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ON A TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC PROBLEM II. NUMERICAL ANALYSIS (*)

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Résumé. — *On considère l'approximation numérique d'un problème de magnétohydrodynamique bidimensionnelle par des techniques standard d'éléments finis. L'analyse numérique est faite dans le cas de solutions régulières du problème continu. On obtient des estimations d'erreur pour la méthode choisie.*

Abstract. — *We consider the numerical approximation of a two-dimensional magnetohydrodynamic problem by standard finite element techniques. The numerical analysis is made for the case of regular solutions of the continuous problem. Error estimates are derived for the selected numerical method.*

1. INTRODUCTION

We have considered in a first paper (Rappaz-Touzani [1]) the development of a mathematical model and its mathematical analysis for two-dimensional magnetohydrodynamic problems involved in particular in electromagnetic casting processes. The main feature of this problem was the nonlinear coupling between the Navier-Stokes equations and an elliptic equation governing the electromagnetic process. In that paper, we prove that the model admits at least one solution and that this solution is unique if the prescribed total current is small enough.

The present work deals with a numerical method to solve such a nonlinear problem. More precisely, the Navier-Stokes equations are solved by a standard finite element method that is assumed to satisfy the Babuska-Brezzi condition (*cf.* Girault-Raviart [2]) and the electromagnetic problem, which is formulated in the whole plane $\mathbb{R}^2$, is solved by a coupled finite element/boundary element procedure, (Johnson-Nedelec [3]). The analysis of the coupled numerical scheme is based on the theory developed in Crouzeix-Rappaz [4].

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Let us precise the main abstract result we will use in the following : assume we are given two Banach spaces X, Y with respective norms $\| \cdot \|_X$, $\| \cdot \|_Y$ and let us define two mappings

$$G : X \to Y \quad \text{and} \quad T : Y \to X$$

where G is a C^1-mapping and T belongs to $\mathscr{L}(Y;X)$ where $\mathscr{L}(Y;X)$ denotes the space of all linear continuous mappings from Y into X equipped with the norm

$$\|T\|_{\mathscr{L}(Y;X)} = \sup_{u \in Y,\ \|u\|_Y = 1} \|Tu\|_X .$$

We begin by assuming that TG possesses a fixed point Φ in X, i.e., $\Phi \in X$ is such that

$$\Phi = TG(\Phi) . \tag{1.1}$$

In order to compute an approximation Φ_h of Φ, we ensure we have got a family of linear operators $(T^h)_h \subset \mathscr{L}(Y;X)$ with finite dimension ranges and we solve the approximate problems consisting in finding $\Phi_h \in X$ such that

$$\Phi_h = T^h G(\Phi_h) . \tag{1.2}$$

By using Theorem 3.1 of Crouzeix-Rappaz [4] with $F_h(\lambda, \Phi) = \Phi - T^h G(\Phi)$ (here F_h is independent of λ) and $\tilde{u}_h = \Phi_h$, the reader will easily check the following result :

THEOREM 1.1 : *We assume that the following hypotheses are satisfied :*

$$\lim_{h \to 0} \|T - T^h\|_{\mathscr{L}(Y;X)} = 0 , \tag{1.3}$$

$$(I - T\,DG(\Phi)) \text{ is an isomorphism from } X \text{ onto } X , \tag{1.4}$$

There exist $\delta > 0$, $C > 0$, *such that*

$$\|DG(\Phi) - DG(\Psi)\|_{\mathscr{L}(Y;X)} \leq C \|\Phi - \Psi\|_X$$

$$\text{for all } \Psi \in X \text{ satisfying } \|\Phi - \Psi\|_X \leq \delta , \tag{1.5}$$

where I *is the identity operator in* X. *Then, there exist* $\varepsilon > 0$, $\tilde{C} > 0$ *and* $h_0 > 0$ *such that for all* $0 < h \leq h_0$ *there is a unique* $\Phi_h \in X$ *satisfying*

$$\Phi_h = T^h G(\Phi_h) , \tag{1.6}$$

and

$$\| \Phi - \Phi_h \|_X \leqslant \varepsilon . \tag{1.7}$$

Moreover, we have the bound

$$\| \Phi - \Phi_h \|_X \leqslant \tilde{C} \| (T - T^h) G(\Phi) \|_X . \quad \square \tag{1.8}$$

In fact Theorem 1.1 claims the existence of a fixed point Φ_h of the mapping $T^h G$ (See (1.6)), its uniqueness in a neighbourhood of Φ (See (1.7)) and gives some error estimate (See (1.8)) under the consistency hypothesis (1.3) and the stability assumption (1.4) when the derivative DG is lipschitz continuous at Φ (See (1.5)).

We now introduce some notations concerning the Sobolev spaces that will be used throughout this paper. In the following, we denote for $p \geqslant 1$ by $L^p(\Omega)$, $W^{m,p}(\Omega)$, $H^m(\Omega)$ the classical Sobolev spaces respectively equipped with the norms $\| \cdot \|_{0,p,\Omega}$, $\| \cdot \|_{m,p,\Omega}$, $\| \cdot \|_{m,\Omega}$. Moreover, $| \cdot |_{m,\Omega}$ stands for the semi-norm of the space $H^m(\Omega)$; $H_0^1(\Omega)$ is the space of functions of $H^1(\Omega)$ the trace of which is vanishing, $L_0^2(\Omega)$ is the space of functions of $L^2(\Omega)$ the integral of which is vanishing and $H^2_{\text{loc}}(\mathbb{R}^2)$ is the space of functions defined on $\mathbb{R}^2$ which are $H^2(\mathcal{O})$ for all bounded domains $\mathcal{O} \subset \mathbb{R}^2$.

The outline of the paper is as follows : in Section 2, we recall the nonlinear problem to solve and state the continuous problem in an operator form that will be used for numerical approximation. Section 3 sets the approximate problem using appropriate finite dimension spaces. At this point, we shall precise that in order to avoid technical difficulties mainly related to isoparametric finite elements, we assume we are given abstract finite-dimension subspaces of the spaces in which the continuous problem is defined and assume standard approximability and stability properties on these subspaces. Section 4 is devoted to the approximation of the associated linear problems using standard tools of finite element analysis and to the main convergence result of the paper for the nonlinear magnetohydrodynamic problem.

2. THE CONTINUOUS PROBLEM

Let us first briefly recall the mathematical model (for more details, see [1]).

Let Ω_0, Ω_1, Ω_2 denote three disconnected bounded domains of $\mathbb{R}^2$ with respective boundaries Γ_0, Γ_1 and Γ_2, which are assumed to be of class C^1. We define $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.

