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LOCAL ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF THE STOKES EQUATIONS (*)

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Abstract. — *Local error estimates are derived which apply to most stable mixed finite element discretizations of the stationary Stokes equations.*

Résumé. — *Nous prouvons des estimations locales d'erreur qui s'appliquent à la plupart des discrétisations stables par éléments finis mixtes du problème de Stokes stationnaire.*

Key words : Stokes equations, mixed finite element method, local error estimates, interior error estimates.
AMS(MOS) subject classifications (1985 revision), 65N30, 65N15, 76M10, 76D07.

1. INTRODUCTION

In this article we establish local error estimates for finite element approximations to solutions of the Stokes equations. To fix ideas, consider a finite element approximation to the Stokes equations on a polygonal domain. Suppose that the velocity space contains (at least) all continuous piecewise polynomials of degree $r \geqslant 1$ subordinate to some triangulation of the domain which satisfy any essential boundary conditions, and that the pressure space contains all continuous piecewise polynomials of degree r or of degree $r - 1$ (in which case the continuity is dropped for $r = 1$). Suppose also that the usual stability condition for Stokes elements is fulfilled. Specific examples for $r = 1$ include the MINI finite element (continuous piecewise linears and bubble functions for the velocity and continuous piecewise linears for the pressure) [1] and the $P_2 - P_0$ finite element (continuous piecewise quadratics for the velocity and discontinuous piecewise constants for the pressure) [6].

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For $r = 2$ the Hood-Taylor element (continuous piecewise quadratics for the velocity and continuous piecewise linears for the pressure) [8] and the augmented $P_2 - P_1$ element (continuous piecewise quadratics plus bubbles for velocity and discontinuous piecewise linears for pressure) [3], [9] fulfill these hypotheses. Many other examples are known as well. The usual global error estimate asserts that for such discretizations the finite element approximation converges in $\underset{\sim}{H}^1 \times L^2(\Omega)$ with the same rate as the best approximation in the finite element spaces, namely the error is $O(h^r)$ if the exact solution is smooth enough. Measured in $\underset{\sim}{L}^2 \times H^{-1}$ the error converges to zero with one higher order. These estimates require that the exact velocity field belong to $\underset{\sim}{H}^{r+1}$ and the exact pressure field to H^r. However, such smoothness will generally not hold if the domain of the equations is not smoothly bounded or if the boundary or forcing data is not smooth. In such a case the solution, while not globally smooth, will usually be smooth in large subdomains, namely any interior regions a positive distance from the singular points of the data. It is therefore important to ask whether the optimal order convergence holds in such subdomains, or whether the singularities degrade the convergence globally. In this paper we establish local error estimates which assert that the rate of convergence in subdomains of smoothness is indeed optimal. The precise statement is contained in Theorem 5.3, which is the major result of the paper.

Local estimates (often called interior estimates because of their application to problems with boundary singularities), were first studied in 1974 by Nitsche and Schatz [10] for second order elliptic problems. Through this and subsequent works, the local convergence theory is reasonably well understood for such problems. See Wahlbin's handbook article [12, Chapter III] for an extensive treatment. In 1985 Douglas and Milner adapted the Nitsche-Schatz approach to the Raviart-Thomas mixed method for scalar second order elliptic problems [5]. The present work adapts it to analyze a wide class of methods for the Stokes equations. Although the general approach is not new, there are a number of significant difficulties which arise for the Stokes system that are not present in previous works. Recently, Lucia Gastaldi [7] obtained interior error estimates for some finite element methods for the Reissner-Mindlin plate model. This work is related to local error analysis of the Stokes equations since the Reissner-Mindlin model can be reformulated as a decoupled system of two Laplace equations and a perturbed Stokes system. (Indeed the methods we develop here will be used in a subsequent work to analyze the method of Arnold and Falk [2] for the Reissner-Mindlin model.) However Gastaldi's work depends strongly on special properties of mixed methods which arise from particular methods for the plate (especially a *commuting diagram property*), and so would not easily adapt to a general analysis of mixed methods for the Stokes equations.

After the preliminaries of the next section, we set out the hypotheses for the finite element spaces in section 3. In addition to the natural requirements of approximability and stability alluded to above, here, as in much of the theory of interior estimates, the superapproximation property plays a crucial role. In section 4, we introduce the local equations and derive some basic properties of their solutions. Section 5 gives the precise statement of our main result and its proof. We close with a short application in section 6.

2. NOTATIONS AND PRELIMINARIES

Let Ω denote a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We shall use the usual standard L^2-based Sobolev spaces $H^m = H^m(\Omega)$, $m \in \mathbb{Z}$, with the norm $\| \cdot \|_{m,\Omega}$. Recall that for $m \in \mathbb{N}$, H^{-m} denotes the normed dual of $\mathring{H}^m$, the closure of $C_0^\infty(\Omega)$ in H^m. We use the notation $(\,\cdot\,,\cdot\,)$ for both the $L^2(\Omega)$-innerproduct and its extension to a pairing of $\mathring{H}^m$ and H^{-m}. If X is any subspace of L^2, then $\hat{X}$ denotes the subspace of elements with average value zero. We affix an undertilde to a space to denote the 2-vector-valued analogue. The undertilde is also affixed to vector-valued functions and operators, and double undertildes are used for matrix-valued objects. This is illustrated in the definitions of the following standard differential operators :

$$\operatorname{div} \underset{\sim}{\phi} = \partial\phi_1/\partial x + \partial\phi_2/\partial y\,, \quad \underset{\sim}{\operatorname{grad}}\, p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix},$$

$$\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi} = \begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_1/\partial y \\ \partial\phi_2/\partial x & \partial\phi_2/\partial y \end{pmatrix}.$$

The letter C denotes a generic constant, not the same in each occurrence, but always independent of the meshsize parameter h.

Let G be an open subset of Ω and s an integer. If $\phi \in H^s(G)$, $\psi \in H^{-s}(G)$, and $w \in C_0^\infty(G)$, then

$$|(w\phi, \psi)| \leqslant C\, \|\phi\|_{s,G} \|\psi\|_{-s,G}\,,$$

with the constant C depending only on G, w, and s. For $\underset{\sim}{\Phi} \in \underset{\sim}{H}^s(G)$, $\underset{\sim}{\Psi} \in \underset{\sim}{H}^{-s+1}(G)$ define

$$R(w, \underset{\sim}{\Phi}, \underset{\sim}{\Psi}) = (\underset{\sim}{\Phi}(\underset{\sim}{\operatorname{grad}}\, w)^t, \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\Psi}) - (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\Phi}, \underset{\sim}{\Psi}(\underset{\sim}{\operatorname{grad}}\, w)^t)\,. \quad (2.1)$$

Then

$$|R(w, \underset{\sim}{\Phi}, \underset{\sim}{\Psi})| \leq C \|\underset{\sim}{\Phi}\|_{s,G} \|\underset{\sim}{\Psi}\|_{-s+1,G} . \tag{2.2}$$

If, moreover, $\underset{\sim}{\Psi} \in \underset{\sim}{H}^{-s+2}$, we have identity

$$(\underset{\approx}{\text{grad}}\,(w\underset{\sim}{\Phi}), \underset{\approx}{\text{grad}}\,\underset{\sim}{\Psi}) = (\text{grad}\,\underset{\sim}{\Phi}, \underset{\approx}{\text{grad}}\,(w\underset{\sim}{\Psi})) + R(w, \underset{\sim}{\Phi}, \underset{\sim}{\Psi}) .$$

The following lemma states the well-posedness and regularity of the Dirichlet problem for the generalized Stokes equations on smooth domains. (Because we are interested in local estimates we really only need this results when the domain is a disk.) For the proof see [11, Chapter I, § 2].

LEMMA 2.1 : *Let G be a smoothly bounded plane domain and m a nonnegative integer. Then for any given functions $\underset{\sim}{F} \in \underset{\sim}{H}^{m-1}(G)$, $K \in H^m(G) \cap \hat{L}^2(G)$, there exist uniquely determined functions*

$$\underset{\sim}{\phi} \in \underset{\sim}{H}^{m+1}(G) \cap \mathring{\underset{\sim}{H}}^1(G) , \quad p \in H^m(G) \cap \hat{L}^2(G) ,$$

such that

$$\begin{aligned} (\underset{\approx}{\text{grad}}\,\underset{\sim}{\phi}, \underset{\approx}{\text{grad}}\,\underset{\sim}{\psi}) - (\text{div}\,\underset{\sim}{\psi}, p) = (\underset{\sim}{F}, \underset{\sim}{\psi}) , &\quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{H}}^1(G) , \\ (\text{div}\,\underset{\sim}{\phi}, q) = (K, q) , &\quad \text{for all} \quad q \in \hat{L}^2(G) . \end{aligned}$$

Moreover,

$$\|\underset{\sim}{\phi}\|_{m+1,G} + \|p\|_{m,G} \leq C(\|\underset{\sim}{F}\|_{m-1,G} + \|K\|_{m,G}) ,$$

where the constant C is independent of $\underset{\sim}{F}$ and K.

