M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 30, nº 2 (1996), p. 185-213

<http://www.numdam.org/item?id=M2AN_1996__30_2_185_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 30, n° 2, 1996, p. 185 à 213)

ON THE ACCURACY OF ASYMPTOTIC APPROXIMATIONS FOR LONGITUDINAL DEFORMATION OF A THIN PLATE (*)

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Abstract. — *We estimate the difference between a solution of the three-dimension problem about longitudinal deformation of a thin plate and solutions of two-dimensional problems modelling it. By construction of three initial asymptotic terms, including boundary layers, we form the "high precision" problem, the solution of which gives more precise approximations of three-dimensional displacement and stress fields than the usual one. Different types of loading are under consideration.*

Résumé. — *On étudie la différence entre la solution d'un problème tri-dimensionnel de déformation longitudinale d'une plaque mince et les solutions d'un problème bi-dimensionnel modélisant cette déformation. On construit un problème précis dont la solution donne des informations plus précises que les modèles habituels sur le déplacement tri-dimensionnel et le tenseur des contraintes.*

1. PRELIMINARY DESCRIPTION OF THE RESULTS

It is a well-known fact that longitudinal deformation of a thin three-dimensional isotropic plate $Q_h = \Omega \times (-h/2, h/2)$ can be described approximately by a solution of the two-dimensional elasticity problem for the Lâme operator with a new Poisson ratio. In the classics of the theory of elasticity the correspondance between these formulations of the plate problem is concluded by the hypotheses which are based on certain physical reasons and assert the stress field in Q_h to depend only on the variables (x_1, x_2) and to satisfy the restrictions

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \quad \text{in} \quad Q_h$$

(*) Manuscript received November 5, 1994.
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M² AN Modélisation mathématique et Analyse numérique 0764-583X/96/02/$ 4.00

(x_3-axis is perpendicular to the bases $S_h^{\pm}$ of Q_h). Such assumptions, of course, are sufficient to derive the equations in Ω from the ones in Q_h. However, in frames of this heuristic approach boundary value conditions on the lateral surface S_h^0 can be fulfilled only in some integral sense and, hence, in the vicinity of S_h^0 the stress state loses the above-mentioned plane properties. This perturbation of the state is interpreted in mechanics as an abnormal influence of S_h^0; it results in so-called plate edge effects.

Several mathematical approaches have been developed to perform the derivation in question (see [1-5], etc.). Most of them are based principally on the direct asymptotic analysis of the elasticity problem, since plate's thickness h is to be regarded as a small parameter. Leading terms of the asymptotics in Q_h prove to coincide with a solution of the problem in Ω, while among terms of higher orders there are component solutions of boundary layer type which are closely connected with the plate edge effects. The procedure to investigate the boundary layer phenomenon was developed in [6-9] and others. In our paper we follow the way indicated in [10, 1] and slightly modified in accordance with [8] (see Sect. 4 and 5).

The question of justification comes up, apart from formal asymptotic constructions. In [11, 2] it was proved that in certain natural sense solutions of the problem in Q_h converge as $h \to +0$ to a solution of the corresponding problem in Ω (both longitudinal deformation and bending of plates are under consideration). The results of the paper [12], which treats self-adjoint elliptic problems in thin domains, can be applied also to the plate problems in virtue of Korn's inequality with a correct distribution of powers of h at terms in $H^1(Q_1)$-norm. The necessary inequality was obtained in [13] where the theory of bending of thin plates was justified (see also [14-16] with variants of Korn's inequalities including weighted ones). This inequality becomes asymptotically sharp for bending (*cf.* [17] where the precision of the estimate verified in [13] is confirmed indirectly). However, avoiding principal specifications of plane stress state it loses the sharpness in the case of longitudinal deformation. Thus, the estimates derived from the general ones in [12] can not be precise. In Section 1, by a new distribution of multipliers h^m, we fit up Korn's inequality for our case and then, in Section 6, use it to justify the asymptotics.

To check up the asymptotic precision of an estimate, one should, first of all, find out asymptotic terms of higher order. For a plate with the clamped lateral surface such correcting terms were constructed in [18] (this paper does not contain complete proofs, but the estimates needed to conclude them can be derived, of course, by the same considerations as in Sect. 6). It happens that boundary layer component solutions are of the main importance for the corrections while the solution of the problem in Ω with Dirichlet conditions on $\partial\Omega$ determines the displacement and stress fields in Q_h up to $O(h)$ and $O(hd_h^{-1})$ (here $d_h(x) = \max\{h, \operatorname{dist}(x, S_h^0)\}$ and the correct interpretation of the symbol O is given in Sect. 6). The precision of the two-dimensional approximation for the plate Q_h with loaded lateral surface S_h^0 is unpredictably high : if the load on S_h^0 does not depend on x_3, the correcting terms in the displacement and stress asymptotics are, respectively, equal to $O(h^3)$ and $O(h^3 d_h^{-1})$! For other types of loading (mass forces, compression loads applied at plate's bases, etc.) there appear additional asymptotic terms which have the same structure as the leading one, but are generated by another solutions of the previous problem in Ω. Owing to the outlined similarity we sum up all the solutions in Ω which have figured in the asymptotics and form the "high precision" problem. The data of the last problem, of course, depend on the small parameter. The main point is that although the solution $v^* = v^0 + hv^1 + h^2 v^2$ can be obtained by the same means as v^0, the two-dimensional field v^* gives rise to high order approximations of displacements and stresses (see Sect. 7 and 8). In this connection, we mention that analogous modelling of deformation of plates with clamped edges is realized by treating problems in a slightly perturbed domain $\Omega(h)$ with elastic clamping conditions on $\partial\Omega(h)$ (see [18], and [17] in case of bending).

As usual, the asymptotic procedures applied here need supplementary assumptions on smoothness of the problem data. Paying attention to simplification of proofs we do not search for the sharpest restrictions and choose ones, sufficient in plenty to conclude precise estimates. Nevertheless, basing on that in engineering it is very unreal to distinguish between Sobolev spaces $H^{\varepsilon+9/2}(\Omega)$ and $H^{\varepsilon+7/2}(\Omega)$ (see Remark 3), we are hoping that this incompletion will not deny possible applications of the results.

2. FORMULATION OF THE PROBLEM ON A THIN PLATE

Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a smooth simple contour. We introduce the cylinder

$$Q_h = \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}^1 : y \in \Omega, |z| < h/2\}$$

with the bases $S_h^{\pm} = \Omega \times \{\pm h/2\}$ and the lateral side $S_h^0 = \partial\Omega \times (-h/2, h/2)$; here $h/\mathrm{diam}\,\Omega$ is a small parameter. Henceforth we achieve $\mathrm{diam}\,\Omega = 1$ by rescaling, the h being small itself. We assume the thin plate Q_h to be made from an isotropic homogeneous elastic material and regard the three-dimensional problem of the linear theory of elasticity

$$L(\nabla_x)\, u^h(x) \equiv -\mu\Delta_x u^h(x) - (\lambda+\mu)\nabla_x \nabla_x \cdot u^h(x) = F(y), x \in Q_h\,; \quad (1)$$

$$\sigma^{(3)}(u^h; x) \equiv \sigma(u^h; x)\, e^3 = hP^{\pm}(y), x \in S_h^{\pm}\,; \quad (2)$$

$$\sigma^{(n)}(u^h; x) \equiv \sigma(u^h; x)\, n(x) = P^0(y), x \in S_h^0, \quad (3)$$

where L is the Lâme operator with the constants λ, μ; $\nabla_x = \text{grad}$, $\Delta_x = \nabla_x \cdot \nabla_x$, u^h is a displacement vector, $\sigma(u)$ is a stress tensor with the Cartesian components

$$\sigma_{ij}(u) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{i,j}\,\lambda \nabla_x \cdot u\,, \quad i, j = 1, 2, 3\,; \tag{4}$$

$\delta_{i,j}$ means the Kronecker symbol, e^i is the unit vector of x_i-axis, $n = (n', 0)$ and n' imply the unit outward normals on S_h^0 and $\partial\Omega$, respectively. In order to treat the pure longitudinal deformation of the plate we suppose, in addition, that

$$F(y) = (f(y), 0)\,, \quad P^{\pm}(y) = (\pm q(y)/2, 0)\,, \quad P^0(y) = (p(y), 0)\,, \tag{5}$$

where the mass force f and the loads q, p are two-dimensional vectors.

Due to the assumptions on the data symmetry we always may choose a solution of (1)-(3) with the properties

$$u_i^h(y, z) = u_i^h(y, -z)\,, \quad i = 1, 2\,, \quad u_3^h(y, z) = -u_3^h(y, -z)\,. \tag{6}$$

Further we deal only with solutions satisfying (6). It is a well-known fact that the field u^h is determined up to rigid displacements, the lineal of which, under the restrictions (6), takes the form

$$\mathcal{R} = \{c_1 e^1 + c_2 e^2 + c_0(e^1 x_2 - e^2 x_1) : c_p \in \mathbb{R}\}\,.$$

Thus, in order to treat unique solutions we must add to (6) certain orthogonality conditions — we select the following ones :

$$(\bar{u}^h, \psi')_\omega = 0 \quad \forall \psi \in \mathcal{R}\,. \tag{7}$$

Here ω is a subdomain of Ω, $\text{mes}_2\,\omega > 0$ and $\text{dist}(\omega, \Omega) \geqslant d > 0$; $(\ ,\ )_\Theta$ means the inner product in $L_2(\Theta)$; ψ' implies the two-dimensional vector (ψ_1, ψ_2) obtained from ψ;

$$\bar{v}(y) = h^{-1}\int_{-h/2}^{h/2} v(y, z)\, dz\,.$$

We shall need variants of Korn's inequality adapted for thin domains. The first one was obtained in [13] and we present here the original proof, simple and short.