The above three domains stand for the intersection with the plane $Ox_1 x_2$ of three infinite parallel cylindrical conductors Λ_0, Λ_1, Λ_2 with a generating line which is orthogonal to the plane $Ox_1 x_2$. Actually, Λ_1 and Λ_2 represent a solid inductor surrounding a liquid metal conductor enclosed in a fixed domain

Λ_0. An alternating current of frequency $\omega/2\pi$ and total intensity $J \geqslant 0$ flows in the inductor and gives rise to a magnetic field **b**. Since all the electric currents flow in the orthogonal direction to Ω_0, Ω_1 and Ω_2, the magnetic field **b** lies in the plane $Ox_1 x_2$ and depends only on the variables x_1, x_2. From $\nabla \cdot \mathbf{b} = 0$ and since the currents have a sinusoidal time behaviour, there exists a function $\varphi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\mathbf{b} = \mathrm{Re}(e^{i\omega t}\, \mathbf{curl}\, \varphi) \quad \text{with} \quad \mathbf{curl}\, \varphi := \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\right).$$

The magnetic field **b** interacts with the electric currents and produces Lorentz forces which cause a motion in the liquid region Λ_0. Since we suppose that the frequency $\omega/2\pi$ is large enough, we admit that only a time-averaged Lorentz force is responsible for the fluid motion which is assumed to be stationary.

Denoting by **u**, p, ν, ρ respectively the velocity, the pressure, the kinematic viscosity and the density of the liquid and by μ_0 the magnetic permeability of the vacuum, by σ_k the electric conductivity of Λ_k which is assumed to be constant, we can see that **u**, p depend only on the point $x = (x_1, x_2) \in \Omega_0$, **u** has only two components in the plane $Ox_1 x_2$ and the unknowns $\varphi : \mathbb{R}^2 \to \mathbb{C}$ and $(\mathbf{u}, p) : \Omega_0 \to \mathbb{R}^2 \times \mathbb{R}$ satisfy the system of partial differential equations (See [1]) :

$$-\Delta\varphi + \mu_0 \sigma_k \mathbf{u} . \nabla\varphi + i\mu_0 \omega \sigma_k(\varphi - I_k(\varphi)) = \mu_0 J_k \quad \text{in } \Omega_k, k = 0, 1, 2, \quad (2.1)$$

$$\Delta\varphi = 0 \quad \text{in } \Omega' = \mathbb{R}^2 \backslash \overline{\Omega}, \quad (2.2)$$

$$\varphi(x) = O(|x|^{-1}) \quad |x| \to +\infty, \quad (2.3)$$

$$[\varphi] = \left[\frac{\partial \varphi}{\partial n}\right] = 0 \quad \text{on } \Gamma_k, k = 0, 1, 2, \quad (2.4)$$

$$-\nu \Delta\mathbf{u} + \mathbf{u} . \nabla\mathbf{u} + \nabla p - \frac{\sigma_0 \omega}{2\rho}(\varphi_I \nabla\varphi_R - \varphi_R \nabla\varphi_I)$$

$$+ \frac{\sigma_0}{2\rho}((\mathbf{u} . \nabla\varphi_R)\nabla\varphi_R + (\mathbf{u} . \nabla\varphi_I)\nabla\varphi_I) = 0 \quad \text{in } \Omega_0, \quad (2.5)$$

$$\nabla . \mathbf{u} = 0 \quad \text{in } \Omega_0, \quad (2.6)$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0, \quad (2.7)$$

where, in (2.1), we have extended the velocity **u** by zero in the domains Ω_1 and Ω_2 and where

$$I_k(\varphi) := \frac{1}{|\Omega_k|} \int_{\Omega_k} \varphi(x)\, dx,$$

and

$$J_k := \begin{cases} \dfrac{(-1)^k J}{|\Omega_k|} & \text{if } k = 1, 2\,, \\ 0 & \text{if } k = 0\,. \end{cases}$$

Here above, the functions φ_R and φ_I stand respectively for the real and imaginary part of φ, the brackets [.] denote the jump of a function through the curves Γ, $|\Omega_k|$ is the measure of Ω_k and $J \geqslant 0$ is a given total current intensity imposed in the inductor $\Lambda_1 \cup \Lambda_2$. Notice that, unlike in [1], we have chosen a formulation where the magnetic potential φ is an $O(|x|^{-1})$ when $|x| \to \infty$ which removes the condition $\int_{\Omega_0} \varphi \, dx = 0$.

In order to give an approximation of Problem (2.1)-(2.7) we introduce a new formulation of it ; we start by the electromagnetic problem.

Let $\mathbf{u}$ denote a given function of the space $H_0^1(\Omega_0)^2$ such that $\nabla \cdot \mathbf{u} = 0$. We consider the following problem.

Find $\varphi \in H^2_{\text{loc}}(\mathbb{R}^2)$ such that :

$$- \Delta\varphi + i\mu_0 \, \omega\sigma_k(\varphi - I_k(\varphi)) + \mu_0 \, \sigma_k \, \mathbf{u} \, . \nabla\varphi = \mu_0 \, J_k \quad \text{in } \Omega_k, \; k = 0, 1, 2\,, \quad (2.8)$$

$$\Delta\varphi = 0 \quad \text{in } \Omega' := \mathbb{R}^2 \backslash \overline{\Omega}\,, \quad (2.9)$$

$$\varphi(x) = O(|x|^{-1}) \quad |x| \to +\infty\,, \quad (2.10)$$

where $\mathbf{u}$ is zero in the domains Ω_1 and Ω_2.

Following Rappaz-Touzani [1] we can prove that this problem has a unique solution that differs by an additive constant from the problem given in [1]. In fact, as mentioned earlier, we do not require here that $I_0(\varphi) = 0$ but impose, instead, that φ vanishes at the infinity. In order to give a variational formulation of (2.8)-(2.10) that is well adapted to numerical discretization, we represent the function $\varphi|_{\Omega'}$ as a solution of an integral equation on Γ. In other words, eqs. (2.9), (2.10) give (*cf.* Nedelec [5]) :

$$\varphi(x) = \int_\Gamma \lambda(y) \, K(x, y) \, ds_y - \int_\Gamma \varphi(y) \, K_n(x, y) \, ds_y\,, \quad x \in \Omega'\,, \quad (2.11)$$

$$\frac{1}{2} \varphi(x) = \int_\Gamma \lambda(y) \, K(x, y) \, ds_y - \int_\Gamma \varphi(y) \, K_n(x, y) \, ds_y\,, \quad x \in \Gamma\,, \quad (2.12)$$

where

$$\lambda = \frac{\partial \varphi}{\partial n} \quad \text{on } \Gamma ,$$

$$K(x, y) := \frac{1}{2\pi} \log |x - y| ,$$

$$K_n(x, y) := \frac{\partial}{\partial n_y} K(x, y) = -\frac{1}{2\pi} \frac{n_y \cdot (x - y)}{|x - y|^2} ,$$

the vector n_y standing for the outer unit normal at y.

It is clear that, using (2.9) and eq. (2.10) we obtain $\int_\Gamma \lambda \, ds = 0$. Following [3], we define the space

$$\tilde{H}^{-\frac{1}{2}}(\Gamma) := \{\mu \in H^{-\frac{1}{2}}(\Gamma) \, ; \langle \mu, 1 \rangle = 0\}$$

where the brackets $\langle \cdot , \cdot \rangle$ denote the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ and define the following « reduced » problem.

