M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN - Modélisation mathématique et analyse numérique, tome 29, nº 5 (1995), p. 605-627
<http://www.numdam.org/item?id=M2AN_1995__29_5_605_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 29, n° 5, 1995, p. 605 à 627)

SOLUTION OF DOUBLY NONLINEAR AND DEGENERATE PARABOLIC PROBLEMS BY RELAXATION SCHEMES (*)

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Communicated by R. GLOWINSKI

Abstract. — *A numerical method is proposed to solve degenerate double nonlinear parabolic problems. The degeneracy include both : locally fast and slow diffusion. The proposed method is based on a nonstandard time discretization including two relaxation functions by means of which the fast or slow diffusion is controled. The relaxation functions are determined by iterations. A large scale of diffusion problems with free boundary or their approximations are included in the present setting.*

Résumé. — *On propose une méthode numérique pour résoudre des problèmes doublement non linéaires paraboliques et dégénérés. La méthode proposée est fondée sur une discrétisation en temps non standard qui permet de contrôler les diffusions lente et rapide au moyen de deux fonctions de relaxation. Les fonctions de relaxation sont déterminées de manière itérative. Cet article présente un large spectre de problèmes de diffusion à frontières libres ou leur approximation.*

1. INTRODUCTION

We consider the following degenerate doubly nonlinear parabolic problem

(1.1)
$$\partial_t b(x, u) - \nabla a(t, x, u, \nabla\beta(u)) = f(t, x, u) \quad \text{in} \quad I \times \Omega, \quad I \equiv (0, T), T < \infty$$

with the mixed boundary conditions

(1.2)
$$u = 0 \quad \text{on} \quad \Gamma_1 \times I, \quad a(t, x, u, \nabla\beta(u)) \cdot \nu = g(t, x, u) \quad \text{on} \quad \Gamma_2 \times I$$

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and with the initial condition

$$b(x, u(0)) = b(x, u_0)\,. \tag{1.3}$$

Here $\Omega \subset R^N$ is a bounded domain with the Lipschitz continuous boundary $\partial\Omega$ and Γ_1, $\Gamma_2 \subset \partial\Omega$ are open with $\text{mes}_{N-1}\Gamma_1 + \text{mes}_{N-1}\Gamma_2 = \text{mes}_{N-1}\partial\Omega$, $\Gamma_1 \cap \Gamma_2 = 0$.

The functions $b(x, s)$, $\beta(s)$ are Lipschitz continuous and strictly increasing in s and $a(t, x, \eta, \xi)$ is monotone, coercive in ξ. The degeneracy $b'_s(x, s) = 0$, $\beta'(s) = 0$ is included, thus (1.1)-(1.3) represents a large class of free boundary models, or their approximations, e.g., flows in porous media, infiltrations, phase change - see [1], [2], [4], [6-8] etc. The existence of variational solutions to (1.1)-(1.3) has been studied in [2], [18], [3], [13], [5] etc.

Our main goal is to approximate (1.1)-(1.3) from a numerical point of view and to prove its convergence. We use a nonstandard time discretization with a relaxation function. Then we reduce (1.1) to a sequence of regular elliptic problems coupled with an algebraic condition for the relaxation function. Problem (1.1)-(1.3) includes locally both : fast and slow diffusion. Neither $b(x, \beta^{-1}(s))$ nor $\beta(b^{-1}(x, s))$ needs to be Lipschitz continuous in s (x being fixed). Our approximation scheme reduces to the ones used in [11], [14] where $b(x, s) \equiv s$, respectively.

We suggest the following approximation scheme :

Let $\tau = T/n$, $n \in \mathbb{N}$ be time step, $u_i \approx u(t_i, x)$ for $t_i = i \cdot \tau$, $\Theta_i \approx \beta(u_i)$. On the time level $t = t_i$ we determine Θ_i from the elliptic problem

$$\lambda_i(\Theta_i - \beta(u_{i-1})) - \tau\,\nabla a(t, x, u_{i-1}, \nabla\Theta_i) = \tau f(t_i, x, u_{i-1}) \equiv \tau f_i$$

(1.4)

$$\Theta_i = 0 \text{ on } \Gamma_1\,, \quad a(t_i, x, u_{i-1}, \nabla\Theta_i)\cdot\nu = g(t_i, x, \beta^{-1}(\Theta_{i-1})) \equiv g_i \text{ on } \Gamma_2$$

where $\lambda_i \in L_\infty(\Omega)$ has to satisfy the convergence conditions

$$\left|\lambda_i - \frac{b_n(x, u_{j-1} + \mu_i(\Theta_i - \beta(u_{i-1}))) - b_n(x, u_{i-1})}{\Theta_i - \beta(u_{i-1})}\right| < \tau \tag{1.5}$$

with additional relaxation function μ_i satisfying

$$\frac{\alpha}{L_\beta} \le \mu_i \le \min\left\{K, \frac{\beta^{-1}(\beta(u_{i-1}) + \alpha(\Theta_i - \beta(u_{i-1}))) - u_{i-1}}{\Theta_i - \beta(u_{i-1})}\right\} \tag{1.6}$$

where

$$b_n(x, s) = b(x, s) + \tau^d s, \quad d \in (0, 1/2).$$

$K, \alpha \in (0, 1)$ are parameters of the method and L_β is the Lipschitz constant of β. We define

$$(1.7) \qquad u_i = u_{i-1} + \mu_i(\Theta_i - \beta(u_{i-1})) \quad \text{for} \quad i = 1, \ldots, n.$$

When $\Theta_i = \beta(u_{i-1})$, then we replace the difference quotients in (1.5), (1.6) by $\partial_s b_n(x, u_{i-1}) \cdot \min\left\{K, \alpha / \beta'(u_{i-1})\right\}$, $\min\left\{K, \alpha / \beta'(u_{i-1})\right\}$, respectively. By means of relaxation functions λ_i, μ_i we control the degeneracy of both : $b'_s(x, s) = 0$ and $\beta'(s) = 0$. The conditions (1.5), (1.6) include also a regularization of b, β in a neighbourhood of the degeneracy. When b is linear in s, then we can take $\lambda_i \equiv \mu_i$ and if β is linear, we can take $\mu_i \equiv \alpha$ (α is close to 1) - see [11], [12].

If we put $\mu_i \equiv \alpha / L_\beta$, then (1.6) is satisfied. However from the numerical point of view the equality on R.H.S. in (1.6) is desirable and the numerical experiments in [12] confirm it.

A special case, $b(x, s) \equiv s$, $\lambda_i \equiv \mu_i \equiv \mu$-constant, $\mu \in \left(0, 1 / L_\beta\right)$ has been studied in a series of papers [20], [21], [23] etc.

To determine λ_i, μ_i we can use the iterations with the parameter $k = 1, \ldots,$ where we replace λ_i, Θ_i in (1.4) by $\lambda_{i,k-1}$, $\Theta_{i,k}$ and we define $\lambda_{i,k}$, $\mu_{i,k}$ by equalities with the corresponding difference quotients. We shall discuss the determination of λ_i, μ_i in (1.5), (1.6) in Section 2. In Section 3 we prove the convergence of our method. We discuss the convergence of the full discretization scheme (when (1.4) is projected on a finite dimensional space) in Section 4.