3. FINITE ELEMENT SPACES

In this section we collect assumptions on the mixed finite element spaces that will be used in the paper. In addition to the usual approximation and stability properties required for the finite elements spaces, we need the so-called *superapproximation property,* which was first introduced by Nitsche and Schatz [10].

Let $\Omega \subset \mathbb{R}^2$ be the bounded open set on which we solve the Stokes equations and let h denote a mesh size parameter. We denote by $\underset{\sim}{V}_h$ the finite element subspace of $\underset{\sim}{H}^1(\Omega)$, and by W_h the finite element subspace of $L^2(\Omega)$. For $\Omega_0 \subseteq \Omega$, define

$$\underset{\sim}{V}_h(\Omega_0) = \{\underset{\sim}{\phi}\,_{|\Omega_0|}\ \underset{\sim}{\phi} \in \underset{\sim}{V}_h\}, \quad W_h(\Omega_0) = \{p_{|\Omega_0|}\ p \in W_h\}$$

$$\mathring{\underset{\sim}{V}}_h(\Omega_0) = \{\underset{\sim}{\phi} \in \underset{\sim}{V}_h | \operatorname{supp} \underset{\sim}{\phi} \subseteq \overline{\Omega}_0\}, \quad \mathring{W}_h(\Omega_0) = \{p \in W_h | \operatorname{supp} p \subseteq \overline{\Omega}_0\}.$$

Let G_0 and G be concentric open disks with $G_0 \Subset G \Subset \Omega$, i.e., $\overline{G}_0 \subset G$ and $\overline{G} \subset \Omega$. We assume that there exists a positive real number h_0 and positive integers k_1 and k_2, such that for $h \in (0, h_0]$, the following properties hold.

A1. *Approximation property.*

(1) If $\underset{\sim}{\phi} \in \underset{\sim}{H}^m(G)$ for some positive integer m, then there exists a $\underset{\sim}{\phi}^I \in \underset{\sim}{V}_h$ such that

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}^I \|_{1,G} \leqslant C h^{r_1 - 1} |\underset{\sim}{\phi}|_{m,G}, \quad r_1 = \min(k_1 + 1, m).$$

(2) If $p \in H^l(G)$ for some nonnegative integer l, then there exists a $p^I \in W_h$, such that

$$\|p - p^I\|_{0,G} \leqslant C h^{r_2} \|p\|_{l,G}, \quad r_2 = \min(k_2 + 1, l).$$

Furthermore, if $\underset{\sim}{\phi}$ and p vanish on $G \setminus \overline{G}_0$, respectively, then $\underset{\sim}{\phi}^I$ and p^I can be chosen to vanish on $\Omega \setminus \overline{G}$.

A2. *Superapproximation property.* Let $w \in C_0^\infty(G)$, $\underset{\sim}{\phi} \in \underset{\sim}{V}_h$, and $p \in W_h$. Then there exist $\underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(G)$ and $q \in \mathring{W}_h(G)$, such that

$$\| w\underset{\sim}{\phi} - \underset{\sim}{\psi} \|_{1,\Omega} \leqslant Ch\ \|\underset{\sim}{\phi}\|_{1,G}$$

$$\| wp - q \|_{0,\Omega} \leqslant Ch\ \|p\|_{0,G},$$

where C depends only on G and w.

A3. *Inverse property.* For each $h \in (0, h_0]$, there exists a set G_h, $G_0 \Subset G_h \Subset G$, such that for each nonnegative integer m there is a constant C for which

$$\|\underset{\sim}{\phi}\|_{1,G_h} \leqslant C h^{-1-m} \|\underset{\sim}{\phi}\|_{-m,G_h}, \quad \text{for all} \quad \underset{\sim}{\phi} \in \underset{\sim}{V}_h,$$

$$\|p\|_{0,G_h} \leqslant C h^{-m} \|p\|_{-m,G_h}, \quad \text{for all} \quad p \in W_h.$$

A4. *Stability property.* There is a positive constant γ, such that for all $h \in (0, h_0]$ there is a domain G_h, $G_0 \Subset G_h \Subset G$ for which

$$\inf_{\substack{p \in \hat{W}_h(G_h) \\ p \neq 0}} \sup_{\substack{\underset{\sim}{\phi} \in \mathring{V}_h(G_h) \\ \underset{\sim}{\phi} \neq 0}} \frac{(\operatorname{div} \underset{\sim}{\phi}, p)_{G_h}}{\|\underset{\sim}{\phi}\|_{1, G_h} \|p\|_{0, G_h}} \geqslant \gamma .$$

When $G_h = \Omega$, property A4 is the standard stability condition for Stokes elements. It will usually hold as long as G_h is chosen to be a union of elements. The standard stability theory for mixed methods then gives us the following result.

LEMMA 3.1 : *Let G_h be a subdomain for which the stability inequality in A4 holds. Then for $\underset{\sim}{\phi} \in \mathring{\underset{\sim}{H}}{}^1(G_h)$ and $p \in L^2(G_h)$, there exist unique $\pi\underset{\sim}{\phi} \in \mathring{\underset{\sim}{V}}_h(G_h)$ and $\pi p \in W_h(G_h)$ with $\int_{G_h} \pi p = \int_{G_h} p$ such that*

$$(\underset{\approx}{\operatorname{grad}} (\underset{\sim}{\phi} - \pi\underset{\sim}{\phi}), \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) - (\operatorname{div} \underset{\sim}{\psi}, p - \pi p) = 0 , \quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(G_h),$$
$$(\operatorname{div} (\underset{\sim}{\phi} - \pi\underset{\sim}{\phi}), q) = 0 , \quad \text{for all} \quad q \in W_h(G_h).$$

Moreover,

$$\|\underset{\sim}{\phi} - \pi\underset{\sim}{\phi}\|_{1, G_h} + \|p - \pi p\|_{0, G_h} \leqslant$$
$$\leqslant C\left(\inf_{\underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(G_h)} \|\underset{\sim}{\phi} - \underset{\sim}{\psi}\|_{1, G_h} + \inf_{q \in W^h(G_h)} \|p - q\|_{0, G_h} \right) .$$

The approximation properties A1 are typical of finite element spaces $\underset{\sim}{V}_h$ and W_h constructed from polynomials of degrees at least k_1 and k_2, respectively. (It does not matter that the subdomain G is not a union of elements since $\underset{\sim}{\phi}$ and p can be extended beyond G.) The superapproximation property is discussed as Assumptions 7.1 and 9.1 in [12]. Many finite element spaces are known to have the superapproximation property. In particular, it was verified in [10] for Lagrange and Hermite elements. To end this section we shall verify the superapproximation for the MINI element.

Let b_T denote the cubic bubble on the triangle T, so on $T\, b_T$ is the cubic polynomial satisfying $b_T|_{\partial T} = 0$ and $\int_T b_T = 1$. We extend b_T outside T by zero. For a given triangulation $\mathcal{T}_h$ let V_h denote the span of the continuous piecewise linear functions and the bubble functions b_T, $T \in \mathcal{T}_h$. The MINI element uses $V_h \times V_h$ as the finite element space for velocities. We wish to show that if $\phi \in V_h$ and $w \in C_0^\infty(G)$ then $\|w\phi - \psi\|_{1,G} \leqslant Ch \|\phi\|_{1,G}$ for some $\psi \in \mathring{V}_h(G)$. We begin by writing $\phi = \phi_l + \phi_b$ with ϕ_l piecewise linear and $\phi_b = \Sigma_{T \in \mathcal{T}_h} \beta_T b_T$ for some $\beta_T \in \mathbb{R}$.