Given $(g, q) \in L^2(\Omega) \times H^{1/2}(\Gamma)$, find $(\varphi, \lambda) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$ such that :

$$a(\varphi, \psi) - \langle \lambda, \psi \rangle = \int_\Omega g \psi^* \, dx \quad \forall \psi \in H^1(\Omega) , \quad (2.13)$$

$$b(\lambda, \mu) + \langle \mu, \varphi \rangle^* = \langle \mu, q \rangle^* \quad \forall \mu \in \tilde{H}^{-\frac{1}{2}}(\Gamma) , \quad (2.14)$$

where α^* denotes the complex conjugate of a complex number α. Here above :

$$a(\varphi, \psi) := \int_\Omega \nabla \varphi \cdot \nabla \psi^* \, dx + i\omega\mu_0 \sum_{k=0}^{2} \sigma_k \int_{\Omega_k} \varphi \psi^* \, dx ,$$

$$b(\lambda, \mu) := -2 \int_\Gamma \int_\Gamma \lambda(y) \, \mu^*(x) \, K(x, y) \, ds_y \, ds_x .$$

Using results of [3] it can be shown that if (φ, λ) is a solution of (2.13)-(2.14) with :

$$\varphi \in H^2(\Omega),$$

$$g|_{\Omega_k} = \mu_0 J_k + i\mu_0 \omega\sigma_k I_k(\varphi) - \mu_0 \sigma_k \mathbf{u} . \nabla\varphi , \quad k = 0, 1, 2 ,$$

$$q = -2 \int_\Gamma \varphi(y)\, K_n(\,.\,, y)\, ds_y ,$$

then φ is a solution of (2.8)-(2.10).

THEOREM 2.1 : *Problem (2.13)-(2.14) admits a unique solution.*

Proof : Let us multiply the equations (2.13)-(2.14) by the complex number $(1 - \alpha i)$ where α is a positive number to be precised later. We have a new equivalent variational problem.

Find $(\varphi, \lambda) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$ such that :

$$\mathscr{B}((\varphi, \lambda), (\psi, \mu)) = (1 - \alpha i)\left(\int_\Omega g\psi^*\, dx + \langle \mu, q \rangle^*\right)$$

$$\forall (\psi, \mu) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\Gamma) \qquad (2.15)$$

where

$$\mathscr{B}((\varphi, \lambda), (\psi, \mu)) := (1 - \alpha i)(a(\varphi, \psi) - \langle \lambda, \psi \rangle + b(\lambda, \mu) + \langle \mu, \varphi \rangle^*) .$$

We have

$$\operatorname{Re} \mathscr{B}((\psi, \mu), (\psi, \mu)) = \int_\Omega |\nabla\psi|^2\, dx + \alpha\omega\mu_0 \sum_{k=0}^{2} \sigma_k \int_{\Omega_k} |\psi|^2\, dx$$

$$+ i\alpha(\langle \mu, \psi \rangle - \langle \mu, \psi \rangle^*) + b(\mu, \mu) .$$

From Nedelec [5], the coerciveness of b implies the existence of a real number $\gamma > 0$ such that :

$$b(\mu, \mu) \geqslant \gamma \|\mu\|^2_{-\frac{1}{2}, \Gamma} \quad \forall \mu \in \tilde{H}^{-\frac{1}{2}}(\Gamma) .$$

Therefore, if $\sigma_m := \min(\sigma_0, \sigma_1, \sigma_2) > 0$ we have

$$\operatorname{Re} \mathscr{B}((\psi, \mu), (\psi, \mu)) \geq \int_\Omega |\nabla \psi|^2 \, dx + \alpha\omega\mu_0 \sigma_m \int_\Omega |\psi|^2 \, dx$$

$$- 2\alpha \|\mu\|_{-\frac{1}{2}, \Gamma} \|\psi\|_{\frac{1}{2}, \Gamma} + \gamma \|\mu\|^2_{-\frac{1}{2}, \Gamma} .$$

Choosing α such that $\alpha\mu_0 \omega\sigma_m \leq 1$ and using the trace inequality

$$\|\psi\|_{\frac{1}{2}, \Gamma} \leq \kappa \|\psi\|_{1, \Omega}$$

for some constant $\kappa > 0$, we obtain

$$\operatorname{Re} \mathscr{B}((\psi, \mu), (\psi, \mu)) \geq \alpha\mu_0 \omega\sigma_m \|\psi\|^2_{1, \Omega} + \gamma \|\mu\|^2_{-\frac{1}{2}, \Gamma}$$

$$- 2\alpha\kappa \|\mu\|_{-\frac{1}{2}, \Gamma} \|\psi\|_{1, \Omega} .$$

In addition, the inequality

$$ab \leq \frac{a^2}{2\gamma} + \frac{\gamma}{2} b^2$$

yields

$$\operatorname{Re} \mathscr{B}((\psi, \mu), (\psi, \mu)) \geq \alpha\left(\mu_0 \omega\sigma_m - 2\frac{\alpha\kappa^2}{\gamma}\right) \|\psi\|^2_{1, \Omega} + \frac{\gamma}{2} \|\mu\|^2_{-\frac{1}{2}, \Gamma} .$$

It is then sufficient to choose $0 < \alpha < \min(1/\mu_0 \omega\sigma_m, \mu_0 \omega\gamma\sigma_m / 2\kappa^2)$ in order to guarantee the coerciveness of the form $\mathscr{B}$.

The sesquilinearity and continuity of $\mathscr{B}$ are obvious. Moreover, the conjugate linearity and continuity of the form defined by the right hand side of (2.15) are obvious. The Lax-Milgram theorem gives then the existence and uniqueness of the solution. □

The previous theorem allows us to define a linear and continuous operator :

$$T_E : (g, q) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma) \mapsto \varphi \in H^1(\Omega) ,$$

where (φ, λ) is the solution of Problem (2.13)-(2.14).

LEMMA 2.1 : *The linear operator* T_E *maps continuously* $L^2(\Omega) \times H^{3/2}(\Gamma)$ *into* $H^2(\Omega)$ *and consequently*

$$T_E : L^2(\Omega) \times H^{\frac{3}{2}}(\Gamma) \to W^{1,4}(\Omega)$$

is a compact linear operator.

Proof : It suffices to follow the same reasoning as in Johnson-Nedelec [3] and use the compactness of the imbedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. □

Remark 2.1 : In the previous lemma : we have an analogous result when we assume that the domains Ω_0, Ω_1 and Ω_2 are convex polygonal domains. In fact, in this case, if $(g, q) \in L^2(\Omega) \times H^{3/2}(\Gamma)$ (in the sense that q is the restriction of an H^2-function to Γ), then $\varphi = T_E(g, q) \in H^{1+5/6-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. This result can be found in Costabel-Dauge [6].

Now, by using (2.12) and denoting

$$H_\varphi := -2 \int_\Gamma \varphi(y)\, K_n(\,.\,, y)\, ds_y ,$$

Problem (2.8)-(2.10) can be written as the following one :

$$\text{Find } \varphi \in W^{1,4}(\Omega) \text{ such that } \quad \varphi = T_E(g(\varphi, \mathbf{u}), H\varphi) \tag{2.16}$$

where

$$g(\varphi, \mathbf{u}) := \mu_0 J_k + i\omega\mu_0\, \sigma_k\, I_k(\varphi) - \mu_0\, \sigma_k\, \mathbf{u} \,.\, \nabla\varphi \quad \text{in } \Omega_k,\ k = 0, 1, 2\,.$$

Observe that Problem (2.16) is meaningful since if $\varphi \in W^{1,4}(\Omega)$ and if $\mathbf{u} \in H_0^1(\Omega_0)^2$ then $\mathbf{u} \,.\, \nabla\varphi \in L^2(\Omega_0)$ and consequently $g(\varphi, \mathbf{u}) \in L^2(\Omega)$. Furthermore, $\varphi|_\Gamma \in H^{1/2}(\Gamma)$ and a regularity result of Seeley [7] imply $H\varphi \in H^{3/2}(\Gamma)$.