2. ASSUMPTIONS AND DETERMINATION OF λ_i, μ_i

Let us construct (see [2]) the nonnegative function $B^*(x, s)$ by

$$(2.1) \quad B^*(x, s) := b(x, \beta^{-1}(s))\, s - \int_0^s b(x, \beta^{-1}(z))\, dz \quad \text{for} \quad s \in \{y \in R : y = \beta(z)\}.$$

We assume

H_1) $b(x, s)$ is strictly monotone in s satisfying : $b(x, 0) = 0$,

$0 \leq \partial_s b(x, s) \leq M$,

$$|\partial_s b(x, s) - \partial_s b(x', s)| \leq \omega(|x - x'|)(1 + \partial_s b(x, s))$$

where $\omega : R_+ \to R$ is continuous, $\omega(0) = 0$ and

$$|b(x, s)|^2 \leq C_1 B^*(x, \beta(s)) + C_2 \quad \forall s \in R; \tag{2.2}$$

H_2) $\beta : R \to R$ is strictly monotone, Lipschitz continuous with $\beta(0) = 0$, $0 \leq \beta'(s) \leq M$;

H_3) $a(t, x, \eta, \xi) : I \times \Omega \times R \times R^N \to R^N$ is continuous,

$$(a(t, x, \eta, \xi_1) - a(t, x, \eta, \xi_2)) \cdot (\xi_1 - \xi_2) \geq 0,$$

$$|a(t, x, \eta, \xi)| \leq C(1 + (B^*(x, \beta(\eta)))^\gamma + |\xi|^{p-1}),$$

$$p \geq 2, \quad 0 < \gamma < \frac{p-1}{p};$$

$$a(t, x, \eta, \xi) \cdot \xi \geq C_1 |\xi|^p - C_2$$

uniformaly for $(t, x) \in Q_T \equiv I \times \Omega$, $\eta \in R$, $\xi \in R^N$;

H_4) $f(t, x, \eta) : I \times \Omega \times R \to R$ is continuous and

$$|f(t, x, \eta)| \leq C(1 + (B^*(x, \beta(\eta)))^\gamma), \quad 0 < \gamma < \frac{p-1}{p};$$

H_5) $|g(t, x, s)| \leq C(1 + |\beta(s)|^\gamma)$, $0 < \gamma < p - 1$;

H_6) $u_0 \in L_2(\Omega)$, $\beta(u_0) \in V$.

The stronger convergence results will be obtained under the strong monotonicity

H_7) $(a(t, x, \eta, \xi_1) - a(t, x, \eta, \xi_2)) \cdot (\xi_1 - \xi_2) \geq C|\xi_1 - \xi_2|^p$.

In the next we drop the variable x.

Remark 2.3 : The assumptions $p \geq 2$ and (2.2) are the only structural restrictions due to our method. If b is bounded, then (2.2) is satisfied. If $|b(x, s)| \approx c_1 |s|^\alpha$, $|\beta(s)| \approx c_2 |s|^\beta$ (asymptotically for $(|s| \to \infty)$ then (2.2) requires $\alpha \leq \beta$.

We shall use the standard functional spaces $L_\infty \equiv L_\infty(\Omega)$, $L_p \equiv L_p(\Omega)$, $W_p^1(\Omega)$ (Sobolev space), $V = \{u \in W_p^1 : u = 0 \text{ on } \Gamma_1\}$, $C^{0, \beta}(\bar{\Omega})$ (Hölder space), $L_p(I, V)$ - see, e.g. [18], [9]. We denote $x \cdot y = \sum_{i=1}^N x_i y_i$, $(u, v) = \int_\Omega u \cdot v$, $(u, v)_{\Gamma_2} = \int_{\Gamma_2} u \cdot v$ and V^* the dual space to V with the duality $\langle u, v \rangle$ for $u \in V^*$, $v \in V$.

Let $|\cdot|_\infty$, $|\cdot|_p$, $\|\cdot\|$, $\|\cdot\|_*$, $|\cdot|_{\Gamma,p}$, $\|\cdot\|_{L_p(I,V)}$, $\|\cdot\|_{0,\alpha}$ denote the norms in L_∞, L_p, V, V^*, $L_p(\Gamma)$, $L_p(I,V)$, $C^{0,\alpha}$, respectively. By C we denote a generic nonnegative constant.

DEFINITION 2.4 : *A measurable function $u : Q_T \to R$ is a variational solution of* (1.1)-(1.3) *iff*

(i) $b(x,u) \in L_2(Q_T)$, $\partial_t b(x,u) \in L_q(I, V^*)$ $(p^{-1} + q^{-1} = 1)$, $\beta(u) \in L_p(I,V)$;

(ii) $\int_I \langle \partial_t b(x,u), v\rangle = -\int_{Q_T} (b(x,u) - b(0,u_0)) \cdot \partial_t v$, $\forall v \in V \cap L_\infty(Q_T)$ *with* $\partial_t v \in L_\infty(Q_T)$, $v(T) = 0$;

(iii)

$$\int_I \langle \partial_t b(x,u), v\rangle + \int_I (a(t,u,\nabla\beta(u)), \nabla v) + \int_I (g(t,u), v)_{\Gamma_2} = \int_I (f(t,u), v),$$

$\forall v \in L_p(I,V)$.

We consider a variational solution $\Theta_i \in V \cap L_2(\Omega)$of (1.4) in the sense

$$(2.5)\quad (\lambda_i(\Theta_i - \beta(u_{i-1})), v) + \tau(a(t_i, u_{i-1}, \nabla\Theta_i), \nabla v) + \tau(g_i, v)_{\Gamma_2} = \tau(f_i, v), \quad \forall v \in V.$$

If $\lambda_i \in L_\infty(\Omega)$, $u_{i-1} \in L_2$, $\Theta_{i-1} \in V$ then the existence of $\Theta_i \in V$ satisfying (2.5) is guaranteed by the theory of monotone operators - see e.g., [17].

The assumptions H_1)-H_6) then guarantee the existence of $\{\Theta_i\}_{i=1}^n$ for any $\lambda_i \in L_\infty(\Omega)$, $\lambda_i \geq c\tau^d$.

In what follows (in this section) we shall discuss the determination of the couple λ_i, Θ_i satisfying (2.5), (1.5), (1.6). Consider the iteration scheme for $k = 1, \ldots,$

$$(2.6)\quad (\lambda_{i,k-1}(\Theta_{i,k} - \beta(u_{i-1})), v) + \tau(a(t_i, u_{i-1}, \nabla\Theta_{i,k}), \nabla v) + \tau(g_i, v) = \tau(f_i, v) \quad \forall v \in V$$

$$(2.7)\quad \lambda_{i,k} := \frac{b_n(x, u_{i-1} + \mu_{i,k}(\Theta_{i,k} - \beta(u_{i-1}))) - b_n(x, u_{i-1})}{\Theta_{i,k} - \beta(u_{i-1})}$$

with

$$(2.8)\quad \bar{\mu}_{i,k} := \min\left\{K, \frac{\beta^{-1}(\beta(\mu_{i-1}) + \alpha(\Theta_{i,k} - \beta(u_{i-1}))) - u_{i-1}}{\Theta_{i,k} - \beta(u_{i-1})}\right\}$$

$$\mu_{i,k} := \min\{\bar{\mu}_{i,k}, \mu_{i,k-1}\}$$

where

$$\lambda_{i,0} := \partial_s b_n(x, u_{i-1}) \cdot \min\left\{K, \frac{\alpha}{\beta'(u_{i-1})}\right\},$$

$$\mu_{i,0} := \min\left\{K, \frac{\alpha}{\beta'(u_{i-1})}\right\}$$

and in the points x where $\Theta_{i,k} = \beta(u_{i-1})$ we put $\lambda_{i,k} = \lambda_{i,0}$, $\mu_{i,k} = \mu_{i,0}$.

We can expect the convergence $\lambda_{i,k} \to \lambda_i$, $\Theta_{i,k} \to \Theta_i$ in $C(\bar{\Omega})$ norm for $k \to \infty$ (where λ_i, Θ_i satisfy (1.4)-(1.6) under the regularity properties of data, $\{\Theta_{i,k}\}$ provided τ, d are sufficiently small. Only in some special cases we can guarantee the convergence of (2.6)-(2.8), see [14] (Theorem 4.9).

