(\underset{\sim}{w}) : \underset{\approx}{\epsilon}{}_k^*(\underset{\sim}{v}) + r\operatorname{div}_k \underset{\sim}{v} + q\operatorname{div}_k \underset{\sim}{w} - t^2 rq\}\,dx\,,$$

where

$$\underset{\approx}{\epsilon}{}_k^*(\underset{\sim}{v}) := \underset{\approx}{\operatorname{grad}}_k \underset{\sim}{v} - \frac{1}{2}(P_{k-1}\operatorname{rot}_k \underset{\sim}{v})\,\underset{\approx}{\chi},$$

and P_{k-1} is the L^2-orthogonal projection onto $M^0_{-1}(\mathcal{T}_{k-1}) = W_k$. The discrete problem is :

Find $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times W_k$ such that

$$\mathfrak{B}_k^t((\underset{\sim}{u}, p), (\underset{\sim}{v}, q)) = \frac{1}{2\mu}\left[\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v} ds\right] \quad \forall(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k .$$

We use the interpolation operator defined by $\Pi_k(\underset{\sim}{w}, r) = (\underset{\sim}{w}', r^*)$ where $\underset{\sim}{w}'$ is the L^2-orthogonal projection of the $\underset{\sim}{w}^*$ defined by (2.3), and r^* is the $\underset{\sim}{L}^2$-orthogonal projection of r. The stability, interpolation and error estimates in this case are found in [13] and [20].

Timoshenko beam For Example 1.3 we take $\underset{\sim}{V}_k = \underset{\sim}{\mathring{M}}{}_0^1(\mathcal{T}_k)$ and $W_k = M_{-1}^0(\mathcal{T}_k)$. Since both $\underset{\sim}{V}_k$ and W_k are conforming, we take $\mathfrak{B}_k^t$ to be the restriction of $\mathfrak{B}^t$ to $\underset{\sim}{V}_k \times W_k$, and the discrete problem is just the restriction of the continuous problem to $\underset{\sim}{V}_k \times W_k$. The interpolation operator is defined by $\Pi_k(\underset{\sim}{w}, r) = (\underset{\sim}{w}^*, r^*)$ where $\underset{\sim}{w}^*$ is the Lagrange interpolant of $\underset{\sim}{w}$ and r^* is the L^2-orthogonal projection of r. The interpolation error estimate is standard (*cf.* [21], [14]). The stability and error estimates in this case are found in [1] and [6].

Reissner-Mindlin plate problem For Example 1.4 we take $\underset{\sim}{V}_k = \underset{\sim}{\mathring{N}}{}_0^1(\mathcal{T}_k)$ and $W_k = \hat{M}_0^1(\mathcal{T}_k)$. Since both $\underset{\sim}{V}_k$ and W_k are conforming, we take $\mathfrak{B}_k^t$ to be the restriction of $\mathfrak{B}^t$ to $\underset{\sim}{V}_k \times W_k$, and the discrete problem is the restriction of the continuous problem to $\underset{\sim}{V}_k \times W_k$. In this case,

$$\Pi_k : [\underset{\sim}{H}^2(\Omega) \cap \underset{\sim}{\mathring{H}}{}^1(\Omega)] \times \hat{H}^1(\Omega) \to \underset{\sim}{\mathring{M}}{}_0^1(\mathcal{T}_k) \times \hat{M}_0^1(\mathcal{T}_k)$$

is just the standard Lagrange interpolation operator. The stability and error estimates in this case are found in [27].

Remarks : In the case of conforming finite elements (Examples 1.3 and 1.4) the error estimates (A.7) follow from the stability estimate (A.5), the elliptic regularity estimate (A.3) and duality arguments. In the case of nonconforming finite elements (Examples 1.1 and 1.2) one must estimate additional consistency terms which measure the effect of the nonconformity.

Mesh-Dependent Norms

Each $\underset{\sim}{V}_k \times W_k$ is equipped with an inner-product $(\cdot, \cdot)_k$ such that

$$\text{(A.8)} \quad ((\underset{\sim}{v}, q), (\underset{\sim}{v}, q))_k \sim \|\underset{\sim}{v}\|^2_{\underset{\sim}{L}^2(\Omega)} + h_k^2(1 + t^{2\alpha} h_k^{-2\alpha}) \|q\|^2_{L^2(\Omega)} .$$

(The constants defining the equivalence are independent of k and t.)

Let $B_k^t : \underset{\sim}{V}_k \times W_k \to \underset{\sim}{V}_k \times W_k$ be defined by

(2.4)
$$(B_k^t(\underset{\sim}{w}, r), (\underset{\sim}{v}, q))_k = \mathfrak{B}_k^t((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) \quad \forall (\underset{\sim}{w}, r), (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k .$$

Clearly B_k^t is symmetric and nonsingular. By a standard inverse estimate (*cf.* [21], [14]) and (A.4) we have

$$\text{spectral radius of } B_k^t \leqslant C h_k^{-2} . \tag{2.5}$$

The mesh-dependent norm $|||\,.\,|||_{s,k}$ on $\underset{\sim}{V}_k \times W_k$ is defined by

$$|||(\underset{\sim}{v}, q)|||_{s,k} := \sqrt{((B_k^t B_k^t)^{s/2}(\underset{\sim}{v}, q), (\underset{\sim}{v}, q))_k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k . \tag{2.6}$$

It follows from (A.8), (2.4) and (2.6) that for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k$

$$|||(\underset{\sim}{v}, q)|||_{0,k} \sim \|\underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} + h_k(1 + t^{\alpha} h_k^{-\alpha}) \|q\|_{L^2(\Omega)} , \tag{2.7}$$

and

$$|||(\underset{\sim}{v}, q)|||_{2,k} = \sup_{(\underset{\sim}{v}', q') \in \underset{\sim}{V}_k \times W_k} \frac{|\mathfrak{B}_k^t((\underset{\sim}{v}, q), (\underset{\sim}{v}', q'))|}{|||(\underset{\sim}{v}', q')|||_{0,k}} . \tag{2.8}$$

Examples : Pure displacement and pure traction problems In both Examples 1.1 and 1.2 we take

$$((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))_k = (\underset{\sim}{v}_1, \underset{\sim}{v}_2)_{\underset{\sim}{L}^2(\Omega)} + h_k^2(q_1, q_2)_{L^2(\Omega)} .$$

Timoshenko beam problem The mesh-dependent inner product for Example 1.3 is defined by

$$((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))_k = h_k \sum_p \underset{\sim}{v}_1(p) \cdot \underset{\sim}{v}_2(p) + h_k^2(q_1, q_2)_{L^2(\Omega)} ,$$

where p ranges over all internal vertices of $\mathcal{T}_k$.

Reissner-Mindlin plate problem Let $\underset{\sim}{v}_i = \underset{\sim}{\phi}_i + \underset{\sim}{\psi}_i$ where $\underset{\sim}{\phi}_i \in \overset{\circ}{\underset{\sim}{M}}{}^1_0(\mathcal{T}_k)$ and $\underset{\sim}{\psi}_i \in \underset{\sim}{B}^3(\mathcal{T}_k)$ for $i = 1, 2$. Then

$$((\underset{\sim}{v}_1, q_1), (\underset{\sim}{v}_2, q_2))_k = h_k^2 \sum_p \underset{\sim}{\phi}_1(p) \cdot \underset{\sim}{\phi}_2(p) +$$

$$+ (\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2)_{\underset{\sim}{L}^2(\Omega)} + h_k^2(h_k^2 + t^2) \sum_p c_k(p)\, q_1(p)\, q_2(p)$$

where p ranges over all vertices of $\mathcal{T}_k$ and $c_k(p)$ is chosen so that

$$h_k^2 \sum_p c_k(p)\, q(p) = \int_\Omega q\, dx \quad \forall q \in M_0^1(\mathcal{T}_k)\,.$$

(Since our family of triangulations is obtained by a regular subdivision, $c_k(\cdot)$, $k = 1, 2, 3, \ldots$, only assume finitely many different values. The constants in (A.8) are therefore independent of k.)