We are now able to give a new formulation of the magnetohydrodynamic problem. Let us consider the following Stokes problem :

$$\text{Given } \mathbf{g} \in L^{\frac{4}{3}}(\Omega_0)^2,$$

$$\text{find } (\mathbf{u}, p) \in H_0^1(\Omega_0)^2 \times L_0^2(\Omega_0) \text{ such that :}$$

$$-\ \nu\, \Delta\mathbf{u} + \nabla p = \mathbf{g} \quad \text{in } \Omega_0\,, \quad (2.17)$$
$$\nabla \,.\, \mathbf{u} = 0 \quad \text{in } \Omega_0\,. \quad (2.18)$$

Existence and uniqueness of a solution to this problem (*cf.* Temam [8]) enable us to define a linear and continuous operator

$$T_H : \mathbf{g} \in L^{\frac{4}{3}}(\Omega_0)^2 \mapsto \mathbf{u} \in H_0^1(\Omega_0)^2\,.$$

From [8] we have the regularity result :

$$T_H \mathbf{g} \in W^{2,p}(\Omega_0)^2 \quad \text{if} \quad \mathbf{g} \in L^p(\Omega_0)^2 , \tag{2.19}$$

for $1 < p < +\infty$. We then deduce that the operator $T_H \in \mathscr{L}(L^{4/3}(\Omega_0)^2 ; H_0^1(\Omega_0)^2)$ is compact.

Let us define the mapping :

$$\mathbf{f}_H : (\varphi, \mathbf{u}) \in W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2 \mapsto \mathbf{f}_H(\varphi, \mathbf{u}) \in L^{\frac{4}{3}}(\Omega_0)^2 ,$$

where

$$\mathbf{f}_H : (\varphi, \mathbf{u}) := -(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\sigma_0 \omega}{2\rho}(\varphi_I \nabla\varphi_R - \varphi_R \nabla\varphi_I)$$

$$- \frac{\sigma_0}{2\rho}((\mathbf{u} \cdot \nabla\varphi_R)\nabla\varphi_R + (\mathbf{u} \cdot \nabla\varphi_I)\nabla\varphi_I) .$$

Problem (2.1)-(2.7) can now be formulated as the following one :

Find $(\varphi, \mathbf{u}) \in W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2$ such that :

$$\varphi = T_E(g(\varphi, \mathbf{u}), H\varphi) , \tag{2.20}$$

$$\mathbf{u} = T_H \mathbf{f}_H(\varphi, \mathbf{u}) . \tag{2.21}$$

In order to be in the framework of Theorem 1.1 we introduce the notations :

$$X := W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2 , \quad Y := L^2(\Omega) \times H^{\frac{3}{2}}(\Gamma) \times L^{\frac{4}{3}}(\Omega_0)^2$$

and define the mappings

$$G : \Phi = (\varphi, \mathbf{u}) \in X \mapsto G(\Phi) := (g(\varphi, \mathbf{u}), H\varphi, \mathbf{f}_H(\varphi, \mathbf{u})) \in Y , \tag{2.22}$$

$$T : (g, q, \mathbf{r}) \in Y \mapsto T(g, q, \mathbf{r}) := (T_E(g, q), T_H \mathbf{r}) \in X .$$

Problem (2.20)-(2.21) can then be written in the form

$$\text{Find } \Phi \in X \text{ such that } \Phi = TG(\Phi) , \tag{2.23}$$

DEFINITION 2.1 : *We shall say that $\Phi \in X$ is a regular solution of Problem (2.23) (or equivalently (2.20)-(2.21)) if the operator $I - TDG(\Phi)$ is an isomorphism from X onto X.*

THEOREM 2.2 : *Problem (2.20)-(2.21) (or (2.23)) has at least one solution $\Phi = (\varphi, \mathbf{u})$. Moreover, there is a constant $J_0 > 0$ such that if $J \leqslant J_0$ then this solution is unique and is a regular solution of Problem (2.23).*

Proof : By using Theorem 4.3 together with Theorem 4.1 of the first part of this paper (*cf.* [1]), we can prove that Problem (2.20)-(2.21) has at least one solution $(\varphi, \mathbf{u})$. In fact, it is sufficient for this end to take a solution $(\phi, \mathbf{u}, p, \alpha, \beta)$ of (2.23)-(2.30) in [1] (see Theorem 4.3 of [1]), to remark that $\alpha = 0$ (see Theorem 4.1 of [1]) and to check that $(\varphi, \mathbf{u})$ with $\varphi = \phi - \beta$ is a solution of (2.1)-(2.7). In this case we obtain $I_0(\varphi) = -\beta$ since $I_0(\phi) = 0$. The uniqueness for small values of J results from Theorem 5.2 of [1].

In order to prove that the mapping $(I - T\,DG(\Phi))$ is an isomorphism from X onto X, we first define the linear operator $\mathscr{G} : X \to Y$ by

$$\mathscr{G}\Psi := (i\omega\mu_0\,\sigma I(\psi), H\psi, 0),$$

where $\Psi = (\psi, \mathbf{v}) \in X$ and $\sigma I(\psi) := \sigma_k I_k(\psi)$ in Ω_k, $k = 0, 1, 2$. By differentiating G we obtain if $\Psi = (\psi, \mathbf{v}) \in X$:

$$DG(\Phi)\,\Psi = (i\omega\mu_0\,\sigma I(\psi) - \mu_0\,\sigma\mathbf{u}\,.\,\nabla\psi - \mu_0\,\sigma\mathbf{v}\,.\,\nabla\varphi, H\psi, A(\Phi)\,\Psi + B(\Phi)\,\mathbf{v}), \quad (2.24)$$

where

$$A(\Phi)\,\Psi = -\frac{\sigma_0\,\omega}{2\rho}(\varphi_I\,\nabla\psi_R + \psi_I\,\nabla\varphi_R - \varphi_R\,\nabla\psi_I - \psi_R\,\nabla\varphi_I) - \frac{\sigma_0}{2\rho}((\mathbf{u}\,.\,\nabla\varphi_R)\,\nabla\psi_R + (\mathbf{u}\,.\,\nabla\psi_R)\,\nabla\varphi_R + (\mathbf{u}\,.\,\nabla\varphi_I)\,\nabla\psi_I + + (\mathbf{u}\,.\,\nabla\psi_I)\,\nabla\varphi_I,$$

and

$$B(\Phi)\,\mathbf{v} = -\frac{\sigma_0}{2\rho}((\mathbf{v}\,.\,\nabla\varphi_R)\,\nabla\varphi_R + (\mathbf{v}\,.\,\nabla\varphi_I)\,\nabla\varphi_I) - \mathbf{v}\,.\,\nabla\mathbf{u} - \mathbf{u}\,.\,\nabla\mathbf{v}\,.$$