We reformulate this result in our case.

Let us consider a linear elliptic operator in the form

$$(2.9)\quad -\nabla a(t, x, u, \nabla v) \equiv -\sum_{i,j=1}^{N} a_{ij}(x)\, \partial^2_{x_i x_j} v + \sum_{i=1}^{N} b_i(x)\, \partial_{x_i} v + q(x)\, v$$

with a positive definite matrix $\{a_{ij}(x), i, j = 1, \dots, N\}$ and $a_{ij} \in C^1(\bar{\Omega})$, b_j, $q \in C(\bar{\Omega})$. Moreover we assume $\Gamma_2 = 0$ and

$$(2.10)\qquad \beta(u_{i-1}),\quad f_i \in C^{0,\gamma}(\bar{\Omega}),\quad \gamma \in (0, 1)$$

$$(2.11)\qquad |\partial_s^2 b(x, s)| \leqslant K.$$

PROPOSITION 2.12 : *If* (2.9)-(2.11) *are satisfied and* τ, d *are sufficiently small, then* $\mu_{i,k} \to \mu_i$, $\lambda_{i,k} \to \lambda_i$, $\Theta_{i,k} \to \Theta_i$ *in* $C(\bar{\Omega})$ *where* λ_i, Θ_i, μ_i *satisfy* (2.5), (1.5), (1.6). *If* k_0 *is sufficiently large, then* $\Theta_i := \Theta_{i,k_0}$, $\mu_i := \mu_{i,k_0-1}$, $\lambda_i = \lambda_{i,k_0-1}$ *satisfy* (2.5), (1.5), (1.6).

From the construction of $\{\mu_{i,k}\}$ we have that $\mu_{i,k} \to \mu_i$ in $L_s(\Omega)$ $\forall s > 1$. If d and τ are sufficiently small then $\Theta_{i,k} \to \Theta_i$ in $L_2(\Omega)$ (see [14], Theorem 4.10). Then regularity results ([19], Theorem 13.1, 14.1) guarantee $C^{0,\alpha}(\bar{\Omega})$ boundedness of $\{\Theta_{i,k}\}$ and hence we deduce $C(\bar{\Omega})$ convergence of $\mu_{i,k} \to \mu_i$. The rest of the proof goes along the same lines as that one of Theorem 4.10 in [14].

Remark 2.13 : To determine λ_i, μ_i, Θ_i in (1.4)-(1.6) we have to solve the nonlinear (regular) elliptic problem

(2.14)
$$b_n(x, u_{i-1} + \mu_i(\Theta_i - \beta(u_{i-1}))) - b_n(x, u_{i-1}) - \tau\, \nabla a(t_i, u_{i-1}, \nabla \Theta_i) = \tau f_i$$

$$\Theta_i = 0 \quad \text{on} \quad \Gamma_1, a(t_i, u_{i-1}, \nabla \Theta_i) \cdot \nu = g_i \quad \text{on} \quad \Gamma_2$$

with the algebraic condition (1.6).

The condition (1.6) is evidently satisfied if we put $\mu_i := \alpha / L_\beta$. However, from the numerical point of view the equality on R.H.S. in (1.6) is desirable. Thus we can solve iteratively (2.14), (1.6) (see [9]) choosing the monotone (and hence convergent) modification (2.8). If Θ_{i,k_0} is a "good" approximation of Θ_i in (2.14) corresponding to μ_{i,k_0-1} then we can take $\mu_i := \mu_{i,k_0-1}$, $\Theta_i := \Theta_{i,k_0}$,

$$\lambda_i := \frac{b_n(x, u_{i-1} + \mu_{i,k_0-1}(\Theta_{i,k_0} - \beta(u_{i-1}))) - b_n(x, u_{i-1})}{\Theta_{i,k_0} - \beta(u_{i-1})}$$

and in (2.5) we shall have $f_i + \tilde{f}_i$ in the place of f_i where $\|\tilde{f}_i\|_* \le c(\tau)$ with $c(\tau) \to 0$ for $\tau \to 0$. In such a case we obtain the same convergence results as in the case (2.5), (1.5), (1.6).

Remark 2.14 : In many practical implementations of the method (1.4)-(1.7) it is sufficient to replace λ_i by λ_{i-1} in (2.5) provided τ is sufficiently small. We can expect the proof of convergence also in some special cases.

In this direction see [24], [25] where $b(s) \equiv s$, $a(t, x, u, \nabla u) \equiv \nabla u$ and $\beta \in C^{1,1}(R)$.

3. CONVERGENCE OF THE METHOD

By means of Θ_i, u_i from (2.5), (1.5)-(1.7) we construct approximate solutions (Rothe's functions)

$$\Theta^n(t) := \Theta_{i-1} + (t - t_{i-1})\tau^{-1}(\Theta_i - \Theta_{i-1}), \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n$$

$$\bar{\Theta}^n := \Theta_i \quad for \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n \tag{3.1}$$

$$\bar{\Theta}^n(0) \equiv \Theta_0 \equiv \beta(u_0).$$

Analogously we define u^n and $\bar{u}^n$ by means of u_i ($i = 1, ..., n$). By $\{\bar{n}\}$ we denote a subsequence of $\{n\}$. Our main result is

THEOREM 3.2 : *Let the assumptions H_1)-H_6) be fulfilled.*

Then $b_{\bar{n}}(x, \bar{u}^{\bar{n}}) \to b(x, u)$, $b(x, \bar{u}^{\bar{n}}) \to b(x)$ *in* $L_s(Q_T)$ $\bar{\Theta}^{\bar{n}} \to \Theta \equiv \beta(u)$ *in* $L_s(Q_T)$ $\forall s < 2$, *where u is a variational solution of* (1.1)-(1.3) *and $\{\bar{n}\}$ is a suitable subsequence of $\{n\}$. If the variational solution u of* (1.1)-(1.3) *is unique then the original sequences $\{b_n(x, \bar{u}^n)\}$, $\{\bar{\Theta}^n\}$ are convergent.*

First we prove some *a priori* estimates, "integration by parts formula"

$$(3.3)\quad \int_0^t \langle \partial_t b(x, u, (t)), \beta(u(t)) \rangle = \int_\Omega B^*(x, \beta(u)) - \int_\Omega B^*(x, \beta(u_0))$$

($B^*(x, \beta(s))$ is from (2.1) - see [2]) and compactness of $\{b_n(x, \bar{u}^n)\}$. Then we take the limit $n \to \infty$ in (1.4).

LEMMA 3.4 : *The following* a priori *estimates hold :*

$$\max_{1 \leqslant i \leqslant n} \int_\Omega B_n^*(x, \beta(u_i)) \leqslant C, \quad \sum_{i=1}^N \| \Theta_i \|^p \tau \leqslant C,$$

$$\sum_{i=1}^N |b_n(x, u_i) - b_n(x, u_{i-1})|_2^2 \leqslant C,$$

$$\sum_{i=1}^N |u_i - u_{i-1}|_2^2 \leqslant C\tau^{-d}$$

uniformly for n.