Remarks : All of the mesh-dependent inner products in the examples are actually also defined on the corresponding finite element spaces without the constraints (imposed by « $\wedge$ » or « $\perp$ »). They have the properties that

(a) the nodal basis of the finite element space without the constraints are orthogonal with respect to the inner product, and

(b) the constraints themselves can be enforced by the inner product. This means that there exists a finite-dimensional subspace S_k of the unconstrained finite element space such that a member of the unconstrained finite element space belongs to $\underset{\sim}{V}_k \times W_k$ if and only if it is orthogonal to S_k with respect to the inner product. This is important for the programming aspect of the multigrid method.

Intergrid Transfer Operators

There is a coarse-to-fine intergrid transfer operator $I_{k-1}^k : \underset{\sim}{V}_{k-1} \times W_{k-1} \to \underset{\sim}{V}_k \times W_k$ defined by

$$I_{k-1}^k(\underset{\sim}{v}, q) = (J_{k-1}^k \underset{\sim}{v}, q), \text{ where } J_{k-1}^k : \underset{\sim}{V}_{k-1} \to \underset{\sim}{V}_k\,.$$

We assume that

(A.9*a*) $$\| J_{k-1}^k \underset{\sim}{v} \|_{\underset{\sim}{L}^2(\Omega)} \leqslant C \| \underset{\sim}{v} \|_{\underset{\sim}{L}^2(\Omega)} \quad \text{for all} \quad \underset{\sim}{v} \in \underset{\sim}{V}_{k-1}\,.$$

(A.9*b*) $$\| \underset{\sim}{v} - J_{k-1}^k \underset{\sim}{v} \|_{\underset{\sim}{L}^2(\Omega)} \leqslant C h_k \| \underset{\sim}{v} \|_{k-1} \text{ for all } \underset{\sim}{v} \in \underset{\sim}{V}_{k-1}\,.$$

(A.9*c*)
$$|\!|\!| I_{k-1}^k \Pi_{k-1}(\underset{\sim}{w}, r) - \Pi_k(\underset{\sim}{w}, r) |\!|\!|_{0,k} \leqslant C h_k^2 \{ |\underset{\sim}{w}|_{\underset{\sim}{H}^2(\Omega)} + |r|_{H^1(\Omega)} + t^\alpha |r|_{H^{1+\alpha}(\Omega)} \}$$

$$\text{for all } (\underset{\sim}{w}, r) \in (\underset{\sim}{H}^2(\Omega) \times H^{1+\alpha}(\Omega)) \cap (\underset{\sim}{V} \times W)\,.$$

The fine-to-coarse intergrid transfer operator I_k^{k-1} : $\underset{\sim}{V}_k \times W_k \to \underset{\sim}{V}_{k-1} \times W_{k-1}$ is then defined by

$$(2.9) \qquad (I_k^{k-1}(\underset{\sim}{w}, r), (\underset{\sim}{v}, q))_{k-1} = ((\underset{\sim}{w}, r), I_{k-1}^k(\underset{\sim}{v}, q))_k$$

for all $(\underset{\sim}{w}, r) \in \underset{\sim}{V}_k \times W_k$, $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$.

Examples : Pure displacement and pure traction problems In both of these examples, J_{k-1}^k is just the L^2-orthogonal projection.

Timoshenko beam problem Here the finite element spaces are nested, so we can take J_{k-1}^k to be the natural injection operator.

Reissner-Mindlin plate problem Let $\underset{\sim}{v} \in \underset{\sim}{V}_{k-1}$ be written as $\underset{\sim}{\phi} + \underset{\sim}{\psi}$ where $\underset{\sim}{\phi} \in \overset{\circ}{\underset{\sim}{M}}{}_0^1(\mathcal{T}_{k-1})$ and $\underset{\sim}{\psi} \in \underset{\sim}{B}^3(\mathcal{T}_{k-1})$. Then $J_{k-1}^k \underset{\sim}{v} = \underset{\sim}{\phi}$.

Remarks : The estimate (A.9*a*) is trivial for these examples. The estimate (A.9*b*) is trivial for Example 1.3, which is conforming and nested. This estimate for Examples 1.1, 1.2 and 1.4 can be easily established by a simple computation. We refer to [11] for more details. The estimate (A.9*c*) in these examples follows easily from (A.8) and the interpolation error estimate (A.6). Moreover, in Example 1.1, J_{k-1}^k can be a more general type of averaging operator that requires less effort in programming (*cf.* [12]).

In the original paper of Huang (*cf.* [27]), $J_{k-1}^k \underset{\sim}{v}$ is defined to be $\underset{\sim}{\phi} + \underset{\sim}{\psi}^*$, where $\underset{\sim}{\psi}^* \in \underset{\sim}{B}^3(\mathcal{T}_k)$ is the L^2-orthogonal projection of $\underset{\sim}{\psi}$. The fact that one can drop the bubbles in the intergrid transfer operator was first observed in [36].

Another feature of the intergrid transfer operator in these examples is that they are actually defined on the corresponding finite element spaces without constraints. The constraints themselves are preserved by I_{k-1}^k and I_k^{k-1}. This is tied to the fact that the constraints can be enforced by the mesh-dependent inner product. (For more details, see [12] and [13].)

3. THE MULTIGRID ALGORITHM

We will describe an iteration scheme for each level k. The k^{th} level iteration with initial guess $(\underset{\sim}{y}_0, z_0) \in \underset{\sim}{V}_k \times W_k$ yields $MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))$ as an approximate solution to the following equation in $\underset{\sim}{V}_k \times W_k$:

$$(3.1) \qquad B_k^t(\underset{\sim}{y}, z) = (\underset{\sim}{w}, r) .$$

For $k = 1$, $MG(1, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))$ is the solution obtained from a direct method. In other words,

$$MG(1, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r)) = (B_1^t)^{-1}(\underset{\sim}{w}, r).$$

For $k > 1$, there are two steps.

Smoothing Step. Let $(\underset{\sim}{y}_l, z_l) \in \underset{\sim}{V}_k \times W_k$ be defined recursively by the equations

(3.2)
$$(\underset{\sim}{y}_l, z_l) = (\underset{\sim}{y}_{l-1}, z_{l-1}) + \frac{1}{\Lambda_k} B_k^t((\underset{\sim}{w}, r) - B_k^t(\underset{\sim}{y}_{l-1}, z_{l-1})), \quad 1 \leqslant l \leqslant m,$$

where m is a positive integer independent of k and t and $\Lambda_k := Ch_k^{-4}$ (*cf.* (2.5)) dominates the spectral radius of $(B_k^t)^2$.

Correction Step. Let $(\underset{\sim}{\bar{w}}, \bar{r}) := I_k^{k-1}((\underset{\sim}{w}, r) - B_k^t(\underset{\sim}{y}_m, z_m))$. Let $(\underset{\sim}{v}_i, q_i) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ be defined recursively by

(3.3) $(\underset{\sim}{v}_0, q_0) = (\underset{\sim}{0}, 0)$ and

$$(\underset{\sim}{v}_i, q_i) = MG(k-1, (\underset{\sim}{v}_{i-1}, q_{i-1}), (\underset{\sim}{\bar{w}}, \bar{r})), \quad i = 1, 2.$$

Then $MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))$ is defined to be

$$(\underset{\sim}{y}_m, z_m) + I_{k-1}^k(\underset{\sim}{v}_2, q_2).$$

Remarks :

1. The algorithm defined above is a one-sided $\mathscr{W}$-cycle algorithm. The amount of work of the method is $\mathcal{O}(n_k)$, where n_k is the dimension of $\underset{\sim}{V}_k \times W_k$.
2. Since the constraints on the finite element spaces can be enforced using the mesh-dependent inner product $(\,.\,,\,.\,)_k$ and the constraints are preserved by the intergrid transfer operators I_{k-1}^k and I_k^{k-1}, except on the first level one can actually use the finite element spaces without any constraints in the construction of the multigrid algorithm. This is beneficial for programming since there is a natural nodal basis for the finite element spaces without constraints (*cf.* [12] and [13]).
3. For Examples 1.1, 1.2 and 1.3, one can use a nested iteration of these schemes to obtain a full multigrid algorithm. In Example 1.4 one can couple these schemes with nonconforming multigrid methods for the Laplace equation (*cf.* [7], [9], [10]) to solve the full Reissner-Mindlin plate problem.