It is then easy to show that there exists a constant C_1, independent of J, such that for all $\Phi \in X$:

$$\| DG(\Phi) - \mathscr{G} \|_{\mathscr{L}(X;Y)} \leqslant C_1 \|\Phi\|_X (1 + \|\Phi\|_X)\,.$$

Moreover we can prove, by using the same technique as in the proof of Theorem 4.2 in [1], that there is another constant C_2, also independent of J, such that if Φ is a solution of (2.23), then

$$\|\Phi\|_X \leqslant C_2 J .$$

It follows that for all $\varepsilon > 0$, there exists $J_0 > 0$ such that if $J \leqslant J_0$ and if Φ is a solution of (2.23) then

$$\|DG(\Phi) - \mathcal{G}\|_{\mathcal{L}(X;Y)} \leqslant \varepsilon .$$

By writing $I - T\,DG(\Phi) = (I - T\mathcal{G}) - T(DG(\Phi) - \mathcal{G})$ and by noticing that $I - T\mathcal{G}$ is an isomorphism from X onto X since $T\mathcal{G}$ is compact and $\operatorname{Ker}(I - T\mathcal{G}) = \{0\}$, we easily prove that $I - T\,DG(\Phi)$ is an isomorphism from X onto X when $J \leqslant J_0$ is small enough. □

We shall now be concerned with the numerical approximation of Problem (2.20)-(2.21), or equivalently (2.23).

3. THE DISCRETE PROBLEM

In order to introduce a numerical method to solve Problem (2.1)-(2.7). We define W_h, M_h, V_h Q_h as finite-dimension subspaces of the spaces $W^{1,4}(\Omega)$, $\tilde{H}^{-1/2}(\Gamma)$, $H_0^1(\Omega_0)^2$, $L_0^2(\Omega_0)$ respectively. To simplify the presentation, we suppose that W_h, M_h, V_h, Q_h are piecewise polynomial subspaces. The numerical approximation of Problem (2.20)-(2.21) in a variational form is then defined by the discret problem :

Find $(\varphi_h, \lambda_h, \mathbf{u}_h, p_h) \in W_h \times M_h \times V_h \times Q_h$ such that :

$$a(\varphi_h, \psi) - \langle \lambda_h, \psi \rangle = \int_\Omega g(\varphi_h, \mathbf{u}_h)\, \psi^*\, dx \qquad \forall \psi \in W_h , \quad (3.1)$$

$$b(\lambda_h, \mu) + \langle \mu, \varphi_h \rangle^* - \langle \mu, H\varphi_h \rangle^* = 0 \qquad \forall \mu \in M_h , \quad (3.2)$$

$$\nu(\nabla \mathbf{u}_h | \nabla \mathbf{v})_0 - (p_h, \nabla \cdot \mathbf{v})_0 - \int_{\Omega_0} \mathbf{f}_H(\varphi_h, \mathbf{u}_h) \cdot \mathbf{v}\, dx = 0 \quad \forall \mathbf{v} \in V_h , \quad (3.3)$$

$$(q, \nabla \cdot \mathbf{u}_h)_0 = 0 \qquad \forall q \in Q_h , \quad (3.4)$$

where $(\cdot\,, \cdot)_0$ is the $L^2(\Omega_0)$-scalar product and

$$(\nabla \mathbf{v} | \nabla \mathbf{w})_0 := \sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial x_j}, \frac{\partial w_i}{\partial x_j} \right)_0 , \quad \forall \mathbf{v}, \mathbf{w} \in H_0^1(\Omega_0)^2 .$$

Let us notice that in Besson *et al.* [9], a particular choice of the spaces W_h, M_h, V_h, Q_h as finite element spaces was made to build an approximation based on the above formulation.

Following the methodology defined in [4], we transform the above problem in order to write it as a discrete analogue of the continuous problem (2.20)-(2.21) (or equivalently (2.23)). To this end, we consider the discrete version of Problem (2.13)-(2.14). To each pair $(g, q) \in L^2(\Omega) \times H^{1/2}(\Gamma)$ we associate the pair $(\varphi_h, \lambda_h) \in W_h \times M_h$ where (φ_h, λ_h) is the unique solution of the discrete version of (2.13)-(2.14), (*cf.* Theorem 2.1), i.e.

Find $(\varphi_h, \lambda_h) \in W_h \times M_h$ such that:

$$a(\varphi_h, \psi) - \langle \lambda_h, \psi \rangle = \int_\Omega g\psi^* \, dx \quad \forall \psi \in W_h\,, \quad (3.5)$$

$$b(\lambda_h, \mu) + \langle \mu, \varphi_h \rangle^* = \langle \mu, q \rangle^* \qquad \forall \mu \in M_h\,. \quad (3.6)$$

The mapping $(g, q) \mapsto \varphi_h$ defines an operator $T_E^h \in \mathscr{L}(L^2(\Omega) \times H^{1/2}(\Gamma)\,; H^1(\Omega))$ whose range is included in W_h. Notice that since $W_h \subset W^{1,4}(\Omega)$, the operator T_E^h is also an element of $\mathscr{L}(L^2(\Omega) \times H^{1/2}(\Gamma)\,; H^{1,4}(\Omega))$.

A discrete approximation of the operator T_H can be defined in an analogous way. It is well known (*cf.* [2]) that if the following inf-sup condition holds:

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega_0} q \nabla \cdot \mathbf{v} \, dx}{\|q\|_{0,\Omega_0} |\mathbf{v}|_{1,\Omega_0}} \geqslant \beta > 0\,,$$

then to each function $\mathbf{g} \in L^{4/3}(\Omega_0)^2$ we can associate the unique pair $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ solution of the discrete Stokes problem:

Find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that:

$$\nu(\nabla \mathbf{u}_h | \nabla \mathbf{v})_0 - (p_h, \nabla \cdot \mathbf{v})_0 = (\mathbf{g}, \mathbf{v})_0 \quad \forall \mathbf{v} \in V_h\,, \quad (3.7)$$

$$(q, \nabla \cdot \mathbf{u}_h)_0 = 0 \qquad \forall q \in Q_h\,. \quad (3.8)$$

The mapping $\mathbf{g} \mapsto \mathbf{u}_h$ defines an operator

$$T_H^h \in \mathscr{L}(L^{4/3}(\Omega_0)^2\,; H_0^1(\Omega_0)^2)$$

whose range is included in V_h.

The fully discrete problem corresponding to (2.20)-(2.21) can now be given in the following way:

Find $(\varphi_h, \mathbf{u}_h) \in W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2$ such that:

$$\varphi_h = T_E^h(g(\varphi_h, \mathbf{u}_h), H\varphi_h)\,, \quad (3.9)$$

$$\mathbf{u}_h = T_H^h \mathbf{f}_H(\varphi_h, \mathbf{u}_h)\,. \quad (3.10)$$

It is easy to check that a solution of Problem (3.1)-(3.4) is also a solution of Problem (3.9)-(3.10). Conversely, if $(\varphi_h, \mathbf{u}_h) \in W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2$ is a solution of Problem (3.9)-(3.10), then $(\varphi_h, \mathbf{u}_h) \in W_h \times V_h$ since the range of T_E^h is included in W_h and the range of T_H^h is included in V_h. Moreover, there exist $\lambda_h \in M_h$, $p_h \in Q_h$ such that $(\varphi_h, \lambda_h, \mathbf{u}_h, p_h)$ satisfy Problem (3.1)-(3.4).

