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FOURIER-Chebyshev PSEUDOSPECTRAL METHODS FOR THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS (*)

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Abstract. — *Fourier-Chebyshev pseudospectral approximation for the two-dimensional unsteady Navier-Stokes equations is analyzed. The generalized stability and convergence are proved strictly. The numerical results are presented.*

Résumé. — *Dans cet article, on propose une méthode combinant l'approche pseudo-spectrale de type Fourier et l'approche pseudo-spectrale de type Chebyshev pour résoudre les équations de Navier-Stokes en dimension 2. Une équation de type Poisson est utilisée pour calculer la pression. On démontre la stabilité et la convergence. Les résultats numériques sont présentés.*

Key words: Navier-Stokes equations, Fourier-Chebyshev pseudospectral approximation.
Subject Classification. AMS(MOS): 65N30, 76D99.

1. INTRODUCTION

Since spectral and pseudospectral approximations have the accuracy of « infinite order », they have become two of the most efficient numerical methods for solving the nonlinear partial differential equations arising in fluid dynamics (see [1-7]).

In studying the channel flow, parallel boundary layer and some of related topics, we often meet semi-periodic flows. We could use mixed spectral-finite difference approximation or finite element approximation. But the accuracy is limited due to the finite difference approximation or finite element approximation. Thus in order to keep the accuracy of « infinite order », some authors proposed mixed Fourier-Chebyshev spectral or pseudospectral approximation, e.g., see [8-10]. In this paper, we consider the mixed Fourier-Chebyshev pseudospectral approximation for the two-dimensional Navier-Stokes equations.

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Let $x = (x_1, x_2)^T$ and $\Omega = I_1 \times I_2$ where $I_1 = \{x_1 / -1 < x_1 < 1\}$, $I_2 = \{x_2 / -\pi < x_2 < \pi\}$. We denote by Γ the boundary $\{x / |x_1| = 1, x_2 \in I_2\}$. The speed and the pressure are denoted by $U(x, t) = (U_1(x, t), U_2(x, t))^T$ and $P(x, t)$. $\nu > 0$ is the kinetic viscosity. $U_0(x)$ and $f(x, t)$ are given functions with the period 2π for the variable x_2 . Let $\partial_t = \partial / \partial t$ and $\partial_j = \partial / \partial x_j$, $j = 1, 2$. The Navier-Stokes equations are as follows (see [7])

$$\begin{cases} \partial_t U + (U \cdot \nabla) U - \nu \nabla^2 U + \nabla P = f, & \text{in } \Omega \times (0, T], \\ \nabla^2 P + \Phi(U) = \nabla \cdot f, & \text{in } \Omega \times (0, T], \\ U(x, 0) = U_0(x), & \text{in } \overline{\Omega} \end{cases} \quad (1.1)$$

where

$$\Phi(U) = 2(\partial_2 U_1 \partial_1 U_2 - \partial_1 U_1 \partial_2 U_2).$$

Suppose that U and P have the period 2π for the variable x_2 , and that the boundary is a fixed non-slip wall and so $U = 0$ on Γ . There is no boundary condition for the pressure generally. But if we use the second equation to evaluate the value of the pressure, then we need a non-standard boundary condition. Since the boundary is non-slip, we can assume approximately that $\partial P / \partial n|_{\Gamma} = 0$ (see [11]). Then the second equation of (1.1) is a Neumann problem for P at each time. It requires the consistency condition, i.e.,

$$\int_{\Omega} (\Phi(U) - \nabla \cdot f) dx = 0.$$

Since

$$\Phi(U) = 2[\partial_2(U_1 \partial_1 U_2) - \partial_2(U_1 \partial_2 U_2)]$$

and the boundary conditions for U , we know that

$$\int_{\Omega} \Phi(U) dx = 0.$$

Moreover, by the symmetry of the boundary Γ , we can assume that

$$\int_{I_2} (f(1, x_2, t) - f(-1, x_2, t)) dx_2 = 0, \quad \forall t \in [0, T],$$

and so

$$\int_{\Omega} \nabla \cdot f dx = 0.$$

Thus this Neumann problem for P is consistent. For fixing the value of pressure, we require that for all $t \leq T$,

$$\int_{\Omega} P(x, t) dx = 0.$$

This model is often used in fluid dynamics.

We shall propose a scheme for the above model, by using Fourier pseudospectral approximation in the periodic direction and Chebyshev pseudospectral approximation in the other direction. This algorithm keeps the convergence rate of « infinite order » as Fourier-Chebyshev spectral approximation, but it is much easier to be performed. In particular, it is easy to deal with nonlinear terms and saves a lot of computations. On the other hand, since we calculate the pressure based on the second formula of (1.1), we avoid the very difficult job of choosing the trial function space in which the divergence of every element vanishes everywhere. We construct the scheme in the next section and present the numerical results in Section III, which show the advantages of this method. We list some lemmas in Section IV. Then we analyze the generalized stability in Section V and the convergence in Section VI respectively. As we know, the theoretical analysis of Chebyshev pseudospectral approximation is much more complicated than the spectral or pseudospectral approximation based on Fourier expansion. So the technique used in this paper is quite precise. It is also very useful for other related problems.

2. THE SCHEME

Let D be an interval (or a domain) in R (or R^2). $L^2(D)$ and $H^r(D)$ ($r > 0$) denote the usual Hilbert spaces with the usual inner product and norm, etc. Let $\omega(x_1) = (1 - x_1^2)^{-\frac{1}{2}}$, and u and v be scalar or vector function. Define

$$(u, v)_{\omega, I_1} = \int_{I_1} uv\omega dx_1, \quad \|v\|_{\omega, I_1} = (v, v)_{\omega, I_1}^{\frac{1}{2}},$$

$$L^2_{\omega}(I_1) = \{v/v \text{ is measurable and } \|v\|_{\omega, I_1} < \infty\}.$$

Furthermore,

$$(u, v)_{\omega} = \frac{1}{2\pi} \int_{\Omega} uv\omega dx, \quad \|v\|_{\omega} = (v, v)_{\omega}^{\frac{1}{2}},$$

$$L^2_{\omega}(\Omega) = \{v/v \text{ is measurable and } \|v\|_{\omega} < \infty\}.$$

Now, let M and N be positive integers. We denote by \mathcal{P}_M the set containing all polynomials with degree $\leq M$, defined on I_1 . Let

$$V_M = \{v(x_1) \in \mathcal{P}_M / v(-1) = v(1) = 0\},$$

$$W_M = \left\{v(x_1) \in \mathcal{P}_M / \frac{dv}{dx_1}(-1) = \frac{dv}{dx_1}(1) = 0\right\}.$$

Set

$$\tilde{V}_N = \text{span} \{e^{ilx_2} / |l| \leq N\}.$$

We denote by V_N the subset of \tilde{V}_N , containing all real-valued functions. Let

$$V_{M,N} = V_M \times V_N, \quad W_{M,N} = W_M \times V_N.$$

Let $P_M^{(1)}$ be the orthogonal projection operator from $L^2_\omega(I_1)$ to \mathcal{P}_M and $P_N^{(2)}$ be the orthogonal projection operator from $L^2(I_2)$ to V_N . Furthermore, $P_{M,N} = P_M^{(1)} P_N^{(2)}$.

Next, let $\{x_1^{(j)}, \omega^{(j)}\}$ be the nodes and weights of Gauss-Lobatto integration, i.e.,

$$x_1^{(j)} = \cos \frac{j\pi}{M}, \quad \text{for } 0 \leq j \leq M,$$

$$\omega^{(0)} = \omega^{(M)} = \frac{\pi}{2M}, \quad \omega^{(j)} = \frac{\pi}{M}, \quad \text{for } 1 \leq j \leq M-1.$$

Let $h = 2\pi/2N+1$ be the mesh size for the variable x_2 , and set

$$\Omega_{M,N} = \{(x_1, x_2) / x_1 = x_1^{(j)}, x_2 = kh, 1 \leq j \leq M-1, |k| \leq N\},$$

$$\bar{\Omega}_{M,N} = \{(x_1, x_2) / x_1 = x_1^{(j)}, x_2 = kh, 0 \leq j \leq M, |k| \leq N\}.$$

We introduce the following discrete inner products and norms

$$(u, v)_{M,\omega} = \sum_{j=0}^M \omega^{(j)} u(x_1^{(j)}) v(x_1^{(j)}), \quad \|v\|_{M,\omega} = (v, v)_{M,\omega}^{\frac{1}{2}},$$

$$(u, v)_{M,N,\omega} = \frac{1}{2N+1} \sum_{j=0}^M \sum_{|k| \leq N} \omega^{(j)} u(x_1^{(j)}, kh) v(x_1^{(j)}, kh),$$

$$\|u, v\|_{M,N,\omega} = (v, v)_{M,N,\omega}^{\frac{1}{2}}, \quad |v|_{1,M,N,\omega} = (\|\partial_1 v\|_{M,N,\omega}^2 + \|\partial_2 v\|_{M,N,\omega}^2)^{\frac{1}{2}}.$$

Let $P_C^{(1)}$ be the interpolation from $C(\bar{I}_1)$ to \mathcal{P}_M , and $P_C^{(2)}$ be the interpolation from $C(\bar{I}_2)$ to V_N , such that

$$P_C^{(1)} v(x_1^{(j)}, x_2) = v(x_1^{(j)}, x_2),$$

$$P_C^{(2)} v(x_1, x_2^{(k)}) = v(x_1, x_2^{(k)}) = v(x_1, x_2^{(k)}), \quad 0 \leq j \leq M, |k| \leq N.$$

Moreover $P_C = P_C^{(1)} P_C^{(2)}$ and then

$$P_C v(x_1, x_2) = v(x_1, x_2), \quad \text{in } \bar{\Omega}_{M,N}.$$

Let τ be the step of time t , and

$$S_\tau = \left\{ t/t = k\tau, 0 \leq k \leq \left[\frac{T}{\tau} \right] \right\}.$$

For simplicity, we denote $u(x, t)$ by $u(t)$ or u sometimes. Let

$$u_\tau(t) = \frac{1}{\tau}(u(t + \tau) - u(t)).$$

To approximate the nonlinear terms, we define

$$d_c(u, v) = \sum_{j=1}^2 P_C(v_j \partial_j u),$$

$$\Phi_c(u) = 2[P_C(\partial_2 u_1 \partial_1 u_2) - P_C(\partial_1 u_1 \partial_2 u_2)].$$

Now, let u and p be the approximations to U and P respectively. The Fourier-Chebyshev pseudospectral scheme for problem (1.1) with the boundary condition mentioned in the above, is to find $u \in V_{M,N}$, $p \in W_{M,N}$ for $t \in S_\tau$, such that

$$\begin{cases} u_\tau(t) + d_c(u(t), u(t)) - v \nabla^2(u(t) + \sigma r u(t)) + \nabla p(t) = P_C f(t), & \text{in } \Omega_{M,N} \times S_\tau, \\ \nabla^2 p(t) + \Phi_C(u(t)) = P_C(\nabla \cdot f(t)), & \text{in } \Omega_{M,N} \times S_\tau, \\ u(0) = P_C U_0, & \text{in } \bar{\Omega}_{M,N} \end{cases}$$

where σ is a parameter and $0 \leq \sigma \leq 1$. In addition, we require that for all $t \in S_\tau$,

$$\int_{\Omega} p(x, t) dx = 0 .$$

3. THE NUMERICAL RESULTS

In this section, we present some numerical results. The test functions are

$$\begin{cases} U_1 = A e^{Bt} (x_1^2 - 1)^2 \sin 2 x_2 , \\ U_2 = 2 A e^{Bt} (x_1^3 - x_1) \cos 2 x_2 , \\ P = C e^{Bt} (3 x_1 - x_1^3) \sin 2 x_2 . \end{cases}$$

For describing the errors, define

$$E(z(t)) = \left(\frac{\sum_{x \in \Omega_{M,N}} |z(x, t) - \zeta(x, t)|^2}{\sum_{x \in \Omega_{M,N}} |z(x, t)|^2} \right)^{\frac{1}{2}}$$

where $z = U$ or P , $\zeta = u$ or p .