We know that there exists a piecewise linear function ψ_l supported in G for which

$$\| w\phi_l - \psi_l \|_{1,\Omega} \leqslant Ch \, \| \phi_l \|_{1,G} .$$

Turning to the bubble function term ϕ_b define

$$\psi_b = \sum_{T \subset G} (\beta_T P_T w) \, b_T \in \mathring{V}_h(G)$$

where $P_T w \in \mathbb{R}$ is the average value of w on T. Now if T intersects supp w then $T \subset G$, at least for h sufficiently small. Hence

$$\| w\phi_b - \psi_b \|_{0,\Omega}^2 = \sum_{T \subset G} \| w\phi_b - \psi_b \|_{0,T}^2 = \sum_{T \subset G} \| \beta_T b_T (w - P_T w) \|_{0,T}^2$$

$$\leqslant \sum_{T \subset G} \| w - P_T w \|_{L^\infty(T)}^2 \| \beta_T b_T \|_{0,T}^2 \leqslant Ch^2 \| \phi_b \|_{0,G}^2 ,$$

where the constant C depends on w. Moreover,

$$\begin{aligned} &\| \underset{\sim}{\text{grad}} \, (w\phi_b - \psi_b) \|_{0,\Omega}^2 \\ &= \sum_{T \subset G} \| \underset{\sim}{\text{grad}} \, (w\phi_b - \psi_b) \|_{0,T}^2 \\ &= \sum_{T \subset G} \| \underset{\sim}{\text{grad}} \, (\beta_T b_T (w - P_T w)) \|_{0,T}^2 \\ &= \sum_{T \subset G} \| \beta_T (w - P_T w) \, \underset{\sim}{\text{grad}} \, b_T + \beta_T b_T \underset{\sim}{\text{grad}} \, (w - P_T w) \|_{0,T}^2 \\ &\leqslant C \left(h^2 \sum_{T \subset G} \| \underset{\sim}{\text{grad}} \, w \|_{\infty,T}^2 \| \beta_T \underset{\sim}{\text{grad}} \, b_T \|_{0,T}^2 + \| \underset{\sim}{\text{grad}} \, w \|_{\infty,T}^2 \sum_{T \subset G} \| \beta_T b_T \|_{0,T}^2 \right) \\ &\leqslant Ch^2 \| \phi_b \|_{1,G}^2 , \end{aligned}$$

where we used the fact that

$$\| b_T \|_{0,T} \leqslant Ch \| b_T \|_{1,T} .$$

Taking $\psi_h = \psi_b + \psi_l \in \mathring{V}_h(G)$ we thus have

$$\| w\phi_h - \psi_h \|_{1,\Omega} \leq Ch(\|\phi_b\|_{1,G} + \|\phi_l\|_{1,G}) .$$

We complete the proof by showing that

$$\|\phi_b\|_{1,T} + \|\phi_l\|_{1,T} \leq C\|\phi_b + \phi_l\|_{1,T}$$

for any triangle T with the constant C depending only on the minimum angle of T. Since $\int_T \underset{\sim}{\text{grad}}\, \phi_b \cdot \underset{\sim}{\text{grad}}\, \phi_l = 0$, it suffices to prove that

$$\|\phi_b\|_{0,T} + \|\phi_l\|_{0,T} \leq C\|\phi_b + \phi_l\|_{0,T} .$$

If T is the unit triangle this hold by equivalence of all norms on the finite dimensional space of cubic polynomials, and the extension to an arbitrary triangle is accomplished by scaling.

4. INTERIOR DUALITY ESTIMATE

Let $(\underset{\sim}{\phi}, p) \in \underset{\sim}{H}^1(\Omega) \times L^2(\Omega)$ be some solution to the generalized Stokes equations

$$-\underset{\sim}{\Delta}\underset{\sim}{\phi} + \underset{\sim}{\text{grad}}\, p = \underset{\sim}{F} ,$$
$$\text{div}\, \underset{\sim}{\phi} = K .$$

Regardless of the boundary conditions used to specify the particular solution, $(\underset{\sim}{\phi}, p)$ satisfies

$$(\underset{\approx}{\text{grad}}\, \underset{\sim}{\phi}, \underset{\approx}{\text{grad}}\, \underset{\sim}{\psi}) - (\text{div}\, \underset{\sim}{\psi}, p) = (\underset{\sim}{F}, \underset{\sim}{\psi}) , \quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{H}}^1(\Omega) ,$$
$$(\text{div}\, \underset{\sim}{\phi}, q) = (K, q) , \quad \text{for all} \quad q \in L^2(\Omega) .$$

Similarly, regardless of the particular boundary conditions, the finite element solution $(\underset{\sim}{\phi}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ satisfies

$$(\underset{\approx}{\text{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\text{grad}}\, \underset{\sim}{\psi}) - (\text{div}\, \underset{\sim}{\psi}, p_h) = (\underset{\sim}{F}, \underset{\sim}{\psi}) , \quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(\Omega) ,$$
$$(\text{div}\, \underset{\sim}{\phi}_h, q) = (K, q) , \quad \text{for all} \quad q \in W_h .$$

Therefore

$$(\underset{\approx}{\text{grad}}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\text{grad}}\, \underset{\sim}{\psi}) - (\text{div}\, \underset{\sim}{\psi}, p - p_h) = 0 , \quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(\Omega) , \tag{4.1}$$

$$(\text{div}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), q) = 0 , \quad \text{for all} \quad q \in W_h . \tag{4.2}$$

The local interior error analysis starts from these local discretization equations.

THEOREM 4.1 : *Let $G_0 \Subset G$ be concentric open disks with closures contained in Ω and s an arbitrary nonnegative integer. Then there exists a constant C such that if $(\underset{\sim}{\phi}, p) \in \underset{\sim}{H}^1(\Omega) \times L^2(\Omega)$, and $(\underset{\sim}{\phi}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ satisfy (4.1) and (4.2), we have*

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{0, G_0} + \|p - p_h\|_{-1, G_0} \leq C(h\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1, G} + h\|p - p_h\|_{0, G} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-s, G} + \|p - p_h\|_{-1-s, G}) . \tag{4.3}$$

In order to prove the theorem we first establish two lemmas.

LEMMA 4.2 : *Under the hypotheses of Theorem 4.1, there exists a constant C for which*

$$\|p - p_h\|_{-s-1, G_0} \leq C(h\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1, G} + h\|p - p_h\|_{0, G} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-s-1, G} + \|p - p_h\|_{-s-2, G}) .$$

Proof: Choose a function $w \in C_0^\infty(G)$ which is identically 1 on G_0. Also choose a function $\delta \in C_0^\infty(G_0)$ with integral 1. Then

$$\|p - p_h\|_{-s-1, G_0} \leq \|w(p - p_h)\|_{-s-1, G} = \sup_{\substack{g \in \mathring{H}^{s+1}(G) \\ g \neq 0}} \frac{(w(p - p_h), g)}{\|g\|_{s+1, G}} . \tag{4.4}$$

Now

$$(w(p - p_h), g) = \left(w(p - p_h), g - \delta \int_G g \right) + (w(p - p_h), \delta) \int_G g$$

and clearly

$$\left| (w(p - p_h), \delta) \int_G g \right| \leq C\|p - p_h\|_{-s-2, G}\|g\|_{0, G} .$$

Since $g - \delta \int_G g \in H^{s+1}(G) \cap \hat{L}^2(G)$ it follows from Lemma 2.1 that there exist

$$\underset{\sim}{\Phi} \in \underset{\sim}{H}^{s+2}(G) \cap \underset{\sim}{\mathring{H}}^1(G) \quad \text{and} \quad P \in H^{s+1}(G) \cap \hat{L}^2(G)$$

such that

$$(\underset{\approx}{\text{grad}}\ \underset{\sim}{\Phi}, \underset{\approx}{\text{grad}}\ \underset{\sim}{\psi}) - (\text{div}\ \underset{\sim}{\psi}, P) = 0\,, \quad \text{for all} \quad \underset{\sim}{\psi} \in \underset{\sim}{H}^1(G)\,, \tag{4.5}$$

$$(\text{div}\ \underset{\sim}{\Phi}, q) = \left(g - \delta \int_G g, q \right), \quad \text{for all} \quad q \in L^2(G)\,. \tag{4.6}$$

Furthermore,

$$\| \underset{\sim}{\Phi} \|_{s+2,G} + \| P \|_{s+1,G} \leq C \| g \|_{s+1,G}\,. \tag{4.7}$$