Notice that the discrete problem (3.9)-(3.10) takes account exactly of the nonlinear terms $g(\varphi_h, \mathbf{u}_h)$, $H\varphi_h$ and $\mathbf{f}_H(\varphi_h, \mathbf{u}_h)$. This is possible since these terms involve the unknown functions φ_h and $\mathbf{u}_h$ and their derivatives. Consequently if a finite element piecewise polynomial approximation is used, the calculation of these terms requires the integration of polynomial functions over finite elements and rational or logarithmic functions on edges (boundaries of finite elements). All these calculations can be performed exactly.

As in the continuous case, with the following notations :

$$\begin{aligned} &\Phi_h = (\varphi_h, \mathbf{u}_h), \\ &T^h : (g, q, \mathbf{r}) \in Y \mapsto T^h(g, q, \mathbf{r}) := (T_E^h(g, q), T_H^h \mathbf{r}) \in X, \end{aligned} \tag{3.11}$$

the discrete problem (3.9)-(3.10) becomes :

$$\text{Find } \Phi_h \in X \text{ such that } \Phi_h = T^h G(\Phi_h). \tag{3.12}$$

The remaining part of this paper is devoted to the analysis of Problem (3.9)-(3.10) — or equivalently (3.12) — and the estimation of the error functions $\varphi - \varphi_h$ and $\mathbf{u} - \mathbf{u}_h$ in suitable norms. For this, Theorem 1.1 will be applied with the notations introduced for Problem (2.23) and Problem (3.12).

We now assume that the solution $\Phi = (\varphi, \mathbf{u})$ of (2.20)-(2.21) (or equivalently (2.23)) is such that (1.4) is satisfied, i.e., the solution is regular. Let us notice that this hypothesis is not void since it is satisfied in particular when the current intensity J is small enough (see Theorem 2.2). The following section is devoted to checking hypotheses (1.3)-(1.5) under suitable conditions on the spaces W_h, M_h, V_h and Q_h and for deriving error estimates.

4. ERROR ESTIMATES

In order to derive error estimates we shall now assume some approximability conditions on the spaces W_h, M_h, V_h and Q_h. In fact, to avoid some technical difficulties related to the regularity of boundaries of the domains Ω_j,

$j = 0, 1, 2$, we shall not introduce concrete finite element spaces which must be isoparametric elements but rather restrict ourselves to an abstract setting of the problem. Namely we assume the following hypotheses :

(i) There exist

$$r_E^h \in \mathscr{L}(H^2(\Omega)\,;\, W_h),\ \pi_E^h \in \mathscr{L}(H^{\frac{1}{2}}(\Gamma) \cap \tilde{H}^{-\frac{1}{2}}(\Gamma)\,;\, M_h)$$

such that :

$$h^{\frac{1}{2}} \| \psi - r_E^h \psi \|_{1,4,\Omega} + \| \psi - r_E^h \psi \|_{1,\Omega} + \| \mu - \pi_E^h \mu \|_{-\frac{1}{2},\Gamma} \leqslant$$

$$\leqslant Ch(\| \psi \|_{2,\Omega} + \| \mu \|_{\frac{1}{2},\Gamma}) \quad \forall \psi \in H^2(\Omega), \forall \mu \in H^{\frac{1}{2}}(\Gamma) \cap \tilde{H}^{-\frac{1}{2}}(\Gamma)\,. \tag{4.1}$$

(ii) We have :

$$\| \psi \|_{1,4,\Omega} \leqslant Ch^{-\frac{1}{2}} \| \psi \|_{1,\Omega} \quad \forall \psi \in W_h\,. \tag{4.2}$$

(iii) There exist

$$r_H^h \in \mathscr{L}(H_0^1(\Omega_0)^2 \cap W^{2,4/3}(\Omega_0)^2\,;\, V_h),\ \pi_H^h \in \mathscr{L}(L^2(\Omega_0)\,;\, Q_h)$$

such that :

$$| \mathbf{v} - r_H^h \mathbf{v} |_{1,\Omega_0} + \| q - \pi_H^h q \|_{0,\Omega_0} \leqslant Ch(\| \mathbf{v} \|_{2,\Omega_0} + \| q \|_{1,\Omega_0})$$

$$\forall \mathbf{v} \in H^2(\Omega_0)^2 \cap H_0^1(\Omega_0)^2, \forall q \in H^1(\Omega_0)\,. \tag{4.3}$$

$$| \mathbf{v} - r_H^h \mathbf{v} |_{1,\Omega_0} + \| q - \pi_H^h q \|_{0,\Omega_0} \leqslant Ch^{\frac{1}{2}}(\| \mathbf{v} \|_{2,\frac{4}{3},\Omega_0} + \| q \|_{1,\frac{4}{3},\Omega_0})$$

$$\forall \mathbf{v} \in W^{2,\frac{4}{3}}(\Omega_0)^2 \cap H_0^1(\Omega_0)^2, \forall q \in W^{1,\frac{4}{3}}(\Omega_0)\,. \tag{4.4}$$

$$\| \mathbf{v} - r_H^h \mathbf{v} \|_{0,\infty,\Omega_0} \leqslant Ch \| \mathbf{v} \|_{2,\Omega_0} \quad \forall \mathbf{v} \in H^2(\Omega_0)^2 \cap H_0^1(\Omega_0)^2\,. \tag{4.5}$$

(iv) There exists a constant $\beta > 0$ independent of h such that :

$$\sup_{\mathbf{v} \in V_h} \frac{\displaystyle\int_{\Omega_0} q \nabla \cdot \mathbf{v}\, dx}{\| q \|_{0,\Omega_0} | \mathbf{v} |_{1,\Omega_0}} \geqslant \beta\,, \quad \forall q \in Q_h, q \neq 0\,. \tag{4.6}$$

(v) We have :

$$\| \mathbf{v} \|_{0,\infty,\Omega_0} \leqslant C |\ln h|\, | \mathbf{v} |_{1,\Omega_0} \quad \forall \mathbf{v} \in V_h\,. \tag{4.7}$$

Notice that Hypotheses (i), (ii), (iii) and (v) are classical in finite element approximation (*cf.* [10], [2]). They assume the validity of an inverse inequality (in the context of finite element approximation, this means that the mesh is quasi-uniform). Hypothesis (iv) is the classical inf-sup condition for the Stokes equations.

Let us first prove that the solutions of the linear discrete problems (3.5)-(3.6) and (3.7)-(3.8) converge respectively to the solutions of the problems (2.13)-(2.14) and (2.17)-(2.18).