We solve (1.1) by scheme (2.1) with $\sigma = 0$, $M = N = 4$ and $\tau = 0.005$. We obtain very accurate results, see Table I and Table II. For comparison, we also consider the Fourier pseudospectral-finite element scheme (FPSFE). In scheme FPSFE, we approximate the speed and the pressure by linear elements in the $x_1 -$ direction. The interval I_1 is uniformly

Table I. — The errors of (2.1), $A = 0.5$, $B = 0.1$, $C = 0.2$, $\nu = 0.001$.

t	$E(U(t))$	$E(P(t))$
0.5	0.1602E-3	0.1748E-3
1.0	0.3967E-3	0.3653E-3
1.5	0.7814E-3	0.5823E-3
2.0	0.1423E-2	0.8793E-3
2.5	0.2559E-2	0.1405E-2

partitioned with the mesh size $h^* = 2/M^*$. The calculation is carried out with $\sigma = 0$, $M^* = N = 4$ and $\tau = 0.005$. The numerical results show that (2.1) gives much better results than scheme FPSFE, see Table III and Table IV.

Table II. — The errors of (2.1), $A = 0.5$, $B = 0.1$, $C = 0.2$, $\nu = 0.0001$.

t	$E(U(t))$	$E(P(t))$
0.5	0.1614E-3	0.1769E-3
1.0	0.3962E-3	0.3707E-3
1.5	0.7840E-3	0.5902E-3
2.0	0.1425E-2	0.8829E-3
2.5	0.2573E-2	0.1408E-2

Table III. — The errors $E(U(t))$, $A = B = 0.1$, $C = 0$, $\nu = 0.001$.

t	Scheme (2.1)	FPSFE
0.5	0.1283E-3	0.2254E-1
1.0	0.2528E-3	0.4549E-1
1.5	0.3745E-3	0.6922E-1
2.0	0.4979E-3	0.9442E-1
2.5	0.6629E-3	0.1219

Table IV. — The errors $E(U(t))$, $A = B = 0.1$, $C = 0$, $\nu = 0.0001$.

t	Scheme (2.1)	FPSFE
0.5	0.1288E-3	0.2264E-1
1.0	0.2548E-3	0.4602E-1
1.5	0.3779E-3	0.7050E-1
2.0	0.5025E-3	0.9682E-1
2.5	0.6806E-3	0.1258

4. SOME LEMMAS

We first introduce some notations. For any integer $r \geq 0$, define

$$|v|_{r, \omega, I_1} = \left\| \frac{d^r v}{dx_1^r} \right\|_{\omega, I_1}, \quad \|v\|_{r, \omega, I_1} = \left(\sum_{k=0}^r |v|_{k, \omega, I_1}^2 \right)^{\frac{1}{2}},$$

$$H_\omega^r(I_1) = \{v / \|v\|_{r, \omega, I_1} < \infty\}.$$

For any real $r > 0$, $H_\omega^r(I_1)$ is defined by the complex interpolation between the spaces $H_\omega^{[r]}(I_1)$ and $H_\omega^{[r+1]}(I_1)$. Let B be a Banach space with the norm $\|\cdot\|_B$. Set

$$L^2(D, B) = \{v(z) : D \rightarrow B/v \text{ is strongly measurable, and } \|v\|_{L^2(D, B)} < \infty\},$$

$$C(D, B) = \{v(z) : D \rightarrow B/v \text{ is strongly measurable, and } \|v\|_{C(D, B)} < \infty\}$$

where

$$\|v\|_{L^2(D, B)} = \left(\int_D \|v(z)\|_B^2 dz \right)^{\frac{1}{2}}, \quad \|v\|_{C(D, B)} = \max_{z \in D} \|v(z)\|_B.$$

Moreover, for all integer $\mu \geq 0$,

$$H^\mu(D, B) = \{v(z) \in L^2(D, B) / \|v\|_{H^\mu(D, B)} < \infty\}$$

equipped with

$$\|v\|_{H^\mu(D, B)} = \left(\sum_{k=0}^{\mu} \left\| \frac{\partial^k v}{\partial z^k} \right\|_{L^2(D, B)}^2 \right)^{\frac{1}{2}}.$$

For real $\mu > 0$, we can define $H^\mu(D, B)$ by the complex interpolation. We now introduce some non-isotropic spaces as follows

$$H_\omega^{r, s}(\Omega) = L^2(I_2, H_\omega^r(I_1)) \cap H^s(I_2, L_\omega^2(I_1)), \quad r, s \geq 0,$$

$$M_\omega^{r, s}(\Omega) = H_\omega^{r, s}(\Omega) \cap H^1(I_2, H_\omega^{r-1}(I_1)) \cap H^{s-1}(I_2, H_\omega^1(I_1)), \quad r, s \geq 1,$$

with the norms

$$\|v\|_{H_{\omega}^{r,s}(\Omega)} = (\|v\|_{L^2(I_2, H_{\omega}^r(I_1))}^2 + \|v\|_{H^s(I_2, L_{\omega}^2(I_1))}^2)^{\frac{1}{2}},$$

$$\|v\|_{M_{\omega}^{r,s}(\Omega)} = (\|v\|_{H_{\omega}^{r,s}(\Omega)}^2 + \|v\|_{H^1(I_2, H_{\omega}^{r-1}(I_1))}^2 + \|v\|_{H^{s-1}(I_2, H_{\omega}^1(I_1))}^2)^{\frac{1}{2}}.$$

Also let

$$X_{1,\omega}^{r,s}(\Omega) = H^s(I_2, H^{r+1}(I_1)) \cap H^{s+1}(I_2, H^r(I_1)), \quad r, s \geq 0,$$

$$X_{2,\omega}^{r,s}(\Omega) = H^s(I_2, H^{r+4}(I_1)) \cap H^{s+1}(I_2, H^{r+2}(I_1)) \cap H^{s+2}(I_2, H^r(I_1)),$$

$r, s \geq 0.$

Their norms are defined similarly. Let $C_p^{\infty}(\Omega)$ be the set of all infinitely differential functions defined on $\bar{\Omega}$, with the period 2π for the variable x_2 . $C_{0,p}^{\infty}(\Omega)$ denotes the subset of $C_p^{\infty}(\Omega)$, whose elements vanish on Γ . Furthermore, $H_{p,\omega}^{r,s}(\Omega)$ and $H_{0,p,\omega}^{r,s}(\Omega)$ are the closures of $C_p^{\infty}(\Omega)$ and $C_{0,p}^{\infty}(\Omega)$ in $H_{\omega}^{r,s}(\Omega)$, etc. Besides let

$$\|v\|_{\infty} = \sup_{x \in \bar{\Omega}} |v|, \quad |v|_{1,\infty} = \max_{1 \leq j \leq 2} \|\partial_j v\|_{\infty}, \quad \|v\|_{1,\infty} = \|v\|_{\infty} + |v|_{1,\infty}.$$

Next, we list some lemmas. Throughout the paper, c will be a positive constant which may be different in different cases.

LEMMA 1 : *If $u(x, t) \in C(\bar{\Omega})$ for all $t \in S_{\tau}$, then*

$$2(u(t), u_t(t))_{M,N,\omega} = (\|u(t)\|_{M,N,\omega}^2)_t - \tau \|u_t(t)\|_{M,N,\omega}^2.$$

LEMMA 2 (Lemma 1 of [10]) : *If $u \in C(\bar{\Omega})$ and $v \in \mathcal{P}_M \times V_N$, then*

$$\|v\|_{\omega} \leq \|v\|_{M,N,\omega} \leq \sqrt{2} \|v\|_{\omega},$$

$$|(u, v)_{M,N,\omega} - (u, v)_{\omega}| \leq c(\|u - P_{M-1,N} u\|_{\omega} + \|u - P_C u\|_{\omega}) \|v\|_{\omega}.$$

Furthermore if also $u \in \mathcal{P}_M \times V_N$, then

$$|(u, v)_{M,N,\omega} - (u, v)_{\omega}| \leq cM^{-r} \|u\|_{H_{\omega}^{r,0}(\Omega)} \|v\|_{\omega}.$$

LEMMA 3 : *For all $v \in \mathcal{P}_M \times V_N$,*

$$\|v\|_{\infty} \leq c(MN)^{\frac{1}{2}} \|v\|_{\omega}.$$

Proof: Let

$$v_i(x_1) = \frac{1}{2\pi} \int_{I_2} v(x_1, x_2) e^{-ix_2} dx_2. \tag{4.1}$$

Then $v_j(x_1) \in \mathcal{P}_M$, and

$$v(x_1, x_2) = \sum_{|l| \leq N} v_l(x_1) e^{ilx_2}.$$

By (9.5.3) of [7],

$$\|v\|_\infty \leq \sum_{|l| \leq N} \|v_l\|_{\infty, I_1} \leq cM^{\frac{1}{2}} \sum_{|l| \leq N} \|v_l\|_{\omega, I_1} \leq c(MN)^{\frac{1}{2}} \|v\|_\omega.$$

LEMMA 4 : If $v \in H_{p, \omega}^{r, s}(\Omega)$ and $r, s \geq 0$, then

$$\|v - P_{M, N} v\|_\omega \leq c(M^{-r} + N^{-s}) \|v\|_{H_{\omega}^{r, s}(\Omega)}.$$

If in addition

$$v \in H^\beta(I_2, H_\omega^r(I_1)) \cap H^s(I_2, H_\omega^\alpha(I_1)) \cap H^{s'}(I_2, H_\omega^{r'}(I_1)),$$

$$0 \leq \alpha \leq r, r', \quad 0 \leq \beta \leq s, s', \quad r, r' > \frac{1}{2}, \quad s, s' > \frac{1}{2},$$

then

$$\begin{aligned} \|v - P_C v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))} &\leq cM^{2\alpha-r} \|v\|_{H^\beta(I_2, H_\omega^r(I_1))} + cN^{\beta-s} \|v\|_{H^s(I_2, H_\omega^\alpha(I_1))} \\ &\quad + cq(\beta) M^{2\alpha-r'} N^{\beta-s'} \|v\|_{H^{s'}(I_2, H_\omega^{r'}(I_1))}, \end{aligned}$$

where $q(\beta) = 0$ for $\beta > \frac{1}{2}$ and $q(\beta) = 1$ for $\beta \leq \frac{1}{2}$.