Then, taking $q = w(p - p_h)$ in (4.6), we obtain

$$\left(g - \delta \int_G g, w(p - p_h) \right)$$

$$= (\text{div}\ \underset{\sim}{\Phi}, w(p - p_h)) = (\text{div}\ (w\underset{\sim}{\Phi}), p - p_h) - (\underset{\sim}{\text{grad}}\ w, (p - p_h)\ \underset{\sim}{\Phi})$$

$$= (\text{div}\ (w\underset{\sim}{\Phi})^I, p - p_h) + \{ (\text{div}\ [w\underset{\sim}{\Phi} - (w\underset{\sim}{\Phi})^I], p - p_h) -$$

$$- (\underset{\sim}{\text{grad}}\ w, (p - p_h)\ \underset{\sim}{\Phi}) \}$$

$$=: A_1 + B_1\,. \tag{4.8}$$

Here the superscript I is the approximation operator specified in property A1 of section 3. Choosing $\underset{\sim}{\psi} = (w\underset{\sim}{\Phi})^I$ in (4.1), we get

$$A_1 := (\text{div}\ (w\underset{\sim}{\Phi})^I, p - p_h) = (\underset{\approx}{\text{grad}}\ (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\text{grad}}\ (w\underset{\sim}{\Phi})^I)$$

$$= (\underset{\approx}{\text{grad}}\ (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\text{grad}}\ (w\underset{\sim}{\Phi})) + (\underset{\approx}{\text{grad}}\ (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h),$$

$$\underset{\approx}{\text{grad}}\ [(w\underset{\sim}{\Phi})^I - (w\underset{\sim}{\Phi})])$$

$$= (\underset{\approx}{\text{grad}}\ [w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)], \underset{\approx}{\text{grad}}\ \underset{\sim}{\Phi}) + \{ R(w, \underset{\sim}{\Phi}, \underset{\sim}{\phi} - \underset{\sim}{\phi}_h)$$

$$+ (\underset{\approx}{\text{grad}}\ (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\text{grad}}\ [(w\underset{\sim}{\Phi})^I - w\underset{\sim}{\Phi}]) \} =: A_2 + B_2\,, \tag{4.9}$$

where R is defined in (2.1). Next, setting $\psi = w(\phi - \phi_h)$ in (4.5), we obtain

$$A_2 := (\text{grad}\,[w(\phi - \phi_h)], \text{grad}\,\Phi) = (\text{div}\,[w(\phi - \phi_h)], P)$$

$$= (\text{div}(\phi - \phi_h), wP) + (\text{grad}\,w, P(\phi - \phi_h))$$

$$= (\text{div}\,(\phi - \phi_h), wP - (wP)^I) + (\text{grad}\,w, P(\phi - \phi_h)),$$

where we applied (4.2) in the last step.

Applying the approximation property A1 and (2.2) we get

$$|B_1| \leq C(h\|\Phi\|_{2,G}\|p - p_h\|_{0,G} + \|\Phi\|_{s+2,G}\|p - p_h\|_{-s-2,G}),$$

$$|B_2| \leq C(\|\phi - \phi_h\|_{-s-1,G}\|\Phi\|_{s+2,G} + h\,\|\phi - \phi_h\|_{1,G}\|\Phi\|_{2,G}), \tag{4.10}$$

$$|A_2| \leq C(h\|\phi - \phi_h\|_{1,G}\|P\|_{1,G} + \|\phi - \phi_h\|_{-s-1}\|P\|_{s+1,G}).$$

Substituting (4.7) into (4.10) and combining the result with (4.4), (4.8), and (4.9), we arrive at (4.3). □

Now we state the second lemma to be used in the proof of Theorem 4.1.

LEMMA 4.3 : *Under the hypotheses of Theorem 4.1, there exists a constant C for which*

$$\|\phi - \phi_h\|_{-s,G_0} \leq C(h\|\phi - \phi_h\|_{1,G} + h\|p - p_h\|_{0,G}$$
$$+ \|\phi - \phi_h\|_{-s-1,G} + \|p - p_h\|_{-s-2,G}).$$

Proof: Given $F \in H^s(G)$, define $\Phi \in H^{s+2}(G) \cap \mathring{H}^1(G)$ and $P \in H^{s+1}(G) \cap \hat{L}^2(G)$ by

$$(\text{grad}\,\Phi, \text{grad}\,\psi) - (\text{div}\,\psi, P) = (F, \psi), \quad \text{for all} \quad \psi \in \mathring{H}^1(G), \tag{4.11}$$

$$(\text{div}\,\Phi, q) = 0, \quad \text{for all} \quad q \in L^2(G). \tag{4.12}$$

Then, by Lemma 2.1,

$$\|\Phi\|_{s+2,G} + \|P\|_{s+1,G} \leq C\|F\|_{s,G}, \quad C = C(G_0, G).$$

Now

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{-s, G_0} \leqslant \| w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h) \|_{-s, G} = \sup_{\substack{\underset{\sim}{F} \in \mathring{\underset{\sim}{H}}^s(G) \\ \underset{\sim}{F} \neq 0}} \frac{(w(\underset{\sim}{\phi} - \underset{\sim}{\phi}), \underset{\sim}{F})}{\| \underset{\sim}{F} \|_{s, G}}$$

with w as in the proof of the previous lemma. Setting $\underset{\sim}{\psi} = w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)$ in (4.11), we get

$$\begin{aligned} (w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\sim}{F}) &= (\underset{\approx}{\operatorname{grad}}\, \Phi, \underset{\approx}{\operatorname{grad}}\, [w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)]) - (\operatorname{div} [w(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)], P) \\ &= \{ (\underset{\approx}{\operatorname{grad}}\, (w \underset{\sim}{\Phi}), \underset{\approx}{\operatorname{grad}}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)) - (\operatorname{div}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), wP) \} \\ &\quad - \{ R(w, \underset{\sim}{\Phi}, \underset{\sim}{\phi} - \underset{\sim}{\phi}_h) + \\ &\quad + (\underset{\sim}{\operatorname{grad}}\, w, P(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)) \} =: E_1 + F_1 , \end{aligned}$$

To estimate E_1, we set $q = (wP)^I$ in (4.2) and obtain

$$\begin{aligned} E_1 = (\underset{\approx}{\operatorname{grad}}\, (w \underset{\sim}{\Phi})^I, \underset{\approx}{\operatorname{grad}}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)) &- \{ (\operatorname{div}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), wP - (wP)^I) \\ &- (\underset{\approx}{\operatorname{grad}}\, [w \underset{\sim}{\Phi} - (w \underset{\sim}{\Phi})^I], \underset{\approx}{\operatorname{grad}}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)) \} =: E_2 + F_2 . \end{aligned}$$

Taking $\underset{\sim}{\psi} = (w \underset{\sim}{\Phi})^I$ in (4.1), we arrive at

$$\begin{aligned} E_2 &= (\underset{\approx}{\operatorname{grad}}\, (w \underset{\sim}{\Phi})^I, \underset{\approx}{\operatorname{grad}}\, (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)) \\ &= (\operatorname{div}\, (w \underset{\sim}{\Phi})^I, p - p_h) \\ &= (\operatorname{div}\, (w \underset{\sim}{\Phi}), p - p_h) + (\operatorname{div}\, [(w \underset{\sim}{\Phi})^I - (w \underset{\sim}{\Phi})], p - p_h) \\ &= (\underset{\sim}{\operatorname{grad}}\, w, (p - p_h)\, \underset{\sim}{\Phi}) + (\operatorname{div}\, [(w \underset{\sim}{\Phi})^I - (w \underset{\sim}{\Phi})], p - p_h) , \end{aligned}$$

where we applied (4.12) in the last step. Applying (2.2) and the approximation property A1, we have

$$\begin{aligned} |F_1| &\leqslant C(\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{-s-1, G} \| \underset{\sim}{\Phi} \|_{s+2, G} + \| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{-s-1, G} \| P \|_{s+1, G}) , \\ |F_2| &\leqslant Ch(\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{1, G} \| P \|_{1, G} + \| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{1, G} \| \underset{\sim}{\Phi} \|_{2, G}) , \\ |E_2| &\leqslant C(\| p - p_h \|_{-s-2, G} \| \underset{\sim}{\Phi} \|_{s+2, G} + h \| p - p_h \|_{0, G} \| \underset{\sim}{\Phi} \|_{2, G}) . \end{aligned}$$

From these bounds we get the desired result. □

Proof of Theorem 4.1 : Let $G_0 \Subset G_1 \Subset \dots G_s = G$ be concentric disks. First applying Lemma 4.2 and Lemma 4.3 with s replaced by 0 and G replaced by G_1, we obtain

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \leqslant C(h\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1,G_1} + h\|p - p_h\|_{0,G_1} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-1,G_1} + \|p - p_h\|_{-2,G_1}) .$$

To estimate $\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-1,G_1}$ and $\|p - p_h\|_{-2,G_1}$, we again apply Lemma 4.2 and Lemma 4.3, this time with G_0 and G being replaced by G_1 and G_2 and s replaced by 1. Thus, we get

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{0,G_0} + \|p - p_h\|_{-1,G_0} \leqslant C(h\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1,G_2} + h\|p - p_h\|_{0,G_2} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-2,G_2} + \|p - p_h\|_{-3,G_2}) .$$

Continuing in this fashion, we obtain (4.3). □

5. INTERIOR ERROR ESTIMATES

In this section we state and prove the main result of this paper, Theorem 5.3. First we obtain in Lemma 5.1 a bound on solutions of the homogeneous discrete system. In Lemma 5.2 this bound is iterated to get a better bound, which is then used to establish the desired local estimate on disks. Finally Theorem 5.3 is extends this estimate to arbitrary interior domains.