4. THREE LEMMAS

Let $P_k^{k-1} : \underset{\sim}{V}_k \times W_k \to \underset{\sim}{V}_{k-1} \times W_{k-1}$ be defined by

$$\mathfrak{B}_{k-1}^t(P_k^{k-1}(\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \mathfrak{B}_k^t((\underset{\sim}{w}, r), I_{k-1}^k(\underset{\sim}{v}, q)) \tag{4.1}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$, $(\underset{\sim}{w}, r) \in \underset{\sim}{V}_k \times W_k$.

From (A.9*a*), the definition of I_{k-1}^k, (2.1) and (2.7) we know that

$$|||I_{k-1}^k(\underset{\sim}{v}, q)|||_{0,k} \leqslant C\,|||(\underset{\sim}{v}, q)|||_{0,k-1} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}\,. \tag{4.2}$$

It follows from (4.2) and (2.8) that

$$|||P_k^{k-1}(\underset{\sim}{w}, r)|||_{2,k-1} \leqslant C\,|||(\underset{\sim}{w}, r)|||_{2,k} \quad \forall (\underset{\sim}{w}, r) \in \underset{\sim}{V}_k \times W_k\,. \tag{4.3}$$

LEMMA 4.1 : *Let $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$. Assume that $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k \times W_k$ satisfies*

$$\mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k\,, \tag{4.4}$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ *satisfies*

$$\mathfrak{B}_{k-1}^t((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}, q)) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}\,, \tag{4.5}$$

Then there exists a positive constant C such that

$$|||(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)|||_{0,k-1} \leqslant C h_k^2 \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\,. \tag{4.6}$$

Proof : Let

$$(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k) = (\underset{\sim}{\eta}, \tau) \in \underset{\sim}{V}_{k-1} \times W_{k-1}\,. \tag{4.7}$$

Recall that (*cf.* (2.7))

$$|||(\underset{\sim}{\eta}, \tau)|||_{0,k-1} \sim \|\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)} + h_{k-1}(1 + t^\alpha h_{k-1}^{-\alpha})\, \|\tau\|_{L^2(\Omega)}\,.$$

We will estimate $\|\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)}$ and $h_{k-1}(1 + t^\alpha h_{k-1}^{-\alpha})\, \|\tau\|_{L^2(\Omega)}$ by duality arguments.

Let $(\underset{\sim}{\phi}, \psi) \in \underset{\sim}{V} \times W$ satisfy

$$\mathfrak{B}^t((\underset{\sim}{\phi}, \psi), (\underset{\sim}{v}, q)) = \int_\Omega \underset{\sim}{\eta} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W \tag{4.8}$$

and $(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ satisfy

$$\mathfrak{B}^t_{k-1}((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{v}, q)) = \int_\eta \underset{\sim}{\eta} \cdot \underset{\sim}{v}\, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1} . \tag{4.9}$$

The elliptic regularity estimate (A.3a) and the discretization error estimate (A.7a) imply that

$$\| \underset{\sim}{\phi} \|_{H^2(\Omega)} + \| \psi \|_{H^1(\Omega)} + t^\alpha \| \psi \|_{H^{1+\alpha}(\Omega)} \leq C \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \tag{4.10}$$

and

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)} \leq C h_k^2 \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} . \tag{4.11}$$

Using (4.9), (4.1), (4.7), (4.4), (4.5), (A.6), (4.11), (4.10) and (A.9), we have

$$\begin{aligned}
\int_\Omega |\underset{\sim}{\eta}|^2\, dx &= \mathfrak{B}^t_{k-1}((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\eta}, \tau)) \\
&= \mathfrak{B}^t_{k-1}((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})) \\
&\quad - \mathfrak{B}^t_k(I^k_{k-1}(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\zeta}_k, \xi_k)) \\
&= \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{\phi}_{k-1}\, dx - \int_\Omega \underset{\sim}{f} \cdot J^k_{k-1}\, \underset{\sim}{\phi}_{k-1}\, dx \\
&\leq \| \underset{\sim}{f} \|_{\underset{\sim}{L}^2(\Omega)} \| \underset{\sim}{\phi}_{k-1} - J^k_{k-1}\, \underset{\sim}{\phi}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)} \\
&\leq \| \underset{\sim}{f} \|_{\underset{\sim}{L}^2(\Omega)} \{ \| \underset{\sim}{\phi}_{k-1} - \Pi^v_k\, \underset{\sim}{\phi} \|_{\underset{\sim}{L}^2(\Omega)} + \| \Pi^v_k\, \underset{\sim}{\phi} - J^k_{k-1}\, \Pi^v_{k-1}\, \underset{\sim}{\phi} \|_{\underset{\sim}{L}^2(\Omega)} \\
&\quad + \| J^k_{k-1}(\Pi^v_{k-1}\, \underset{\sim}{\phi} - \underset{\sim}{\phi}_{k-1}) \|_{\underset{\sim}{L}^2(\Omega)} \} \\
&\leq \| \underset{\sim}{f} \|_{\underset{\sim}{L}^2(\Omega)} \{ C h_k^2 \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} + C \| \Pi^v_{k-1}\, \underset{\sim}{\phi} - \underset{\sim}{\phi}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)} \} \\
&\leq C h_k^2 \| \underset{\sim}{f} \|_{\underset{\sim}{L}^2(\Omega)} \| \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} .
\end{aligned}$$

Therefore

$$\|\underset{\sim}{\eta}\|_{L^2(\Omega)} \leqslant C h_k^2 \|\underset{\sim}{f}\|_{L^2(\Omega)} . \tag{4.12}$$

Let $(\underset{\sim}{\omega}, \chi) \in \underset{\sim}{V} \times W$ satisfy

$$\mathfrak{B}^t((\underset{\sim}{\omega}, \chi), (\underset{\sim}{v}, q)) = \int_\Omega \tau q \, dx \quad \forall (\underset{\sim}{v}, q) \times \underset{\sim}{V} \times W \tag{4.13}$$

and $(\underset{\sim}{\omega}_{k-1}, \chi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ satisfy

(4.14)

$$\mathfrak{B}^t_{k-1}((\underset{\sim}{\omega}_{k-1}, \chi_{k-1}), (\underset{\sim}{v}, q)) = \int_\Omega \tau q \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1} .$$

From (A.5) we have

$$\|\underset{\sim}{\omega}_{k-1}\|_{k-1} \leqslant C \|\tau\|_{L^2(\Omega)} . \tag{4.15}$$

From (4.14), (4.1), (4.7), (4.4), (4.5), (A.9*b*) and (4.15) we have

$$\begin{aligned} \int_\Omega \tau^2 \, dx &= \mathfrak{B}^t_{k-1}((\underset{\sim}{\omega}_{k-1}, \chi_{k-1}), (\underset{\sim}{\eta}, \tau)) \\ &= \mathfrak{B}^t_{k-1}((\underset{\sim}{\omega}_{k-1}, \chi_{k-1}), (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})) \\ &\quad - \mathfrak{B}^t_k(I^k_{k-1}(\underset{\sim}{\omega}_{k-1}, \chi_{k-1}), (\underset{\sim}{\zeta}_k, \xi_k)) \\ &= \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{\omega}_{k-1} \, dx - \int_\Omega \underset{\sim}{f} \cdot J^k_{k-1} \underset{\sim}{\omega}_{k-1} \, dx \\ &\leqslant \|\underset{\sim}{f}\|_{L^2(\Omega)} \|\underset{\sim}{\omega}_{k-1} - J^k_{k-1} \underset{\sim}{\omega}_{k-1}\|_{L^2(\Omega)} \\ &\leqslant C h_k \|\underset{\sim}{f}\|_{L^2(\Omega)} \|\tau\|_{L^2(\Omega)} . \end{aligned} \tag{4.16}$$

Therefore

$$h_k \|\tau\|_{L^2(\Omega)} \leqslant C h_k^2 \|\underset{\sim}{f}\|_{L^2(\Omega)} . \tag{4.17}$$

For the case $\alpha = 0$ the proof of the lemma is completed by combining (4.12) and (4.17).