Proof : We rewrite (2.5) by means of (1.5) in the form

$$(3.5)\qquad (b_n(x, u_i) - b_n(x, u_{i-1}), \varphi) + \tau(\omega_i(\Theta_i - \beta(u_{i-1})), \varphi) +$$

$$+ \tau(a(t_i, u_{i-1}, \nabla\Theta_i), \nabla\varphi) + \tau(g(t_i, \beta^{-1}(\Theta_{i-1})), \varphi)_{\Gamma_2} =$$

$$\tau(f(t_i, u_{i-1}), \varphi), \quad \forall \varphi \in V$$

where

$$\tau\omega_i = \frac{b_n(x, u_{i-1} + \mu_i(\Theta_i - \beta(u_{i-1}))) - b_n(x, u_{i-1})}{\Theta_i - \beta(u_{i-1})} - \lambda_i$$

with $|\omega_i| \leqslant 1$.

We put $\varphi = \Theta_i$ into (3.5) and sum up for $i = 1, \dots, j$. We denote the corresponding equality by $J_1 + J_2 + J_3 + J_4 = J_5$. We rearrange the first term in the form

$$J_1 = \sum_{i=1}^{j} (b_n(x, u_i) - b_n(x, u_{i-1}), \Theta_i - \beta(u_{i-1}))$$

$$(3.6) \qquad + \sum_{i=1}^{j} (b_n(x, u_i) - b_n(x, u_{i-1}), \beta(u_i))$$

$$- \sum_{i=1}^{j} (b_n(x, u_i) - b_n(x, u_{i-1}), \beta(u_i) - \beta(u_{i-1})) \equiv J_1^1 + J_1^2 - J_1^3 .$$

We can check easily that the signs of $u_i - u_{i-1}$, $\Theta_i - \beta(u_{i-1})$, $b_n(x, u_i) - b_n(x, u_{i-1})$ and $\beta(u_i) - \beta(u_{i-1})$ are the same. Then, from (2.8) we conclude

$$|\beta(u_i) - \beta(u_{i-1})| \leqslant \alpha |\Theta_i - \beta(u_{i-1})| .$$

Inserting it in J_1^3 we have

$$(3.7) \qquad J_1^3 \leqslant \alpha J_1^1 , \quad \alpha \in (0, 1) .$$

In J_1^2 we use Abel's summation. Denoting $z_i := \beta(u_i)$ we obtain

$$J_1^2 = (b_n(x, \beta^{-1}(z_j)), z_j) - (b_n(x, \beta^{-1}(z_0)), z_0)$$

$$(3.8) \qquad - \sum_{i=1}^{j} (b_n(x, \beta^{-1}(z_{i-1})), z_i - z_{i-1}) \geqslant (b_n(x, u_j), \beta(u_j)) -$$

$$- (b_n(x, u_0), \beta(u_0)) - \sum_{i=1}^{j} \int_\Omega \int_{\beta(u_{j-1})}^{\beta(u_j)} b_n(x, \beta^{-1}(z))\, dz =$$

$$= \int_\Omega B^*(x, \beta(u_j)) - \int_\Omega B^*(x, \beta(u_0))$$

where the monotonicity of b, β has been used. We estimate the penalty term J_2 by

$$|J_2| \leqslant \varepsilon \sum_{i=1}^{j} |\Theta_i|_2^2 \tau + C_\varepsilon \tau \sum_{i=1}^{j} |u_i - u_{i-1}|_2^2 \tag{3.9}$$

and take into account $p \geqslant 2$. The elliptic term gives us

$$J_3 \geqslant C \sum_{i=1}^{j} |\nabla \Theta_i|_p^p \tau - C . \tag{3.10}$$

We estimate the boundary term J_4 by

$$|J_4| \leqslant \varepsilon \sum_{i=1}^{j} \int_{\Gamma_2} \tau |\Theta_i|^p + C_\varepsilon \leqslant \varepsilon C \sum_{i=1}^{j} \| \Theta_i \|^p \tau + C_\varepsilon \tag{3.11}$$

where Young's inequality $(xy \leqslant x^p / p + y^q / q)$, H_5) and the imbedding $V \hookrightarrow L_p(\partial\Omega)$ has been used. Using H_4) we estimate

$$|J_5| \leqslant C_\varepsilon \sum_{i=1}^{j} \tau \int_{\Omega} B^*(x, \beta(u_i)) + \varepsilon \sum_{i=1}^{j} \tau |\Theta_i|_p^p + C_\varepsilon . \tag{3.12}$$

Moreover (1.7) and Lipschitz continuity of b_n imply

$$\begin{aligned} J_1^1 - J_1^3 &\geqslant \frac{1-\alpha}{K} \sum_{i=1}^{j} (b_n(x, u_i) - b_n(x, u_{i-1}), u_i - u_{i-1}) \\ &\geqslant \frac{1-\alpha}{K} \sum_{i=1}^{j} |b_n(x, u_i) - b_n(x, u_{i-1})|_2^2 . \end{aligned} \tag{3.13}$$

The first inequality in (3.13) implies also

$$J_1^1 - J_1^3 \geqslant \frac{1-\alpha}{K} \sum_{i=1}^{j} |u_i - u_{i-1}|_2^2 \tau^d . \tag{3.14}$$

Collecting the estimates (3.7)-(3.14) and using the Growall's argument, we obtain the estimates of Lemma 3.4.

Denote by

$$(3.15)\quad \hat{b}_n(x,\bar{u}^n) := b_n(x,u_{i-1}) + \frac{t-t_{i-1}}{\tau}\,(b_n(x,u_i) - b_n(x,u_{i-1}))$$

$$\text{for}\quad t\in[t_{i-1},t_i]\,,\quad i=1,\dots,n\,.$$

LEMMA 3.16 : *The estimates*

$$\|\partial_t\hat{b}_n(x,\bar{u}^n)\|_{L_q(I,V^*)} \leqslant C\,,\quad (q^{-1}+p^{-1}=1)$$

$$\int_0^{T-z}\Big(b_n(x,\bar{u}^n(t+z)) - b_n(x,\bar{u}^n(t)),\beta(\bar{u}^n(t+z)) - \beta(\bar{u}^n(t))\Big)$$

$$\leqslant C(z+\tau^{(1-d)/2})$$

hold uniformly for $n\in\mathbb{N}$, $0<z\leqslant z_0$.

Proof : We estimate the first term in (3.5) by means of the rest terms using the *a priori* estimates of Lemma 3.4. By duality we obtain

$$(3.17)\quad \left\|\frac{b_n(x,u_i) - b_n(x,u_{i-1})}{\tau}\right\|_{V^*} \leqslant$$

$$C\left(1+|u_i-u_{i-1}|_2 + \left(\int_\Omega B^*(x,\beta(u_{i-1}))\right)^{1/q} + \|\Theta_i\|^{p/q}\right)$$

which implies the first estimate in Lemma 3.16. To obtain the second estimate, we sum up (3.5) for $i=j+1,\dots,j+k$ and then we put $\varphi=\tau(\Theta_{j+k}-\Theta_j)$ and sum up for $j=1,\dots,n-k$. Using the *a priori* estimates of Lemma 3.4 we successively obtain

$$\sum_{j=1}^{n-k}(b_n(x,u_{j+k}) - b_n(x,u_j),\Theta_{j+k}-\Theta_j) \leqslant C\,k\tau$$

which can be rewritten into the form

$$(3.18)\int_0^{T-z}(b_n(x,\bar{u}^n(t+z)) - b_n(x,\bar{u}^n(t)),\bar{\Theta}^n(t+z) - \bar{\Theta}^n(t)) \leqslant C(z+\tau)$$

$$(k\tau<z\leqslant(k+1)\,\tau)\,.$$

Now we use (1.7) and replace Θ^n by $\beta(\bar{u}^n)$ in (3.18) where

$$\bar{\Theta}^n(t) = \frac{1}{\bar{\mu}^n(t)}(\bar{u}^n(t) - \bar{u}^n_\tau(t)) + \beta(\bar{u}^n_\tau(t))$$

$$\bar{u}^n_\tau \equiv \bar{u}^n(t-\tau)\,.$$

Using (2.2), Lemma 3.4, (3.18) and Lipschitz continuity of β we estimate

$$\int_0^{T-z}(b_n(x, \bar{u}^n(t+z)) - b_n(x, \bar{u}^n(t)), \beta(\bar{u}^n(t+z)) - \beta(\bar{u}^n(t))) \leqslant$$

$$C\left[\int_0^T\left(\left\{\int_\Omega B^*(x, \beta(\bar{u}^n(t+z)))\right\}^{1/2} + \left\{\int_\Omega B^*(x, \beta(\bar{u}^n(t)))\right\}^{1/2}\right)\right] \times$$

$$\left[\left\{\int_0^T|\bar{u}^n - \bar{u}^n_\tau(t)|_2^2\,dt\right\}^{1/2} + \left\{\int_0^{T-z}|\bar{u}^n(t+z) - \bar{u}^n(t+z)|_2^2\right\}^{1/2}\right] +$$

$$C(z+\tau) \leqslant C(z + \tau^{(1-d)/2})$$

which implies the second estimate in Lemma 3.16.