LEMMA 5.1 : *Suppose* $(\underset{\sim}{\phi}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ *satisfies*

$$(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\psi}) - (\operatorname{div} \underset{\sim}{\psi}, p_h) = 0 , \quad \text{for all} \quad \underset{\sim}{\psi} \in \mathring{\underset{\sim}{V}}_h(\Omega) , \tag{5.1}$$

$$(\operatorname{div} \underset{\sim}{\phi}_h, q) = 0 , \quad \text{for all} \quad q \in \mathring{W}_h(\Omega) . \tag{5.2}$$

Then for any concentric disks $G_0 \Subset G \Subset \Omega$, and any nonnegative integer t, we have

$$\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \leqslant C\,(h\|\underset{\sim}{\phi}_h\|_{1,G} + h\|p_h\|_{0,G} + \|\underset{\sim}{\phi}_h\|_{-t,G} + \|p_h\|_{-t-1,G}) , \tag{5.3}$$

where $C = C(t, G_0, G)$.

Proof : Let G_h, $G_0 \Subset G_h \Subset G$, be as in Assumption A4. Let G' be another disk concentric with G_0 and G, such that $G_0 \Subset G' \Subset G_h$, and construct

$w \in C_0^\infty(G')$ with $w \equiv 1$ on G_0. Set $\tilde{\underset{\sim}{\phi}}_h = w\underset{\sim}{\phi}_h \in \overset{\circ}{\underset{\sim}{H}}{}^1(G')$, $\tilde{p}_h = wp_h \in L^2(G')$. By Lemma 3.1, we may define functions $\pi\tilde{\underset{\sim}{\phi}}_h \in \overset{\circ}{\underset{\sim}{V}}_h(G_h)$ and $\pi\tilde{p}_h \in W_h(G_h)$ by the equations

$$(\underset{\approx}{\operatorname{grad}}\,(\tilde{\underset{\sim}{\phi}}_h - \pi\tilde{\underset{\sim}{\phi}}_h), \underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\psi}) - (\operatorname{div}\underset{\sim}{\psi}, \tilde{p}_h - \pi\tilde{p}_h) = 0\,, \quad \text{for all} \quad \underset{\sim}{\psi} \in \overset{\circ}{\underset{\sim}{V}}_h(G_h)\,, \tag{5.4}$$

$$(\operatorname{div}\,(\tilde{\underset{\sim}{\phi}}_h - \pi\tilde{\underset{\sim}{\phi}}_h), q) = 0\,, \quad \text{for all} \quad q \in W_h(G_h)\,, \tag{5.5}$$

together with $\int_{G_h} (\pi\tilde{p}_h - \tilde{p}_h) = 0$. Furthermore, there exists a constant C such that

$$\begin{aligned}
&\|\tilde{\underset{\sim}{\phi}}_h - \pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} + \|\tilde{p}_h - \pi\tilde{p}_h\|_{0,G_h} \\
&\leqslant C\left(\inf_{\underset{\sim}{\psi} \in \overset{\circ}{\underset{\sim}{V}}_h(G_h)} \|\tilde{\underset{\sim}{\phi}}_h - \underset{\sim}{\psi}\|_{1,G_h} + \inf_{q \in W_h(G_h)} \|\tilde{p}_h - q\|_{0,G_h}\right) \\
&\leqslant Ch(\|\underset{\sim}{\phi}_h\|_{1,G'} + \|p_h\|_{0,G'})\,,
\end{aligned} \tag{5.6}$$

where we have used the superapproximation property in the last step.

To prove (5.3), note that

$$\begin{aligned}
\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} &\leqslant \|\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} + \|\tilde{p}_h\|_{0,G_h} \\
&\leqslant \|\tilde{\underset{\sim}{\phi}}_h - \pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} + \|\tilde{p}_h - \pi\tilde{p}_h\|_{0,G_h} + \\
&\quad \|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} + \|\pi\tilde{p}_h\|_{0,G_h} \\
&\leqslant Ch(\|\underset{\sim}{\phi}_h\|_{1,G'} + \|p_h\|_{0,G'}) + \|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} + \|\pi\tilde{p}_h\|_{0,G_h}\,.
\end{aligned} \tag{5.7}$$

Next, we bound $\|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h}$. In (5.4) we take $\underset{\sim}{\psi} = \pi\tilde{\underset{\sim}{\phi}}_h$ to obtain, for a positive constant c,

$$\begin{aligned}
c\|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h}^2 &\leqslant (\underset{\approx}{\operatorname{grad}}\,\pi\tilde{\underset{\sim}{\phi}}_h, \underset{\approx}{\operatorname{grad}}\,\pi\tilde{\underset{\sim}{\phi}}_h) = \\
&= (\underset{\approx}{\operatorname{grad}}\,\tilde{\underset{\sim}{\phi}}_h, \operatorname{grad}\pi\tilde{\underset{\sim}{\phi}}_h) - (\operatorname{div}\pi\tilde{\underset{\sim}{\phi}}_h, \tilde{p}_h - \pi\tilde{p}_h)\,.
\end{aligned} \tag{5.8}$$

For the first term on the right hand side of (5.8), we have

$$
\begin{aligned}
(\underset{\approx}{\operatorname{grad}}\, \widetilde{\underset{\sim}{\phi}_h}, \underset{\approx}{\operatorname{grad}}\, \pi \widetilde{\underset{\sim}{\phi}_h}) &= (\underset{\approx}{\operatorname{grad}}\,(w \underset{\sim}{\phi}_h), \underset{\approx}{\operatorname{grad}}\, \pi \widetilde{\underset{\sim}{\phi}_h}) \\
&= (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}}\,(w \pi \widetilde{\underset{\sim}{\phi}_h})) - R(w, \pi \widetilde{\underset{\sim}{\phi}_h}, \underset{\sim}{\phi}_h) \\
&= (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}}\,(w \pi \widetilde{\underset{\sim}{\phi}_h})^I) + \\
&\quad + \{(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}}\,[w \pi \widetilde{\underset{\sim}{\phi}_h} - (w \pi \widetilde{\underset{\sim}{\phi}_h})^I]) \\
&\quad - R(w, \pi \widetilde{\underset{\sim}{\phi}_h}, \underset{\sim}{\phi}_h)\} =: G_1 + H_1 . \qquad (5.9)
\end{aligned}
$$

To bound G_1, we take $\underset{\sim}{\psi} = (w \pi \widetilde{\underset{\sim}{\phi}_h})^I$ in (5.1) and get

$$
\begin{aligned}
G_1 &= (\operatorname{div}\,(w \pi \widetilde{\underset{\sim}{\phi}_h})^I, p_h) \\
&= (\operatorname{div}\,(w \pi \widetilde{\underset{\sim}{\phi}_h}), p_h) + (\operatorname{div}\,[(w \pi \widetilde{\underset{\sim}{\phi}_h})^I - w \pi \widetilde{\underset{\sim}{\phi}_h}], p_h) \\
&= (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, w p_h) + (\underset{\sim}{\operatorname{grad}}\, w, p_h\, \pi \widetilde{\underset{\sim}{\phi}_h}) + (\operatorname{div}\,[(w \pi \widetilde{\underset{\sim}{\phi}_h})^I - w \pi \widetilde{\underset{\sim}{\phi}_h}], p_h) \\
&= (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \widetilde{p_h}) + (\underset{\sim}{\operatorname{grad}}\, w, p_h\, \pi \widetilde{\underset{\sim}{\phi}_h}) + (\operatorname{div}\,[(w \pi \widetilde{\underset{\sim}{\phi}_h})^I - w \pi \widetilde{\underset{\sim}{\phi}_h}], p_h) \\
&=: (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \widetilde{p_h}) + H_2 . \qquad (5.10)
\end{aligned}
$$

Combining (5.7), (5.8), (5.9) and (5.10), we obtain

$$
\begin{aligned}
c \| \pi \widetilde{\underset{\sim}{\phi}_h} \|^2_{1, G_h} &\leqslant (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \widetilde{p_h}) + H_1 + H_2 - (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \widetilde{p_h} - \pi \widetilde{p_h}) \qquad (5.11) \\
&= (\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \pi \widetilde{p_h}) + H_1 + H_2 .
\end{aligned}
$$