LEMMA 4.1 : *If* $(g, q) \in L^2(\Omega) \times H^{3/2}(\Gamma)$ *then*

$$\|(T_E - T_E^h)(g, q)\|_{1,\Omega} + h^{\frac{1}{2}}\|(T_E - T_E^h)(g, q)\|_{1,4,\Omega} \leq Ch(\|g\|_{0,\Omega} + \|q\|_{0,\frac{3}{2},\Gamma}), \quad (4.8)$$

where C *is a constant independent of* h.

Proof: Let (g, q) denote an arbitrary element of $L^2(\Omega) \times H^{3/2}(\Gamma)$ and let (φ, λ) be the unique solution of Problem (2.13)-(2.14) corresponding to the data (g, q), i.e., $\varphi = T_E(g, q) \in H^2(\Omega)$ (*cf.* Lemma 2.1). Let in addition $\varphi_h = T_E^h(g, q)$. We have from (4.1) and [3] the error estimate

$$\|\varphi - \varphi_h\|_{1,\Omega} \leq Ch(\|\varphi\|_{2,\Omega} + \|\lambda\|_{\frac{1}{2},\Gamma}). \quad (4.9)$$

The triangle inequality gives :

$$\|\varphi - \varphi_h\|_{1,4,\Omega} \leq \|\varphi - r_E^h \varphi\|_{1,4,\Omega} + \|r_E^h \varphi - \varphi_h\|_{1,4,\Omega}.$$

Using (4.1) we have

$$\|\varphi - r_E^h \varphi\|_{1,4,\Omega} \leq Ch^{\frac{1}{2}}\|\varphi\|_{2,\Omega}.$$

Furthermore, from (4.2) and (4.9) we get :

$$\begin{aligned} \|r_E^h \varphi - \varphi_h\|_{1,4,\Omega} &\leq C_1 h^{-\frac{1}{2}}\|r_E^h \varphi - \varphi_h\|_{1,\Omega} \\ &\leq C_1 h^{-\frac{1}{2}}(\|\varphi - r_E^h \varphi\|_{1,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega}) \\ &\leq C_2 h^{\frac{1}{2}}(\|\varphi\|_{2,\Omega} + \|\lambda\|_{\frac{1}{2},\Gamma}). \end{aligned}$$

Finally, the continuity of the mapping $(g, q) \in L^2(\Omega) \times H^{3/2}(\Gamma) \mapsto (\varphi, \lambda) \in H^2(\Omega) \times H^{1/2}(\Gamma)$ achieves the proof. □

LEMMA 4.2 : *There exists a constant C independent of h such that :*
(1) *If* $\mathbf{g} \in L^2(\Omega_0)^2$ *then*

$$|(T_H - T_H^h)\,\mathbf{g}|_{1,\Omega_0} \leqslant Ch \|\mathbf{g}\|_{0,\Omega_0}, \tag{4.10}$$

(2) *If* $\mathbf{g} \in L^{4/3}(\Omega_0)^2$ *then*

$$|(T_H - T_H^h)\,\mathbf{g}|_{1,\Omega_0} \leqslant Ch^{\frac{1}{2}} \|\mathbf{g}\|_{0,\frac{4}{3},\Omega_0}. \tag{4.11}$$

Proof: Let $\mathbf{g} \in L^p(\Omega_0)^2$ for $p = 2$ or $4/3$ and let $(\mathbf{u}, p)$ be the unique solution of Problem (2.17)-(2.18), i.e., $\mathbf{u} = T_H \mathbf{g} \in W^{2,p}(\Omega_0)^2$. Let in addition $\mathbf{u}_h = T_H^h \mathbf{g}$. We have from [2] by using (4.6) the inequality :

$$|\mathbf{u} - \mathbf{u}_h|_{1,\Omega_0} \leqslant C(|\mathbf{u} - r_H^h \mathbf{u}|_{1,\Omega_0} + \|p - \pi_H^h p\|_{0,\Omega_0}) .$$

Using (4.4), (4.3) and (4.6) and the continuity of the mappings

$$\mathbf{g} \in L^p(\Omega_0)^2 \mapsto (\mathbf{u}, p) \in W^{2,p}(\Omega_0)^2 \times W^{1,p}(\Omega) ,$$

we obtain the desired bounds. □

The previous lemmas allow now to prove the following convergence result.

THEOREM 4.1 : *Let* $(\varphi, \mathbf{u}) \in W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2$ *denote a regular solution of Problem (2.20)-(2.21) (Theorem 2.2 shows that if J is small enough, a such* $(\varphi, \mathbf{u})$ *exists). Then, under Hypotheses (4.1)-(4.7), there exist* $\varepsilon > 0$, $h_0 > 0$, $C > 0$ *such that for all* $h \leqslant h_0$ *there is a unique solution* $(\varphi_h, \mathbf{u}_h)$ *of Problem (3.9)-(3.10) in a ball with radius* ε *and center* $(\varphi, \mathbf{u})$ *in* $W^{1,4}(\Omega) \times H_0^1(\Omega_0)^2$. *Moreover, we have the error estimate :*

$$\|\varphi - \varphi_h\|_{1,4,\Omega} + |\mathbf{u} - \mathbf{u}_h|_{1,\Omega_0} \leqslant Ch^{\frac{1}{2}}. \tag{4.12}$$

Proof: We apply Theorem 1.1 with the notations introduced in (2.22) and (3.11). Clearly, Hypothesis (1.3) holds because of (4.8) and (4.11). Hypothesis (1.4) holds because $\Phi = (\varphi, \mathbf{u})$ is assumed to be a regular solution. The Lipschitz continuity of DG at Φ is obvious and (1.5) holds. It follows that the existence of the unique solution $(\varphi_h, \mathbf{u}_h)$ of Problem (3.9)-(3.10) in a neighborhood of $(\varphi, \mathbf{u})$ is a consequence of (1.6).

In order to prove (4.12), we use (1.8), (4.8) and (4.11). □

It is worth noting that Theorem 4.1 gives an error estimate in the $W^{1,4}$-norm for φ which explains the small rate of convergence. The following theorem shows that it is possible to obtain a reasonable rate of convergence when estimating the error in the H^1-norm.

THEOREM 4.2 : *Under the same hypotheses as in Theorem 4.1, there exists a constant C independent of h such that :*

$$\| \varphi - \varphi_h \|_{1, \Omega} + | \mathbf{u} - \mathbf{u}_h |_{1, \Omega_0} \leq Ch .$$

Proof : Let $\Phi = (\varphi, \mathbf{u})$ and $\Phi_h = (\varphi_h, \mathbf{u}_h)$ denote the solutions invoked in Theorem 4.1. We know that since $\mathbf{f}_H(\Phi) \in L^{4/3}(\Omega_0)^2$ then $\mathbf{u} \in W^{2, 4/3}(\Omega_0)^2 \hookrightarrow L^{\infty}(\Omega_0)^2$. From (2.24) we deduce for $\Psi = (\psi, \mathbf{v}) \in X$ the bound :

$$\begin{aligned} \| DG(\Phi)\, \Psi \|_Y \leq C(& \| \psi \|_{1, \Omega} + \| \mathbf{u} \|_{0, \infty, \Omega_0} \| \psi \|_{1, \Omega} + \| \varphi \|_{1, 4, \Omega} | \mathbf{v} |_{1, \Omega_0} \\ & + \| \varphi \|_{1, 4, \Omega} \| \psi \|_{1, \Omega} + \| \mathbf{u} \|_{0, \infty, \Omega_0} \| \varphi \|_{1, 4, \Omega} \| \psi \|_{1, \Omega} \\ & + \| \varphi \|^2_{1, 4, \Omega} | \mathbf{v} |_{1, \Omega_0} + | \mathbf{u} |_{1, \Omega_0} | \mathbf{v} |_{1, \Omega_0}) , \end{aligned}$$

where C is a generic constant.