Proof: The first conclusion comes from Lemma 2 of [8]. We now prove the second one. Let ϑ be the identify operator, then

$$\begin{aligned} \|v - P_C v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))} &\leq \\ \|v - P_C^{(2)} v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))} + \|P_C^{(2)} v - P_C v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))} \\ &\leq D_1 + D_2, \end{aligned}$$

where

$$D_1 = \|v - P_C^{(1)} v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))} + \|v - P_C^{(2)} v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))},$$

$$D_2 = \|(P_C^{(2)} - \vartheta)(\vartheta - P_C^{(1)}) v\|_{H^\beta(I_2, H_\omega^\alpha(I_1))}.$$

By (9.7.7) and (9.7.26) of [7], we have

$$D_1 \leq cM^{2\alpha-r} \|v\|_{H^\beta(I_2, H_\omega^2(I_1))} + cN^{\beta-s} \|v\|_{H^r(I_2, H_\omega^2(I_1))}.$$

If $\beta > \frac{1}{2}$, then

$$D_2 \leq c \|(\vartheta - P_C^{(1)})v\|_{H^\beta(I_2, H_\omega^2(I_1))} \leq cM^{2\alpha-r} \|v\|_{H^\beta(I_2, H_\omega^2(I_1))}.$$

If $\beta > \frac{1}{2}$, then

$$D_2 \leq cN^{\beta-s'} \|(\vartheta - P_C^{(1)})v\|_{H^r(I_2, H_\omega^2(I_1))} \leq cM^{2\alpha-r'} N^{\beta-s'} \|v\|_{H^r(I_2, H_\omega^2(I_1))}.$$

In order to obtain the optimal error estimations, we introduce the projection operator $P_{M,N}^*$ from $H_{0,p,\omega}^{1,1}(\Omega)$ (or $((H_{0,p,\omega}^{1,1}(\Omega))^2)$) onto $V_{M,N}$ (or $(V_{M,N})^2$), i.e., for any $u \in H_{0,p,\omega}^{1,1}(\Omega)$ (or $(H_{0,p,\omega}^{1,1}(\Omega))^2$),

$$(\nabla(u - P_{M,N}^*u), \nabla(v\omega)) = 0, \quad \forall v \in V_{M,N} \text{ (or } V_{M,N}^2).$$

We also assume that there exist positive and suitably big constants c_1 and c_2 such that

$$c_1 N^{\frac{4}{3}} \leq M \leq c_2 N^{\frac{4}{3}}. \quad (4.2)$$

LEMMA 5: If $v \in M_{0,p,\omega}^{r,s}(\Omega)$ and $r, s \geq 1$, then

$$\|v - P_{M,N}^*v\|_{1,\omega} \leq c(M^{1-r} + N^{1-s}) \|v\|_{M_{\omega}^{r,s}(\Omega)}.$$

If $v \in M_{0,p,\omega}^{r+\frac{1}{4},s}(\Omega)$, then

$$\|v - P_{M,N}^*v\|_{\omega} \leq c(M^{-r} + N^{-s}) \|v\|_{M_{\omega}^{r+\frac{1}{4},s}(\Omega)}.$$

Proof: The first conclusion comes from Lemma 3 of [8]. By means of the duality as in [12], we have from (4.2) that

$$\begin{aligned} \|v - P_{M,N}^*v\|_{\omega} &\leq c(M^{-1} + N^{-1}) \|v - P_{M,N}^*v\|_{1,\omega} \\ &\leq c(M^{-1} + N^{-1})(M^{1-r-\frac{1}{4}} + N^{1-s}) \|v\|_{M_{\omega}^{r+\frac{1}{4},s}(\Omega)} \\ &\leq c(M^{-r} + N^{-s}) \|v\|_{M_{\omega}^{r+\frac{1}{4},s}(\Omega)}. \end{aligned}$$

LEMMA 6: If $v \in H_{0,p,\omega}^{r,s}(\Omega) \cap X_{\mu,\omega}^{r,s}(\Omega)$, $\gamma > \frac{1}{2}$, $s > \frac{1}{2}$, and $\mu = 1, 2$, then

$$\|P_{M,N}^* v\|_{\mu,\infty} \leq c \|v\|_{X_{\mu,\omega}^{r,s}(\Omega)}.$$

Proof: Let $v_l(x_1)$ be the same as in (4.1) and

$$P_{M,N}^* v = \sum_{|l| \leq N} v_l^*(x_1) e^{ilx_2}.$$

Then $v_l^* \in V_M$, and $a_l(v_l - v_l^*, u) = 0$, for all $u \in V_M$, where

$$a_l(w, u) = (\partial_1 w, \partial_1(u\omega))_{L^2(I_1)} + l^2(w, u)_{\omega, I_1}, \quad |l| \leq N.$$

If $u \in H_{\omega}^1(I_1)$ and $u(-1) = u(1) = 0$, then

$$a_l(u, u) \geq \frac{1}{4} \|u\|_{l,\omega,I_1}^2 + l^2 \|u\|_{\omega,I_1}^2,$$

and

$$|a_l(w, u)| \leq c(\|w\|_{1,\omega,I_1} + |l| \|w\|_{\omega,I_1})(\|u\|_{1,\omega,I_1} + |l| \|u\|_{\omega,I_1}).$$

Thus

$$\begin{aligned} \frac{1}{4} \|v_l - v_l^*\|_{1,\omega,I_1}^2 + l^2 \|v_l - v_l^*\|_{\omega,I_1}^2 &\leq a_l(v_l - v_l^*, v_l - v_l^*) \\ &\leq c \inf_{u \in V_M} \left(\frac{1}{4} \|v_l - u\|_{1,\omega,I_1}^2 + l^2 \|v_l - u\|_{\omega,I_1}^2 \right). \end{aligned}$$

We take $u = v_{l,*}$, $v_{l,*}$ being the $H_{\omega}^1(I_1)$ -projection of v_l onto V_M . Then by (4.2) and (9.5.17) of [7],

$$\frac{1}{4} \|v_l - v_l^*\|_{1,\omega,I_1}^2 + l^2 \|v_l - v_{l,*}\|_{\omega,I_1}^2 \leq cM^{2-2r} \|v_l\|_{r,\omega,I_1}^2.$$

Moreover by means of the duality,

$$\|v_l - v_l^*\|_{\mu,\omega,I_1} \leq cM^{\mu-r} \|v_l\|_{r,\omega,I_1}, \quad \mu = 0, 1.$$

We now estimate $\|P_{M,N}^* v\|_{1,\infty}$. Obviously

$$\|P_{M,N}^* v\|_{1,\infty} \leq \sum_{|l| \leq N} (|v_l^*|_{1,\infty,I_1} + (1 + |l|) \|v_l^*\|_{\infty,I_1})$$

and

$$\|v_l^*\|_{1, \infty, I_1} \leq \|\partial_1 v_l\|_{\infty, I_1} + \|\partial_1 v_l - P_C^{(1)}(\partial_1 v_l)\|_{\infty, I_1} + \|P_C^{(1)}(\partial_1 v_l) - \partial_1 v_l^*\|_{\infty, I_1}.$$

By embedding theory,

$$\|\partial_1 v_l\|_{\infty, I_1} \leq c \|v_l\|_{r+1, \omega, I_1}, \quad r > \frac{1}{2}.$$

Next let $x_1 = \cos \theta$, $I_\theta = (0, 2\pi)$ and

$$v_l(x_1) = \sum_{j=0}^{\infty} v_l^{(j)} T_j(x_1).$$

Then

$$v_l(x_1) = \hat{v}_l(\theta) = \sum_{j=0}^{\infty} v_l^{(j)} \cos j\theta, \quad \theta \in I_\theta.$$

Let \hat{P}_C be the trigonometric interpolation on I_θ . Then $\widehat{P_C v_l} = \hat{P}_C \hat{v}_l$. We have that for $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned} \|\hat{P}_C \hat{v}_l - \hat{v}_l\|_{\infty, I_\theta} &\leq c \|\hat{P}_C \hat{v}_l - \hat{v}_l\|_{\frac{1}{2}-\varepsilon, I_\theta}^{\frac{1}{2}} \|\hat{P}_C \hat{v}_l - \hat{v}_l\|_{\frac{1}{2}+\varepsilon, I_\theta}^{\frac{1}{2}} \\ &\leq c M^{2^{-r}} \|\hat{v}_l\|_{r, I_\theta}, \quad r > \frac{1}{2}. \end{aligned}$$

By the continuity of the mapping $v_l \rightarrow \hat{v}_l$ from $H_\omega^r(I_1)$ to $H^r(I_\theta)$,

$$\|v_l - P_C^{(1)} v_l\|_{\infty, I_1} \leq c M^{2^{-r}} \|v_l\|_{r, \omega, I_1}, \quad r > \frac{1}{2}. \quad (4.3)$$

Thus

$$\|\partial_1 v_l - P_C^{(1)}(\partial_1 v_l)\|_{\infty, I_1} \leq c M^{2^{-r}} \|v_l\|_{r+1, \omega, I_1}.$$

We have also from the inverse inequality that

$$\begin{aligned} \|P_C^{(1)}(\partial_1 v_l) - \partial_1 v_l^*\|_{\infty, I_1} &\leq cM^{\frac{1}{2}}(\|P_C^{(1)}(\partial_1 v_l) - \partial_1 v_l\|_{\omega, I_1} + \|\partial_1 v_l - \partial_1 v_l^*\|_{\omega, I_1}) \\ &\leq cM^{\frac{1}{2}-r} \|v_l\|_{r+1, \omega, I_1}. \end{aligned}$$

The previous statements lead to

$$\begin{aligned} \sum_{|l| \leq N} |v_l^*|_{1, \infty, I_1} &\leq c \left(\sum_{|l| \leq N} (1+l^2)^s \|v_l\|_{r+1, \omega, I_1}^2 \right)^{\frac{1}{2}} \left(\sum_{|l| \leq N} (1+l^2)^{-s} \right)^{\frac{1}{2}} \\ &\leq c \|v\|_{H^s(I_2, H_w^{s+1}(I_1))}. \end{aligned}$$

We can estimate $\sum_{|l| \leq N} (1+|l|) |v_l^*|_{\infty, I_1}$ similarly and get the first conclusion.

For the second conclusion, we have to consider $\|\partial_1^2 P_{M,N}^* v\|$, etc. Clearly

$$\|\partial_1^2 P_{M,N}^* v\|_{\infty} \leq \sum_{|l| \leq N} (|v_l|_{2, \infty, I_1} + |v_l - P_C^{(1)} v_l|_{2, \infty, I_1} + |P_C^{(1)} v_l - v_l^*|_{2, \infty, I_1}).$$

By embedding theory,

$$|v_l|_{2, \infty, I_1} \leq c \|v_l\|_{r+2, \infty, I_1}, \quad r > \frac{1}{2}.$$

Next, we have

$$\begin{aligned} |v_l - P_C^{(1)} v_l|_{2, \infty, I_1} &\leq \|\partial_1^2 v_l - P_C^{(1)}(\partial_1^2 v_l)\|_{\infty, I_1} \\ &\quad + \|P_C^{(1)}(\partial_1^2 v_l) - \partial_1^2(P_C^{(1)} v_l)\|_{\infty, I_1}. \end{aligned}$$

Moreover by (4.3),

$$\|\partial_1^2 v_l - P_C^{(1)}(\partial_1^2 v_l)\|_{\infty, I_1} \leq cM^{\frac{1}{2}-r} \|v_l\|_{r+2, \omega, I_1},$$

and by (9.5.3) and (9.5.20) of [7],

$$\begin{aligned} \|P_C^{(1)}(\partial_1^2 v_l) - \partial_1^2(P_C^{(1)} v_l)\|_{\infty, I_1} &\leq cM^{\frac{1}{2}}(\|P_C^{(1)}(\partial_1^2 v_l) - \partial_1^2 v_l\|_{\omega, I_1} \\ &\quad + \|\partial_1^2 v_l - \partial_1^2(P_C^{(1)} v_l)\|_{\omega, I_1}) \\ &\leq cM^{\frac{1}{2}-r} \|v_l\|_{r+4, \omega, I_1}. \end{aligned}$$

Furthermore we have from (9.5.3) and (9.5.4) of [7] that

$$\|P_C^{(1)} v_l - v_l^*\|_{2, \infty, I_1} \leq cM^{\frac{1}{2}} \|P_C^{(1)} v_l - v_l^*\|_{2, \omega, I_1} \leq cM^{\frac{9}{2}} \|P_C^{(1)} v_l - v_l^*\|_{\omega, I_1},$$

while

$$\begin{aligned} \|P_C^{(1)} v_l - v_l^*\|_{\omega, I_1} &\leq \|P_C^{(1)} v_l - v_l\|_{\omega, I_1} + \|v_l - v_l^*\|_{\omega, I_1} \\ &\leq cM^{-r-4} \|v_l\|_{r+4, \omega, I_1}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\partial_1^2 P_{M,N}^* v\|_{\infty} &\leq c \left(\sum_{|l| \leq N} (1+l^2)^s \|v_l\|_{r+4, \omega, I_1} \right)^{\frac{1}{2}} \left(\sum_{|l| \leq N} (1+l^2)^{-s} \right)^{\frac{1}{2}} \\ &\leq c \|v\|_{H^s(I_2, H_{\omega}^{r+4}(I_1))}. \end{aligned}$$