LEMMA 3.10: *There exist* $u : Q_T \to R$ *with* $\beta(u) \in L_p(I, V)$, $b(x, u) \in L_\infty(I, L_2(\Omega))$, $\partial_t b(x, u) \in L_q(I, V^*)$ *and a subsequence* $\{\bar{n}\}$ *of* $\{n\}$ *such that*

$$\bar{u}^{\bar{n}} \to u \quad a.e. \quad in \quad Q_T$$

$$b_{\bar{n}}(x, \bar{u}^{\bar{n}} \to b(x, u)) \quad in \quad L_s(Q_T)\,, \quad s < 2$$

$$\bar{\Theta}^{\bar{n}} \to \Theta \equiv \beta(u) \quad in \quad L_s(Q_T)\,, \quad s < 2$$

$$\bar{\Theta}^{\bar{n}} \rightharpoonup \beta(u) \quad in \quad L_p(I, V)$$

$$\partial_t \hat{b}_{\bar{n}}(x, \bar{u}^{\bar{n}}) \rightharpoonup \partial_t b(x, u) \quad in \quad L_q(I, V^*)\,.$$

Proof: Let us construct $\rho(x,s) := \min\{\partial_s b(x,s), \beta'(s)\}$ and $W(x,s) := \int_0^s \rho(x,z)\,dz$. The function $W(x,s)$ is strictly monotone in s and satisfies

$$|W(x,s_1) - W(x,s_2)| \leq \min\{|b(x,s_1) - b(x,s_2)|, |\beta(s_1) - \beta(s_2)|\}.$$

Then as a consequence of Lemma 3.16 we have

$$(3.20) \qquad \int_0^{T-z}\int_\Omega (W(x,\bar{u}^n(t+z)) - W(x,\bar{u}^n(t)))^2 \leq$$

$$\int_0^{T-z}\Big(b(x,\bar{u}^n(t+z)) - b(x,\bar{u}^n(t)), \beta(\bar{u}^n(t+z)) - \beta(\bar{u}^n(t))\Big) \leq$$

$$C(z + \tau^{(d-1)/2}), \quad \forall n, \quad 0 < z \leq z_0$$

because in the second *a priori* estimate in Lemma 3.16 we can replace b_n by

$$b(\operatorname{sign}(\bar{u}^n(t) - \bar{u}^n_\tau) = \operatorname{sgn}(b(x,\bar{u}^n) - b(x,\bar{u}^n_\tau)) = \operatorname{sgn}(\beta(\bar{u}^n) - \beta(\bar{u}^n_\tau))).$$

Now we prove

(3.21)
$$\int_I\int_\Omega |W(x+y,\bar{u}^n(t,x+y)) - W(x,\bar{u}^n(t,x))|\,dx\,dt \leq \bar{\omega}(|y|) + c\,.\,\tau^{1-d}$$

where $\bar{\omega} : R_+ \to R_+$ is continuous and $\bar{\omega}(0') = 0$. First, $\int_I \|\bar{\Theta}^n(t)\|^2 \leq C$ implies (see, e.g. [21])

$$(3.22) \qquad \int_I\int_\Omega |\bar{\Theta}^n(t,x+y) - \bar{\Theta}^n(t,x)|^p \leq \omega_1(|y|)$$

where ω_1 has the same properties as $\bar{\omega}$. Then (1.7) and the estimates of Lemma 3.2 give us

$$(3.23) \qquad \int_I |\bar{\Theta}^n - \beta(\bar{u}^n)|_2^2 \leq C\tau^{1-d}.$$

As a consequence of (3.22), (3.23) we have

$$J \equiv \int_I |\beta(\bar{u}^n(t, x+y)) - \beta(\bar{u}^n(t, x))|_2^2 \leqslant C(\omega_1(|y|) + \tau^{1-d})$$

and hence

$$(3.24) \quad \int_I \int_\Omega |W(x, \bar{u}^n(t, x+y)) - W(x, \bar{u}^n(t, x))| \, dx \, dt \leqslant CJ^{1/2} \leqslant$$

$$\leqslant C(\omega_1(|y|) + \tau^{1-d})^{1/2}.$$

The continuity of $b(x, s)$ in s (see H_1)) and (2.2) imply

$$\int_I \int_\Omega |W(x+y, \bar{u}^n(t, x)) - W(x, \bar{u}^n(t, x))| \leqslant C\omega(|y|)$$

and similarly when $\bar{u}^n(t, x)$ is replaced by $\bar{u}^n(t, x+y)$. Hence and from (3.22)-(3.24) it follows (3.21). The estimates (3.20), (3.21) then imply (Kolmogorov's compactness criterium) compactness of $\{W(x, \bar{u}^n)\}_{n=1}^\infty$ in $L_1(Q_T)$.

Since $W(x, s)$ is strictly monotone, we have $\bar{u}^n \to u$ a.e. in Q_T. Thus $\beta(\bar{u}^n) \to \beta(u)$ in $L_s(Q_T)$, $s < 2$ ($\{\beta(\bar{u}^n)\}$ is bounded in $L_2(Q_T)$ - see (3.23) and Lemma 3.4). Then $\bar{\Theta}^n \to \Theta \equiv \beta(u)$ in $L_s(Q_T)$ because of (3.23). Since $b_n(x, s) \to b(x, s)$ locally uniformly in s and $\bar{u}^n \to u$ a.e. in Q_T, we have $b_{\bar{n}}(x, \bar{u}^n) \to b(x, u)$ a.e. in Q_T. Then, $b_{\bar{n}}(x, \bar{u}^n) \to b(x, u)$ in $L_s(Q_T)$ because of Lemma 3.4 and (2.2). Lemma 3.16 implies $\partial_t \hat{b}_{\bar{n}}(x, \bar{u}^n) \rightharpoonup \chi$ (weakly) in $L_q(I, V^*)$. Then (3.15) and Lemma 3.4 imply

$$\int_I |\hat{b}_{\bar{n}}(x, \bar{u}^n) - b_{\bar{n}}(x, \bar{u}^n)|_2^2 \leqslant 2\tau \sum_{i=1}^n |b_n(x, u_i) - b_n(x, u_{i-1})|_2^2 \leqslant c\tau^{1-d}$$

and hence $\chi \equiv \partial_t b(x, u)$ (see, e.g. [9]). The rest of the proof is a consequence of Lemma 3.4 and (2.2).

To prove Theorem 3.2, we substantially make use of the integration by parts formula (see also [2]).