Denoting by Z the space $H^1(\Omega) \times H^1_0(\Omega_0)^2$, it is easily seen that, in the previous inequality, Ψ can be chosen in Z and that $DG(\Phi)$ can be continuously extended as an operator of $\mathscr{L}(Z ; Y)$. Consequently we have

$$DG(\Phi) \in \mathscr{L}(Z ; Y) . \tag{4.13}$$

Using analogous arguments we check that if $\tilde{\Phi} = (\tilde{\varphi}, \tilde{\mathbf{u}})$ is such that $\tilde{\varphi} \in W^{1,4}(\Omega)$, $\tilde{\mathbf{u}} \in H^1_0(\Omega_0)^2 \cap L^{\infty}(\Omega_0)^2$ with

$$\| \tilde{\varphi} \|_{1, 4, \Omega} + | \tilde{\mathbf{u}} |_{1, \Omega_0} + \| \tilde{\mathbf{u}} \|_{0, \infty, \Omega_0} \leq C ,$$

then there is a constant $\tilde{C}$, that depends on C but not on $\tilde{\Phi}$, such that

$$\| DG(\Phi) - DG(\tilde{\Phi}) \|_{\mathscr{L}(Z ; Y)} \leq \tilde{C} \| \Phi - \tilde{\Phi} \|_{W^{1,4}(\Omega) \times (H^1(\Omega_0)^2 \cap L^{\infty}(\Omega_0)^2)} . \tag{4.14}$$

In what follows, Θ will stand for the space $W^{1,4}(\Omega) \times (H^1_0(\Omega_0)^2 \cap L^{\infty}(\Omega_0)^2)$ equipped with the norm :

$$\| \Phi \|_{\Theta} := \| \varphi \|_{1, 4, \Omega} + | \mathbf{u} |_{1, \Omega_0} + \| \mathbf{u} \|_{0, \infty, \Omega_0} .$$

Let us now prove that

$$\lim_{h \to 0} \| \mathbf{u} - \mathbf{u}_h \|_{H^1_0(\Omega_0)^2 \cap L^{\infty}(\Omega_0)^2} = 0 . \tag{4.15}$$

For this end, (4.12) shows that it is sufficient to prove that

$$\lim_{h \to 0} \|\mathbf{u} - \mathbf{u}_h\|_{0, \infty, \Omega_0} = 0 .$$

This inequality holds thanks to inequality (4.7). Indeed we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{0, \infty, \Omega_0} \leqslant \|\mathbf{u} - r_H^h \mathbf{u}\|_{0, \infty, \Omega_0} + \|r_H^h \mathbf{u} - \mathbf{u}_h\|_{0, \infty, \Omega_0} .$$

Inequality (4.7) yields

$$\|r_H^h \mathbf{u} - \mathbf{u}_h\|_{0, \infty, \Omega_0} \leqslant C_1 |\ln h| \, |r_H^h \mathbf{u} - \mathbf{u}_h|_{1, \Omega_0}$$

$$\leqslant C_1 |\ln h| \, (|\mathbf{u} - r_H^h \mathbf{u}|_{1, \Omega_0} + |\mathbf{u} - \mathbf{u}_h|_{1, \Omega_0}) .$$

Using (4.5) and noticing that $\mathbf{u} \in H^2(\Omega_0)^2$, we obtain (4.15) and

$$\lim_{h \to 0} \|\Phi - \Phi_h\|_{\Theta} = 0 . \tag{4.16}$$

Let us now define for $\Psi = (\psi, \mathbf{v}) \in X$:

$$F(\Psi) := \Psi - TG(\Psi) , \quad F^h(\Psi) := \Psi - T^h G(\Psi) .$$

We have since $F^h(\Phi_h) = 0$:

$$\Phi - \Phi_h = DF^h(\Phi)^{-1} (DF^h(\Phi)(\Phi - \Phi_h) + F^h(\Phi) -$$

$$- F^h(\Phi_h)) - DF^h(\Phi)^{-1} F^h(\Phi) .$$

Hence

$$\Phi - \Phi_h = DF^h(\Phi)^{-1} \int_0^1 (DF^h(\Phi) - DF^h(s\Phi + (1-s)\Phi_h))(\Phi - \Phi_h) \, ds$$

$$- DF^h(\Phi)^{-1} F^h(\Phi) .$$

By using Lemmas 4.1 and 4.2 we have $\lim_{h \to 0} \|T - T^h\|_{\mathcal{L}(Y;Z)} = 0$ and consequently $\lim_{h \to 0} \|DF(\Phi) - DF^h(\Phi)\|_{\mathcal{L}(Z;Z)} = 0$. Moreover, applying

Fredholm's alternative and the compactness of T it is easy to show that if $\Phi = (\varphi, \mathbf{u})$ is a regular solution of Problem (2.20)-(2.21), then $DF(\Phi)$ is an isomorphism from Z onto itself. It follows that for h small enough, $DF^h(\Phi)$ is an isomorphism of Z onto itself and $\|DF^h(\Phi)^{-1}\|_{\mathscr{L}(Z;Z)}$ is bounded, i.e.

$$\|DF^h(\Phi)^{-1}\|_{\mathscr{L}(Z;Z)} \leqslant C .$$

We thus have

$$\begin{aligned} \|\Phi - \Phi_h\|_Z \leqslant\ & \|DF^h(\Phi)^{-1}\|_{\mathscr{L}(Z;Z)} \|T^h\|_{\mathscr{L}(Y;Z)} \\ & \times \sup_{s \in [0,1]} \|DG(\Phi) - DG(s\Phi + (1-s)\Phi_h)\|_{\mathscr{L}(Z;Y)} \|\Phi - \Phi_h\|_Z \\ & + \|DF^h(\Phi)^{-1}\|_{\mathscr{L}(Z;Z)} \|F^h(\Phi)\|_Z ; \end{aligned}$$

and since $\|T^h\|_{\mathscr{L}(Y;Z)}$ is bounded with respect to h (see Lemmas 4.1 and 4.2) we obtain by using (4.14) :

$$\|\Phi - \Phi_h\|_Z \leqslant C(\|\Phi - \Phi_h\|_\Theta \|\Phi - \Phi_h\|_Z + \|F^h(\Phi)\|_Z) .$$

By using (4.16), we conclude that there exists $h_0 > 0$ such that if $h \leqslant h_0$ then

$$\begin{aligned} \|\Phi - \Phi_h\|_Z & \leqslant C \|F^h(\Phi)\|_Z \\ & \leqslant C \|(T - T^h) G(\Phi)\|_Z . \end{aligned}$$

Finally, making use of the bounds (4.8) and (4.10) yields the desired result. □

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