Taking $q = \pi \widetilde{p_h}$ in (5.5), we get

$$
\begin{aligned}
(\operatorname{div}\, \pi \widetilde{\underset{\sim}{\phi}_h}, \pi \widetilde{p_h}) &= (\operatorname{div}\, \widetilde{\underset{\sim}{\phi}_h}, \pi \widetilde{p_h}) = (\operatorname{div}\,(w \underset{\sim}{\phi}_h), \pi \widetilde{p_h}) \\
&= (\operatorname{div}\, \underset{\sim}{\phi}_h, w \pi \widetilde{p_h}) + (\underset{\sim}{\operatorname{grad}}\, w, \pi \widetilde{p_h}\, \underset{\sim}{\phi}_h) \\
&= (\operatorname{div}\, \underset{\sim}{\phi}_h, w \pi \widetilde{p_h} - (w \pi \widetilde{p_h})^I) + (\underset{\sim}{\operatorname{grad}}\, w, \pi \widetilde{p_h}\, \underset{\sim}{\phi}_h) =: H_3 , \qquad (5.12)
\end{aligned}
$$

where we used (5.2) at the last step. Applying the Schwarz inequality (2.2), and the superapproximation property A2, we get

$$|H_1| \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G'} + \|\underset{\sim}{\phi}_h\|_{0,G'})\,\|\pi\widetilde{\underset{\sim}{\phi}_h}\|_{1,G_h},$$

$$|H_2| \leqslant C(\|p_h\|_{-1,G'} + h\|p_h\|_{0,G'})\,\|\pi\widetilde{\underset{\sim}{\phi}_h}\|_{1,G_h},$$

$$|H_3| \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G'} + \|\underset{\sim}{\phi}_h\|_{0,G'})\,\|\pi\widetilde{p_h}\|_{0,G_h}.$$

Combining the above three inequalities with (5.11) and (5.12), and using the arithmetic-geometry mean inequality, we arrive at

$$\|\pi\widetilde{\underset{\sim}{\phi}_h}\|_{1,G_h}^2 \leqslant C_1(h^2\|\underset{\sim}{\phi}_h\|_{1,G'}^2 + \|\underset{\sim}{\phi}_h\|_{0,G'}^2 + h^2\|p_h\|_{0,G'}^2 + \|p_h\|_{-1,G'}^2) \tag{5.13}$$
$$+ C_2(\|\underset{\sim}{\phi}_h\|_{0,G'} + h\|\underset{\sim}{\phi}_h\|_{1,G'})\,\|\pi\widetilde{p_h}\|_{0,G_h}.$$

Next we estimate $\|\pi\widetilde{p_h}\|_{0,G_h}$. By the triangle inequality,

$$\|\pi\widetilde{p_h}\|_{0,G_h} \leqslant \left\|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\text{meas}\,(G_h)}\right\|_{0,G_h} +$$
$$+\ \text{meas}\,(G_h)^{-1}\left\|\int_{G_h}(\pi\widetilde{p_h} - \widetilde{p_h})\right\|_{0,G_h}$$
$$+\ \text{meas}\,(G_h)^{-1}\left\|\int_{G_h}\widetilde{p_h}\right\|_{0,G_h}. \tag{5.14}$$

Notice that the second term on the right hand side of (5.14) is bounded above by the right hand side of (5.6), and, for the last term,

$$\left\|\int_{G_h}\widetilde{p_h}\right\|_{0,G_h} = \left\|\int_{G_h}wp_h\right\|_{0,G_h} \leqslant C\|p_h\|_{-1,G'}. \tag{5.15}$$

To estimate the first term, we use the inf-sup condition,

$$\left\|\pi\widetilde{p_h} - \frac{\int_{G_h}\pi\widetilde{p_h}}{\text{meas}\,(G_h)}\right\|_{0,G_h} \leqslant C \sup_{\substack{\underset{\sim}{\psi}\in \mathring{V}_h(G_h)\\ \underset{\sim}{\psi}\neq 0}} \frac{(\text{div}\,\underset{\sim}{\psi}, \pi\widetilde{p_h})_{G_h}}{\|\underset{\sim}{\psi}\|_{1,G_h}}. \tag{5.16}$$

To deal with the numerator on the right hand side of (5.16), we apply (5.4),

$$
\begin{aligned}
(\operatorname{div} \underset{\sim}{\psi}, \pi \widetilde{p_h}) &= (\operatorname{div} \underset{\sim}{\psi}, \widetilde{p_h}) - (\underset{\approx}{\operatorname{grad}} (\widetilde{\underset{\sim}{\phi}_h} - \pi \widetilde{\underset{\sim}{\phi}_h}), \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) \\
&= (\operatorname{div} \underset{\sim}{\psi}, w p_h) - (\underset{\approx}{\operatorname{grad}} (\widetilde{\underset{\sim}{\phi}_h} - \pi \widetilde{\underset{\sim}{\phi}_h}), \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) \\
&= (\operatorname{div} (w \underset{\sim}{\psi}), p_h) - (\underset{\approx}{\operatorname{grad}} (\tilde{\underset{\sim}{\phi}}_h - \pi \widetilde{\underset{\sim}{\phi}_h}), \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) - \\
&\quad - (\underset{\sim}{\operatorname{grad}} \underset{\sim}{w}, p_h \underset{\sim}{\psi}) \\
&= (\operatorname{div} (w \underset{\sim}{\psi})^I, p_h) - (\underset{\approx}{\operatorname{grad}} (\widetilde{\underset{\sim}{\phi}_h} - \pi \widetilde{\underset{\sim}{\phi}_h}), \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) \\
&\quad + (\operatorname{div} (w \underset{\sim}{\psi} - (w \underset{\sim}{\psi})^I), p_h) - (\underset{\sim}{\operatorname{grad}} w, p_h \underset{\sim}{\psi}) . \qquad (5.17)
\end{aligned}
$$

We use (5.1) to treat $(\operatorname{div} (w \underset{\sim}{\psi})^I, p_h)$ and get

$$
\begin{aligned}
&(\operatorname{div} (w \underset{\sim}{\psi})^I, p_h) = (\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \ \underset{\approx}{\operatorname{grad}} (w \underset{\sim}{\psi})^I) \\
&= (\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \ \underset{\approx}{\operatorname{grad}} (w \underset{\sim}{\psi})) + (\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \ \underset{\approx}{\operatorname{grad}} [(w \underset{\sim}{\psi})^I - w \underset{\sim}{\psi}]) \\
&= (\underset{\approx}{\operatorname{grad}} (w \underset{\sim}{\phi}_h), \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) + \{R(w, \underset{\sim}{\psi}, \underset{\sim}{\phi}_h) + \\
&\quad + (\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \ \underset{\approx}{\operatorname{grad}} [(w \underset{\sim}{\psi})^I - w \underset{\sim}{\psi}])\} \\
&=: (\underset{\approx}{\operatorname{grad}} \widetilde{\underset{\sim}{\phi}_h}, \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) + M_1 . \qquad (5.18)
\end{aligned}
$$

Combining (5.17) and (5.18), we get

$$
\begin{aligned}
(\operatorname{div} \underset{\sim}{\psi}, \pi \widetilde{p_h}) &= (\underset{\approx}{\operatorname{grad}} \pi \widetilde{\underset{\sim}{\phi}_h}, \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) + \{(\operatorname{div} (w \underset{\sim}{\psi} - (w \underset{\sim}{\psi})^I, p_h) - \\
&\qquad - (\underset{\sim}{\operatorname{grad}} w, p_h \underset{\sim}{\psi})\} + M_1 \\
&=: (\underset{\approx}{\operatorname{grad}} \pi \widetilde{\phi_h}, \ \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) + M_1 + M_2 . \qquad (5.19)
\end{aligned}
$$

Then applying the superapproximation property, the Schwarz inequality, and (2.2), we arrive at

$$
\begin{aligned}
|M_1| &\leqslant C(\| \underset{\sim}{\phi}_h \|_{0, G'} + h \| \underset{\sim}{\phi}_h \|_{1, G'}) \| \underset{\sim}{\psi} \|_{1, G_h} , \\
|M_2| &\leqslant C(h \| p_h \|_{0, G'} + \| p_h \|_{-1, G'}) \| \underset{\sim}{\psi} \|_{1, G_h} , \\
|(\underset{\approx}{\operatorname{grad}} \pi \widetilde{\underset{\sim}{\phi}_h}, \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi})| &\leqslant \| \pi \widetilde{\underset{\sim}{\phi}_h} \|_{1, G_h} \| \underset{\sim}{\psi} \|_{1, G_h} .
\end{aligned}
$$

Combining (5.14), (5.15), (5.16), and (5.19) with the above three inequalities, we obtain

$$\|\pi \tilde{p}_h\|_{0,G_h} \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G'} + \|\underset{\sim}{\phi}_h\|_{0,G'} + h\|p_h\|_{0,G'} + \\ + \|p_h\|_{-1,G'} + \|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h}) . \quad (5.20)$$