Similarly

$$\begin{aligned} \|\partial_1 \partial_2 P_{M,N}^* v\|_{\infty} &\leq c \|v\|_{H^{s+1}(I_2, H_{\omega}^{r+1}(I_1))}, \\ \|\partial_2^2 P_{M,N}^* v\|_{\infty} &\leq c \|v\|_{H^{s+2}(I_2, H_{\omega}^r(I_1))}. \end{aligned}$$

LEMMA 7: If $v \in \mathcal{P}_M \times V_N$, then

$$|v|_{1, \omega}^2 \leq (2M^4 + N^2) \|v\|_{\omega}^2.$$

Proof: Let $u \in \mathcal{P}_M$, u_q and w_q denote the coefficients of Chebyshev expansions of u and $\partial_1 u$. By (2.4.22) of [7],

$$c_q w_q = 2 \sum_{\substack{m=q+1 \\ m+q \text{ odd}}}^M m u_m, \quad c_0 = 2, \quad c_q = 1 \quad \text{for } q \geq 1.$$

Thus

$$c_q w_q^2 \leq \frac{4}{c_q} \sum_{\substack{m=q+1 \\ m+q \text{ odd}}}^M \frac{m^2}{c_m} \sum_{\substack{m=q+1 \\ m+q \text{ odd}}}^M c_m u_m^2 \leq 2M^3 \sum_{m=0}^M c_m u_m^2$$

and

$$\|\partial_1 u\|_{\omega, I_1}^2 \leq 2M^4 \|u\|_{\omega, I_1}^2.$$

Then the conclusion follows.

LEMMA 8 : Let $v \in W_{M,N}$ and $g \in \mathcal{P}_M \times V_N$ satisfy the following equation

$$-(\nabla^2 v, u)_{M,N,\omega} = (g, u)_{M,N,\omega}, \quad \forall u \in V_{M,N}. \tag{4.4}$$

Then

$$\|v\|_{1,\omega} \leq c \|g\|_{\omega}.$$

To prove this lemma, we need preliminary knowledge. Let $\eta = (1 - x_1^2)^{1/2}$. Define the spaces $L_\eta^2(I_1)$ and $H_\eta^r(I_1)$ in the same way as $L_\omega^2(I_1)$ and $H_\omega^r(I_1)$, etc. From Lemma 3.1 of [14], we know that for any $\xi \in L_\eta^2(I_1)$ and $l \neq 0$, there exists a unique function $w \in H_\eta^2(I_1)$ such that

$$\begin{cases} Lw = -\frac{d^2 w}{dx_1^2} + l^2 w = \xi, & \text{in } I_1, \\ \frac{dw}{dx_1}(-1) = \frac{dw}{dx_1}(1) = 0. \end{cases} \tag{4.5}$$

Let $H^{-s}(I_1)$ be the dual space of $H^s(I_1)$. By an argument as in the proof of Lemma 3.1 of [14], we have from the Lax-Milgram lemma that

$$\|w\|_{H^1(I_1)}^2 \leq c \|\xi\|_{H^{-1}(I_1)}^2.$$

Also (4.5) leads to

$$|w|_{H^1(I_1)}^2 + l^2 \|w\|_{L^2(I_1)}^2 \leq \|w\|_{H^1(I_1)} \|\xi\|_{H^{-1}(I_1)}$$

and so

$$l^2 \|w\|_{L^2(I_1)}^2 \leq c \|\xi\|_{H^{-1}(I_1)}^2.$$

Thus

$$\|w\|_{H^1(I_1)}^2 + l^2 \|w\|_{L^2(I_1)}^2 \leq c \|\xi\|_{H^{-1}(I_1)}^2.$$

On the other hand, if $\xi \in L^2(I_1)$, then $w \in H^2(I_1)$. Furthermore, by multiplying (4.5) by w , d^2w/dx_1^2 and integrating by parts, we obtain that

$$\|w\|_{H^2(I_1)}^2 + l^2 \|w\|_{H^1(I_1)}^2 \leq c \|\xi\|_{L^2(I_1)}^2.$$

By the theory of interpolation of spaces, we have that if $\xi \in H^{-s}(I_1)$ with $0 < s < 1$, then $w \in H^{2-s}(I_1)$ and

$$\|w\|_{H^{2-s}(I_1)}^2 + l^2 \|w\|_{H^{1-s}(I_1)}^2 \leq c \|\xi\|_{H^{-s}(I_1)}^2. \tag{4.6}$$

According to Theorem 4.2 of [14], we know that for any real $s \geq 0$, $H_\eta^s(I_1) \subset H^{s-1/4}(I_1)$. Therefore $\xi \in L_\eta^2(I_1) \subset H^{-1/4}(I_1)$ and $w \in H_\eta^2(I_1) \subset H^{7/4}(I_1)$. So (4.6) implies

$$\|w\|_{H_\eta^2(I_1)}^2 + l^2 \|w\|_{H_\eta^3(I_1)}^2 \leq c \|\xi\|_{H^{-1/4}(I_1)}^2 \leq c \|\xi\|_{\eta, I_1}^2. \tag{4.7}$$

Moreover, Theorem 4.1 of [14] tells us that for any real $s \geq 1/4$, $H^s(I_1) \subset H_\omega^{s-1/4}(I_1)$. Hence (4.7) leads to

$$\|w\|_{H_\omega^3(I_1)}^2 + l^2 \|w\|_{H_\omega^2(I_1)}^2 \leq c \|\xi\|_{\eta, I_1}^2. \tag{4.8}$$

Now, we prove Lemma 8. Let

$$v = \sum_{|l| \leq N} v_l(x_1) e^{ilx_2}, \quad g = \sum_{|l| \leq N} g_l(x_1) e^{ilx_2}$$

where $v_l \in W_M$ and $g_l \in \mathcal{P}_M$. Firstly, we consider the case with $l \neq 0$. Let $u = (1 - x_1^2) z_l e^{ilx_2}$ in (4.4), with $z_l \in \mathcal{P}_{M-2}$. Since $u \partial_1^2 v \in \mathcal{P}_{2M-2} \times V_{2N}$, we have

$$(\partial_1^2 v, u)_{M, N, \omega} = (\partial_1^2 v, u)_\omega.$$

Thus we have from (4.4) that

$$\begin{aligned} (Lv_l, z_l)_{\eta, I_1} &= \left(-\frac{d^2 v_l}{dx_1^2} + l^2 v_l, z_l \right)_{\eta, I_1} = \\ &= (g_l, z_l)_{\eta, I_1} + E_1(z_l) + E_2(z_l), \quad \forall z_l \in \mathcal{P}_{M-2} \end{aligned} \tag{4.9}$$

where

$$E_1(z_l) = l^2(v_p, (1 - x_1^2) z_l)_{\omega, I_1} - l^2(v_p, (1 - x_1^2) z_l)_{M, \omega},$$

$$E_2(z_l) = (g_p, (1 - x_1^2) z_l)_{M, \omega} - (g_p, (1 - x_1^2) z_l)_{\omega, I_1}.$$

Let $z_l = P_{M-2}^{(1)} Lv_l$ in (4.9) where $P_{M-2}^{(1)}$ is given in Section II and L is the same as in (4.5). We have that

$$z_l = Lv_l - l^2(v_l - P_{M-2}^{(1)} v_l) \in \mathcal{P}_{M-2}. \quad (4.10)$$

So (4.9) reads

$$\begin{aligned} & \|Lv_l\|_{\eta, I_1}^2 - l^2(Lv_p, v_l - P_{M-2}^{(1)} v_l)_{\eta, I_1} \\ &= (g_p, Lv_l)_{\eta, I_1} - l^2(g_p, v_l - P_{M-2}^{(1)} v_l)_{\eta, I_1} + E_1(P_{M-2}^{(1)} Lv_l) + E_2(P_{M-2}^{(1)} Lv_l). \end{aligned} \quad (4.11)$$

By (9.3.5) of [7], we have

$$\begin{aligned} |E_1(P_{M-2}^{(1)} Lv_l)| &\leq cl^2 \|(1 - x_1^2) z_l\|_{\omega, I_1} \|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1} \\ &\leq cl^2 \|z_l\|_{\eta, I_1} \|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1}. \end{aligned}$$

Moreover by (4.10),

$$|E_1(P_{M-2}^{(1)} Lv_l)| \leq cl^2 \|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1} (\|Lv_l\|_{\eta, I_1} + l^2 \|v_l - P_{M-2}^{(1)} v_l\|_{\eta, I_1}).$$

Similarly, we have

$$\begin{aligned} |E_2(P_{M-2}^{(1)} Lv_l)| &\leq c \|g_l\|_{\omega, I_1} \|(1 - x_1^2) z_l - P_{M-1}^{(1)}((1 - x_1^2) z_l)\|_{\omega, I_1} \\ &\leq c \|g_l\|_{\omega, I_1} \|(1 - x_1^2) z_l\|_{\omega, I_1} = c \|g_l\|_{\omega, I_1} \|z_l\|_{\eta, I_1} \end{aligned}$$

and so (4.10) leads to

$$|E_2(P_{M-2}^{(1)} Lv_l)| \leq c \|g_l\|_{\omega, I_1} (\|Lv_l\|_{\eta, I_1} + l^2 \|v_l - P_{M-2}^{(1)} v_l\|_{\eta, I_1}).$$

By substituting the above estimations into (4.11), we obtain

$$\begin{aligned} & \frac{1}{2} \|Lv_l\|_{\eta, I_1}^2 - cl^2 \|Lv_l\|_{\eta, I_1} (\|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1} + \|v_l - P_{M-2}^{(1)} v_l\|_{\eta, I_1}) \\ & - cl^4 \|v_l - P_{M-2}^{(1)} v_l\|_{\eta, I_1}^2 \\ & - cl^4 \|v_l - P_{M-2}^{(1)} v_l\|_{\eta, I_1} \|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1} \leq c \|g_l\|_{\omega, I_1}^2. \end{aligned} \quad (4.12)$$

Clearly $\|v\|_{\eta, I_1} \leq \|v\|_{\omega, I_1}$ and by Theorem 2.2 of [13],

$$\|v_l - P_{M-1}^{(1)} v_l\|_{\omega, I_1} \leq cM^{-s} \|v_l\|_{H_{\omega(I_1)}^s}, \quad \text{etc.}$$

Therefore by letting $\xi = Lv_l$ and $w = v_l$ in (4.8), we have from (4.12) that

$$\begin{aligned} & \left[\frac{1}{4} (\|v_l\|_{H_{\omega}^{\frac{3}{2}}(I_1)} + |l| \|v_l\|_{H_{\omega}^{\frac{1}{2}}(I_1)}) - cl^2 M^{-\frac{3}{2}} \|v_l\|_{H_{\omega}^{\frac{1}{2}}(I_1)} \right] \|Lv_l\|_{\eta, I_1} \\ & - cl^4 M^{-3} \|v_l\|_{H_{\omega}^{\frac{3}{2}}(I_1)}^2 + \frac{1}{4} (\|v_l\|_{H_{\omega}^{\frac{3}{2}}(I_1)}^2 + l^2 \|v_l\|_{H_{\omega}^{\frac{1}{2}}(I_1)}^2) \leq c \|g_l\|_{\omega, I_1}^2. \end{aligned}$$

By (4.2), we have that for sufficiently large M , $1 - 16cl^4 M^{-3} \geq 4\alpha > 0$. Thus

$$\alpha \|v_l\|_{H_{\omega}^{\frac{3}{2}}(I_1)}^2 + \frac{1}{4} l^2 \|v_l\|_{H_{\omega}^{\frac{1}{2}}(I_1)}^2 \leq c \|g_l\|_{\omega, I_1}^2.$$

Next, we deal with the case with $l = 0$. By taking $u = -(1 - x_1^2) \partial_1^2 v_l$ in (4.4), we get

$$\|\partial_1^2 v_l\|_{\eta, I_1} \leq c \|g_l\|_{\omega, I_1}.$$

Moreover,

$$\begin{aligned} \int_{I_1} \omega(\partial_1 v_l)^2 dx_1 &= \int_{I_1} \omega(x_1) \left(\int_{-1}^{x_1} \partial_1^2 v_l(\zeta) d\zeta \right)^2 dx_1 \\ &\leq c \int_{I_1} \omega(x_1) \left[\int_{I_1} \eta(\zeta) (\partial_1^2 v_l(\zeta))^2 d\zeta \int_{I_1} \omega(\zeta) d\zeta \right] dx_1 \\ &\leq c \|\partial_1^2 v_l\|_{\eta, I_1}. \end{aligned}$$

Thus for $l = 0$,

$$|v_l|_{1, \omega, I_l} \leq c \|g_l\|_{\omega, I_l}.$$

Finally,

$$\begin{aligned} |v|_{1, \omega}^2 &= \sum_{|l| \leq N} (|v_l|_{1, \omega, I_l}^2 + l^2 \|v_l\|_{\omega, I_l}^2) \\ &\leq c \sum_{|l| \leq N} \|g_l\|_{\omega, I_l}^2 = c \|g\|_{\omega}^2. \end{aligned}$$

This completes the proof of Lemma 8.