LEMMA 1 : *Let u satisfy* (6) *and* (7). *Then*

$$E(u) \equiv \frac{1}{\mu} \sum_{i,j=1}^{3} \int_{Q_h} \left(\sigma_{ij}(u)^2 - \frac{\lambda}{\mu + 3\lambda} \sigma_{ii}(u)\, \sigma_{jj}(u) \right) dx$$

$$\geqslant c_\Omega [\, \| \nabla_y u' \|_h^2 + h^2 \| \partial_z u' \|_h^2 + \| u' \|_h^2 \tag{8}$$

$$+ h^2 \| \nabla_y u_3 \|_h^2 + h^4 \| \partial_z u_3 \|_h^2 + h^2 \| u_3 \|_h^2]$$

where $\partial_z = \partial/\partial z$, $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2)$, c_Ω depends neither on $h \in (0, 1]$ nor on u, and $\| \cdot \|_h$ stands for $L_2(Q_h)$-norm.

Proof: Since $\lambda(\mu + 3\lambda)^{-1} < 1/3$, in virtue of (4) we get

$$cE(u) \geqslant \int_{Q_h} \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right|^2 dx . \tag{9}$$

We perform the changes of variables

$$x \mapsto (y, \zeta) = (y, h^{-1} z) , \quad u \mapsto (v, w) = (u', h u_3) \tag{10}$$

in the last integral I and obtain

$$I = h \int_{Q_1} \left[\sum_{i=1}^{2} \left(\left| \frac{\partial v_i}{\partial y_i} \right|^2 + h^{-2} \left| \frac{\partial v_i}{\partial \zeta} + \frac{\partial w}{\partial y_i} \right|^2 \right) + \left| \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right|^2 + h^{-4} \left| \frac{\partial w}{\partial \zeta} \right|^2 \right] dy\, d\zeta .$$

We replace h by 1 in the integrand and apply Korn's inequality (e.g. [19, 14]) in the fixed domain Q_1 (note that $h \leqslant 1$ and conditions (6), (7) at $h = 1$ are fulfilled by (v, w)). Thus,

$$I \geqslant c(\Omega)\, h(\| v ; H^1(Q_1) \|^2 + \| w ; H^1(Q_1) \|^2) .$$

It suffices to get back to x and u by inverting (10). ■

Due to (9) with $i = j = 3$ one can omit the multiplier h^4 at $\| \partial_z u_3 \|_h^2$ in (8). Nevertheless, inequality (8) stands to ignore specifications of restrictions (6) (bending of the plate is impossible) and it may be improved. First, since $\overline{u}_3 = 0$ due to (6), we appeal to the Poincare inequality

$$\int_{-h/2}^{h/2} | \partial_z u_3(y, z) |^2 dz \geqslant \frac{\pi^2}{h^2} \int_{-h/2}^{h/2} | u_3(y, z) |^2 dz ,$$

integrated over Ω, and change $h^2 \| u_3 \|_h^2$ for $h^{-2} \| u_3 \|_h^2$ in (8). Next, in view of two-dimensional Korn's inequality completed by (7),

$$\sum_{i,j=1}^{2} \int_{Q_h} \left| \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right|^2 dy\, dz \geqslant h \sum_{i,j=1}^{2} \int_{\Omega} \left| \frac{\partial \bar{u}_i}{\partial y_j} + \frac{\partial \bar{u}_j}{\partial y_i} \right|^2 dy \geqslant$$

$$\geqslant c(\Omega)\, h \| \bar{u}' ; H^1(\Omega) \|^2 = c(\Omega)\, \| \bar{u}' ; H^1(Q_h) \|^2 .$$

This relation together with

$$h^2 \| \partial_z u' \|_h^2 = h^2 \| \partial_z (u' - \bar{u}') \|_h^2 \geqslant \pi^2 \| u' - \bar{u}' \|_h^2$$

allows us to eliminate h^2 at $\| u' \|_h^2$ in (8). We formulate now the resulting inequality.

LEMMA 2 : *If u satisfies* (6) *and* (7), *then*

$$E(u) \geqslant c_\Omega [\, \| \nabla_y u' \|_h^2 + h^2 \| \partial_z u' \|_h^2 + \| u' \|_h^2 +$$

$$+ h^2 \| \nabla_y u_3 \|_h^2 + \| \partial_z u_3 \|_h^2 + h^{-2} \| u_3 \|_h^2] \equiv c_\Omega | u |_h^2 . \quad \blacksquare \qquad (11)$$

Multiplying (1) by u^h, integrating by parts and taking (2), (3) into account, we arrive at

$$E(u^h) = (f, u^{h\prime})_{Q_h} + (p, u^{h\prime})_{S_h^0} + \frac{1}{2} \sum_{\pm} h(q, u^{h\prime})_{S_h^\pm}$$

where

$$(f, u^{h\prime})_{Q_h} \leqslant h^{1/2} \| f ; L_2(\Omega) \| \cdot \| u^{h\prime} \|_h ,$$

$$(p, u^{h\prime})_{S_h^0} \leqslant h^{1/2} \| p ; L_2(\partial\Omega) \| \cdot \| u^{h\prime} ; L_2(S_h^0) \| \leqslant$$

$$\leqslant c h^{1/2} \| p ; L_2(\partial\Omega) \| \, (\| u^{h\prime} \|_h^2 + \| \nabla_g u^{h\prime} \|_h^2) ,$$

$$\frac{1}{2} \sum_{\pm} h(q, u^{h\prime})_{S_h^\pm} = (q, \partial_z (z u^{h\prime}))_{Q_h} \leqslant$$

$$\leqslant h^{1/2} \| q ; L_2(\Omega) \| \, (\| u^{h\prime} \|_h^2 + (h/2)^2 \| \partial_z u^{h\prime} \|_h^2) .$$

Applying (11), we conclude

$$| u^h |_h \leqslant c h^{1/2} (\| f ; L_2(\Omega) \| + \| q ; L_2(\Omega) \| + \| p ; L_2(\partial\Omega) \|) . \qquad (12)$$

PROPOSITION 1 : If the compatibility condition

$$(f+q, \psi')_{\Omega} + (p, \psi')_{\partial\Omega} = 0 \quad \forall \psi \in \mathscr{R} \tag{13}$$

is fulfilled, then problem (1)-(3) has the unique solution $u^h \in H^1(Q_h)$ subject to (6), (7), and the estimate (12) is valid with the constant c depending neither on $h \leqslant 1$ nor on f, p, q.

Proof: Taking account of (5) we derive (13) from the usual compatibility conditions for an elasticity problem in a spatial domain. They are six (the main vector and moment of loading have to vanish), however (7) contains only three linearly independent conditions — the three are fulfilled spontaneously in virtue of (6). Uniqueness of a solution follows from that a rigid displacement becomes trivial, since it satisfies both (6) and (7). ■

We note that (12) and (13) were the very reasons to put h into the right-hand side of (2).

3. THE ASYMPTOTICS OF u^h AT A DISTANCE FROM S_h^0

Following the general approach for constructing asymptotic decompositions of solutions of elliptic problems in thin domains (see [1, 20], etc.) we take

$$u^h(x) \sim \sum_{j=0}^{\infty} h^j[(v^j(y), 0) + V^j(y, \zeta)] \tag{14}$$

as the asymptotic form for a solution of (1)-(3), (6), (7). We use just the same notations as in (10) and prescribe

$$\overline{V}^j(y) \equiv \int_{-1/2}^{1/2} V^j(y, \zeta)\, d\zeta = 0\,, \quad j = 0, 1, \dots \tag{15}$$

While going over to the coordinates (y, ζ) we get the representations

$$L(\nabla_x) = h^{-2} L^0(\partial_\zeta) + h^{-1} L^1(\nabla_y, \partial_\zeta) + h^0 L^2(\nabla_y)\,, \tag{16}$$

$$B(\nabla_x) = h^{-1} B^0(\partial_\zeta) + h^0 B^1(\nabla_y)$$

where $\quad B(\nabla_x)\, u \equiv \sigma^{(3)}(u), \quad \xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2),$

$$L^0(\xi_3) = -M\xi_3^2\,, \quad B^0(\xi_3) = M\xi_3\,, \quad M = \operatorname{diag}\{\mu, \mu, \lambda + 2\mu\}\,,$$

$$L_{j3}^1(\xi) = L_{3j}^1(\xi) = -(\lambda + \mu)\, \xi_j\, \xi_3\,, \quad L_{33}^2(\xi') = -\mu|\xi'|^2\,,$$

$$L_{jk}^2(\xi') = -\mu|\xi'|^2\, \delta_{j,k} - (\lambda + \mu)\, \xi_j\, \xi_k\,,$$

$$B_{j3}^1(\xi') = \mu\xi_j\,, \quad B_{3k}^1(\xi') = \lambda\xi_k \quad (j, k = 1, 2)\,. \tag{17}$$

In the list (17) we have omitted zero entries of 3×3-matrices L^k and B^k. We put (14), (16) into (1) and (2) and pick up coefficients at h^{j-2} and h^{j-1}, respectively. As the result, we obtain the Neumann problems for ordinary differential (in ζ) equations with the parameter $y \in \Omega$:

$$L^0(\partial_\zeta)\, V^j(y, \zeta) = - L^1(\nabla_y, \partial_\zeta)\, V^{j-1}(y, \zeta) - L^2(\nabla_y)\, V^{j-2}(y, \zeta) -$$

$$- L^2(\nabla_y)\,(v^{j-2}(y), 0) + \delta_{j,2}\, F(y),$$

$$\zeta \in (-1/2, 1/2)\,,$$

$$B^0(\partial_\zeta)\, V^j(y, \pm 1/2) = - B^1(\nabla_y)\, [V^{j-1}(y, \pm 1/2) +$$

$$(v^{j-1}(y), 0)] + \delta_{j,2}\, P^{\pm}(y). \quad (18)$$

Here $V^\ell = 0$ and $v^\ell = 0$ in the case $\ell < 0$. It is clear that the formula

$$\int_{-1/2}^{1/2} \mathcal{F}(y, \zeta)\, d\zeta + g^+(y) - g^-(y) = 0\,, \quad y \in \Omega\,, \quad (19)$$

implies the compatibility condition for the problem

$$L^0(\partial_\zeta)\, \mathcal{V}(y, \zeta) = \mathcal{F}(y, \zeta), \zeta \in (-1/2, 1/2)\,; \quad B^0(\partial_\zeta)\, \mathcal{V}(y, \pm 1/2) = g^{\pm}(y)\,.$$

Our immediate objective is to solve, step by step, problems (18) with $j = 0, \dots, 4$ and write down corresponding compatibility conditions which will be intended to define v^k in (14).