In the case $\alpha = 1$ it follows from the elliptic regularity estimate (A.3b) and the discretization error estimate (A.7b) that

$$(4.18) \qquad \|\underset{\sim}{\omega}\|_{\underset{\sim}{H}^2(\Omega)} + \|\chi\|_{H^1(\Omega)} + t\|\chi\|_{H^2(\Omega)} \leqslant Ct^{-1}\|\tau\|_{L^2(\Omega)}$$

and

$$(4.19) \qquad \|\underset{\sim}{\omega} - \underset{\sim}{\omega}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \leqslant Ct^{-1} h_k^2\|\tau\|_{L^2(\Omega)}\,.$$

We obtain from (4.16) that

$$\begin{aligned}\int_\Omega \tau^2\,dx &\leqslant \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\|\underset{\sim}{\omega}_{k-1} - J_{k-1}^k\,\underset{\sim}{\omega}_{k-1}\|_{\underset{\sim}{L}^2(\Omega)} \\ &\leqslant \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\{\|\underset{\sim}{\omega}_{k-1} - \Pi_k^v\,\underset{\sim}{\omega}\|_{\underset{\sim}{L}^2(\Omega)} + \|\Pi_k^v\,\underset{\sim}{\omega} - J_{k-1}^k\,\Pi_{k-1}^v\,\underset{\sim}{\omega}\|_{\underset{\sim}{L}^2(\Omega)} \\ &\quad + \|J_{k-1}^k(\Pi_{k-1}^v\,\underset{\sim}{\omega} - \omega_{k-1})\|_{\underset{\sim}{L}^2(\Omega)}\} \\ &\leqslant \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\{Ct^{-1}h_k^2\|\tau\|_{L^2(\Omega)} + C\|\Pi_{k-1}^v\,\underset{\sim}{\omega} - \omega_{k-1}\|_{\underset{\sim}{L}^2(\Omega)}\} \\ &\leqslant Ct^{-1}h_k^2\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\|\tau\|_{L^2(\Omega)}\end{aligned}$$

by using (A.6), (4.18), (4.19) and (A.9). Therefore

$$(4.20) \qquad t\|\tau\|_{L^2(\Omega)} \leqslant Ch_k^2\|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)}\,.$$

For the case $\alpha = 1$ the proof of the lemma is completed by combining (4.12), (4.17) and (4.20). □

LEMMA 4.2 : *There exists a positive constant C such that*

$$(4.21) \qquad \|\underset{\sim}{v} - J_{k-1}^k\,\underset{\sim}{v}\|_{\underset{\sim}{L}^2(\Omega)} \leqslant Ch_k^2|||(\underset{\sim}{v}, q)|||_{2,k-1}$$

for all $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$.

Proof: Given any

$$(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1},$$

let $(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k \times W_k$ satisfy

$$(4.22) \quad \mathfrak{B}_k^t((\underset{\sim}{\xi}_k, \xi_k), (\underset{\sim}{v}', q')) = \int_\Omega (\underset{\sim}{v} - J_{k-1}^k\,\underset{\sim}{v})\cdot\underset{\sim}{v}'\,dx \quad \forall(\underset{\sim}{v}', q') \in \underset{\sim}{V}_k \times W_k$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ satisfy

$$\text{(4.23)} \quad \mathfrak{B}^t_{k-1}((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}', q')) = \int_\Omega (\underset{\sim}{v} - J^k_{k-1} \underset{\sim}{v}) \cdot \underset{\sim}{v}' \, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_{k-1} \times W_{k-1} .$$

Therefore using (4.22), (4.23), (4.1), (2.8), and Lemma 4.1 we have

$$\begin{aligned} \| \underset{\sim}{v} - J^k_{k-1} \underset{\sim}{v} \|^2_{\underset{\sim}{L}^2(\Omega)} &= \int_\Omega (\underset{\sim}{v} - J^k_{k-1} \underset{\sim}{v}) \cdot \underset{\sim}{v} \, dx - \int_\Omega (\underset{\sim}{v} - J^k_{k-1} \underset{\sim}{v}) \cdot J^k_{k-1} \underset{\sim}{v} \, dx \\ &= \mathfrak{B}^t_{k-1}((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}, q)) \\ &\quad - \mathfrak{B}^t_k((\underset{\sim}{\zeta}_k, \xi_k), I^k_{k-1}(\underset{\sim}{v}, q)) \\ &= \mathfrak{B}^t_{k-1}((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P^{k-1}_k(\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)) \\ &\leqslant |||(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P^{k-1}_k(\underset{\sim}{\zeta}_k, \xi_k)|||_{0,k-1} |||(\underset{\sim}{v}, q)|||_{2,k-1} \\ &\leqslant C h^2_k \| \underset{\sim}{v} - J^k_{k-1} \underset{\sim}{v} \|_{\underset{\sim}{L}^2(\Omega)} |||(\underset{\sim}{v}, q)|||_{2,k-1} . \quad \square \end{aligned}$$

LEMMA 4.3: *Let* $g \in L^2(\Omega)$, $(\underset{\sim}{v}_k, q_k) \in \underset{\sim}{V}_k \times W_k$ *satisfy*

$$\text{(4.24)} \qquad \mathfrak{B}^t_k((\underset{\sim}{v}_k, q_k), (\underset{\sim}{v}, q)) = \int_\Omega g q \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k$$

and

$$\text{(4.25)} \quad \mathfrak{B}^t_{k-1}((\underset{\sim}{v}_{k-1}, q_{k-1}), (\underset{\sim}{v}, q)) = \int_\Omega g q \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1} .$$

Then there exists a positive constant C such that

$$\text{(4.26)} \qquad |||(\underset{\sim}{v}_k, q_k) - I^k_{k-1}(\underset{\sim}{v}_{k-1}, q_{k-1})|||_{0,k} \leqslant C h_k \| g \|_{L^2(\Omega)} .$$

Proof: Recall that (*cf.* (2.7))

$$\begin{aligned} |||(\underset{\sim}{v}_k, q_k) - I^k_{k-1}(\underset{\sim}{v}_{k-1}, q_{k-1})|||_{0,k} &\sim \| \underset{\sim}{v}_k - J^k_{k-1} \underset{\sim}{v}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)} \\ &\quad + h_k (1 + t^\alpha h_k^{-\alpha}) \| q_k - q_{k-1} \|_{L^2(\Omega)} . \end{aligned}$$

First we estimate $\| \underset{\sim}{v}_k - J_{k-1}^k \, \underset{\sim}{v}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)}$ by a duality argument.
Let $(\underset{\sim}{\zeta}, \xi) \in \underset{\sim}{V} \times W$ satisfy

(4.27)
$$\mathfrak{B}^t((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) = \int_\Omega (\underset{\sim}{v}_k - J_{k-1}^k \, \underset{\sim}{v}_{k-1}) \cdot \underset{\sim}{v} \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W,$$