LEMMA 9 (Lemma 9 of [10]): For any $v \in V_{M, N}$,

$$-(\nabla^2 v, v)_{M, N, \omega} \geq \frac{1}{4} \|v\|_{1, \omega}^2.$$

LEMMA 10 (Lemma 4.16 of [15]): Assume that the following conditions are fulfilled

- (i) $E(t)$ and $E_1(t)$ are non-negative functions defined on S_τ ;
- (ii) ρ, M_1 and M_2 are non-negative constants, and $M_3 > 0$;
- (iii) $A(z)$ is a function such that if $z \leq M_3$, then $A(z) \leq 0$;
- (iv) for all $t \in S_\tau$,

$$E(t) \leq \rho + M_1 \tau \sum_{\zeta \in S_\tau, \zeta < t} (E(\zeta) + M_2 E^2(\zeta) + A(E(\zeta)) E_1(\zeta));$$

- (v) $E(0) \leq \rho$, and for some $t_1 \in S_\tau$,

$$\rho e^{2M_1 t_1} \leq \min\left(\frac{1}{M_2}, M_3\right).$$

Then for all $t \in S_\tau, t \leq t_1$,

$$E(t) \leq \rho e^{2M_1 t}.$$

5. THE ANALYSIS OF GENERALIZED STABILITY

This section is dedicated to the generalized stability of scheme (2.1). The scheme (2.1) is equivalently written as

$$\left\{ \begin{aligned} &(u_t + d_c(u, u) - v \nabla^2(u + \sigma \tau u_t) + \nabla p, v)_{M, N, \omega} = (P_C f, v)_{M, N, \omega}, \\ &\forall v \in (V_{M, N})^2, t \in S_\tau, \\ &-(\nabla^2 p, v)_{M, N, \omega} = (\Phi_c(u) - P_C(\nabla \cdot f), v)_{M, N, \omega}, \quad \forall v \in V_{M, N}, t \in S_\tau, \\ &u(0) = P_C U_0, \quad \text{in } \bar{\Omega}_{M, N}. \end{aligned} \right. \tag{5.1}$$

If $u(0)$ and f have the errors $\tilde{u}(0)$ and \tilde{f} , then u and p have the errors \tilde{u} and \tilde{p} . They satisfy

$$\left\{ \begin{array}{l} (\tilde{u}_t + d_c(u, \tilde{u}) + d_c(\tilde{u}, u + \tilde{u}) - \nu \nabla^2(\tilde{u} + \sigma\tau\tilde{u}_t) + \nabla\tilde{p}, v)_{M, N, \omega} = \\ \quad (P_C\tilde{f}, v)_{M, N, \omega}, \quad \forall v \in (V_{M, N})^2, t \in S_\tau, \\ - (\nabla^2\tilde{p}, v)_{M, N, \omega} = (\Phi_c(\tilde{u}) + \Phi^*(u, \tilde{u}) - P_C(\nabla \cdot \tilde{f}), v)_{M, N, \omega}, \\ \quad \forall v \in V_{M, N}, t \in S_\tau, \end{array} \right. \quad (5.2)$$

where

$$\begin{aligned} \Phi^*(u, \tilde{u}) = & 2[P_C(\partial_2 u_1 \partial_1 \tilde{u}_2) + P_C(\partial_2 \tilde{u}_1 \partial_1 u_2) - \\ & - P_C(\partial_1 u_1 \partial_2 \tilde{u}_2) - P_C(\partial_1 \tilde{u}_1 \partial_2 u_2)]. \end{aligned}$$

Let $\varepsilon > 0$, and m be an undetermined positive constant. By taking $v = 2\tilde{u} + m\tau\tilde{u}_t$ in the first formula of (5.2), we have from Lemma 1, Lemma 2 and Lemma 9 that

$$\begin{aligned} & (\|\tilde{u}\|_{M, N, \omega}^2)_t + \tau(m - 1 - \varepsilon) \|\tilde{u}_t\|_{M, N, \omega}^2 + \frac{\nu}{2} \|\tilde{u}\|_{1, \omega}^2 \\ & + \frac{\nu\sigma m\tau^2}{4} \|\tilde{u}_t\|_{1, \omega}^2 - 2\nu\sigma\tau(\nabla^2\tilde{u}_t, \tilde{u})_{M, N, \omega} - \nu m\tau(\nabla^2\tilde{u}, \tilde{u}_t)_{M, N, \omega} + \sum_{j=1}^7 F_j \\ & \leq 2\|\tilde{u}\|_{\omega}^2 + \left(2 + \frac{\tau m^2}{\varepsilon}\right) \|P_C\tilde{f}\|_{\omega}^2 \quad (5.3) \end{aligned}$$

where

$$F_1 = (d_c(u, \tilde{u}), 2\tilde{u} + m\tau\tilde{u}_t)_{M, N, \omega},$$

$$F_2 = 2(d_c(\tilde{u}, u), \tilde{u})_{M, N, \omega},$$

$$F_3 = m\tau(d_c(\tilde{u}, u), \tilde{u}_t)_{M, N, \omega},$$

$$F_4 = 2(d_c(\tilde{u}, \tilde{u}), \tilde{u})_{M, N, \omega},$$

$$F_5 = m\tau(d_c(\tilde{u}, \tilde{u}), \tilde{u}_t)_{M, N, \omega},$$

$$F_6 = 2(\nabla\tilde{p}, \tilde{u})_{M, N, \omega},$$

$$F_7 = m\tau(\nabla\tilde{p}, \tilde{u}_t)_{M, N, \omega}.$$

It is easy to verify that if $u, v \in \mathcal{P}_M \times V_N$ and $uv \in \mathcal{P}_{2M-1} \times V_{2N}$, then (see [10])

$$(u, v)_{M, N, \omega} = (u, v)_\omega. \quad (5.4)$$

Hence

$$-2 \nu \sigma \tau (\nabla^2 \tilde{u}_t, \tilde{u})_{M, N, \omega} = A_1 + A_2 + A_3,$$

$$- \nu m \tau (\nabla^2 \tilde{u}, \tilde{u}_t)_{M, N, \omega} = B_1 + B_2 + B_3$$

where

$$A_1 = 2 \nu \sigma \tau (\partial_1 \tilde{u}_t, \partial_1 \tilde{u})_\omega,$$

$$A_2 = 2 \nu \sigma \tau (\partial_2 \tilde{u}_t, \partial_2 \tilde{u})_{M, N, \omega},$$

$$A_3 = 2 \nu \sigma \tau (\partial_1 \tilde{u}_t, x_1 \omega^2 \tilde{u})_\omega,$$

$$B_1 = \nu m \tau (\partial_1 \tilde{u}, \partial_1 \tilde{u}_t)_\omega,$$

$$B_2 = \nu m \tau (\partial_2 \tilde{u}, \partial_2 \tilde{u}_t)_{M, N, \omega},$$

$$B_3 = \nu m \tau (\partial_1 \tilde{u}, x_1 \omega^2 \tilde{u}_t)_\omega.$$

It can be shown that

$$\begin{aligned} A_1 + B_1 &= \nu \tau \left(\sigma + \frac{m}{2} \right) [(\|\partial_1 \tilde{u}\|_\omega^2)_t - \tau \|\partial_1 \tilde{u}_t\|_\omega^2] \\ &= \nu \tau \left(\sigma + \frac{m}{2} \right) [(\|\partial_1 \tilde{u}\|_{M, N, \omega}^2)_t - \tau \|\partial_1 \tilde{u}_t\|_{M, N, \omega}^2], \\ A_2 + B_2 &= \nu \tau \left(\sigma + \frac{m}{2} \right) [(\|\partial_2 \tilde{u}\|_{M, N, \omega}^2)_t - \tau \|\partial_2 \tilde{u}_t\|_{M, N, \omega}^2]. \end{aligned}$$

By Lemma 1 of [16], we have that

$$\|v \omega^2\|_{\omega, I_1} \leq |v|_{1, \omega, I_1}, \quad \forall v \in H_{0, \omega}^1(I_1),$$

and so

$$\begin{aligned} |A_3| &\leq 2\nu\sigma\tau \|\partial_1 \tilde{u}_t\|_\omega \|\partial_1 \tilde{u}\|_\omega \\ &\leq \frac{\nu\sigma}{4} \|\partial_1 \tilde{u}\|_\omega^2 + 4\nu\sigma\tau^2 \|\partial_1 \tilde{u}_t\|_\omega^2, \\ |B_3| &\leq \nu m\tau \|\partial_1 \tilde{u}\|_\omega \|\partial_1 \tilde{u}_t\|_\omega \\ &\leq \frac{\nu m}{8} \|\partial_1 \tilde{u}\|_\omega^2 + 2\nu m\tau^2 \|\partial_1 \tilde{u}_t\|_\omega^2. \end{aligned}$$

Thus (5.3) reads

$$\begin{aligned} &(\|\tilde{u}\|_{M,N,\omega}^2)_t + \tau(m-1-\epsilon) \|\tilde{u}_t\|_{M,N,\omega}^2 + \frac{\nu}{8}(4-m-2\sigma) |\tilde{u}|_{1,\omega}^2 \\ &+ \nu\tau\left(\sigma + \frac{m}{2}\right) (|\tilde{u}|_{1,M,N,\omega}^2)_t + \frac{\nu\sigma m\tau^2}{4} \|\tilde{u}_t\|_{1,\omega}^2 - 5\nu\tau^2\left(\sigma + \frac{m}{2}\right) |\tilde{u}_t|_{1,M,\omega}^2 \\ &+ \sum_{j=1}^7 F_j \leq 2\|\tilde{u}\|_\omega^2 + \left(2 + \frac{\tau m^2}{\epsilon}\right) \|P_C \tilde{f}\|_\omega^2. \end{aligned}$$

By Lemma 2 and Lemma 7, we get

$$\begin{aligned} &(\|\tilde{u}\|_{M,N,\omega}^2)_t + \tau\left[m-1-\epsilon - \nu\tau\left(5(m+2\sigma) - \frac{\sigma m}{4}\right)(2M^4 + N^2)\right] \|\tilde{u}_t\|_\omega^2 \\ &+ \frac{\nu}{8}(4-m-2\sigma) |\tilde{u}|_{1,\omega}^2 + \nu\tau\left(\sigma + \frac{m}{2}\right) (|\tilde{u}|_{1,M,N,\omega}^2)_t + \sum_{j=1}^7 F_j \\ &\leq 2\|\tilde{u}\|_\omega^2 + \left(2 + \frac{\tau m^2}{\epsilon}\right) \|P_C \tilde{f}\|_\omega^2. \quad (5.5) \end{aligned}$$