We denote the right-hand sides of (18) by $\mathcal{F}^j, g^{j\pm}$. Since $\mathcal{F}^0 = 0$, $g^{0\pm} = 0$, according to (15)

$$V^0 = 0\,. \quad (20)$$

Further, $\mathcal{F}^1 = 0$, $g^{1\pm} = - \lambda \nabla_y \cdot v^0 e^3$ and obviously

$$V^1(y, \zeta) = - \frac{\lambda \zeta}{\lambda + 2\mu} \nabla_y \cdot v^0(y)\, e^3\,. \quad (21)$$

By (20), (21) and (5), (17) we get

$$\mathcal{F}^{2\prime} = - \frac{\lambda(\lambda + \mu)}{\lambda + 2\mu} \nabla_y \nabla_y \cdot v^0 + \mu \Delta_y v^0 + (\lambda + \mu) \nabla_y \nabla_y \cdot v^0 + f\,,$$

$$\mathcal{F}^2_3 = 0\,, \quad g^{2\pm\prime} = \pm \frac{1}{2} \frac{\lambda \mu}{\lambda + 2\mu} \nabla_y \nabla_y \cdot v^0 \pm \frac{q}{2}\,, \quad g^{2\pm}_3 = - \lambda \nabla_y \cdot v^1$$

and that is why the problem (18) with $j = 2$ has a solution if and only if the following equation with $k = 0$ is valid:

$$L'(\nabla_y)\, v^k(y) \equiv -\mu \Delta_y\, v^k(y) - (\lambda' + \mu)\, \nabla_y \nabla_y \cdot v^k(y) = f^k(y), \quad y \in \Omega . \tag{22}$$

Here L' is the two-dimensional Láme operator with the same shear modulus μ as in (1) and the new second Láme constant

$$\lambda' = 2\mu\lambda(\lambda + 2\mu)^{-1} . \tag{23}$$

Moreover,

$$f^0 = f + q \tag{24}$$

and the solution V^2 of (18), (15) takes the form

$$V^2(y, z) = \left(\left[\frac{\zeta^2}{2} - \frac{1}{24}\right] \left[\frac{\lambda}{\lambda + 2\mu} \nabla_y \nabla_y \cdot v^0(y) + \frac{1}{\mu} q(y)\right], 0 \right) - \\ - \frac{\lambda\zeta}{\lambda + 2\mu} \nabla_y \cdot v^1(y)\, e^3 . \tag{25}$$

We have to continue our procedure. The vectors $\mathscr{F}^j$, $g^{j\pm}$, where $j = 3, 4$, become very cumbersome; we omit them and write down only formulae related to v^1 and v^2. First of all, with help of (21), (25) and (17) we transform the condition (19), where $\mathscr{F}$, $g^{\pm}$ are replaced by $\mathscr{F}^4$, $g^{4\pm}$, into system (22) with $k = 1$ and

$$f^1 = 0 .$$

Besides,

$$V^3(y, \zeta) = -\frac{\zeta}{6} \left\{ \left[\zeta^2 - \frac{1}{4}\right] \frac{\lambda^2}{(\lambda + 2\mu)^2} \Delta_y \nabla_y \cdot v^0(y) + \right. \\ \left. + \frac{1}{\mu} \left[\frac{\lambda + \mu}{\lambda + 2\mu} \zeta^2 - \frac{\lambda + 3\mu}{\lambda + 2\mu} \cdot \frac{1}{4}\right] \nabla_y \cdot q(y) \right\} e^3 \\ + \left(\frac{\lambda}{\lambda + 2\mu} \left[\frac{\zeta^2}{2} - \frac{1}{24}\right] \nabla_y \nabla_y \cdot v^1(y), 0 \right) \\ - \frac{\lambda\zeta}{\lambda + 2\mu} \nabla_y \cdot v^2(y)\, e^3 \tag{27}$$

(compare the last line with (25)). Finally, the equality (19) at $j = 4$ coincides with (22) where $k = 2$ and

$$f^2 = \lambda[12(\lambda + 2\mu)]^{-1} \nabla_y \nabla_y \cdot q . \tag{28}$$

4. THE BOUNDARY LAYER

In Section 3 we derived systems (22) intended to define the functions v^j in (14). In order to supply them with boundary conditions we investigate the boundary layer phenomenon near the lateral surface.

Owing to (20), (21), (25), (27)

$$\sigma_{jk}(u^h) = \tau_{jk}(v^0) + h\tau_{jk}(v^1) + h^2 \tau_{jk}(v^2) - \delta_{j,k} \frac{h^2}{2} \frac{\lambda}{\lambda + 2\mu} \nabla_y \cdot q +$$

$$+ \frac{h^2}{2} \left[\zeta^2 - \frac{1}{12}\right] \tau_{jk}\left(\frac{\lambda}{\lambda + 2\mu} \nabla_y \nabla_y \cdot v^0 + \frac{1}{\mu} q\right) + \cdots \quad (j, k = 1, 2), \tag{29}$$

$$\sigma_{j3}(u^h) = h\zeta q_j + h^2 O + \cdots ,$$

$$\sigma_{33}(u^h) = -\frac{h^2}{2} \left[\zeta^2 - \frac{1}{4}\right] \nabla_y \cdot q + \cdots ,$$

where dots stand for inessential terms and $\tau_{jk}(v)$ are the Cartesian components of the two-dimensional stress tensor $\tau(v)$,

$$\tau_{jk}(v) = \mu\left(\frac{\partial v_j}{\partial y_k} + \frac{\partial v_k}{\partial y_j}\right) + \delta_{j,k} \lambda' \nabla_y \cdot v . \tag{30}$$

Stresses (29) have to satisfy certain conditions on plate's lateral side. According to (5) and (29) we eliminate in (3) discrepancies of order $O(1)$ by setting

$$\tau^{(n)}(v^0 ; y) \equiv \tau(v^0 ; y)\, n'(y) = p(y) , \quad y \in \partial\Omega . \tag{31}$$

Discrepancies of higher orders are compensated by the solution of boundary layer type

$$\chi(n) \sum_{j=2}^{\infty} h^j w^j\left(s, \frac{n}{h}, \frac{z}{h}\right) . \tag{32}$$

Here (s, n) are natural coordinates in the neighbourhood $\mathscr{U}$ of $\partial\Omega$, s and n mean the are length on $\partial\Omega$ and the distance from $\partial\Omega$ along the outward normal, respectively ; $\chi \in C_0^\infty(\mathcal{U})$ is a cut-off function $\chi = 1$ near $\partial\Omega$. We introduce the "rapid" coordinates $\eta = (\eta_1, \eta_2)$, where $\eta_1 = h^{-1} n$ and $\eta_2 = h^{-1} z$. To go over to the coordinates (s, η_1, η_2) in (1)-(3), we recall the following form of homogeneous equilibrium equations :

$$A[\partial_s \sigma_{ss} + 2k\sigma_{ns}] + \partial_n \sigma_{ns} + \partial_z \sigma_{sz} = 0 ,$$

$$A[\partial_s \sigma_{ns} - k(\sigma_{ss} - \sigma_{nn})] + \partial_n \sigma_{nn} + \partial_z \sigma_{nz} = 0 , \tag{33}$$

$$A[\partial_s \sigma_{sz} + k\sigma_{nz}] + \partial_n \sigma_{nz} + \partial_z \sigma_{zz} = 0 .$$

Here $\partial_s = \partial/\partial s$, etc. ; $A(s, n) = [1 + nk(s)]^{-1}$, k is the curvature of $\partial\Omega$ (positive for convex domains) ; lastly the components of the tensor $\sigma = \sigma(u)$ are defined by

$$\sigma_{ss} = A(\lambda + 2\mu)\,[\partial_s u_s + ku_n] + \lambda[\partial_n u_n + \partial_z u_z] ,$$

$$\sigma_{nn} = (\lambda + 2\mu)\,\partial_n u_n + \lambda[\partial_z u_z + A(\partial_s u_s + ku_n)] ,$$

$$\sigma_{zz} = (\lambda + 2\mu)\,\partial_z u_z + \lambda[\partial_n u_n + A(\partial_s u_s + ku_n)] , \tag{34}$$

$$\sigma_{sn} = \mu[A(\partial_s u_n - ku_s) + \partial_n u_s] ,$$

$$\sigma_{nz} = \mu[\partial_n u_z + \partial_z u_n] , \quad \sigma_{zs} = \mu[\partial_z u_s + A\partial_s u_z] ,$$

while $\sigma_{sn} = \sigma_{ns}$ and so on.