$(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k \times W_k$ satisfy

(4.28)
$$\mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)) = \int_\Omega (\underset{\sim}{v}_k - J_{k-1}^k \, \underset{\sim}{v}_{k-1}) \cdot \underset{\sim}{v} \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k$$

and $(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ satisfy

$$(4.29) \quad \mathfrak{B}_{k-1}^t((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})) = \int_\Omega (\underset{\sim}{v}_k - J_{k-1}^k \, \underset{\sim}{v}_{k-1}) \cdot \underset{\sim}{v} \, dx$$
$$\forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}.$$

By applying the discretization error estimate (A.7a) to (4.27)-(4.29) we have

$$(4.30) \qquad \| \xi_k - \xi_{k-1} \|_{L^2(\Omega)} \leq C h_k \| \underset{\sim}{v} - J_{k-1}^k \; \underset{\sim}{v}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)}.$$

Let $(\underset{\sim}{\eta}, \tau) = (\underset{\sim}{\zeta}_k, \xi_k) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)$. By (4.28), (4.1), (4.24) and (4.25), we have

(4.31)
$$\begin{aligned} \| \underset{\sim}{v}_k - J_{k-1}^k \, \underset{\sim}{v}_{k-1} \|_{\underset{\sim}{L}^2(\Omega)}^2 &= \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_k, q_k) - I_{k-1}^k(\underset{\sim}{v}_{k-1}, q_{k-1})) \\ &= \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_k, q_k)) \\ &\quad - \mathfrak{B}_{k-1}^t(P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}_{k-1}, q_{k-1})) \\ &= \int_\Omega g \tau \, dx \leq \| g \|_{L^2(\Omega)} \| \tau \|_{L^2(\Omega)}. \end{aligned}$$

Since

$$(\underset{\sim}{\eta}, \tau) = (\underset{\sim}{\zeta}_k, \xi_k) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) + ((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)),$$

it follows from (2.7), (4.30) and Lemma 4.1 that

(4.32)

$$\begin{aligned}\| \tau \|_{L^2(\Omega)} &\leqslant \| \xi_k - \xi_{k-1} \|_{L^2(\Omega)} + \frac{1}{h_k} |||(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - P_k^{k-1}(\underset{\sim}{\zeta}_k, \xi_k)|||_{0,k-1} \\ &\leqslant Ch_k \| \underset{\sim}{v}_k - \underset{\sim}{J}_{k-1}^k \underset{\sim}{v}_{k-1} \|_{L^2(\Omega)} .\end{aligned}$$

Combining (4.31) and (4.32) we obtain

$$\| \underset{\sim}{v}_k - \underset{\sim}{J}_{k-1}^k \underset{\sim}{v}_{k-1} \|_{L^2(\Omega)} \leqslant Ch_k \| g \|_{L^2(\Omega)} . \tag{4.33}$$

On the other hand, the stability estimate (A.5) for the discrete problem implies that

$$\| q_k \|_{L^2(\Omega)} + \| q_{k-1} \|_{L^2(\Omega)} \leqslant C \| g \|_{L^2(\Omega)} . \tag{4.34}$$

Therefore we have

$$h_k \| q_k - q_{k-1} \|_{L^2(\Omega)} \leqslant Ch_k \| g \|_{L^2(\Omega)} . \tag{4.35}$$

In the case $\alpha = 0$, the proof of the lemma is completed by combining (4.33) and (4.35).

In the case $\alpha = 1$ we compare q_k and q_{k-1} to r where $(\underset{\sim}{w}, r) \in \underset{\sim}{V} \times W$ satisfies

$$\mathfrak{B}'((\underset{\sim}{w}, r), (\underset{\sim}{v}, q)) = \int_\Omega gq \, dx \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V} \times W . \tag{4.36}$$

It follows from (A.7*b*) that

$$t(\| r - q_k \|_{L^2(\Omega)} + \| r - q_{k-1} \|_{L^2(\Omega)}) \leqslant Ch_k \| g \|_{L^2(\Omega)} . \tag{4.37}$$

Therefore we have

$$t \| q_k - q_{k-1} \|_{L^2(\Omega)} \leqslant Ch_k \| g \|_{L^2(\Omega)} . \tag{4.38}$$

The lemma now follows from (4.33), (4.35) and (4.38). □

5. CONVERGENCE ANALYSIS

We follow the methodology of [5]. The first step is to establish the convergence of the two-grid algorithm where the residual equation is solved exactly on the coarser grid. The final output of the two-grid algorithm for (3.1) is $(\underset{\sim}{y}^{\#}, z^{\#}) := (\underset{\sim}{y}_m, z_m) + I_{k-1}^{k}(\underset{\sim}{v}^{\#}, q^{\#})$, where

$$(5.1) \qquad \begin{aligned}(\underset{\sim}{v}^{\#}, q^{\#}) &= (B_{k-1}^{t})^{-1}(\bar{\underset{\sim}{w}}, \bar{r}) \\ &= (B_{k-1}^{t})^{-1} I_{k}^{k-1}((\underset{\sim}{w}, r) - B_k^t(\underset{\sim}{y}_m, z_m)) \\ &= (B_{k-1}^{t})^{-1} I_{k}^{k-1} B_k^t(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m).\end{aligned}$$

LEMMA 5.1 : $(\underset{\sim}{v}^{\#}, q^{\#}) = P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m)$.

Proof: Given any $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$, it follows from (2.4), (5.1), (2.9) and (4.1) that

$$\begin{aligned}\mathfrak{B}_{k-1}^{t}((\underset{\sim}{v}^{\#}, q^{\#}), (\underset{\sim}{v}, q)) &= (B_{k-1}^{t}(\underset{\sim}{v}^{\#}, q^{\#}), (\underset{\sim}{v}, q))_{k-1} \\ &= (I_k^{k-1} B_k^t(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q))_{k-1} \\ &= (B_k^t(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^{k}(\underset{\sim}{v}, q))_k \\ &= \mathfrak{B}_k^t((\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^{k}(\underset{\sim}{v}, q)) \\ &= \mathfrak{B}_{k-1}^{t}(P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q)). \quad \square\end{aligned}$$

Let R_k be defined by

$$(5.2) \qquad R_k := I - \frac{1}{\Lambda_k}(B_k^t)^2.$$

It follows from the definition of the mesh dependent norms that

$$(5.3) \qquad |||R_k(\underset{\sim}{v}, q)|||_{s,k} \leqslant |||(\underset{\sim}{v}, q)|||_{s,k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k \text{ and } s \in \mathbb{R}.$$

From the smoothing step (3.2), we obtain

$$(5.4) \qquad (\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m) = R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0).$$

Combining Lemma 5.1 and (5.4) we have the following relation between the initial error and the final error of the two-grid algorithm :

$$(5.5)\qquad (\underset{\sim}{y} - \underset{\sim}{y}^{\#}, z - z^{\#}) = (I - I_{k-1}^{k} P_k^{k-1}) R_k^m (\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0) .$$

The effect of R_k^m is measured by the following lemma on the smoothing property. Its proof is standard (*cf.* [5]) and will be omitted.

LEMMA 5.2 (Smoothing Property) : *There exists a positive constant C such that*

$$(5.6)\quad |||R_k^m(\underset{\sim}{v}, q)|||_{2,k} \leqslant C h_k^{-2} m^{-1/2} |||(\underset{\sim}{v}, q)|||_{0,k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k .$$

The following lemma on the approximation property measures the effect of the operators $I - I_{k-1}^{k} P_k^{k-1}$.