LEMMA 3.25 : *Let u be from Lemma 3.19. Then*

$$\int_0^t \langle \partial_t b(x,u), \beta(u) \rangle =$$

$$= \int_\Omega B^*(x, \beta(u(t))) - \int_\Omega B^*(x, \beta(u_0)) \quad \text{for} \quad \text{a.e. } t \in I .$$

Proof: In the proof we follow [2] (Lemma 1.3). For a.e. $(x,t) \in \Omega \times (0, T-\tau)$ we have $(u(t) \equiv u_0$ for $t \in (-\tau, 0))$

$$B^*(x, \beta(u(t))) - B^*(x, \beta(u(t-\tau))) \leqslant (b(x,u(t)) - b(x,u(t-\tau)))\, \beta(u(t)) \tag{3.26}$$

and

$$B^*(x, \beta(u(t+\tau))) - B^*(x, \beta(u(t))) \geqslant (b(x,u(t+\tau)) - b(x,u(t)))\, \beta(u(t)) \tag{3.27}$$

because

$$J \equiv \int_{\beta(u(t))}^{\beta(u(t+\tau))} b(x, \beta^{-1}(z))\, dz \leqslant b(x, u(t+\tau))\, (\beta(u(t+\tau)) - \beta(u(t)))$$

and

$$J \geqslant b(x, u(t))\, (\beta(u(t+\tau)) - \beta(u(t))) .$$

Now we multiply (3.26) by τ^{-1} and integrate it over $(0,t) \times \Omega$ $(\beta(u) \in L_p(Q_T),\ b(x,u) \in L_2(Q_T),\ p \geqslant 2)$ and obtain

$$\frac{1}{\tau} \int_{t-\tau}^t \int_\Omega B^*(x, \beta(u(s)))ds - \int_\Omega B^*(x, \beta(u_0)) \leqslant \tag{3.28}$$

$$\leqslant \int_0^t \int_\Omega \frac{b(x,u(s)) - b(x,u(s-\tau))}{\tau} \beta(u(s))\, ds\, dx = \int_0^t \langle \delta_t^\tau b(x,u), \beta(u) \rangle$$

where $\delta_t^\tau b(x,u) \equiv \big(b(x,u(s)) - b(x,u(s-\tau))\big)\tau^{-1}$ and $u(t) = 0$ for $t \in (-\tau, 0)$.

Since $\partial_t b(x, u) \in L_q(I, V^*)$ we have $\delta_t^\tau b(x, u) \to \partial_t b(x, u)$ in $L_q(I, V^*)$ for $\tau \to 0$ (see, e.g. [13], [9]). Since $\beta(u) \in L_p(I, V)$ we can take $\tau \to 0$ in (3.28) and obtain

$$\int_0^t \langle \partial_t b(x, u), \beta(u) \rangle \geq \int_\Omega B^*(x, \beta(u(t))) - \int_\Omega B^*(x, \beta(u_0))$$

for a.e. $t \in I$. To prove the inverse inequality we integrate (3.27) over $(0, t) \times \Omega$ and proceed analogously as above. To estimate

$$\int_\Omega B^*(x, \beta(u_0)) \leq \underline{\lim}\, \tau^{-1} \int_0^\tau \int_\Omega B^*(x, \beta(u(s)))\, dx\, ds$$

we follow the argumentation in [2] (Lemma 1.3). Thus the proof is complete.

Proof of Theorem 3.2 : We follow the idea of [13]. We integrate (3.5) over $(0, t)$ and obtain

$$\int_0^t \langle \partial_t \hat{b}(x, \bar{u}^n), \varphi \rangle + \tau \int_0^t \left(\frac{\overline{\omega}^n}{\overline{\mu}^n} (\bar{u}^n - \bar{u}_\tau^n), \varphi \right) +$$

(3.29) $$\int_0^t (a_n(s, \bar{u}_\tau^n, \nabla \overline{\Theta}^n), \nabla \varphi) + \int_0^t (g_n(t, \beta^{-1}(\overline{\Theta}_\tau^n)), \varphi)_\Gamma =$$

$$\int_0^t (f_n(t, \bar{u}_\tau^n), \varphi), \quad \forall \varphi \in L_p(I, V)$$

where $a_n(t, x, \eta, \xi) = a(t_i, x, \eta, \xi)$ for $t \in (t_{i-1}, t_i)$, $i = 1, \dots, n$ and similarly we define f_n, g_n. We use $\overline{\omega}^n(t) := \omega_i$ for $t \in [t_{i-1}, t_i]$, $i = 1, \dots, n$. Due to the *a priori* estimates of Lemma 3.4 and H_3) we have

$$\| a_n(t, \bar{u}_\tau^n, \nabla \overline{\Theta}^n) \|_{[L_q(Q_T)]^N} \leq C$$

which implies $a_n(t, \bar{u}_\tau^n, \nabla \overline{\Theta}^n) \rightharpoonup \chi$ in $[L_q(Q_T)]^N$ (through a subsequence of n denoted again by n). Then taking the limit as $n \to \infty$ in (3.29) we obtain (see Lemma 3.19)

(3.30) $$\int_0^t \langle \partial_t b(x, u), \varphi \rangle + \int_0^t (\chi, \nabla \varphi) + \int_0^t (g(t, u), \varphi)_{\Gamma_2} =$$

$$\int_0^t (f(t, u), \varphi) \quad \forall \varphi \in L_p(I, V)$$

because $\overline{\Theta}^n \to \beta(u)$ in $L_s(Q_T)$ and $\int_0^T \|\overline{\Theta}^n\|^p \leq C$ imply $\overline{\Theta}^n \to \beta$ in $L_p(I, L_p(\Gamma_2))$ (see, [16]). Inserting $\varphi = \beta(u)$ in (3.20) and taking into account Lemma 3.25 we obtain

$$(3.31)\quad \int_0^t (\chi, \nabla\beta(u))\,dt = -\int_0^t (g(t,u), \beta(u))_{\Gamma_2} + \int_0^t (f(t,u), \beta(u)) - \int_\Omega B^*(x, \beta(u)) + \int_\Omega B^*(x, \beta(u_0))$$

for a.e. $t \in I$. On the other hand, from (3.6), $B_n^*(x,s) \geq B^*(x,s)$ and from Fatou's Lemma we obtain

$$(3.32)\quad \underline{\lim} \int_0^t \langle \partial_t \hat{b}(x, \overline{u}^n), \overline{\Theta}^n \rangle \geq \underline{\lim} \int_\Omega B_n^*(x, \beta(\overline{u}^n(t))) - \int_\Omega B^*(x, \beta(u_0)) \geq \int_\Omega B^*(x, \beta(u(t))) - \int_\Omega B^*(x, \beta(u_0))$$

for a.e. $t \in I$. We put $\varphi = \overline{\Theta}^n$ into (3.29) and then we make use of (3.32). We obtain

$$(3.33)\quad \overline{\lim} \int_0^t (a_n(s, \overline{u}_\tau^n, \nabla\overline{\Theta}^n), \nabla\overline{\Theta}^n) \leq -\underline{\lim} \int_0^t \langle \partial_t \hat{b}(x, \overline{u}^n), \nabla\overline{\Theta}^n \rangle - \int_0^t (g(t,u), \beta(u))_{\Gamma_2} + \int_0^t (f(t,u), \beta(u)) \leq \int_0^t (\chi, \nabla\beta(u))$$

because of (3.31), H_4), H_5) and Lemma 3.19.

As a consequence of the monotonicity argument we have

$$\int_0^t (a_n(s, \overline{u}_\tau^n, \nabla\overline{\Theta}^n) - a_n(s, \overline{u}_\tau^n, w), \nabla\overline{\Theta}^n - w) \geq 0$$

for all $w \in [L_p(Q_T)]^N$.