We turn to estimate $|F_j|$. By Lemma 2,

$$\begin{aligned} |F_1| &\leq c \left(\sum_{j=1}^2 \|P_C(\tilde{u}_j, \partial_j u)\|_{M,N,\omega} \right) (\|\tilde{u}\|_{M,N,\omega} + m\tau \|\tilde{u}_t\|_{M,N,\omega}) \\ &\leq c \|u\|_{1,\infty} \|\tilde{u}\|_{M,N,\omega} (\|\tilde{u}\|_{M,N,\omega} + m\tau \|\tilde{u}_t\|_{M,N,\omega}) \\ &\leq \epsilon\tau \|\tilde{u}_t\|_\omega^2 + c \left(\|u\|_{1,\infty} + \frac{\tau m^2}{\epsilon} \|u\|_{1,\infty}^2 \right) \|\tilde{u}\|_\omega^2, \\ |F_2| &\leq c \left(\sum_{j=1}^2 \|P_C(u_j, \partial_j \tilde{u})\|_{M,N,\omega} \right) \|\tilde{u}\|_{M,N,\omega} \\ &\leq c \|u\|_\infty |\tilde{u}|_{1,M,N,\omega} \|\tilde{u}\|_{M,N,\omega} \\ &\leq \frac{\epsilon\nu}{8} |\tilde{u}|_{1,\omega}^2 + \frac{c}{\epsilon\nu} \|u\|_\infty^2 \|\tilde{u}\|_\omega^2. \end{aligned}$$

Similarly,

$$\begin{aligned} |F_3| &\leq cm\tau \|u\|_\infty |\tilde{u}|_{1, M, N, \omega} \|\tilde{u}_t\|_{M, N, \omega} \\ &\leq \epsilon\tau \|\tilde{u}_t\|_\omega^2 + \frac{c\tau m^2}{\epsilon} \|u\|_\infty^2 |\tilde{u}|_{1, \omega}^2. \end{aligned}$$

By Lemma 2 and Lemma 3, we have

$$\begin{aligned} |F_4| &\leq c \|\tilde{u}\|_\infty \|\tilde{u}\|_{M, N, \omega} |\tilde{u}|_{1, M, N, \omega} \\ &\leq c(MN)^{\frac{1}{2}} \|\tilde{u}\|_\omega^2 |\tilde{u}|_{1, \omega} \\ &\leq \frac{\epsilon\nu}{8} |\tilde{u}|_{1, \omega}^2 + \frac{cMN}{\epsilon\nu} \|\tilde{u}\|_\omega^4, \\ |F_5| &\leq cm\tau(MN)^{\frac{1}{2}} \|\tilde{u}\|_\omega \|\tilde{u}_t\|_\omega |\tilde{u}|_{1, \omega} \\ &\leq \epsilon\tau \|\tilde{u}_t\|_\omega^2 + \frac{c\tau m^2 MN}{\epsilon} \|\tilde{u}\|_\omega^2 |\tilde{u}|_{1, \omega}^2. \end{aligned}$$

Next, we apply Lemma 8 to the second formula of (5.2), and obtain that

$$|\tilde{p}|_{1, \omega} \leq c(\|\Phi_c(\tilde{u})\|_\omega + \|P_C(\nabla \cdot \tilde{f})\|_\omega + \|\Phi^*(u, \tilde{u})\|_\omega).$$

By Lemma 2 and Lemma 3, we have that

$$\|\Phi_c(\tilde{u})\|_\omega \leq \|\Phi_c(\tilde{u})\|_{M, N, \omega} \leq c|\tilde{u}|_{1, \infty} |\tilde{u}|_{1, \omega} \leq c(MN)^{\frac{1}{2}} |\tilde{u}|_{1, \omega}^2$$

and

$$\|\Phi^*(u, \tilde{u})\|_\omega \leq c|u|_{1, \infty} |\tilde{u}|_{1, \omega}.$$

Therefore,

$$\begin{aligned} |F_6| &\leq 2 |\tilde{p}|_{1, M, N, \omega} \|\tilde{u}\|_{M, M, \omega} \leq 4 |\tilde{p}|_{1, \omega} \|\tilde{u}\|_\omega \\ &\leq \frac{\epsilon\nu}{8} |\tilde{u}|_{1, \omega}^2 + \left(c + \frac{c}{\epsilon\nu} |u|_{1, \infty}^2\right) \|\tilde{u}\|_\omega^2 \\ &\quad + \frac{cMN}{\epsilon\nu} \|\tilde{u}\|_\omega^2 |\tilde{u}|_{1, \omega}^2 + c \|P_C(\nabla \cdot \tilde{f})\|_\omega^2, \\ |F_7| &\leq \epsilon\tau \|\tilde{u}_t\|_\omega^2 + \frac{c\tau m^2}{\epsilon} |u|_{1, \infty}^2 |\tilde{u}|_{1, \omega}^2 \\ &\quad + \frac{c\tau m^2 MN(M^4 + N^2)}{\epsilon} \|\tilde{u}\|_\omega^2 |\tilde{u}|_{1, \omega}^2 + \frac{c\tau m^2}{\epsilon} \|P_C(\nabla \cdot \tilde{f})\|_\omega^2. \end{aligned}$$

Let $\|u\|_{1,\infty} = \max_{t \in \mathcal{S}_t} \|u(t)\|_{1,\infty}$, etc. By substituting the above estimations into (5.5), we get

$$\begin{aligned} & (\|\tilde{u}\|_{M,N,\omega}^2)_t + \tau \left[m - 1 - 5\epsilon - \nu\tau \left(5(m + 2\sigma) - \frac{\sigma m}{4} \right) (2M^4 + N^2) \right] \|\tilde{u}_t\|_\omega^2 \\ & + \frac{\nu}{8} (4 - m - 2\sigma - 3\epsilon) |\tilde{u}|_{1,\omega}^2 + \nu\tau \left(\sigma + \frac{m}{2} \right) (|\tilde{u}|_{1,M,N,\omega}^2)_t \\ & \leq M_1 \|\tilde{u}\|_\omega^2 + M_2 \|\tilde{u}\|_\omega^4 + B_1(\|\tilde{u}\|_\omega) |\tilde{u}|_{1,\omega}^2 + G_1 \end{aligned} \quad (5.6)$$

where

$$M_1 = c + c \|u\|_{1,\infty} + \frac{c}{\epsilon} \left(\tau m^2 + \frac{1}{\nu} \right) \|u\|_{1,\infty}^2, \quad M_2 = \frac{cMN}{\epsilon\nu},$$

$$B_1(\|\tilde{u}\|_\omega) = \frac{c\tau m^2}{\epsilon} \|u\|_{1,\infty} + \frac{cMN}{\epsilon} \left(\frac{1}{\nu} + \tau m^2 M^4 + \tau m^2 N^2 \right) \|\tilde{u}\|_\omega^2,$$

$$G_1 = \left(2 + \frac{\tau m^2}{\epsilon} \right) \|P_{\mathcal{O}} \tilde{f}\|_\omega^2 + \left(c + \frac{c\tau m^2}{\epsilon} \right) \|P_c(\nabla \cdot \tilde{f})\|_\omega^2.$$

Let ϵ be suitably small, and suppose that

$$\nu\tau(2M^4 + N^2) < \frac{1}{\lambda \left(5 - \frac{\sigma}{4} \right)}, \quad \lambda > 1. \quad (5.7)$$

We take

$$m = \frac{\frac{33}{32} + 5\epsilon + 10\nu\sigma\tau(2M^4 + N^2)}{1 - \frac{1}{\lambda}}.$$

Then the coefficient of the term $\|\tilde{u}_t\|_\omega^2$ in (5.6) is not less than $\tau/32$. Obviously

$$m \leq \left(\frac{33}{32} + 5\epsilon + \frac{10\sigma}{\lambda \left(5 - \frac{\sigma}{4} \right)} \right) \left(1 - \frac{1}{\lambda} \right)^{-1}.$$

Thus if

$$\lambda > \left(\frac{10\sigma}{5-\frac{\sigma}{4}} + \frac{7}{2} - 2\sigma \right) \left(\frac{79}{32} - 2\sigma \right)^{-1}, \tag{5.8}$$

then the coefficient of the term $|\tilde{u}|_{1,\omega}^2$ in (5.6) is not less than $\nu/16$. Thus we have from (5.6) that

$$\begin{aligned} & (\|\tilde{u}\|_{M,N,\omega}^2)_t + \frac{\tau}{32} \|\tilde{u}\|_{\omega}^2 + \frac{\nu}{32} |\tilde{u}|_{1,\omega}^2 + \nu\tau \left(\sigma + \frac{m}{2} \right) (|\tilde{u}|_{1,M,N,\omega}^2)_t \\ & \leq M_1 \|\tilde{u}\|_{\omega}^2 + M_2 \|\tilde{u}\|_{\omega}^4 + B(\|\tilde{u}\|_{\omega}) |\tilde{u}|_{1,\omega}^2 + G_1 \end{aligned} \tag{5.9}$$

with

$$B(\|\tilde{u}\|_{\omega}) = -\frac{\nu}{32} + B_1(\|\tilde{u}\|_{\omega}).$$

By summing (5.9) for $t \in S_{\tau}$, we have

$$\begin{aligned} & \|\tilde{u}(t)\|_{M,N,\omega}^2 + \frac{\tau}{32} \sum_{\zeta \in S_{\tau}, \zeta < t} (\tau \|\tilde{u}_t(\zeta)\|_{\omega}^2 + \nu |\tilde{u}(\zeta)|_{1,\omega}^2) \\ & \leq \|\tilde{u}(0)\|_{M,N,\omega}^2 + \nu\tau \left(\sigma + \frac{m}{2} \right) |\tilde{u}(0)|_{1,M,N,\omega}^2 \\ & + \tau \sum_{\zeta \in S_{\tau}, \zeta < t} [M_1 \|\tilde{u}(\zeta)\|_{\omega}^2 + M_2 \|\tilde{u}(\zeta)\|_{\omega}^4 + B(\|\tilde{u}(\zeta)\|_{\omega}) |\tilde{u}(\zeta)|_{1,\omega}^2 + G_1(\zeta)]. \end{aligned}$$

Let

$$\begin{aligned} E(t) &= \|\tilde{u}(t)\|_{\omega}^2 + \frac{\tau}{32} \sum_{\zeta \in S_{\tau}, \zeta < t} (\tau \|\tilde{u}_t(\zeta)\|_{\omega}^2 + \nu |\tilde{u}(\zeta)|_{1,\omega}^2), \\ \rho(t) &= 2 \|\tilde{u}(0)\|_{\omega}^2 + \nu\tau(2\sigma + m) |\tilde{u}(0)|_{1,\omega}^2 + \tau \sum_{\zeta \in S_{\tau}, \zeta < t} G_1(\zeta). \end{aligned}$$

We get from (5.10) and Lemma 2 that

$$E(t) \leq \rho(t) + \tau \sum_{\zeta \in S_{\tau}, \zeta < t} [M_1 E(\zeta) + M_2 E^2(\zeta) + B(\|\tilde{u}(\zeta)\|_{\omega}) |\tilde{u}(\zeta)|_{1,\omega}^2].$$

Finally we use Lemma 10 to obtain the following result.