LEMMA 5.3 (Approximation Property) : *There exists a positive constant C such that for $k > 1$*

$$(5.7)\quad |||(I - I_{k-1}^{k} P_k^{k-1})(\underset{\sim}{v}, q)|||_{0,k} \leqslant C h_k^2 |||(\underset{\sim}{v}, q)|||_{2,k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k .$$

Proof : Given $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_k \times W_k$, let $(\underset{\sim}{\eta}, \tau) = P_k^{k-1}(\underset{\sim}{v}, q)$. Hence

$$(I - I_{k-1}^{k} P_k^{k-1})(\underset{\sim}{v}, q) = (\underset{\sim}{v} - J_k^{k-1}\underset{\sim}{\eta}, q - \tau) .$$

Recall that (*cf.* (2.7))

$$|||(\underset{\sim}{v} - J_k^{k-1}\underset{\sim}{\eta}, q - r)|||_{0,k} \sim \|\underset{\sim}{v} - J_k^{k-1}\underset{\sim}{\eta}\|^2_{\underset{\sim}{L}^2(\Omega)}$$
$$+ h_k(1 + t^{\alpha} h_k^{-\alpha}) \|q - \tau\|_{L^2(\Omega)} .$$

We shall estimate $\|\underset{\sim}{v} - J_k^{k-1}\underset{\sim}{\eta}\|_{\underset{\sim}{L}^2(\Omega)}$ and

$$h_k(1 + t^{\alpha} h_k^{-\alpha}) \|q - \tau\|_{L^2(\Omega)}$$

by duality arguments.

Let $(\underset{\sim}{\zeta}, \xi) \in \underset{\sim}{V} \times W$ satisfy

$$(5.9)\qquad \mathfrak{B}^t((\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}', q')) = \int_{\Omega} (\underset{\sim}{v} - J_{k-1}^{k}\underset{\sim}{\eta}) \cdot \underset{\sim}{v}' \, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V} \times W ,$$

$(\underset{\sim}{\zeta}_k, \xi_k) \in \underset{\sim}{V}_k \times W_k$ satisfy

$$(5.10)\ \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}', q')) = \int_{\Omega} (\underset{\sim}{v} - J_{k-1}^{k}\underset{\sim}{\eta}) \cdot \underset{\sim}{v}' \, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_k \times W_k ,$$

and $\quad (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1} \quad$ satisfy

$$(5.11)\quad \mathfrak{B}^t_{k-1}((\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), (\underset{\sim}{v}', q')) = \\ = \int_\Omega (\underset{\sim}{v} - J^k_{k-1}\underset{\sim}{\eta}) \cdot \underset{\sim}{v}'\, dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_{k-1} \times W_{k-1}\,.$$

The elliptic regularity estimate (A.3a) implies that

$$(5.12)\quad \|\underset{\sim}{\zeta}\|_{H^2(\Omega)} + \|\xi\|_{H^1(\Omega)} + t^\alpha \|\xi\|_{H^{1+\alpha}(\Omega)} \leqslant C \|\underset{\sim}{v} - J^k_{k-1}\underset{\sim}{\eta}\|_{L^2(\Omega)}$$

and the discretization error estimate (A.7a) implies that

$$(5.13)\quad \|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_k\|_{L^2(\Omega)} + \|\underset{\sim}{\zeta} - \underset{\sim}{\zeta}_{k-1}\|_{L^2(\Omega)} \\ + h_k(1 + t^\alpha h_k^{-\alpha}) (\|\xi - \xi_k\|_{L^2(\Omega)} + \|\xi - \xi_{k-1}\|_{L^2(\Omega)}) \\ \leqslant C h_k^2 \|\underset{\sim}{v} - J^k_{k-1}\underset{\sim}{\eta}\|_{L^2(\Omega)}\,.$$

From the interpolation error estimate (A.6) we also have

$$(5.14)\quad \|(\underset{\sim}{\zeta}, \xi) - \Pi_k(\underset{\sim}{\zeta}, \xi)\|_{L^2(\Omega)} + \|(\underset{\sim}{\zeta}, \xi) - \Pi_{k-1}(\underset{\sim}{\zeta}, \xi)\|_{L^2(\Omega)} \\ \leqslant C(h_k^2 |\underset{\sim}{\zeta}|_{H^2(\Omega)} + \min(h_k |\xi|_{H^1(\Omega)}, h_k^{1+\alpha} |\xi|_{H^{1+\alpha}(\Omega)}))\,.$$

In view of (2.7) and (5.12)-(5.14) we obtain

$$(5.15)\quad |||(\underset{\sim}{\zeta}_k, \xi_k) - \Pi_k(\underset{\sim}{\zeta}, \xi)|||_{0,k} + |||(\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}) - \Pi_{k-1}(\underset{\sim}{\zeta}, \xi)|||_{0,k-1} \\ \leqslant C h_k^2 \|\underset{\sim}{v} - J^k_{k-1}\underset{\sim}{\eta}\|_{L^2(\Omega)}\,.$$

Moreover, it follows from (A.9c) and (5.12) that

$$(5.16)\quad |||I^k_{k-1}\Pi_{k-1}(\underset{\sim}{\zeta}, \xi) - \Pi_k(\underset{\sim}{\zeta}, \xi)|||_{0,k} \leqslant C h_k^2 \|\underset{\sim}{v} - J^k_{k-1}\underset{\sim}{\eta}\|_{L^2(\Omega)}\,.$$

Hence by (5.10), (4.1) and (5.11) we have

(5.17)

$$
\begin{aligned}
\| \underset{\sim}{v} - J_{k-1}^{k} \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}^{2} &= \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)) - \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k), I_{k-1}^k(\underset{\sim}{\eta}, \tau)) \\
&= \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k) - \Pi_k(\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) \\
&\quad + \mathfrak{B}_k^t(\Pi_k(\underset{\sim}{\zeta}, \xi) - I_{k-1}^k \, \Pi_{k-1}(\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) \\
&\quad + \mathfrak{B}_k^t(I_{k-1}^k \, \Pi_{k-1}(\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) \\
&\quad - \int_\Omega (\underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta}) \cdot J_{k-1}^k \, \underset{\sim}{\eta} \, dx \\
&= \mathfrak{B}_k^t((\underset{\sim}{\zeta}_k, \xi_k) - \Pi_k(\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) \\
&\quad + \mathfrak{B}_k^t(\Pi_k(\underset{\sim}{\zeta}, \xi) - I_{k-1}^k \, \Pi_{k-1}(\underset{\sim}{\zeta}, \xi), (\underset{\sim}{v}, q)) \\
&\quad + \mathfrak{B}_{k-1}^t(\Pi_{k-1}(\underset{\sim}{\zeta}, \xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), P_k^{k-1}(\underset{\sim}{v}, q)) \\
&\quad + \int_\Omega (\underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta}) (\underset{\sim}{\eta} - J_{k-1}^k \, \underset{\sim}{\eta}) \, dx \, .
\end{aligned}
$$

Combining (2.8), (5.15), (4.3), (5.16), Lemma 4.2 and the definition of $(\underset{\sim}{\eta}, \tau)$ we find from (5.17) that

(5.18)

$$
\begin{aligned}
\| \underset{\sim}{v} - J_{k-1}^{k} \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)}^{2} &\leqslant |||(\underset{\sim}{\zeta}_k, \xi_k) - \Pi_k(\underset{\sim}{\zeta}, \xi)|||_{0,k} |||(\underset{\sim}{v}, q)|||_{2,k} \\
&\quad + |||\Pi_k(\underset{\sim}{\zeta}, \xi) - I_{k-1}^k \, \Pi_{k-1}(\underset{\sim}{\zeta}, \xi)|||_{0,k} |||(\underset{\sim}{v}, q)|||_{2,k} \\
&\quad + |||\Pi_{k-1}(\underset{\sim}{\zeta}, \xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})|||_{0,k-1} |||P_k^{k-1}(\underset{\sim}{v}, q)|||_{2,k-1} \\
&\quad + \| \underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \| \underset{\sim}{\eta} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \\
&\leqslant C h_k^2 \| \underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} |||(\underset{\sim}{v}, q)|||_{2,k} \\
&\quad + \| \underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \| \underset{\sim}{\eta} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \\
&\leqslant C h_k^2 \| \underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} \{ |||(\underset{\sim}{v}, q)|||_{2,k} \\
&\quad + |||P_k^{k-1}(\underset{\sim}{v}, q)|||_{2,k-1} \} \\
&\leqslant C h_k^2 \| \underset{\sim}{v} - J_{k-1}^k \, \underset{\sim}{\eta} \|_{\underset{\sim}{L}^2(\Omega)} |||(\underset{\sim}{v}, q)|||_{2,k} \, .
\end{aligned}
$$