Taking the limit as $n \to \infty$ using (3.33) and $a_n(t, x, \overline{u}_\tau^n, w) \to a(t, x, u, w)$ in $[L_q(Q_T)]^N$ (see H_3)), we obtain

$$\int_0^t (\chi - a(s, u, w), \nabla\beta(u) - w) \geq 0$$

from which it follows $\chi \equiv a(t, x, u, \nabla\beta(u))$. Thus i) and iii) of Definition 2.4 are satisfied. To prove ii) we consider v as in Definition 2.4, ii). We obtain

$$\int_0^t \langle \delta_t^\tau b(x, u), v \rangle = \int_0^t \int_\Omega \delta_t^\tau b(x, u) \cdot v =$$

$$-\int_0^{t-\tau} \int_\Omega (b(x, u) - b(x, u_0))\, \delta_t^{-\tau} v + \frac{1}{\tau} \int_{t-\tau}^t \int_\Omega (b(x, u) - b(x, u_0)) v$$

and take the limits as $\tau \to 0$. Since $\delta_t^\tau b(x, u) \to \partial_t b(x, u)$ in $L_q(I, V^*)$ and $\delta_t^{-\tau} v \to \partial_t v$ in $L_p(Q_T)$ $(v \in L_p(I, V),\ \partial_t v \in L_p(I, L_2))$ we obtain ii) from Definition 2.4 and the proof is complete.

We obtain a stronger convergence result under the strong monotonicity H_7).

THEOREM 3.34: *Let the assumptions H_1)-H_7) be satisfied. Then $\bar{\Theta}^n \to \beta(u)$ in $L_p(I, V)$.*

Proof: Let us consider $\bar{u}^n$, $\bar{\Theta}^n$, u as in Theorem 3.2. We put $\varphi = \bar{\Theta}^n - \beta(u)$ into (3.29) and take into account that

$$\underline{\lim} \int_0^t \langle \partial_t \hat{b}(x, \bar{u}^n), \bar{\Theta}^n - \beta(u) \rangle \geqslant 0$$

which is a consequence of $\partial_t \hat{b}(x, \bar{u}^n) \rightharpoonup \partial_t b(x, u)$ in $L_q(I, V^*)$, (3.32) and Lemma 3.25. We rearrange the elliptic term in (3.29) in the form

$$\int_0^t (a_n(s, \bar{u}_\tau^n, \nabla\bar{\Theta}), \nabla(\bar{\Theta}^n - \beta(u))) = \tag{3.35}$$

$$\int_0^t (a_n(s, \bar{u}_\tau^n, \nabla\bar{\Theta}^n) - a_n(s, \bar{u}^n, \nabla\beta(u)), \nabla(\bar{\Theta}^n - \beta(u))) +$$

$$\int_0^t (a_n(s, \bar{u}_\tau^n, \nabla\beta(u)), \nabla(\bar{\Theta}^n - \beta(u))) \geqslant$$

$$C \int_0^t \| \bar{\Theta}^n - \beta(u) \|^p + o(1) \quad (o(1) \to 0 \quad \text{for} \quad \tau \to 0)$$

since

$$\bar{\Theta}^n \rightharpoonup \beta(u) \quad \text{in} \quad L_p(I, V)$$

$$a_n(t, \bar{u}_\tau^n, \nabla\beta(u)) \to a(t, u, \nabla\beta(u)) \quad \text{in} \quad [L_q(Q_T)]^N$$

and $\left(\int_\Omega |\nabla v|_p^p\right)^{1/p}$ is an equivalent norm with $\| \cdot \|$.

Due to convergence properties of $\overline{\Theta}^n$, we find out easily

$$\int_0^t (f_n(s, \overline{u}_\tau^n), \overline{\Theta}^n - \beta(u)) \to 0$$

$$\int_0^t (g_n(s, \beta^{-1}(\overline{\Theta_\tau}^n)), \overline{\Theta}^n - \beta(u)) \to 0$$

for $n \to \infty$ because of H_4), H_5) and Lemma 3.19. Thus the proof is complete.

Remark 3.36 : The obtained convergence results in Theorems 3.2 and 3.34 can be extended to the nonhomogeneous Dirichlet boundary condition on Γ_1 and the continuity of a, f, g in their variables can be replaced by Caratheodory conditions. Moreover, the convergence results hold for systems of the form

$$(1.3') \quad \partial_t b^j(x, u^j) - \nabla a^j(t, x, u, \nabla\beta(u)) = f^j(t, x, u)\,, \quad u^j = 0 \quad \text{on} \quad \Gamma_1^j \times I\,,$$

$$a^j(t, x, u, \nabla\beta(u)) \cdot \nu = g^j(t, x, u) \quad on \quad \Gamma_2^j \times I\,, \quad \text{for} \quad j = 1, \dots, m$$

where $u \equiv (u^1, \dots, u^m)$, $\beta(u) \equiv (\beta^1(u^1), \dots, \beta^m(u^m))$ and b^j, β^j satisfy H_1), H_2).

The assumptions H_3)-H_6) have to be rewritten also for the system. We also have $B_j^*(x, \beta_j(s))$ for $j = 1, \dots, m$ constructed as in (2.1). In this case the relaxation functions λ_i, μ_i are also vector functions $\lambda_i \equiv (\lambda_i^1, \dots, \lambda_i^m)$, $\mu_i \equiv (\mu_i^1, \dots, \mu_i^m)$ and each component λ_i^j, μ_i^j has to satisfy (1.5), (1.6) with b^j, β^j, $(\beta^{-1})^j$ in the place of b, β, β^{-1}.

4. FULL DISCRETIZATION SCHEME

The obtained convergence results can be extended to the full discretization scheme where (1.5) (i.e. elliptic equations) is projected on a finite dimensional subspace $V_\lambda \subset V$.

We assume $V_\lambda \to V$ for $\lambda \to 0$ in canonical sense, i.e.

$$(4.1) \quad \forall v \in V\,, \quad \exists v_\lambda \in V_\lambda \text{ such that } v_\lambda \to v \quad \text{for} \quad \lambda \to 0 \text{ in the norm of } V\,.$$

Instead of $\Theta_i \in V$ we look for $\Theta_i^\lambda \in V_\lambda$ such that

$$(4.2)\quad \begin{gathered}(\lambda_i(\Theta_i^\lambda - \beta(u_{i-1})), \varphi) + \tau(a(t_i, u_{i-1}, \nabla\Theta_i^\lambda), \nabla\varphi) + \\ \tau(g(t_i, \beta^{-1}(\Theta_{i-1}^\lambda)), \varphi)_{\Gamma_2} = \tau(f(t_i, u_{i-1}), \varphi) \\ \forall v \in V_\lambda, \quad i = 1, \dots, n\end{gathered}$$

where $\lambda_i \in L_\infty(\Omega)$ (also $(\mu_i \in L_\infty(\Omega))$ have to satisfy the convergence conditions (1.5)-(1.6) where Θ_i has to be replaced by $\Theta_i^\lambda \in V_\lambda$. Then $u_i \equiv u_i^\lambda$ is defined by (1.7). However the elements u_i (generally) are not elements of V_λ.

The existence of $\Theta_i^\lambda \in V_\lambda$ $(i = 1, \dots, n)$ is a consequence of the fixed point argument, provided $\lambda_i \in L_\infty(\Omega)$, $u_{i-1} \in L_2(\Omega)$.