Now we should follow the same way as in Section 3 : to derive decompositions of the operators L and B written in (s, η), to put them together with (32) into (1)-(3), and to collect coefficients at h^{j-2} and h^{j-1}, respectively. As the result, we obtain a row of problems with differential (in η) equations and the parameter s. Besides, in the coordinates η each of cross-sections of the plate Q_h by planes, perpendicular to S_h^0, becomes the semi-strip $\Pi = (-\infty, 0) \times (-1/2, 1/2)$ after putting $h = 0$. Using the notations

$$W^j = (W^{j\prime}, W_3) , \quad W^{j\prime} = (W_1^j, W_2^j) = (w_n^j, w_z^j) , \quad W_3^j = w_s^j ,$$

$$t_{ik}(W) = \mu\left(\frac{\partial W_i}{\partial \eta_k} + \frac{\partial W_k}{\partial \eta_i}\right) + \delta_{i,k}\,\lambda \nabla_\eta \cdot W' , \quad t_{33}(W) = \lambda \nabla_\eta \cdot W' ,$$

$$t_{i3}(W) = t_{3i}(W) = \mu\,\partial W_3/\partial \eta_i , \quad i, k = 1, 2 ,$$

and observing (33), (34) we find that the whole problem for W^j splits into the elasticity problem for $W^{j\prime}$ and the Neumann problem for W_3^j :

$$
\begin{aligned}
& -\mu\Delta_\eta W^{j\prime}(s,\eta) - (\lambda+\mu)\nabla_\eta \nabla_\eta \cdot W^{j\prime}(s,\eta) = H^{j\prime}(s,\eta), \quad \eta \in \Pi, \\
& t_{1i}(W^j; s, 0, \eta_2) = K_i^{j0}(s,\eta_2), \quad |\eta_2| < 1/2, \\
& t_{2i}(W^j; s, \eta_1, \pm 1/2) = K_i^{j\pm}(s,\eta_1), \quad \eta_1 < 0, \quad i = 1, 2;
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
& -\mu\Delta_\eta W_3^j(s,\eta) = H_3^j(s,\eta), \quad \eta \in \Pi, \\
& t_{13}(W^j; s, 0, \eta_2) = K_3^{j0}(s,\eta_2), \quad |\eta_2| < 1/2, \\
& t_{23}(W^j; s, \eta_1, \pm 1/2) = K_3^{j\pm}(s,\eta_1), \quad \eta_1 < 0.
\end{aligned} \tag{36}
$$

Recalling symmetry conditions (6) we add to (35), (36) the analogous ones

$$
W_\ell^j(s,\eta) = W_\ell^j(s,\eta_1,-\eta_2), \quad \ell = 1, 3; \quad W_2^j(s,\eta) = -W_2^j(s,\eta_1,-\eta_2), \tag{37}
$$

the symmetry restrictions on H^j, K^{j0}, $K^{j\pm}$ that ensure (37) are demanded, too.

The main property of the boundary layer functions W^j is to decrease at an exponential rate as $\eta_1 \to -\infty$. The following assertion is a specification of general results on elliptic problems in domains with cylindrical outlets to infinity (*cf.* [21, 22] and Ch. 5 [23]) ; of course, there exist other approaches to prove this well-known result.

PROPOSITION 2 : *Let δ be a positive small number,*

$$
e^{-\delta\eta_1} H^j \in L_2(\Pi), \quad e^{-\delta\eta_1} K^{j\pm} \in L_2(-\infty, 0), \quad K^{j0} \in L_2(-1/2, 1/2).
$$

If the symmetry restrictions on H^j, $K^{j\pm}$, K^{j0} are valid and

$$
\int_\Pi H_\ell^j\, d\eta + \int_{-1/2}^{1/2} K_\ell^{j0}\, d\eta_2 + \sum_\pm \pm \int_{-\infty}^0 K_\ell^{j\pm}\, d\eta_1 = 0, \quad \ell = 1, 3, \tag{38}
$$

then problems (35), (36) *have the unique solutions $W^{j\prime}$, W_3^j satisfying* (37), *$e^{-\delta\eta_1} W^j \in H^1(\Pi)$ and the inequality*

$$
\| e^{-\delta\eta_1} W^j ; H^1(\Pi) \| \leqslant c(\| e^{-\delta\eta_1} H^j ; L_2(\Pi) \| + \\
+ \| K^{j0} ; L_2(-1, 1) \| + \sum_\pm \| e^{-\delta\eta_1} K^{j\pm} ; L_2(-\infty, 0) \|)
$$

is valid with the constant c, *not depending on* H^j, K^{j0}, $K^{j\pm}$. ■

Remark 1 : To preserve the results in Proposition 2 in non-symmetry case, one has to add other two orthogonality conditions to (38).

Remark 2 : Under smoothness assumptions on the data of (35), (36) the solution W^j becomes smoother everywhere in Π with exception of the angular points $\mathcal{P}_{\pm} = (0, \pm 1/2)$. Nevertheless, if in addition to the hypotheses of Proposition 3, $e^{-\delta\eta_1} K^{j\pm} \in H^{1/2}(-\infty, 0)$, $K^{j0} \in H^{1/2}(-1/2, 1/2)$, then

$$e^{-\delta\eta_1} \min\{1, \operatorname{dist}(\eta, \mathcal{P}_{\pm})\}\, \partial^2 W^j / \partial\eta_i\, \partial\eta_k \in L_2(\Pi),$$

the corresponding estimate holding true. One can take more precise results from [23].

We are going to fulfill (38) by fixing certain boundary value conditions on $\partial\Omega$ for v^{j-1}. We recall that we have compensated F, $P^{\pm}$ in (1), (2) and P^0 in (3) while constructing (14) and prescribing (31), respectively. Hence, taking account of $\zeta = \eta_1$ we obtain, in virtue of (29) and (34), the formulae

$$K_1^{20}(s, \eta_2) = -\tau_{nn}(v^1; s, 0), \quad K_2^{20}(s, \eta_2) = -\eta_2\, n'(s) \cdot q(s, 0),$$

$$K_3^{20}(s, \eta_2) = -\tau_{ns}(v^1; s, 0), \quad K^{2\pm} = 0, \quad H^2 = 0, \tag{39}$$

where we use (s, n) instead of y as arguments of v^1, q and n'. Thus, the conditions (38) with $j = 2$ turn into

$$\tau^{(n)}(v^1; y) = 0, \quad y \in \partial\Omega, \tag{40}$$

and, further,

$$W^{2\prime}(s, \eta) = -n'(s) \cdot q(s, 0)\, X(\eta), \quad W_3^2 = 0, \tag{41}$$

where X satisfies (35) with $H^{j\prime} = 0$, $K^{j\pm} = 0$ and $K^{j0}(\eta_2) = \eta_2\, e^2$.

Since differential operators written in (s, η) lose homogeneity property and their coefficients are not constant, the calculation of H^j, $K^{j\pm}$, K^{j0} for $j \geqslant 3$ looks more complicated than calculations we have performed : one should decompose the coefficients into Teylor series in n, make the change

$n \mapsto \eta_2$ and collect all terms of order $O(h^{j-2})$, the number of which increases intensely due to growth of j. Nevertheless, every expression with $j = 3$ is simplified by $W_3^2 = 0$ and, in the end, we arrive at

$$H_1^3 = k(\lambda + 2\mu)\,\partial_1 W_1^2, \quad H_2^3 = k[\mu\partial_1 W_2^2 + (\mu + \lambda)\,\partial_2 W_1^2],$$

$$K_1^{3\pm} = 0, \quad K_2^{3\pm} = -k\lambda W_1^2; \tag{42}$$

$$K_1^{30} = -k\lambda W_1^2 - \tau_{nn}(v^2) - \frac{1}{2}\left[\eta_2^2 - \frac{1}{12}\right]\tau_{nn}\left(\frac{\lambda}{\lambda + 2\mu}\nabla_y\nabla_y \cdot v^0 + \frac{1}{\mu}q\right) + \frac{1}{2}\frac{\lambda}{\lambda + 2\mu}\left[\eta_2^2 - \frac{1}{4}\right]\nabla_y \cdot q, \quad K_2^{30} = 0,$$

and

$$H_3^3 = (\lambda + \mu)\frac{\partial}{\partial s}(\partial_1 W_1^2 + \partial_2 W_2^2), \mathrm{K}_3^{3\pm} - \mu\frac{\partial}{\partial s}W_2^2, \tag{43}$$

$$K_3^{30} = -\mu\partial_s W_s - \tau_{ns}(v^2) - \frac{1}{2}\left[\eta_2^2 - \frac{1}{12}\right]\tau_{ns}\left(\frac{\lambda}{\lambda + 2\mu}\nabla_y\nabla_y \cdot v^0 + \frac{1}{\mu}q\right).$$

To shorten formulae, we put ∂_i in place of $\partial/\partial\eta_i$. We outline that the discrepancy, produced in (3) by (29), was taken into account both in (42) and (43) (see K_1^{30} and K_3^{30}).

We come up at the main difficulty : to calculate integrals in (38) whilst any simple form for W^2 is not available.

LEMMA 3 : *The formulae*

$$I_1 = -(\lambda + 2\mu)\int_\Pi \partial_1 X_1(\eta)\,d\eta + \lambda\int_{-1/2}^{1/2} X_1(0, \eta_2)\,d\eta_2 = \frac{1}{24}\frac{\lambda}{\lambda + \mu},$$

$$I_1 = -(\lambda + 2\mu)\int_\Pi (\partial_1 X_1(\eta) + \partial_2 X_2(\eta))\,d\eta - \mu\int_{-1/2}^{1/2} X_1(0, \eta_2)\,d\eta_2 - \mu\sum_\pm \pm\int_{-\infty}^0 X_2\left(\eta_1, \pm\frac{1}{2}\right)d\eta_1 = \frac{1}{24}\frac{\lambda}{\lambda + \mu}$$

are valid (*the X was introduced in* (41)).