It follows that

$$\| \underset{\sim}{v} - J_{k-1}^{k} \underset{\sim}{\eta} \|_{L^2(\Omega)} \leqslant C h_k^2 |||(\underset{\sim}{v}, q)|||_{2,k} . \tag{5.19}$$

We now estimate $\| q - \tau \|_{L^2(\Omega)}$.
Let $(\underset{\sim}{\phi}, \psi) \in \underset{\sim}{V} \times W$ satisfy

$$\mathfrak{B}^t((\underset{\sim}{\phi}, \psi), (\underset{\sim}{v}', q')) = \int_\Omega (q - \tau) q' dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V} \times W , \tag{5.20}$$

$(\underset{\sim}{\phi}_k, \psi_k) \in \underset{\sim}{V}_k \times W_k$ satisfy

$$\mathfrak{B}_k^t((\underset{\sim}{\phi}_k, \psi_k), (\underset{\sim}{v}', q')) = \int_\Omega (q - \tau) q' dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_k \times W_k , \tag{5.21}$$

and $(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}) \in \underset{\sim}{V}_{k-1} \times W_{k-1}$ satisfy

$$\begin{aligned} \mathfrak{B}_{k-1}^t((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{v}', q')) = \\ = \int_\Omega (q - \tau) q' dx \quad \forall (\underset{\sim}{v}', q') \in \underset{\sim}{V}_{k-1} \times W_{k-1} . \end{aligned} \tag{5.22}$$

Combining (5.21), (5.22), the definition of $(\underset{\sim}{\eta}, \tau)$, (4.1), (2.8) and Lemma 4.3 we have

(5.23)

$$\begin{aligned} \| q - \tau \|_{L^2(\Omega)}^2 &= \int_\Omega (q - \tau) q \, dx - \int_\Omega (q - \tau) \tau \, dx \\ &= \mathfrak{B}_k^t((\underset{\sim}{\phi}_k, \psi_k), (\underset{\sim}{v}, q)) - \mathfrak{B}_{k-1}^t((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{\eta}, \tau)) \\ &= \mathfrak{B}_k^t((\underset{\sim}{\phi}_k, \psi_k), (\underset{\sim}{v}, q)) - \mathfrak{B}_{k-1}^t((\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), P_k^{k-1}(\underset{\sim}{v}, q)) \\ &= \mathfrak{B}_k^t((\underset{\sim}{\phi}_k, \psi_k) - I_{k-1}^k(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}), (\underset{\sim}{v}, q)) \\ &\leqslant |||(\underset{\sim}{\phi}_k, \psi_k) - I_{k-1}^k(\underset{\sim}{\phi}_{k-1}, \psi_{k-1})|||_{0,k} |||(\underset{\sim}{v}, q)|||_{2,k} \\ &\leqslant C h_k \| q - \tau \|_{L^2(\Omega)} |||(\underset{\sim}{v}, q)|||_{2,k} . \end{aligned}$$

Therefore we have

$$h_k \| q - \tau \|_{L^2(\Omega)} \leqslant C h_k^2 |||(\underset{\sim}{v}, q)|||_{2,k} . \tag{5.24}$$

The proof of the lemma for the case $\alpha = 0$ is now completed by combining (5.19) and (5.24).

For the case $\alpha = 1$, we proceed as follows.

From the elliptic regularity estimate (A.3*b*) we know that $(\underset{\sim}{\phi}, \psi) \in \underset{\sim}{H}^2(\Omega) \times H^2(\Omega)$ and

$$\|\underset{\sim}{\phi}\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} + t\|\psi\|_{H^2(\Omega)} \leqslant Ct^{-1}\|q - \tau\|_{L^2(\Omega)}\,. \tag{5.25}$$

From the discretization error estimate (A.7*b*) and (2.7) we have

$$\begin{aligned} &|||(\underset{\sim}{\phi}, \psi) - (\underset{\sim}{\phi}_{k-1}, \psi_{k-1})|||_{0,k} + |||(\underset{\sim}{\phi}, \psi) - (\underset{\sim}{\phi}_{k-1}, \psi_{k-1})|||_{0,k-1} \\ &\qquad \leqslant Ct^{-1} h_k^2 \|q - \tau\|_{L^2(\Omega)}\,. \end{aligned} \tag{5.26}$$

The interpolation error estimate (A.6), (2.7) and (5.25) imply that

$$\begin{aligned} &|||(\underset{\sim}{\phi}, \psi) - \Pi_k(\underset{\sim}{\phi}, \psi)|||_{0,k} + |||(\underset{\sim}{\phi}, \psi) - \Pi_{k-1}(\underset{\sim}{\phi}, \psi)|||_{0,k-1} \\ &\qquad \leqslant Ct^{-1} h_k^2 \|q - \tau\|_{L^2(\Omega)}\,. \end{aligned} \tag{5.27}$$

It follows from (5.26) and (5.27) that

$$\begin{aligned} &|||(\underset{\sim}{\phi}_k, \psi_k) - \Pi_k(\underset{\sim}{\phi}, \psi)|||_{0,k} + |||(\underset{\sim}{\phi}_{k-1}, \psi_{k-1}) - \Pi_{k-1}(\underset{\sim}{\phi}, \psi)|||_{0,k-1} \\ &\qquad \leqslant Ct^{-1} h_k^2 \|q - \tau\|_{L^2(\Omega)}\,. \end{aligned} \tag{5.28}$$

Finally (A.9*c*) and (5.25) imply that

$$|||I_{k-1}^k\, \Pi_{k-1}(\underset{\sim}{\phi}, \psi) - \Pi_k(\underset{\sim}{\phi}, \psi)|||_{0,k} \leqslant Ct^{-1} h_k^2 \|q - \tau\|_{L^2(\Omega)}\,. \tag{5.29}$$

Using (5.23), (5.28) and (5.29) we find

$$\begin{aligned} \|q - \tau\|_{L^2(\Omega)}^2 &\leqslant |||(\underset{\sim}{\phi}_k, \psi_k) - I_{k-1}^k(\underset{\sim}{\phi}_{k-1}, \psi_{k-1})|||_{0,k} |||(\underset{\sim}{v}, q)|||_{2,k} \\ &\leqslant (\,|||(\underset{\sim}{\phi}_k, \psi_k) - \Pi_k(\underset{\sim}{\phi}, \psi)|||_{0,k} \\ &\quad + |||\Pi_k(\underset{\sim}{\phi}, \psi) - I_{k-1}^k\, \Pi_{k-1}(\underset{\sim}{\phi}, \psi)|||_{0,k} \\ &\quad + |||I_{k-1}^k(\Pi_{k-1}(\underset{\sim}{\phi}, \psi) - (\underset{\sim}{\phi}_{k-1}, \psi_{k-1}))|||_{0,k})\; |||(\underset{\sim}{v}, q)|||_{2,k} \\ &\leqslant Ct^{-1} h_k^2 \|q - \tau\|_{L^2(\Omega)} |||(\underset{\sim}{v}, q)|||_{2,k}\,. \end{aligned}$$

Hence

$$t\|q-\tau\|_{L^2(\Omega)} \leqslant Ch_k^2 |||(\underset{\sim}{v}, q)|||_{2,k}. \tag{5.30}$$

The lemma now follows from (5.19), (5.24) and (5.30). □

The convergence of the two-grid algorithm is obtained by combining the smoothing and approximation properties.