To determine numerically efficient relaxation parameters λ_i, μ_i (see (1.5), (1.6)) we can use the iteration procedure as in Section 2 (see (2.6)-(2.8))

$$(4.3)\quad \begin{gathered}(\lambda_{i,k-1}(\Theta_{i,k}^\lambda - \beta(u_{i-1})), \varphi) + \tau(a(t_i, u_{i-1}, \nabla\Theta_{i,k}^\lambda), \nabla\varphi) + \\ \tau(g(t_i, \beta^{-1}(\Theta_{i-1}^\lambda)), \varphi)_{\Gamma_2} = \tau(f(t_i, u_{i-1}), \varphi)\ , \forall v \in V_\lambda\end{gathered}$$

with

$$(4.4)\quad \lambda_{i,k} \equiv \lambda_{i,k}^\lambda := \frac{b_n(x, u_{i-1} + \mu_{i,k}(\Theta_{i,k}^\lambda - \beta(u_{i-1}))) - b_n(x, u_{i-1})}{\Theta_{i,k}^\lambda - \beta(u_{i-1})}$$

where

$$(4.5)\quad \bar{\mu}_{i,k} \equiv \bar{\mu}_{i,k}^\lambda := \left\{K, \frac{\beta^{-1}(u_{i-1}) + \alpha(\Theta_{i,k}^\lambda - \beta(u_{i-1})) - u_{i-1}}{\Theta_{i,k}^\lambda - \beta(u_{i-1})}\right\}$$

$$\mu_{i,k} \equiv \mu_{i,k}^\lambda := \min\{\bar{\mu}_{i,k}, \mu_{i,k-1}\}\ .$$

We obtain the convergence $\lambda_{i,k} \to \lambda_i$, $\Theta_{i,k} \to \Theta_i$ in $C(\bar{\Omega})$ (V_λ is fixed) under the same assumptions as in Proposition 2.12. If $V_\lambda \subset W_2^1 \cap C(\bar{\Omega})$ then the convergence in $C(\bar{\Omega})$ is equivalent to $L_2(\Omega)$ convergence because of finite dimension of V_λ. However, to guarantee $L_2(\Omega)$ convergence of $\{\Theta_{i,k}^\lambda\}_{k=1}^\infty$ we need small τ and smooth $\beta(u_{i-1}) \in C^{0,\alpha}(\bar{\Omega})$-see [14].

Since $V_\lambda \subset V$ we obtain the same *a priori* estimate of Lemma 3.4 and the second *a priori* estimate of Lemma 3.16. Thus, assertion of Lemma 3.10 holds true also for sequences $\{\bar{u}^{\alpha}\}$, $\{\bar{\Theta}^{\alpha}\}$ where $\alpha = (\tau, \lambda)$, $\alpha \to 0$ and $\{\bar{\alpha}\}$ is a subsequence of $\{\alpha\}$.

The Rothe's functions Θ^α, $\overline{\Theta}^\alpha$ are defined by (3.1) where Θ_i has to be replaced by Θ_i^λ and $\Theta_0 \equiv \beta(u_0)$ by $\Theta_0^\lambda \in V_\lambda$ where $\Theta_0^\lambda \to \beta(u_0)$ in L_2 for $\lambda \to 0$. Formally, we denote $u^\alpha \equiv u^n$. We obtain the following convergence result.

THEOREM 4.6 : *Let* (4.1), H_1)-H_6) *be satisfied and let* $\Theta_0^\lambda \to \Theta_0 \equiv \beta(u_0)$ *in* $L_2(\Omega)$. *Then* $b_{\underline{n}}(x, \overline{u}^{\underline{n}}) \to b(x, u)$, $b(x, \overline{u}^{\underline{n}}) \to b(x, u)$ *in* $L_s(Q_T)$, $\overline{\Theta}^\alpha \to \Theta \equiv \beta(u)$ *in* $L_s(Q_T)$ $\forall s < 2$, *where* u *is a variational solution of* (1.1)-(1.3), $\{\overline{n}\}$ *is a subsequence of* $\{n\}$ *and* $\overline{\Theta}^\alpha$, *are from* (4.2), (1.5), (1.6). *If the variational solution is unique then the original sequences* $\{b_{\underline{n}}(x, \overline{u}^n)\}$, $\{\overline{\Theta}^\alpha\}$ *are convergent. Moreover, if* H_7) *is satisfied then* $\overline{\Theta}^\alpha \to \beta(u)$ *in* $L_p(I, V)$.

The proof goes in the same lines as that one in Theorems 3.2 and 3.34. We discuss the differences only. We obtain the *a priori* estimate (see Lemma 3.16)

$$\| \partial_t \hat{b}_n(x, \overline{u}^n) \|_{L_q(I, V_\lambda^*)} \leqslant C \quad \left(\alpha = (\tau, \lambda) \equiv \left(\frac{T}{n}, \lambda\right)\right).$$

We extend the functional $\partial_t \hat{b}_n(x, \overline{u}^n) \in L_q(I, V_\lambda^*)$ to $F_n \in L_q(I, V^*)$ by the prescription

$$\int_I \langle F_n, w\rangle := \int_I \langle \partial_t \hat{b}_n(x, \overline{u}^n), P_\lambda w\rangle = \int_I \int_\Omega \partial_t \hat{b}_n(x, \overline{u}^n) \,.\, P_\lambda w$$

where $P_\lambda : V \to V_\lambda$ is the projector. Thus $\| F_n \|_{L_q(I, V^*)} \leqslant C$ and hence $F_n \rightharpoonup F$ in $L_q(I, V)$. Since $\hat{b}_n(x, \overline{u}^n) \to b(x, u)$ in $L_s(Q_T)$ we obtain $F \equiv \partial_t b(x, u)$ (see, e.g., [9]). Similarly as in Section 3 we prove (3.30) in the following way. We rewrite (4.2) in the form (3.29) where we replace $\overline{\Theta}^n$ by $\overline{\Theta}^\alpha(\alpha = (I/n, \lambda))$ and we replace φ by $\varphi^\alpha \in L_p(I, V_\lambda)$. For arbitrary $\varphi \in L_p(I, V)$ we can choose $\varphi^\alpha \in L_p(I, V_\lambda)$ such that $\varphi^\alpha \to \varphi$ in $L_p(I, V)$. Then we take the limit $\alpha \to 0$ and obtain (3.30) since

$$\int_0^t \langle \partial_t \hat{b}_n(x, \overline{u}^n), \varphi^\alpha\rangle = \int_0^t \langle F_n, P_\lambda \varphi^\alpha\rangle \to \int_0^t \langle F, \varphi\rangle = \int_0^t \langle \partial_t b(x, u), \varphi\rangle \,.$$

Similarly as in Section 3 we obtain

$$\lim_{\alpha \to 0} \int_0^t \partial_t \hat{b}_n(x, \overline{u}^n) \,.\, \overline{\Theta}^\alpha \geqslant \int_\Omega B^*(x, \beta(u(t))) - \int_\Omega B^*(x, \beta(u_0))$$

by means of which we can verify $\chi \equiv a(t, x, u, \nabla\beta(u))$ along the same lines as in the proof of Theorem 3.2. To prove $\overline{\Theta}^\alpha \to \beta(u)$ in $L_{\underline{p}}(I, V)$ we follow the proof of Theorem 3.34. We use the test function $\varphi = \overline{\Theta}^n - w^\alpha$ in

(3.29) (there $\overline{\Theta}^n$ has to be replaced by $\overline{\Theta}^\alpha$) where $w^\alpha \in L_p(I, V_\lambda)$ and $w^\alpha \to \beta(u)$ in $L_p(I, V)$. The rest of the proof is the same as that one in Theorem 3.34. Thus the proof of Theorem 4.6 is complete.

Remark 4.7 : The suggested numerical method has been tested in some special cases of slow or fast diffusion - see [11], [12], [15], [10] where or $b(s) \equiv s$ or $\beta(s) \equiv s$.

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