Proof: Let us define the vector fields

$$Y^1(\eta) = [4\mu(\lambda + \mu)]^{-1}([\lambda + 2\mu]\,\eta_1, -\lambda\eta_2),$$

$$Y^2(\eta) = [2(\lambda + \mu)]^{-1}(\eta_1, \eta_2). \tag{44}$$

It is clear that

$$t_{11}(Y^1)=1\,,\quad t_{22}(Y^1)=0\,,\quad t_{12}(Y^1)=0\,,$$

$$t_{11}(Y^2)=1\,,\quad t_{22}(Y^2)=1\,,\quad t_{12}(Y^2)=0\,.$$

Besides, the normal stress vector $t^{(\nu)}$ coincides with (t_{11}, t_{12}) on the base of the semi-strip Π and with $\pm\,(t_{12}, t_{22})$ on its sides.

Applying Green formula and recalling boundary value conditions for X and Y^2 we obtain

$$I_2=\lambda\int_{-1/2}^{1/2} X_1(0,\eta_2)\,d\eta_2+\lambda\sum_{\pm}\pm\int_{-\infty}^{0} X_2\Big(\eta_1,\pm\frac{1}{2}\Big)\,d\eta_1$$

$$=\lambda\int_{\partial\Pi} X\,.\,t^{(\nu)}(Y^2)\,d\ell_\eta=\lambda\int_{\partial\Pi} Y^2\,.\,t^{(\nu)}(X)\,d\ell_\eta$$

$$=\lambda\int_{-1/2}^{1/2} Y_2^2(0,\eta_2)\,t_{12}(X\,;0,\eta_2)\,d\eta_2=\frac{\lambda}{2(\lambda+\mu)}\int_{-1/2}^{1/2}\eta_2^2\,d\eta_2\,,$$

where $d\ell_\eta$ implies an are length element on $\partial\Pi$. The analogous relations

$$I_1=-2\,\mu\int_{-1/2}^{1/2} X_1(0,\eta_2)\,d\eta_2=-2\,\mu\int_{\partial\Pi} X\,.\,t^{(\nu)}(Y^1)\,d\ell_\eta=$$

$$-2\,\mu\int_{\partial\Pi} Y^1\,.\,t^{(\nu)}(X)\,d\ell_\eta$$

$$=-2\,\mu\int_{-1/2}^{1/2} Y_2^1(0,\eta_2)\,t_{12}(X\,;0,\eta_2)\,d\eta_2=\frac{\lambda}{2(\lambda+\mu)}\int_{-1/2}^{1/2}\eta_2^2\,d\eta_2$$

leads us to the first equality we need verify. ■

We put (42) and (43) into (38), calculate immediately integrals containing v^2, v^0, q and apply Lemma 3 together with representation (41) to finish the derivation of the relations

$$\tau_{nn}(v^2)=\frac{1}{24}\frac{\lambda}{\lambda+\mu}kn'\,.\,q-\frac{1}{12}\frac{\lambda}{\lambda+2\,\mu}\nabla_y\,.\,q\,,\quad \tau_{ns}(v^2)=-\frac{1}{24}\frac{\lambda}{\lambda+\mu}\frac{\partial}{\partial s}n'\,.\,q$$

that are equivalent to (38) with $j = 3$, $\ell = 1, 3$ and can be rewritten in the vector form

$$\tau^{(n)}(v^2 ; y) = -\frac{1}{12}\frac{\lambda}{\lambda + 2\mu} n'(y)\,[\nabla_y \cdot q(y)] - \\ -\frac{1}{24}\frac{\lambda}{\lambda + 2\mu}\frac{\partial}{\partial s}\{s'(y)\,[n'(y) \cdot q(y)]\}, \quad y \in \partial\Omega, \quad (45)$$

where s' is the unit vector, tangent to $\partial\Omega$. We emphasize that (31), (40) and (45) imply boundary value conditions for the systems (22) with $k = 0, 1, 2$. Moreover, the compatibility conditions for (22), $k = 0$, follow from (13) (see (24)) while the ones for (22), $k = 2$, can be verified with help of the equality

$$\int_\Omega f^2(y) \cdot \psi'(y)\, dy = \frac{\lambda}{12(\lambda + 2\mu)} \times \\ \times \int_{\partial\Omega} [n'(y) \cdot \psi'(y)]\,[\nabla_y \cdot q(y)]\, ds_y \quad \forall \psi \in \mathcal{R}$$

(compare (28) with (45)). We submit the solution v^k to the orthogonality conditions

$$(v^k, \psi')_\omega = 0 \quad \forall \psi \in \mathcal{R} \quad (46)$$

(*cf.* (7)) and conclude, according to (26), (40), that $v^1 = 0$.

5. JUSTIFICATION OF THE ASYMPTOTICS

We suppose that

$$f, q \in H^{\varepsilon + 9/2}(\Omega), \quad p \in H^{\varepsilon + 5}(\partial\Omega) \quad (47)$$

with some positive ε. Then in virtue of well-known assertions connected with smoothness of solutions of elliptic problems we get the inequality

$$\| v^0 ; H^{\varepsilon + 13/2}(\Omega)\| + \| v^2 ; H^{\varepsilon + 9/2}(\Omega)\| \leq c\mathcal{N}_\varepsilon \quad (48)$$

for the solutions of (22), (31) and (22), (45). In (48) and further $\mathcal{N}_\varepsilon$ means the sum of the norms of f, q and p in the function spaces indicated in (47). Besides, the formulae

$$\| V^i ; H^{6-i}(Q_1)\| \leq c\mathcal{N}_\varepsilon, \quad i = 1, \ldots, 4, \quad (49)$$

are valid. To derive (49), we refer to (20), (21), (25), (27) in case $i \leq 3$, and at $i = 4$ we recall the structure of the right-hand side of (18) and orders (in

y) of differential operators in (17). We note also that, first, we can utilize arbitrary v^3 while solving problem (18) with $j=4$ and, second, the compatibility conditions for it were fulfilled by (22), (28) (see the very end of Sect. 3).

Observing formulae (41)-(43) and applying Proposition 2 twice, we obtain

$$\| W^j ; H^{6-j}(\partial\Omega ; H^1(\Pi)) \| \leq c\mathcal{N}_\varepsilon \tag{50}$$

where $j = 2, 3$ and

$$\| W ; H^s(\partial\Omega ; \mathcal{H}) \| = \| s \mapsto \| W(s, .) ; \mathcal{H} \| ; H^s(\partial\Omega) \| .$$

We take

$$U^h(x) = (v^0(y), 0) + h^2(v^2(y), 0) + h^3(v^3(y), 0) + \\ + \sum_{j=1}^{4} h^j V^j\left(y, \frac{z}{h}\right) + \chi(y) \sum_{k=2}^{4} h^k w^k\left(s, \frac{n}{h}, \frac{z}{h}\right) \tag{51}$$

as the global approximation field for the solution u^h of the initial problem (1)-(3). The functions v^1, v^2, V^j and w^2, w^3 have been found in Section 3 and 4. We define W^4 (or w^4 that is the same) as a solution of problems (35), (36) at $j = 4$, since we fulfill orthogonality conditions (38) by fixing the field $\tau^{(n)}(v^3)$ on $\partial\Omega$ (*cf.* Sect. 4). We avoid any other restriction on v^3 and, of course, can choose it so as

$$\| v^3 ; H^3(\Omega) \| \leq c\mathcal{N}_\varepsilon \tag{52}$$

and (46) with $k = 3$ is satisfied. Besides, the inequality (46) holds true also for $j = 4$.

Our immediate objective is to calculate and to estimate the discrepancies produced in (1)-(3) by (51). Let $U^{vh} + U^{wh}$ denote the right-hand side of (51). Owing to procedure performed in Section 3 we have

$$LU^{vh} - F = h^3(L^1 V^4 + L^2 V^3 + L^2 v^3) + h^4 L^2 V^4 \equiv \mathcal{F}^v \quad \text{in} \quad Q_h ,$$

$$BU^{vh} - P^{\pm} = h^4 B^1 V^4 \equiv g^{v^{\pm}} \quad \text{on} \quad S^h_{\pm}$$

and in virtue of (49), (52) and (17)

$$\| \mathcal{F}^v ; L_2(Q_h) \| \leq \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon , \quad \| g^{v^{\pm}} ; L_2(S^{\pm}_h) \| \leq \mathrm{ch}^4 \mathcal{N}_\varepsilon .$$

Let us dwell upon U^{wh} and its discrepancies

$$L\chi U^{wh} = [L, \chi]\, U^{wh} + \chi \sum_{m=1}^{3} h^{5-m}\, \mathscr{L}^{(m)}\, w^{5-m} \equiv \mathscr{F}^{w} \quad \text{in} \quad Q_h\,,$$

$$B\chi U^{wh} = [B, \chi]\, U^{wh} + \chi \sum_{m=1}^{3} h^{5-m}\, \mathscr{B}^{(m)}\, w^{5-m} \equiv g^{w^{\pm}} \quad \text{on} \quad S_h^{\pm}\,.$$