Substituting (5.20) into (5.13), we obtain

$$\|\pi\tilde{\underset{\sim}{\phi}}_h\|_{1,G_h} \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G'} + \|\underset{\sim}{\phi}_h\|_{0,G'} + h\|p_h\|_{0,G'} + \|p_h\|_{-1,G'}) . \quad (5.21)$$

Thus, substituting (5.21) back into (5.20), we find that $\|\pi \tilde{p}_h\|_{0,G_h}$ is also bounded above by the right hand side of (5.21). Therefore, from (5.7) we obtain

$$\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G'} + \|\underset{\sim}{\phi}_h\|_{0,G'} + \\ + h\|p_h\|_{0,G'} + \|p_h\|_{-1,G'}) .$$

Applying Theorem 4.1 for the case that $\underset{\sim}{\phi} = p = 0$ and G' in place of G_0, we finally arrive at

$$\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \leqslant C(h\|\underset{\sim}{\phi}_h\|_{,G} + \|\underset{\sim}{\phi}_h\|_{-t,G} + \\ + h\|p_h\|_{0,G} + \|p_h\|_{-t-1,G}) . \quad \square$$

LEMMA 5.2 : *Suppose the conditions of Lemma 5.1 are satisfied. Then*

$$\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \leqslant C(\|\underset{\sim}{\phi}_h\|_{-t,G} + \|p_h\|_{-t-1,G}) . \quad (5.22)$$

Proof: Let $G_0 \Subset G_1 \Subset \dots \Subset G_{t+2} = G$ be concentric disks and apply Lemma 5.1 to each pair $G_j \Subset G_{j+1}$ to get

$$\|\underset{\sim}{\phi}_h\|_{1,G_j} + \|p_h\|_{0,G_j} \leqslant \\ \leqslant C(h\|\underset{\sim}{\phi}_h\|_{1,G_{j+1}} + h\|p_h\|_{0,G_{j+1}} + \|\underset{\sim}{\phi}_h\|_{-t,G_{j+1}} + \|p_h\|_{-t-1,G_{j+1}}) . \quad (5.23)$$

Combining these we obtain

$$\|\underset{\sim}{\phi}_h\|_{1,G_0} + \|p_h\|_{0,G_0} \leqslant C(h^{t+1}\|\underset{\sim}{\phi}_h\|_{1,G_{t+1}} + h^{t+1}\|p_h\|_{0,G_{t+1}} \\ + \|\underset{\sim}{\phi}_h\|_{-t,G_{t+1}} + \|p_h\|_{-t+1,G_{t+1}}) . \quad (5.24)$$

While by A3, we can find G_h, $G_{t+1} \Subset G_h \Subset G_{t+2} = G$, such that

$$h^{t+1} \| \phi_h \|_{1, G_{t+1}} \leqslant h^{t+1} \| \phi_h \|_{1, G_h} \leqslant C \| \phi_h \|_{-t, G_h} \leqslant C \| \phi_h \|_{-t, G} , \tag{5.25}$$

$$h^{t+1} \| p_h \|_{1, G_{t+1}} \leqslant h^{t+1} \| p_h \|_{0, G_h} \leqslant C \| p_h \|_{-t-1, G_h} \leqslant C \| p_h \|_{-t-1, G} .$$

Thus inequality (5.22) follows from (5.23), (5.24), and (5.25). □

We now state the main result of the paper.

THEOREM 5.3 : *Let* $\Omega_0 \Subset \Omega_1 \Subset \Omega$ *and suppose that* $(\phi, p) \in H^1(\Omega) \times L^2(\Omega)$ *(the exact solution) satisfies* $\phi|_{\Omega_1} \in H^{m-1}(\Omega_1)$ $p|_{\Omega_1} \in H^{m-1}(\Omega_1)$ *for some integer* $m > 0$. *Suppose that* $(\phi_h, p_h) \in V_h \times W_h$ *(the finite element solution) is given so that* (4.1) *and* (4.2) *hold. Let t be a nonnegative integer. Then there exists a constant C depending only on* Ω_1, Ω_0, *and t, such that*

$$\| \phi - \phi_h \|_{s, \Omega_0} + \| p - p_h \|_{s-1, \Omega_0} \leqslant C(h^{r_1 - s} \| \phi \|_{m, \Omega_1} + h^{r_2 - s} \| p \|_{m-1, \Omega_1}$$

$$+ \| \phi - \phi_h \|_{-t, \Omega_1} + \| p - p_h \|_{-t-1, \Omega_1}) , \quad s = 0, 1 \tag{5.26}$$

with $r_1 = \min(k_1 + 1, m)$, $r_2 = \min(k_2 + 2, m)$, *and* k_1, k_2 *as in A1.*

The theorem will follow easily from a slightly more localized version.

LEMMA 5.4 : *Suppose the hypotheses of Theorem 5.3 are fulfilled and, in addition, that* $\Omega_0 = G_0$ *and* $\Omega_1 = G$ *are concentric disks. Then the conclusion of the theorem holds.*

Proof : Let $G_0' \Subset G'$ be further concentric disks strictly contained between G_0 and G and let G_h be a union of elements which is strictly contained between G' and G and for which properties A3 and A4 hold. Thus

$$G_0 \Subset G_0' \Subset G' \Subset G_h \Subset G \Subset \Omega .$$

Take $w \in C_0^\infty(G')$ identically 1 on G_0' and set $\tilde{\phi} = w\phi$, $\tilde{p} = wp$. Let $\pi\tilde{\phi} \in \mathring{V}_h(G)$, $\pi\tilde{p} \in W_h(G)$ be defined by

$$(\operatorname{grad}(\tilde{\phi} - \pi\tilde{\phi}), \operatorname{grad} \psi) - (\operatorname{div} \psi, \tilde{p} - \pi\tilde{p}) = 0 ,$$

$$\text{for all} \quad \psi \in \mathring{V}_h(G_h) , \tag{5.27}$$

$$(\operatorname{div}(\tilde{\phi} - \pi\tilde{\phi}), q) = 0 ,$$

$$\text{for all } q \in W_h(G_h) , \tag{5.28}$$

together with $\int_{G_h} \pi\tilde{p} = \int_{G_h} \tilde{p}$. Then using Lemma 3.1 and A1 we have

$$\|\tilde{\underset{\sim}{\phi}} - \pi\tilde{\underset{\sim}{\phi}}\|_{1,G_h} + \|\tilde{p} - \pi\tilde{p}\|_{0,G_h} \leqslant C\left(\inf_{\underset{\sim}{\psi} \in \mathring{V}_h(G_h)} \|\tilde{\underset{\sim}{\phi}} - \underset{\sim}{\psi}\|_{1,G_h} + \right.$$

$$\left. + \inf_{q \in W_h(G_h)} \|\tilde{p} - q\|_{0,G_h}\right)$$

$$\leqslant C(h^{r_1 - 1}\|\underset{\sim}{\phi}\|_{m,G_h} + h^{r_2 - 1}\|p\|_{m-1,G_h}) . \tag{5.29}$$

Let's now estimate $\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1,G_0}$ and $\|p - p_h\|_{0,G_0}$. First, the triangle inequality gives us

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1,G_0} + \|p - p_h\|_{0,G_0}$$

$$\leqslant \|\underset{\sim}{\phi} - \pi + \tilde{\underset{\sim}{\phi}}\|_{1,G_0} + \|p - \pi\tilde{p}\|_{0,G_0} + \|\pi\tilde{\underset{\sim}{\phi}} - \underset{\sim}{\phi}_h\|_{1,G_0} + \|\pi\tilde{p} - p_h\|_{0,G_0}$$

$$\leqslant \|\underset{\sim}{\phi} - \pi + \tilde{\underset{\sim}{\phi}}\|_{1,G_h} + \|\tilde{p} - \pi\tilde{p}\|_{0,G_h} + \|\pi\tilde{\underset{\sim}{\phi}} - \underset{\sim}{\phi}_h\|_{1,G_0} + \|\pi\tilde{p} - p_h\|_{0,G_0}$$

$$\leqslant C(h^{r_1 - 1}\|\underset{\sim}{\phi}\|_{m,G_h} + h^{r_2 - 1}\|p\|_{m-1,G_h}) +$$

$$+ \|\pi\tilde{\underset{\sim}{\phi}} - \underset{\sim}{\phi}_h\|_{1,G_0} + \|\pi\tilde{p} - p_h\|_{0,G_0} . \tag{5.30}$$

From (5.27), (5.28) and (4.1), (4.2) we find

$$(\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \pi\tilde{\underset{\sim}{\phi}}), \underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\psi}) - (\operatorname{div}\underset{\sim}{\psi}, p_h - \pi\tilde{p}) = 0 ,$$

$$\text{for all} \quad \underset{\sim}{\psi} \in \mathring{V}_h(G_0') ,$$

$$(\operatorname{div}(\underset{\sim}{\phi}_h - \pi\tilde{\underset{\sim}{\phi}}), q) = 0 , \quad \text{for all} \quad q \in \mathring{W}_h(G_0') .$$