THEOREM 5.4 (Convergence of the Two-Grid Algorithm) : *There exists a positive constant C such that*

$$|||(\underset{\sim}{y}-\underset{\sim}{y}^{\#}, z-z^{\#})|||_{0,k} \leqslant Cm^{-1/2}|||(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)|||_{0,k},$$

where $(\underset{\sim}{y}, z)$ *solves (3.1),* $(\underset{\sim}{y}_0, z_0)$ *is the initial guess, and* $(\underset{\sim}{y}^{\#}, z^{\#})$ *is the output of the two-grid algorithm. Therefore for m sufficiently large, the two-grid algorithm is a contraction with contraction number bounded away from one, independent of k and t.*

Proof : The case $k=1$ is trivial. Using (5.5), Lemmas 5.3 and 5.2 we have for $k>1$

$$\begin{aligned} |||(\underset{\sim}{y}-\underset{\sim}{y}^{\#}, z-z^{\#})|||_{0,k} &= |||(I-I_{k-1}^{k} P_k^{k-1}) R_k^m(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)|||_{0,k} \\ &\leqslant Ch_k^2 |||R_k^m(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)|||_{2,k} \\ &\leqslant Cm^{-1/2}|||(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)|||_{0,k}. \quad \square \end{aligned}$$

A standard perturbation argument (*cf.* [5], [13]) and (5.3) then yield the following theorem on the convergence of the k^{th} level iteration.

THEOREM 5.5 (Convergence of the k^{th} Level Iteration) : *There exists a positive constant C such that when the* k^{th} *level iteration is applied to (3.1), we have*

$$|||(\underset{\sim}{y}, z) - MG(k, (\underset{\sim}{y}_0, z_0), (\underset{\sim}{w}, r))|||_{0,k} \leqslant Cm^{-1/2}|||(\underset{\sim}{y}-\underset{\sim}{y}_0, z-z_0)|||_{0,k},$$

provided that m is chosen to be large enough. Therefore for m sufficiently large, the k^{th} *level iteration is a contraction with contraction number bounded away from one, independent of k and t.*

6. CONCLUDING REMARKS

We have developed a framework for establishing the convergence of multigrid solvers for mixed finite element methods with penalty term. The penalty term may correspond to a singular perturbation of the boundary value problem.

Besides the well-posedness of the continuous and discrete problems (A.1, A.2, A.4, A.5), we require the elliptic regularity estimate (A.3), the interpolation error estimate (A.6) and the discretization error estimate (A.7). In addition we need to choose appropriate mesh-dependent norms and intergrid transfer operators which satisfy (A.8) and (A.9).

Our framework admits the use of nonconforming or nonnested finite element spaces. In the case of nested/conforming finite elements, some of the assumptions become trivial or redundant. By setting $t = 0 = \alpha$, the results of Verfürth (*cf.* [34], [35]) are recovered. For the Reissner-Mindlin plate problem, our theory can also be applied to the MITC elements (*cf.* [16], [19], [22], [32]).

Numerical experiments for Examples 1.1, 1.2 and 1.3 can be found in [12], [13], and [6] respectively. Convergence has been observed for small m.

APPENDIX

In this paper, undertildes (respectively double-undertildes) are used for vector-valued (respectively matrix-valued) functions, operators, and their associated spaces.

Matrices

$$\underset{\approx}{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underset{\approx}{\sigma} : \underset{\approx}{\tau} = \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} \tau_{ij} .$$

Space of Infinitesimal Rigid Motions

$$\underset{\sim}{\mathrm{RM}} = \left\{ \underset{\sim}{v} : \underset{\sim}{v} = \begin{pmatrix} a + bx_2 \\ c - bx_1 \end{pmatrix}, a, b, c \in \mathbb{R} \right\} .$$

Sobolev Spaces

$H^k(\Omega)$ is the standard L^2-based Sobolev space and

$$\| \underset{\sim}{v} \|_{\underset{\sim}{H}^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha \underset{\sim}{v}|^2 \, dx \right)^{1/2}$$

$$| \underset{\sim}{v} |_{\underset{\sim}{H}^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha \underset{\sim}{v}|^2 \, dx \right)^{1/2}$$

$$\mathring{H}^1 = \{ f \in H^1(\Omega) : f|_{\partial\Omega} = 0 \}$$

$$\hat{H}^k(\Omega) = \left\{ f \in H^k(\Omega) : \int_\Omega f \, dx = 0 \right\}$$

$$\underset{\sim}{H}^k_\perp(\Omega) = \{ \underset{\sim}{f} \in \underset{\sim}{H}^k(\Omega) : \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v} \, dx = 0 \quad \forall \underset{\sim}{v} \in \underset{\sim}{\mathrm{RM}} \} .$$

Finite Element Spaces

Let $\mathcal{T}_h$ be a triangulation of Ω. Denote by $\mathcal{P}_k(T)$ the space of polynomials on a triangle T of degree $\leq k$.

$$M^k_{-1}(\mathcal{T}_h) = \{\eta \in L^2(\Omega) : \eta|_T \in \mathcal{P}_k(T) \quad \forall T \in \mathcal{T}_h\}$$

$$M^k_0(\mathcal{T}_h) = M^k_{-1}(\mathcal{T}_h) \cap H^1(\Omega)$$

$$M^1_*(\mathcal{T}_h) = \{\eta \in L^2(\Omega) : \eta|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h \text{ and}$$

$$\eta \text{ is continuous at the midpoints of interelement boundaries}\}$$

$$B^3(\mathcal{T}_h) = \{\eta \in M^3_0(\mathcal{T}_h) :$$

$$\eta \text{ vanishes on the boundary of every element}\}$$

$$N^1_0(\mathcal{T}_h) = M^1_0(\mathcal{T}_h) \oplus B^3(\mathcal{T}_h)\,.$$

In addition, a finite element space with a superscript « ○ » represents the subspace of functions vanishing at the boundary nodes, a superscript « $\wedge$ » represents the subspace of functions with zero mean on Ω, and a subscript « $\perp$ » on a vector finite element space represents the subspace of functions L^2 orthogonal to $\underset{\sim}{\mathrm{RM}}$.

Differential Operators

$$\underset{\sim}{\mathrm{curl}}\, p = \begin{pmatrix} \partial p/\partial x_2 \\ -\ \partial p/\partial x_1 \end{pmatrix}$$

$$\mathrm{div}\ \underset{\sim}{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2$$

$$\mathrm{rot}\ \underset{\sim}{v} = -\ \partial v_1/\partial x_2 + \partial v_2/\partial x_1$$

$$\underset{\approx}{\mathrm{grad}}\, \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 \end{pmatrix}$$

$$\underset{\approx}{\epsilon}(\underset{\sim}{v}) = \frac{1}{2}\,[\underset{\approx}{\mathrm{grad}}\ \underset{\sim}{v} + (\underset{\approx}{\mathrm{grad}}\ \underset{\sim}{v})^t]\,.$$

On the space $M^k_{-1}(\mathcal{T}_h)$ all of the above differential operators can be defined piecewise, and the resulting operator will have a subscript h. For example,

$$(\mathrm{div}_h\ \underset{\sim}{v})\,|_T = \mathrm{div}\,(\underset{\sim}{v}|_T) \quad \forall \underset{\sim}{v} \in \underset{\sim}{M}^k_{-1}(\mathcal{T}_h) \quad \text{and all } T \in \mathcal{T}_h\,.$$

When there is a family of triangulations $\mathcal{T}_h$, instead of using the subscript h_k, we simply use the subscript k.

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