Here $[P, Q] = PQ - QP$ is the commutator of the operators P and Q; $\mathscr{L}^{(m)}$ and $\mathscr{B}^{(m)}$ are "tails" of decompositions of the operators L and B written in the coordinate (s, n, z). In accordance with Section 4, where initial terms of these decompositions were taken into accoung while we formed problems (35), (36) to find W^j, the operator $\mathscr{L}^{(m)}$ consists of the differential expressions

$$n^{\ell}\, \partial_n^{j}\, \partial_z^{j}\, \partial_s^{k} \quad (\, i + j + k \leqslant 2,\, \ell = i + j - 2 + m\,) \tag{53}$$

with smooth and bounded coefficients. Recalling Remark 2 we conclude, due to (53) and the inequality $\operatorname{dist}(\eta, \mathscr{P}_{\pm}) \leqslant c\eta_1$ for $\eta \in \Pi$, that $\mathscr{F}^{w} \in L_2(\Pi)$. Moreover, in virtue of Proposition 2

$$\begin{aligned} & h^{2(5-m)} \| \chi \mathscr{L}^{(m)}\, w^{5-m}\,;\, L_2(Q_h) \|^2 \leqslant \\ & \leqslant \mathrm{c}h^{2(5-m)} \int_{\partial\Omega} \int_{-d}^{0} \int_{-h/2}^{h/2} \sum_{i+j+k \leqslant 2} n^{2(i+j-2+m)} \times \\ & \times \left| \partial_n^{i}\, \partial_z^{j}\, \partial_s^{k}\, w^{5-m}\!\left(s, \frac{n}{h}, \frac{z}{h} \right) \right|^2 ds\, dn\, dz \leqslant \\ & \leqslant c \sum_{i+j+k \leqslant 2} h^{2[(5-m)+(i+j-2+m)-(i+j)]+2} \times \\ & \times \int_{-\infty}^{0} e^{2\delta\eta_1}\, d\eta_1 \leqslant \mathrm{c}h^{8}\, \mathscr{N}_{\varepsilon}\,. \end{aligned}$$

Since supports of commutator coefficients lay inside the curvilinear strip $\{x \in Q_h : -d < n < -\ell\}$ where $d,\ \ell > 0$, we also obtain

$$\begin{aligned} & h^{2(5-m)} \| [L, \chi]\, w^{5-m}\,;\, L_2(Q_h) \| \leqslant \\ & \qquad \leqslant \mathrm{c}h^{s} \int_{-d}^{-\ell} e^{2\delta n/h}\, dn\, \mathscr{N}_{\varepsilon} \leqslant c\, e^{-\alpha/h}\, \mathscr{N}_{\varepsilon} \quad (\,\alpha > 0\,)\,. \end{aligned}$$

Resuming the relations derived for $\mathscr{F}^w$ and using the same arguments to estimate $g^{w^{\pm}}$ we find that

$$\| \mathscr{F}^w ; L_2(Q_h) \| \leqslant \mathrm{ch}^4 \mathscr{N}_\varepsilon , \quad \| g^{w^{\pm}} ; L_2(S_h^{\pm}) \| \leqslant \mathrm{ch}^{9/2} \mathscr{N}_\varepsilon .$$

According to our choice of boundary value conditions on the end of the semi-strip Π, we arrive at

$$\sigma^{(n)}(U^h) - P^0 = g^0 \quad \text{on} \quad S_h^0 , \quad \| g^0 ; L_2(S_h^0) \| \leqslant \mathrm{ch}^{9/2} \mathscr{N}_\varepsilon .$$

At last, we mention that it possible to fix the $\mathscr{U}$ and the d (see the texts below (32) and (7), respectively) such that supp $\overline{U^{wh}} \subset \overline{\Omega} \setminus \omega$, and, hence, U^h inherits orthogonality conditions (7) from (46).

Our considerations result in the following assertion.

THEOREM 1 : *Under* (47) *and* (5), (13) *the solution* u^h *of* (1)-(3), (6), (7) *and the approximation function* (51) *are related by*

$$\| \sigma(u^h) - \sigma(U^h) \|_h + |u^h - U^h|_h \leqslant$$

$$\leqslant c_\varepsilon h^{7/2} \{ \| f ; H^{\varepsilon + 9/2}(\Omega) \| + \| q ; H^{\varepsilon + 9/2}(\Omega) \| + \| p ; H^{\varepsilon + 5}(\partial \Omega) \| \} , \quad (54)$$

since $| \cdot |_h$ *means the norm indicated in* (11). *The constant* c_ε *depends neither on* f, q, p *nor on* $h \in (0, 1]$.

Proof: Applying the estimates we have just obtained we follow the verification of Proposition 1 which shows, in particular, how to use Korn's inequality (11). ■

Remark 3 : The precision $O(h^{7/2})$ of the asymptotic approximation (51) in the norm $| \cdot |_h$ holds true even under slighter restrictions on f, q, p than in (47) (it is very predictable that in (54) ε can be replaced by $\varepsilon - 1$). To check this point up, one has to estimate $H^1(Q_h)^*$-norms of discrepancies, i.e. to treat an integral identity, to perform a refined integration by parts, and so on. Here, in order to simplify the verification of the inequality (54), we lose this chance.

Since in virtue of (52), (49), (50)

$$|(v^3, 0)|_h + |V^4|_h + h^{1/2} |\chi w^4|_h \leqslant c \mathscr{N}_\varepsilon ,$$

we can exclude these terms from the approximation formula (51).

COROLLARY 1 : Under (47) and (5), (13) the inequality

$$\left| u^h - (v^0, 0) - h^2 (v^2, 0) - \sum_{j=1}^{3} h^j V^j - \chi \sum_{k=2}^{3} h^k w^k \right|_h \leqslant c_\varepsilon h^{7/2} \mathscr{N}_\varepsilon \quad (55)$$

is valid where $\mathcal{N}_\varepsilon$ means the sum from the braces in (51).

Remark 4 : Observing (51) and (11) we find that inequality (12) contains asymptotically precise estimates of each term in the middle of (11) with exception of $h^2 \| \partial_z u^{h\prime} \|_h^2$ and $h^2 \| \nabla_y u_3^h \|_h^2$. These terms turn out to be smaller than $O(h^{1/2})$ because of

$$\| \partial_z U^{h\prime} \|_h = h^2 \| \partial_z V^{2\prime} \|_h + O(h^2) = O(h^{3/2}) ,$$

$$\| \nabla_y U_3^h \| = h \| \nabla_y V_3^1 \|_h + O(h^2) = O(h^{3/2}) .$$

The author does not know whether the multipliers h^2 at the above terms can be eliminated. Moreover, the estimate (12) becomes totally precise only in the case the new multipliers h^{-2} appear in place of h^2. The latter, of course, is impossible in view of (9). It looks like that the relation

$$\| \partial_z u^{h\prime} \|_h + \| \nabla_y u_3^h \| = O(h^{3/2})$$

follows from the additional assumptions on data smoothness and can not be derived from Korn's inequality itself.

6. THE HIGH PRECISION PROBLEM

The lateral side S_h^0 influences the whole stress-strain state of the plate Q_h, first, by appearing of boundary layer component parts in solution asymptotics and, second, by perturbing the right-hand sides of boundary value conditions for component solutions of "smooth type". The last fact shows, in particular, that plate edge effects do no concentrate in the vicinity of S_h^0. While one describes such effects far from S_h^0 it is very natural to unite three resembling problems for the smooth component solutions v^0, v^1 and v^2 into one problem. Although its solution v^* may be computed by the same means as the solution v^0 of (22), (31), it approximates the deformation of the plate more precisely.

The sum

$$v^*(y) = v^0(y) + hv^1(y) + h^2 v^2(y) \tag{56}$$

turns out to be a solution of the problem

$$L'(\nabla_y)\, v^*(y) = f^*(y), y \in \Omega \, ; \quad \tau^{(n)}(v^* \, ; y) = p^*(y), y \in \partial\Omega \, , \tag{57}$$

where L' is the Láme operator (see (22)) and

$$f^* = f + q + \frac{h^2}{12} \frac{\lambda}{\lambda + 2\mu} \nabla_y \nabla_y \cdot q ,$$

$$p^* = p - \frac{h^2}{24} \left\{ \frac{\lambda}{\lambda + 2\mu} n'(\nabla_y \cdot q) + \frac{\lambda}{\lambda + \mu} \frac{\partial}{\partial s} [s'(n' \cdot q)] \right\} .$$

We outline that f^* is obtained by summing (24), (26) and (28) while p^* is the sum of the right-hand sides of (31), (40) and (45). The function v^* enjoies orthogonality conditions (46). Due to (47)

$$f^* \in H^2(\Omega) , \quad p^* \in H^{5/2}(\partial\Omega) , \quad v^* \in H^4(\Omega) .$$

Starting with v^* we introduce the three-dimensional displacement field $U^* = (U^{*\prime}, U_3^*) \in H^1(Q_h)$,

$$U^{*\prime} = v^* + \frac{h^2}{2} \frac{\lambda}{\lambda + 2\mu} \left(\zeta^2 - \frac{1}{12} \right) \nabla_y \nabla_y \cdot v^* , \tag{58}$$

$$U_3^* = - h \frac{\lambda}{\lambda + 2\mu} \zeta \left[\nabla_y \cdot v^* + \frac{h^2}{6} \frac{\lambda}{\lambda + 2\mu} \left(\zeta^2 - \frac{1}{4} \right) \Delta_y \nabla_y \cdot v^* \right] ,$$

and the stress tensor $\sigma^* \in L_2(Q_h)$,

$$\sigma_{jk}^* = \tau_{jk}(U^{*\prime}), \sigma_{j3}^* = 0, \sigma_{33}^* = 0 , \quad (j, k = 1, 2) . \tag{59}$$

In addition,

$$\hat{U}' = \frac{h^2}{2} \frac{1}{\mu} \left[\zeta^2 - \frac{1}{12} \right] q , \quad \hat{U}_3 = - \frac{h^3}{6} \frac{1}{\mu} \zeta \left[\frac{\lambda + \mu}{\lambda + 2\mu} \zeta^2 - \frac{1}{4} \frac{\lambda + 3\mu}{\lambda + 2\mu} \right] \nabla_y \cdot q ,$$

$$\hat{\sigma}_{jk} = h^2 \left\{ \frac{1}{\mu} \left[\frac{\zeta^2}{2} - \frac{1}{12} \right] \tau_{jk}(q) - \left[\frac{\zeta^2}{2} - \frac{1}{8} \right] \frac{\lambda}{\lambda + 2\mu} \delta_{j,k} \nabla_y \cdot q \right\} , \tag{60}$$

$$\hat{\sigma}_{j3} = h\zeta q_j , \quad \hat{\sigma}_{33} = - h^2 \left[\frac{\zeta^2}{2} - \frac{1}{8} \right] \nabla_y \cdot q .$$

The fields $\hat{U} \in H^1(Q_h)$ and $\hat{\sigma} \in L_2(Q_h)$ are constructed directly (without solving any problem).