We next apply Lemma 5.2 to $\underset{\sim}{\phi}_h - \pi\tilde{\underset{\sim}{\phi}}$ and $p_h - \pi\tilde{p}$ wuth G replaced by G_0'. Then it follows from (5.22) that

$$\|\underset{\sim}{\phi}_h - \pi\tilde{\underset{\sim}{\phi}}\|_{1,G_0} + \|p_h - \pi\tilde{p}\|_{0,G_0} \leqslant C(\|\underset{\sim}{\phi}_h - \pi\tilde{\underset{\sim}{\phi}}\|_{-t,G_0'} +$$

$$+ \|p_h - \pi\tilde{p}\|_{-t-1,G_0'})$$

$$\leqslant C(\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t,G_0'} + \|p - p_h\|_{-t-1,G_0'} + \|\underset{\sim}{\phi} - \pi\tilde{\underset{\sim}{\phi}}\|_{-t,G_0'} +$$

$$+ \|p - \pi\tilde{p}\|_{-t-1,G_0'})$$

$$\leqslant C(\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t,G} + \|p - p_h\|_{-t-1,G} + \|\tilde{\underset{\sim}{\phi}} - \pi\tilde{\underset{\sim}{\phi}}\|_{1,G_h} + \|\tilde{p} - \pi\tilde{p}\|_{0,G_h}) .$$

In the light of (5.30), (5.29), and the above inequality, we have

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1, G_0} + \|p - p_h\|_{0, G_0} \leqslant (h^{r_1 - 1}\|\underset{\sim}{\phi}\|_{m, G} + h^{r_2 - 1}\|p\|_{m-1, G} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t, G} + \|p - p_h\|_{-t-1, G}) . \tag{5.31}$$

Thus, we have proved the desired result for $s = 1$. For $s = 0$, we just apply Theorem 4.1 to the disks G_0 and G' and get

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{0, G_0} + \|p - p_h\|_{-1, G_0} \leqslant C(h\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{1, G'} + h\|p - p_h\|_{0, G'} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t, G'} + \|p - p_h\|_{-t-1, G'}) .$$

Then, applying (5.31) with G_0 replaced by G', we obtain the desired result

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{0, G_0} + \|p - p_h\|_{-1, G_0} \leqslant C(h^{r_1}\|\underset{\sim}{\phi}\|_{m, G} + h^{r_2}\|p\|_{m-1, G} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t, G} + \|p - p_h\|_{-t-1, G}) . \quad \square$$

Proof of Theorem 5.3 : The argument here is same as in Theorem 5.1 of [10]. Let $d = d_0/2$ where $d_0 = \text{dist}(\overline{\Omega}_0, \partial\Omega_1)$. Cover $\overline{\Omega}_0$ with a finite number of disks $G_0(x_i)$, $i = 1, 2, \dots, k$ centered at $x_i \in \overline{\Omega}_0$ with $\text{diam}\, G_0(x_i) = d$. Note that the number of disks, k, depends only on Ω_0 and Ω_1. Let $G(x_i)$, $i = 1, 2, \dots, k$ be corresponding concentric disks with $\text{diam}\, G(x_i) = 2\,d$. Applying Lemma 5.4, we have

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{s, G_0(x_i)} + \|p - p_h\|_{s-1, G_0(x_i)} \leqslant C_i(h^{r_1 - s}\|\underset{\sim}{\phi}\|_{m, G(x_i)} + h^{r_2 - s}\|p\|_{m-1, G(x_i)} + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_{-t, G(x_i)} + \|p - p_h\|_{-t-1, G(x_i)}) . \tag{5.32}$$

Then the inequality (5.26) follows by summing (5.32) i. $\square$

6. AN EXAMPLE APPLICATION

As an example, we apply our general result to the Stokes system when the domain is a non-convex polygon, in which case the finite element approximation does not achieve optimal convergence rate in the energy norm on the whole domain, due to the boundary singularity of the exact solution.

Assume that Ω is a non-convex polygon. Then it is known that the solution of the Stokes system satisfies

$$\underset{\sim}{\phi} \in \underset{\sim}{H}^{s+1}(\Omega) \cap \underset{\sim}{H}_0^1(\Omega), \quad p \in H^s(\Omega),$$

$$\underset{\sim}{\phi} \in \underset{\sim}{H}^2(\Omega_1), p \in H^1(\Omega_1), \quad \text{if} \quad \Omega_1 \Subset \Omega,$$

for $s < s_\Omega$, where s_Ω is a constant which is determined by the largest interior angle of Ω [4]. For a non-convex polygonal domain we have $1/2 < s_\Omega < 1$. The value of s_Ω for various angles have been tabulated in [4]. For example, for an L-shaped domain, $s_\Omega \sim 0.544$.

The MINI element was introduced by Arnold, Brezzi and Fortin [1] as a stable Stokes element with few degrees of freedom. Here the velocity is approximated by the space of continuous piecewise linear functions and bubble functions and the pressure is approximated by the space of continuous piecewise linear functions only. Globally we have

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{1,\Omega} + \| p - p_h \|_{0,\Omega} \leqslant C h^s (\| \underset{\sim}{\phi} \|_{s+1,\Omega} + \| p \|_{s,\Omega}),$$

which reflects a loss of accuracy due to the singularity of the solutions.

In order to apply Theorem 5.3, we note that a standard duality argument as in [1] gives us

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{0,\Omega} + \| p - p_h \|_{-1,\Omega} \leqslant C h^{2s} \| \underset{\sim}{F} \|_{0,\Omega}.$$

Hence, according to Theorem 5.3, for $\Omega_0 \Subset \Omega_1 \Subset \Omega$, we have

$$\| \underset{\sim}{\phi} - \underset{\sim}{\phi}_h \|_{1,\Omega_0} + \| p - p_h \|_{0,\Omega_0} \leqslant C(h \| \underset{\sim}{\phi} \|_{2,\Omega_1} + h \| p \|_{1,\Omega_1} + h^{2s} \| \underset{\sim}{F} \|_{0,\Omega}).$$

Since $2s > 1$, the finite element approximation achieves the optimal order of convergence rate in the energy norm in interior subdomains.

REFERENCES

[1] D. N. Arnold, F. Brezzi and M. Fortin, A stable finite element for the Stokes equations, *Calcolo,* **21,** 1984, pp. 337-344.

[2] D. N. Arnold and R. S. Falk, A uniformly accurate finite element method for the Mindlin-Reissner plate, *SIAM J. Numer. Anal.,* **26,** 1989, pp. 1276-1290.

[3] M. Crouzeix and P.-A. Raviart, Conforming and non-conforming finite element methods for solving the stationary Stokes equations, *RAIRO Anal Numér.,* **7** R-3, 1973, pp. 33-76.

[4] M. DAUGE, Stationary Stokes and Navier-Stokes systems on two- or three-dimentional domains with corners. Part I : Linearized equations, *SIAM J. Math. Anal.,* **20,** 1989, pp. 74-97.

[5] J. DOUGLAS, Jr. and F. A. MILNER, Interior and superconvergence estimates for mixed methods for second order elliptic problems, *RAIRO Modél. Math. Anal. Numér.,* **19,** 1985, pp. 397-428.

[6] M. FORTIN, Calcul numérique des écoulements des fluides de Bingham et des fluides Newtoniens incompressible par des méthodes d'éléments finis, Université de Paris VI, Doctoral thesis, 1972.

[7] L. GASTALDI, Uniform interior error estimates for the Reissner-Mindlin plate model, *Math. Comp.,* **61,** 1993, pp. 539-567.

[8] P. HOOD and C. TAYLOR, A numerical solution of the Navier-Stokes equations using the finite element technique, *Comput. & Fluids,* **1,** 1973, pp. 73-100.

[9] L. MANSFIELD, Finite element subspaces with optimal rates of convergence for stationary Stokes problem, *RAIRO Anal. Numér.,* **16,** 1982, pp. 49-66.

[10] J. A. NITSCHE and A. H. SCHATZ, Interior estimate for Ritz-Galerkin methods, *Math. Comp.,* **28,** 1974, pp. 937-958.

[11] R. TÉMAM, *Navier-Stokes Equations,* North-Holland, Amsterdam, 1984.

[12] L. B. WAHLBIN, Local Behavior in Finite Element Methods, in *Handbook of Numerical Analysis,* P. G. Ciarlet and J. L. Lions, eds., Elsevier, Amsterdam-New York, 1991.