THEOREM 1 : *Assume that the following conditions are satisfied*

- (i) (4.2), (5.7) and (5.8) hold ;
- (ii) for certain small positive constant c_3 , $\tau \|\mathbb{u}\|_{1,\infty}^2 \leq c_3 \nu$;

(iii) for some $t_1 \in S_\tau$, there exist positive constants d_1 and d_2 depending only on $\|u\|_{1,\infty}$ and v , such that $\rho(t_1) e^{d_1 t_1} \leq d_2 / MN$.

Then for all $t \in S_\tau$, $t \leq t_1$, we have

$$E(t) \leq \rho(t) e^{d_1 t}.$$

6. THE CONVERGENCE

In this section, we consider the convergence of scheme (2.1). Let $U^* = P_{M,N}^* U$ and $\tilde{u} = u - U^*$. Since $\partial_1 P \in H_{0,p,\omega}^{1,1}(\Omega)$, we follow the idea in Section 10.4 of [7] to define

$$P^* = P_N^{(2)} P(-1, x_2) + \int_{-1}^{x_1} P_{M-1,N}^*(\partial_1 P(\bar{x}_1, x_2)) d\bar{x}_1, \quad \tilde{p} = p - P^*.$$

By (1.1) and (2.1), we have

$$\left\{ \begin{array}{l} (\tilde{u}_t + d_c(U^*, \tilde{u}) + d_c(\tilde{u}, U^* + \tilde{u}) - \nu \nabla^2(\tilde{u} + \sigma \tau \tilde{u}_t) + \nabla \tilde{p}, v)_{M,N,\omega} = \sum_{j=1}^6 A_j, \\ \quad \forall v \in (V_{M,N})^2, \\ - (\nabla^2 \tilde{p}, v)_{M,N,\omega} = (\Phi_c(\tilde{u}) + \Phi^*(U^*, \tilde{u}), v)_{M,N,\omega} + \sum_{j=7}^9 A_j, \quad \forall v \in V_{M,N}, \\ \tilde{u}(0) = P_C U_0 - P_{M,N}^* U_0, \end{array} \right. \tag{6.1}$$

where $A_j = A_j(v, t)$ and

$$\begin{aligned} A_1(v, t) &= (\partial_t U, v)_\omega - (U_t^*, v)_{M,N,\omega}, \\ A_2(v, t) &= ((U \cdot \nabla) U, v)_\omega - (d_c(U^*, U^*), v)_{M,N,\omega}, \\ A_3(v, t) &= -\nu (\nabla^2 U^*, v)_\omega + \nu (\nabla^2 U^*, v)_{M,N,\omega}, \\ A_4(v, t) &= \nu \sigma \tau (\nabla^2 U_t^*, v)_{M,N,\omega}, \\ A_5(v, t) &= (\nabla P, v)_\omega - (\nabla P^*, v)_{M,N,\omega}, \\ A_6(v, t) &= -(f, v)_\omega + (P_C f, v)_{M,N,\omega}, \\ A_7(v, t) &= -(\nabla^2 P, v)_\omega + (\nabla^2 P^*, v)_{M,N,\omega}, \\ A_8(v, t) &= -(\Phi(U), v)_\omega + (\Phi_c(U^*), v)_{M,N,\omega}, \\ A_9(v, t) &= (\nabla \cdot f, v)_\omega - (P_C(\nabla \cdot f), v)_{M,N,\omega}. \end{aligned}$$

We now estimate $|A_j(v, t)|$. Firstly, we take $v = 2\bar{u}$ in $A_j(v, t)$ ($j = 1, \dots, 6$). We have from Lemma 2, Lemma 4 and Lemma 5 that

$$\begin{aligned} |A_1(\bar{u}, t)| &\leq |(\partial_t U - U_t, \bar{u})_\omega| + |(U_t - U_t^*, \bar{u})_{M, N, \omega}| \\ &\quad + |(U_t, \bar{u})_\omega - (U_t, \bar{u})_{M, N, \omega}| \\ &\leq \|\bar{u}\|_\omega (\|\partial_t U - U_t\|_\omega + \|U_t - U_t^*\|_\omega + c\|U_t - P_{M-1, N} U_t\|_\omega + c\|U_t - P_C U_t\|_\omega) \\ &\leq \|\bar{u}\|_\omega^2 + c(M^{-2r} + N^{-2s}) \|U\|_{C^1(0, T; M_\omega^{r+\frac{1}{2}, s}(\Omega))}^2 + c\tau \|U\|_{H^2(t, t+\tau; L_\omega^2(\Omega))}^2. \end{aligned}$$

By embedding theorem, we know that (see [10])

$$\|uv\|_{H_\omega^{r,s}(\Omega)} \leq c \|u\|_{M_\omega^{r+1, s+1}(\Omega)} \|v\|_{M_\omega^{r+1, s+1}(\Omega)}, \quad r, s > 1. \quad (6.2)$$

Let $A_2(\bar{u}, t) = B_1 + B_2$ where

$$B_1 = ((U \cdot \nabla) U - d_c(U^*, U^*), \bar{u})_\omega,$$

$$B_2 = (d_c(U^*, U^*), \bar{u})_\omega - (d_c(U^*, U^*), \bar{u})_{M, N, \omega}.$$

We have

$$\begin{aligned} |B_1| &\leq \|\bar{u}\|_\omega (\|(U \cdot \nabla) U - (U^* \cdot \nabla) U^*\|_\omega \\ &\quad + \|(U^* \cdot \nabla) U^* - P_C((U^* \cdot \nabla) U^*)\|_\omega). \end{aligned}$$

By Lemma 4, Lemma 5, Lemma 6, (4.2) and (6.2), we get that for $r > 5/4$, $s > 1/2$,

$$\begin{aligned} &\|(U^* \cdot \nabla) U^* - P_C((U^* \cdot \nabla) U^*)\|_\omega \\ &\leq \|(\vartheta - P_C)((U^* - U) \cdot \nabla) U^*\|_\omega + \|(\vartheta - P_C)(U \cdot \nabla)(U^* - U)\|_\omega \\ &\quad + \|(\vartheta - P_C)(U \cdot \nabla) U\|_\omega \\ &\leq c(M^{-1} + N^{-1}) \|((U^* - U) \cdot \nabla) U^*\|_{1, \omega} \\ &\quad + cM^{-1} N^{-1} \|((U^* - U) \cdot \nabla) U^*\|_{H^1(I_2, H_\omega^1(I_1))} \end{aligned}$$

$$\begin{aligned}
& + c(M^{-1} + N^{-1}) \|(U \cdot \nabla)(U^* - U)\|_{1, \omega} \\
& + cM^{-1} N^{-1} \|(U \cdot \nabla)(U^* - U)\|_{H^1(I_2, H_{\omega}^1(I_1))} \\
& + c(M^{-r} + N^{-s}) \|(U \cdot \nabla) U\|_{H_{\omega}^{r,s}(\Omega) \cap H^1(I_2, H_{\omega}^{-\frac{3}{4}}(I_1))} \\
\leq & c(M^{-r} + N^{-s}) \|U^*\|_{2, \infty} \|U\|_{M_{\omega}^{r+\frac{1}{4}, s}(\Omega)} + cM^{-1} \|U\|_{X_{\omega}^{s, \beta}(\Omega)} \|U^* - U\|_{2, \omega} \\
& + c(M^{-1} + N^{-1}) \|U\|_{1, \infty} \|U^* - U\|_{2, \omega} \\
& + cM^{-1} N^{-1} \|U\|_{2, \infty} \|\nabla(U^* - U)\|_{H^1(I_2, H_{\omega}^1(I_1))} \\
& + c(M^{-r} + N^{-s}) \|U\|_{M_{\omega}^{r+3, s+2}(\Omega) \cap H^2(I_2, H_{\omega}^r(I_1)) \cap H^s(I_2, H^3(I_1))}.
\end{aligned}$$

Moreover, Lemma 4, Lemma 5 and Lemma 7 lead to

$$\begin{aligned}
\|U^* - U\|_{2, \omega} & \leq \|U^* - P_C U\|_{2, \omega} + \|P_C U - U\|_{2, \omega} \\
& \leq c(M^2 + N) (\|U^* - U\|_{1, \omega} + \|U - P_C U\|_{1, \omega}) + \|P_C U - U\|_{2, \omega}.
\end{aligned}$$

By using Lemma 4 with $r' = r + 10/4$ and $s' = 1$, we have

$$\begin{aligned}
& (M^2 + N) \|U - P_C U\|_{1, \omega} \leq \\
& \leq c(M^{\frac{3}{4}-r} + N^{1-s}) \|U\|_{H_{\omega}^{r+\frac{13}{4}, s+\frac{8}{3}}(\Omega) \cap H^1(I_2, H_{\omega}^{r+\frac{10}{4}}(I_1)) \cap H^{s+\frac{5}{3}}(I_2, H_{\omega}^1(I_1))}.
\end{aligned}$$

Moreover by using Lemma 4 with the same values of r' and s' as in the above, we have

$$\begin{aligned}
\|U - P_C U\|_{L^2(I_2, H_{\omega}^2(I_1))} & \leq c(M^{\frac{3}{4}-r} + N^{1-s}) \times \\
& \times \|U\|_{L^2(I_2, H^{r+\frac{13}{4}}(I_1)) \cap H^1(I_2, H_{\omega}^{r+\frac{10}{4}}(I_1)) \cap H^{s-1}(I_2, H_{\omega}^2(I_1))}, \\
\|U - P_C U\|_{H^2(I_2, L_{\omega}^2(I_1))} & \leq c(M^{\frac{3}{4}-r} + N^{1-s}) \|U\|_{H^2(I_2, H_{\omega}^{-\frac{3}{4}}(I_1)) \cap H^{s+1}(I_2, L_{\omega}^2(I_1))}, \\
\|U - P_C U\|_{H^1(I_2, H_{\omega}^1(I_1))} & \leq c(M^{\frac{3}{4}-r} + N^{1-s}) \|U\|_{H^1(I_2, H_{\omega}^{r+\frac{5}{4}}(I_1)) \cap H^s(I_2, H_{\omega}^1(I_1))}.
\end{aligned}$$

Finally

$$\|U^* - U\|_{2,\omega} \leq c(M^{\frac{3}{4}-r} + N^{1-s}) \times \\ \|U\|_{H_\omega^{r+\frac{13}{4},s+\frac{8}{3}}(\Omega) \cap H^1(I_2, H_\omega^{r+\frac{10}{4}}(I_1)) \cap H^2(I_2, H_\omega^{r-\frac{3}{4}}(I_1)) \cap H^{s-1}(I_2, H_\omega^2(I_1)) \cap H^{s+\frac{5}{3}}(I_2, H_\omega^1(I_1))}.$$

It is easy to see that

$$\|\nabla(U^* - U)\|_{H^1(I_2, H_\omega^1(I_1))} \leq \\ \|\nabla(U^* - P_C U)\|_{H^1(I_2, H_\omega^1(I_1))} + \|\nabla(P_C U - U)\|_{H^1(I_2, H_\omega^1(I_1))} \\ \leq cN\|U^* - P_C U\|_{2,\omega} + \|\nabla(P_C U - U)\|_{H^1(I_2, H_\omega^1(I_1))}$$

where the term $\|U^* - P_C U\|_{2,\omega}$ is estimated as in the above. By Lemma 4,

$$\|\nabla(P_C U - U)\|_{H^1(I_2, H_\omega^1(I_1))} \\ \leq \|P_C U - U\|_{H^2(I_2, H_\omega^1(I_1))} + \|P_C U - U\|_{H^1(I_2, H_\omega^2(I_1))} \\ \leq c(M^{\frac{7}{4}-r} + N^{2-s}) \times \\ \|U\|_{H^1(I_2, H_\omega^{\frac{9}{4}}(I_1)) \cap H^2(I_2, H_\omega^{\frac{1}{4}}(I_1)) \cap H^{s-1}(I_2, H_\omega^2(I_1)) \cap H^s(I_2, H_\omega^1(I_1))}.$$