We are going to estimate the divergences of $U^* + \hat{U}$ and $\sigma^* + \hat{\sigma}$ from u^h and $\sigma(u^h)$, respectively. In virtue of Theorem 1 it suffices to compare (58), (60) with (51).

According to (21), (25), (26) we have

$$U^{vh\prime} - U^{*\prime} - \hat{U}' = h^3(v^3, 0) + h^4 V^4 - h^4 \frac{\lambda}{\lambda + 2\mu} \left[\frac{\zeta^2}{2} - \frac{1}{24}\right] \nabla_y \nabla_y \cdot v^2 ,$$

$$U_3^{vh} - U_3^* - \hat{U}_3 = h^4 V_3^4 + \frac{h^5}{6} \frac{\lambda^2}{(\lambda + 2\mu)^2} \zeta \left[\zeta^2 - \frac{1}{4}\right] \Delta_y \nabla_y \cdot v^2$$

and therefore

$$|U^{vh} - U^* - \hat{U}|_h \leqslant \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon . \tag{61}$$

We compute the stresses corresponded to U^{vh} (see (29)) and, due to (53), (60), conclude that

$$\| \sigma(U^{vh}) - \sigma^* - \hat{\sigma} \|_h \leqslant \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon \tag{62}$$

Since

$$|U^{wh}|_h + \| \sigma(U^{wh}) \|_h = O(h^2) ,$$

the exponent of h in the resultant inequality for $|u^h - U^* - \hat{U}|_h$ becomes smaller than the exponent in (54). To improve the situation, we put powers of the weight multiplier $d_h(x) = \max\{h, \operatorname{dist}(y, \partial\Omega)\}$ into $L_2(Q_h)$-norms of functions in (11), i.e. we treat function spaces with weighted norms. By choosing suitable exponents of d_h we diminish the contribution of U^{wh} to the corresponding weighted norm of u^h :

$$\begin{aligned} |||U^{wh}|||_h \equiv \{ & \| d_h^{3/2} \nabla_y U^{wh\prime} \|_h^2 + h^2 \| d_h^{1/2} \partial_z U^{wh\prime} \|_h^2 + \| d_h^{1/2} U^{wh\prime} \|_h^2 + \\ & + h^2 \| d_h^{1/2} \nabla_y U_3^{wh} \|_h^2 + \| d_h^{3/2} \partial_z U_3^{wh} \|_h^2 + h^{-2} \| d_h^{3/2} U_3^{wh} \|_h^2 \}^{1/2} \leqslant \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon , \end{aligned} \tag{63}$$

$$\| d_h^{3/2} \sigma(U^{wh}) \|_h \leqslant \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon .$$

While verifying (63), one ought to recall (51) and to take into account the relation

$$\begin{aligned} & \| x \mapsto d_h(x)^m \chi(y) T(s) e^{\alpha n/h} \|_h^2 \leqslant \\ & \qquad \leqslant c \int_{-h/2}^{h/2} \int_{-d}^{0} (h^2 + n^2)^m e^{2\alpha n/h} \, dz \, dn \leqslant \mathrm{Ch}^{2(m+1)} , \end{aligned}$$

where $m, \ \alpha > 0$ and $T \in L_2(\partial\Omega)$.

We note that $d_h(x)^k \leqslant C$ for $k = 0$ and reduce (61) and (62) to

$$|||U^{vh} - U^* - \ddot{U}|||_h + \| d_h^{3/2} [\sigma(U^h) - \sigma^* - \hat{\sigma}] \| \leqslant \mathrm{ch}^{7/2} \mathcal{N}_\varepsilon .$$

Thus, in virtue of (43) the following assertion has been proved.

THEOREM 2 : *Under the conditions of Theorem 1 the inequality*

$$\| d_h^{3/2} [\sigma(u^h) - \sigma^* - \hat{\sigma}] \|_h + ||| \; u^h - U^* - \hat{U} \; |||_h \leqslant$$

$$\leqslant c_\varepsilon h^{7/2} (\|f ; H^{\varepsilon + 9/2}(\Omega)\| + \|q ; H^{\varepsilon + 9/2}(\Omega)\| + \|p ; H^{\varepsilon + 5}(\Omega)\|) \quad (64)$$

is valid. Here U^*, σ^* *and* $\hat{U}$, $\hat{\sigma}$ *are defined by the formulae* (58)-(60) *where* v^* *is the solution* (56) *of the two-dimensional problem* (57).

Remark 4 : In the vicinity of the lateral surface of Q_h the approximations $v^0 + hV^1$ and $U^* + \hat{U}$ possess the same asymptotic accuracies (because the boundary value component $h^2 w^2$ is ignored by both of them). Nevertheless, at a distance of S_h^0 (for example, on $(\omega \times (-h/2, h/2))$ the sum $U^* + \hat{U}$ gives more precise approximation than $v^0 + hV^1$. We repeat that to find U^* is to solve the problem (57) of just the same type as the problem for v^0, while formulae (60) for $\hat{U}$ contain only the datum q. The analogous conclusions hold true also for stress fields.

7. GENERALIZATIONS AND SPECIFICATIONS

In previous sections we treated loads of special type (5) ; here we touch upon modifications which do not influence both the approach and results.

i) Let us suppose that the bases $S_h^{\pm}$ of the plate are free of loads and the mass forces are neglectable, too. In other words, $q = 0$ and $f = 0$ in (5) and (1), (2). In this case the high precision problem (57) coincides completely with the problem (22), (31) for v^0 and, by the way, $v^2 = 0$. Moreover, in virtue of (41) the leading term $h^2 w^2$ of the boundary layer disappears and $U^{wh} = h^3 w^3 + h^4 w^4$ in (51). It is just the reason to improve the estimate (64) by diminishing exponents in the weight multipliers.

We assume the formulae (58), (59) with $v^* = v^0$ to hold true, while $\hat{U} = 0$, $\hat{\sigma} = 0$ in accordance with (60). Repeating the same arguments and calculations as in Section 6 we arrive at the inequalities

$$\| d_h^{1/2} [\sigma(u^h) - \sigma^*] \|_h \leqslant c_\varepsilon h^{7/2} \|p ; H^{\varepsilon+5}(\partial\Omega)\| ,$$

$$\| d_h^{1/2} \nabla_y (u^{h\prime} - U^{*\prime}) \|_h + h \| \partial_z (u^{h\prime} - U^{*\prime}) \|_h + \| u^{h\prime} - U^{*\prime} \|_h +$$
$$+ h^2 \| \nabla_y (u_3^h - U_3^*) \|_h + \| d_h^{1/2} \partial_z (u_3^h - U_3^*) \|_h$$
$$+ h^{-1} \| d_h^{3/2} (u_3^h - U_3^*) \|_h \leqslant c_\varepsilon h^{7/2} \|p ; H^{\varepsilon+5}(\partial\Omega)\| . \quad (65)$$

They looks like (64) (or (63)), but each term in the left of them has the better weight multiplier $d_h(x)^{(t-1)_+}$ in place of $d_h(x)^t$.

It follows from (65) that, by applying only the solution v^0 of the usual two-dimensional problem (22), (31) on longitudinal deformation of plate Q_h, we may approximate the three-dimensional displacement field u^h with unpredictably high precision $O(h^3)$ (the symbol O is to be understood in the sense of (65), while $h^{1/2}$ is wasted to compensate plate's thickness h). The representations of stresses lose their precision $O(h^3)$ only in the vicinity of S_h^0.

We emphasize that the assumption on the smoothness of $\partial\Omega$ is decisive for the above-mentioned facts to hold true. For example, in the case of a cracked plate, where angular points of the boundary $\partial\Omega$ appear, the precision of the approximation, even far from the crack, becomes equal to $O(h)$ (see [24, 25]).

ii) Let us introduce a compression components into the loads applied at plate bases $S_h^\pm$. In other words, we replace (2) by

$$\sigma_{3i}(u^h ; x) = \pm \frac{h}{2} q_i(y) , \quad i = 1, 2 , \quad \sigma_{33}(u^h ; x) = q_3(y) , \quad x \in S_h^\pm ,$$

where $q_i \in H^{\varepsilon + 9/2}(\Omega)$, $q_3 \in H^{\varepsilon + 11/2}(\Omega)$ (compare with (47)). Retaining the approach to construct the asymptotics, we restrict ourselves to present only the list of the corrections in the previous formulae : definitions (21), (25) and (27) of V^1, V^2 and V^3 should to be supplemented respectively by the terms

$$(\lambda + 2\mu)^{-1} \zeta q_3(y)\, e^3 ,$$

$$-\frac{1}{\lambda + 2\mu}\left[\frac{\zeta^2}{2} - \frac{1}{24}\right](\nabla_y q_3(y), 0) ,$$

$$\frac{\lambda}{(\lambda + 2\mu)^2}\frac{\zeta}{6}\left[\zeta^2 - \frac{1}{4}\right]\Delta_y q_3(y)\, e^3 ;$$

in stresses (29) there appear the additional addenda

$$\frac{\lambda}{\lambda + 2\mu} q_3 \delta_{j,k} - \frac{h^2}{2}\left[\zeta^2 - \frac{1}{12}\right]\tau_{jk}\Big(\frac{\lambda}{\lambda + 2\mu}\nabla_y q_3\Big) ,$$

$$0 ,$$

$$q_3 ;$$

the right-hand sides of the corresponding high precision problem (57) take the form

$$f^* + \frac{\lambda}{\lambda + 2\mu}\nabla_y q_3 , \quad p^* - \frac{\lambda}{\lambda + 2\mu} q_3 n' .$$

iii) Let us change (2), (3) for

$$\sigma^{(3)}(u;x)=\mathscr{P}^{\pm}(h,s,\eta_1)\,,\quad x\in S_h^{\pm}\,;$$

$$\sigma^{(n)}(u;x)=\mathscr{P}^{0}(h,s,\eta_2)\,,\quad x\in S_h^{0}\,,\tag{66}$$

and set $f=0$ in (1). In (66) the functions $(-\infty,0]\ni\eta_1\mapsto\mathscr{P}^{\pm}(h,s,\eta_1)$ have compact supports ;

$$\mathscr{P}^{0}\in H^{\varepsilon+5}(\partial\Omega\,;H^{1/2}(-1/2,1/2))\,,\quad \mathscr{P}^{\pm}\in H^{\varepsilon+5}(\partial\Omega\,;H^{1/2}(-\infty,0))\,;\tag{67}$$

$$\mathscr{P}^{0'}(h,s,\eta_2)=\mathscr{P}^{0'}(h,s,-\eta_2)\,,\quad \mathscr{P}^{0}_3(h,s,\eta_2)=-\,\mathscr{P}^{0}_3(h,s,-\eta_2)\,,$$

$$\mathscr{P}^{+'}(h,s,\eta_1)=\mathscr{P}^{-'}(h,s,\eta_1)\,,\quad \mathscr{P}^{+}_3(h,s,\eta_1)=-\,\mathscr{P}^{-}_3(h,s,\eta_1)\,;\tag{68}$$

decompositions of $\mathscr{P}^{\pm}$, $\mathscr{P}^{0}$ in powers of h with non-negative integer exponents are available. The problem (1), (66) corresponds to the plate Q_h loaded in the vicinity of its lateral surface (for instance, plate's edge is held by a vice). The symmetry requirements (68) are analogous to (5) and the smoothness requirements (67) are in accordance with (47), (50) and Remark 2.

The asymptotic procedure described in Section 3 and 4 is fit to investigate the problem (1), (66), too. The only modification, needed in addition, touchs upon the boundary layer component solution (32) which has now $h^1\,w^1(s,\eta)$ as the leading term. The vector W^1 with the components $W^1_1=w^1_n$, $W^1_2=w^1_z$, and $W^1_3=w^1_s$ turns out to be a solution of problems (35) and (36) where $j=1$ and

$$H^1_1=0\,,\quad H^1_2=0\,,\quad H^1_3=0\,;$$

$$K^{1^{\pm}}_1(s,\eta_1)=\mathscr{P}^{\pm}_n(0,s,\eta_1)\,,\quad K^{1^{\pm}}_2(s,\eta_1)=\mathscr{P}^{\pm}_z(0,s,\eta_1)\,,$$

$$K^{1^{\pm}}_3(s,\eta_1)=\mathscr{P}^{\pm}_s(0,s,\eta_1)\,;$$

$$K^{10}_1(s,\eta_2)=\mathscr{P}^{0}_n(0,s,\eta_2)-\tau_{nn}(v^0;s,0)\,,\quad K^{10}_2(s,\eta_2)$$
$$=\mathscr{P}^{0}_z(0,s,\eta_2)\,,\quad K^{10}_3(s,\eta_2)$$
$$=\mathscr{P}^{0}_s(0,s,\eta_2)-\tau_{ns}(v^0;s,0)\,.$$

In virtue of Proposition 2 we arrive at the following condition for the solution W^1 to vanish at an exponential rate as $\eta_1\to-\infty$:

$$\tau^{(n)}(v^0;y)=r^0(s)\,,\quad y\in\partial\Omega\,.\tag{69}$$

Here $r^0 = (r_s^0, r_n^0)$; r_s and r_n are components of the main vector of the load $\mathcal{P}$ on $\partial\Pi$ which coincides with $\mathcal{P}^0$ at $\eta_1 = 0$ and $\pm\mathcal{P}^\pm$ at $\eta_2 = \pm 1/2$;

$$r_s^m(s) = \int_{\partial\Pi} \partial_h^m \mathcal{P}_s(0, s, \eta)\, d\ell_\eta , \quad r_n^m(s) = \int_{\partial\Pi} \partial_h^m \mathcal{P}_n(0, s, \eta)\, d\ell_\eta , \quad m = 0, 1 .$$

The expressions of H^2, $K^{2^\pm}$, K^{20} are combersome and we list only the components figuring in the orthogonality conditions (38). As in Section 4, we take (33), (34) into account and derive the representations

$$\begin{aligned}
H_1^2(s,\eta) &= (\lambda+\mu)\,\partial_s\,\partial_1\,W_3^1(s,\eta) + (\lambda+2\mu)\,k(s)\,\partial_s\,W_1^1(s,\eta)\,,\\
H_3^2(s,\eta) &= (\lambda+\mu)\,\partial_s\,\nabla_\eta\,.\,W^{1'}(s,\eta) + \mu k(s)\,\partial_1\,W_3^1(s,\eta)\,,\\
K_1^{2^\pm}(s,\eta_1) &= \partial_h\,\mathcal{P}_n^\pm(0,s,\eta_1)\,,\\
K_3^{2^\pm}(s,\eta_1) &= \partial_h\,\mathcal{P}_s^\pm(0,s,\eta_1) - \mu\partial_s\,W_2^1(s,\eta_1,\pm 1/2)\,,\\
K_3^{2^\pm}(s,\eta_1) &= \partial_h\,\mathcal{P}_s^\pm(0,s,\eta_1) - \lambda\partial_s\,W_3^1(s,0,\eta_2)\\
&\qquad - \lambda k(s)\,W_1^1(s,0,\eta_2) - \tau_{nn}(v^1\,;s,0)\,,\\
K_3^{20}(s,\eta_2) &= \partial_h\,\mathcal{P}_s^0(0,s,\eta_2) - \mu\partial_s\,W_1^1(s,0,\eta_2)\\
&\qquad + \mu k(s)\,W_3^1(s,0,\eta_2) - \tau_{ns}(v^1\,;s,0)\,.
\end{aligned} \tag{70}$$

In order to calculate the integrals in (38) at $j = 2$ we use the following formulae.

LEMMA 4 : *The equalities*

$$-(\lambda+2\mu)\int_\Pi \partial_1\,W_1^1\,d\eta + \lambda\int_{-1/2}^{1/2} W_1^1\,d\eta_2 = -2\mu R_1\,,$$

$$(\lambda+\mu)\int_\Pi \nabla_\eta\,.\,W^{1'}\,d\eta - \mu\int_{-1/2}^{1/2} W_1^1\,d\eta_2 - \mu\sum_\pm \pm\int_{-\infty}^0 W_2^1\,d\eta_1 = \lambda R_2\,,$$

$$(\lambda+\mu)\int_\Pi \partial_1\,W_3^1\,d\eta - \lambda\int_{-1/2}^{1/2} W_3^1\,d\eta_2 = \mu R_3\,,$$

$$\mu\int_\Pi \partial_1\,W_3^1\,d\eta - \mu\int_{-1/2}^{1/2} W_3^1\,d\eta_2 = 0$$

are valid where $W^{1\prime}$, W_3^1 *are solutions of problems* (36), (36) *with right-hand sides* (70),

$$R_i(s) = \int_{\partial\Pi} [Y_1^i(\eta)\,\mathscr{P}_n(0, s, \eta) + Y_2^i(\eta)\,\mathscr{P}_z(0, s, \eta)\,]\, d\ell_\eta\,, \quad i = 1, 2\,,$$

$$R_3(s) = \mu^{-1} \int_{\partial\Pi} \eta_1\, \mathscr{P}_s(0, s, \eta)\, d\ell_\eta\,,$$

and Y^1, Y^2 *mean vector fields* (44).

Proof: As in the proof of Lemma 3, the first and second equalities are obtained by applying the Green formula with Y^i and $W^{1'}$ for the Láme operator in Π. To derive the third equality, one may put $\mu^{-1}\eta_1$ and W_3^1 into Green formula for the Laplace operator. The forth one is obvious. ■

Due to the obtained relations we transform (38) with $j = 2$ into the conditions

$$\tau_{sn}(v^1\,; s, 0) = r_s^1(s) + \lambda\partial_s R_2(s)\,,$$

$$\tau_{nn}(v^1\,; s, 0) = r_n^1(s) + \mu\partial_s R_3(s) + 2\,\mu k(s)\, R_1(s)\,. \qquad (71)$$

Thus, we have got the problems (22), (69) and (22), (71) to define v^0 and v^1 (note that $f^0 = f^1 = 0$ in (22) according to $f = 0$).

We avoid boundless calculations to find boundary conditions for v^3 and we finish consideration of problem (1), (66) with mentioning that both the justification of the asymptotics and the formation of the high precision problem can be performed according to the patterns we use in previous sections.

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