Let

$$Y_\omega^{r,s}(\Omega) = M_\omega^{r+2,s+2}(\Omega) \cap H_\omega^{r+\frac{13}{4},s+\frac{8}{3}}(\Omega) \cap H^1(I_2, H_\omega^{r+\frac{10}{4}}(I_1)) \cap \\ H^2(I_2, H_\omega^{r+\frac{1}{4}}(I_1)) \cap H^s(I_2, H_\omega^2(I_1)) \cap H^{s+\frac{5}{3}}(I_2, H_\omega^1(I_1)),$$

$$\Psi_\omega^{\alpha,\beta}(\Omega) = X_{2,\omega}^{\alpha,\beta}(\Omega) \cap W^{2,\infty}(\Omega)$$

where $\alpha > 1/2$, $\beta > 1/2$. Then

$$|B_1| \leq \|\tilde{u}\|_\omega^2 + c(M^{-2r} + N^{-2s}) \|U\|_{\Psi_\omega^{\alpha,\beta}(\Omega) \cap Y_\omega^{r,s}(\Omega)}^4.$$

By Lemma 2 and an argument similar to that in the estimation of $|B_1|$, we have

$$\begin{aligned} |B_2| &\leq c \|\tilde{u}\|_{\omega} \|P_C((U^* \cdot \nabla) U^*) - P_{M-1,N} P_C((U^* \cdot \nabla) U^*)\|_{\omega} \\ &\leq \|\tilde{u}\|_{\omega}^2 + c(M^{-2r} + N^{-2s}) \|U\|_{\mathcal{V}_{\omega}^{\alpha,\beta}(\Omega) \cap \mathcal{V}_{\omega}^{\gamma}(\Omega)}. \end{aligned}$$

By (5.4) and the integration by parts,

$$A_3(\tilde{u}, t) = \nu(\partial_2 U^*, \partial_2 \tilde{u})_{\omega} - \nu(\partial_2 U^*, \partial_2 \tilde{u})_{M,N,\omega}.$$

Therefore, Lemma 2 and Lemma 5 lead to

$$|A_3(\tilde{u}, t)| \leq \epsilon \nu |\tilde{u}|_{1,\omega}^2 + \frac{c\nu}{\epsilon} (M^{-2r} + N^{-2s}) \|U\|_{M_{\omega}^{r+1,s+1}(\Omega)}^2.$$

Similarly,

$$|A_4(\tilde{u}, t)| \leq \epsilon \nu |\tilde{u}|_{1,\omega}^2 + \frac{c\nu\sigma^2 t^2}{\epsilon} \|U\|_{C^1(0,T;M_{\omega}^1(\Omega))}^2.$$

Moreover, we get from Lemma 2 that

$$\begin{aligned} |A_5(\tilde{u}, t)| &= |(\nabla(P - P^*), \tilde{u})_{\omega} + (\nabla P^*, \tilde{u})_{\omega} - (\nabla P^*, \tilde{u})_{M,N,\omega}| \\ &\leq \|\tilde{u}\|_{\omega} (\|P - P^*\|_{1,\omega} + c\|\nabla P^* - P_{M-1,N} \nabla P^*\|_{\omega}). \end{aligned}$$

By Lemma 4,

$$\begin{aligned} \|\nabla P^* - P_{M-1,N} \nabla P^*\|_{\omega} &\leq \|(\vartheta - P_{M-1,N}) \nabla(P^* - P)\|_{\omega} + \|(\vartheta - P_{M-1,N}) \nabla P\|_{\omega} \\ &\leq c\|P^* - P\|_{1,\omega} + c(M^{-r} + N^{-s}) \|P\|_{M_{\omega}^{r+1,s+1}(\Omega)}. \end{aligned}$$

By the definition of P^* , Lemma 5 and trace theorem,

$$\|P^* - P\|_{1,\omega} \leq c(M^{-r} + N^{-s}) \|\partial_1 P\|_{M_{\omega}^{r+\frac{1}{2},s+1}(\Omega)} + cN^{-s} \|P\|_{H^{s+\frac{3}{2}}(\Omega)}.$$

Let

$$\begin{aligned} Z_{\omega}^{r,s}(\Omega) &= L^2(I_2, H^{r+2}(I_1)) \cap H^1(I_2, H^{r+1}(I_1)) \cap \\ &H^{s+2}(I_2, H^1(I_1)) \cap H^{s+1}(I_2, H^2(I_1)) \cap H^{s+\frac{3}{2}}(\Omega). \end{aligned}$$

Then

$$|A_5(\tilde{u}, t)| \leq \|\tilde{u}\|_{\omega}^2 + c(M^{-2r} + N^{-2s}) \|P\|_{Z_{\omega}^r(\Omega)}^2.$$

Similarly,

$$|A_6(\tilde{u}, t)| \leq \|\tilde{u}\|_{\omega}^2 + c(M^{-2r} + N^{-2s}) \|f\|_{H_{\omega}^{r,s}(\Omega) \cap H^1(I_2, H_{\omega}^{r-\frac{3}{4}}(I_1))}^2.$$

On the other hand, since $\partial_1 P = \partial_1 P^* = 0$ for $|x_1| = 1$, we can still integrate $A_7(v, t)$ by parts, and so

$$\begin{aligned} |A_7(v, t)| &= |(\partial_1 P - \partial_1 P^*, \partial_1(v\omega))_{L^2(\Omega)} + (\partial_2 P, \partial_2 v)_{\omega} - (\partial_2 P^*, \partial_2 v)_{M, N, \omega}| \\ &\leq \|v\|_{1, \omega} (\|P - P^*\|_{1, \omega} + \|(\partial - P_{M-1, N}) \partial_2 P\|_{\omega} + \|(\partial - P_C) \partial_2 P\|_{\omega}). \end{aligned}$$

By similar estimation as in $|A_5(\tilde{u}, t)|$,

$$|A_7(v, t)| \leq \epsilon \|v\|_{\omega}^2 + \frac{c}{\epsilon} (M^{-2r} + N^{-2s}) \|P\|_{Z_{\omega}^r(\Omega)}^2.$$

Similarly,

$$|A_8(v, t)| \leq \epsilon \|v\|_{\omega}^2 + \frac{c}{\epsilon} (M^{-2r} + N^{-2s}) \|U\|_{Y_{\omega}^{\alpha, \beta}(\Omega) \cap Y_{\omega}^r(\Omega)}^4,$$

$$\begin{aligned} |A_9(v, t)| &\leq \epsilon \|v\|_{\omega}^2 + \frac{c}{\epsilon} (M^{-2r} + N^{-2s}) \times \\ &\quad \|f\|_{M_{\omega}^{r+1, s+1}(\Omega) \cap H^1(I_2, H_{\omega}^{r+\frac{1}{4}}(I_1)) \cap H^2(I_2, H_{\omega}^{r-\frac{3}{4}}(I_1))}^2. \end{aligned}$$

Next, we take $v = m\tau\tilde{u}_t$ in $A_j(v, t)$ ($j = 1, \dots, 6$) and estimate the resulting ones as before. We get

$$\begin{aligned} |m\tau A_1(\tilde{u}, t)| &\leq \epsilon\tau \|\tilde{u}_t\|_{\omega}^2 + \frac{cm^2\tau}{\epsilon} (M^{-2r} + N^{-2s}) \|U\|_{C^1(0, T; M_{\omega}^{r+\frac{1}{4}, s}(\Omega))}^2 \\ &\quad + \frac{cm^2\tau^2}{\epsilon} \|U\|_{H^2(t, t+\tau; L_{\omega}^2(\Omega))}^2, \\ |m\tau A_2(\tilde{u}, t)| &\leq \epsilon\tau \|\tilde{u}_t\|_{\omega}^2 + \frac{cm^2\tau}{\epsilon} (M^{-2r} + N^{-2s}) \|U\|_{\Psi_{\omega}^{\alpha, \beta}(\Omega) \cap Y_{\omega}^{\alpha, \beta}(\Omega)}^4, \\ |m\tau A_3(\tilde{u}, t)| &\leq \epsilon\nu\tau^2 |\tilde{u}|_{1, \omega}^2 + \frac{cvm^2}{\epsilon} (M^{-2r} + N^{-2s}) \|U\|_{M_{\omega}^{r+1, s+1}(\Omega)}^2, \\ |m\tau A_4(\tilde{u}, t)| &\leq \epsilon\nu\tau^2 |\tilde{u}_t|_{1, \omega}^2 + \frac{cv\sigma^2 m^2 \tau^2}{\epsilon} \|U\|_{C^1(0, T; M_{\omega}^{1, 1}(\Omega))}^2, \\ |m\tau A_5(\tilde{u}, t)| &\leq \epsilon\tau \|\tilde{u}_t\|_{\omega}^2 + \frac{cm^2\tau}{\epsilon} (M^{-2r} + N^{-2s}) \|P\|_{Z_{\omega}^{\alpha, \beta}(\Omega)}^2, \\ |m\tau A_6(\tilde{u}, t)| &\leq \epsilon\tau \|\tilde{u}_t\|_{\omega}^2 + \frac{cm^2\tau}{\epsilon} (M^{-2r} + N^{-2s}) \|f\|_{H_{\omega}^{\alpha, \beta}(\Omega) \cap H^1(I_2, H_{\omega}^{r-\frac{3}{4}, s}(I_1))}^2. \end{aligned}$$

Moreover, Lemma 4, Lemma 5, Lemma 7 and (5.7) imply

$$\begin{aligned} \|\tilde{u}(0)\|_{\omega}^2 &\leq c(M^{-2r} + N^{-2s}) \|U_0\|_{M_{\omega}^{r+\frac{1}{4}, s}(\Omega)}^2, \\ \nu\tau |\tilde{u}(0)|_{1, \omega}^2 &\leq c(M^{-2r} + N^{-2s}) \|U_0\|_{M_{\omega}^{r+\frac{1}{4}, s}(\Omega)}^2, \end{aligned}$$

and Lemma 6 gives

$$\|U^*\|_{1, \infty} \leq c \|U\|_{X_{1, \omega}^{\alpha, \beta}(\Omega)}, \quad \|U^*\|_{2, \infty} \leq c \|U\|_{X_{2, \omega}^{\alpha, \beta}(\Omega)}, \quad \alpha > \frac{1}{2}, \beta > \frac{1}{2}.$$

Besides, if (4.2) and (5.7) hold, then for $r > 5/4$ and $s > 7/6$,

$$r^2 + M^{-2r} + N^{-2s} = o\left(\frac{1}{MN}\right).$$

Finally by an argument similar to that of Theorem 1, we obtain the following result.

THEOREM 2 : *Assume that*

(i) *condition (i) of Theorem 1 holds ;*

(ii) for $r > 5/4$, $s > 7/6$, $\alpha > 1/2$ and $\beta > 1/2$,

$$U \in C(0, T; Y_{\omega}^{r,s}(\Omega) \cap \Psi_{\omega}^{\alpha,\beta}(\Omega)),$$

$$\partial_t U \in C(0, T; M_{\omega}^{r+\frac{1}{4},s}(\Omega) \cap M_{\omega}^{1,1}(\Omega)),$$

$$\partial_t^2 U \in L^2(0, T; L_{\omega}^2(\Omega)),$$

$$P \in C(0, T; Z_{\omega}^{r,s}(\Omega)),$$

$$f \in C(0, T; M_{\omega}^{r+1,s+1}(\Omega) \cap H^1(I_2, H_{\omega}^{r+\frac{1}{4}}(I_1)) \cap H^2(I_2, H_{\omega}^{-\frac{3}{4}}(I_1))).$$

Then for all $t \leq T$,

$$\|U(t) - u(t)\|_{\omega}^2 \leq d_3(r^2 + M^{-2r} + N^{-2s}),$$

d_3 being a positive constant depending only on the norms of U , P and f in the spaces mentioned in the above.

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