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TWO-DIMENSIONAL MODELS OF FABRICS (*)

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Abstract. — *The aim of this work is to study different two-dimensional models of a fabric (coated or uncoated) where shearing between warp and weft is taken into account.*

In the first section, a model is described, in which we introduce two two-dimensional displacement fields in the areas where warp and weft are superposed.

The second section shows in a classical study of functional analysis that, if warp and weft interact through elastic forces, the boundary value problem has one unique solution, wether there are Neumann, Dirichlet or periodicity boundary conditions on the edge of the sample.

The third section is devoted to the homogenization method of periodic media which is applied to the considered model. It yields different macroscopic models according to the strength of the coupling between warp and weft and the possible presence of coating.

The last section sums up briefly the previous ones and gives future possibilities of development from this work.

Résumé. — *Le but de ce travail est d'étudier différents modèles bidimensionnels de tissés (enduits ou non) prenant en compte le glissement entre fils de chaîne et fils de trame.*

Dans le premier paragraphe, on introduit un modèle bidimensionnel à deux champs de déplacement dans les zones de contact entre chaîne et trame.

Dans le deuxième paragraphe, une étude d'analyse fonctionnelle classique permet de montrer que, pour une interaction chaîne-trame modélisée par un rappel élastique, les problèmes aux limites correspondant à des conditions de périodicité, de Neumann ou de Dirichlet sur le bord de l'échantillon sont bien posés.

Le troisième paragraphe est consacré à l'application de la méthode d'homogénéisation des milieux périodiques au modèle considéré. On obtient ainsi différents modèles macroscopiques fonctions de l'intensité du couplage chaîne-trame et de la présence éventuelle de résine.

Le dernier paragraphe résume brièvement les précédents et énumère les prolongements qu'on peut donner à ce travail.

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1. THE TWO-DIMENSIONAL MODEL INCLUDING INTERACTION BETWEEN WARP AND WEFT

1.1. Introduction

The aim of this work is that it takes into account the shearing between warp and weft in a model which remains two-dimensional. To this aim, we consider two displacement fields in the area of contact between warp and weft. To say that the model is two-dimensional means that the balance and constitutive equations are set in a plane and that the considered displacement fields are two-dimensional.

We consider the case of small deformations. Fibers and coating are supposed to be linearly elastic. Furthermore, the forces between warp and weft are modelled by a linear elastic springback.

1.2. Description of the model

1.2.1. *Coated fabrics (C.F.)*

The medium is described on the figure 1.

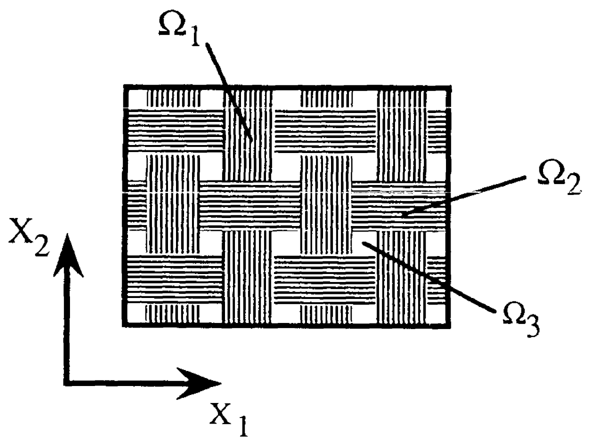

Figure 1. — Modelling of a coated fabric.

The sample of fabrics occupied a domain D of $\mathbb{R}^2$. Ω_1 and Ω_2 are composed of strips parallel to X_2 and X_1 axes, they have a not empty intersection called Ω.

Ω_1 : set of fibers parallel to X_2 (warp)
Ω_2 : set of fibers parallel to X_1 (weft)

Ω_3 : coating (disjoined from Ω_1 and Ω_2)
$\Omega_1 \cap \Omega_2 = \Omega$ (common area of Ω_1 and Ω_2)
$\Gamma^{1e} = \bar{\Omega}_1 \cap \bar{\Omega}_3$ (common boundary of Ω_1 and Ω_3)
$\Gamma^{2e} = \bar{\Omega}_2 \cap \bar{\Omega}_3$
$\Gamma^{1i} = \partial\Omega_1 \cap \Omega_2 = \partial\Omega_1 \cap \partial\Omega$ (Γ^{1i} represents the part of the boundary of Ω_1 which is also a part of Ω_2).
$\Gamma^{2i} = \partial\Omega_2 \cap \Omega_1 = \partial\Omega_2 \cap \partial\Omega$
$D = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Gamma^{1e} \cup \Gamma^{2e}$.

With the assumption of linear elasticity and small perturbations, the constitutive model of each constituent (warp, weft and coating) noted with the figure n $(n = 1, 2, 3)$ is :

$$\sigma^n_{ij} = a^n_{ijkh}\, e^x_{kh}(\underline{u}^n) \tag{1}$$

$$e^x_{kh}(\underline{u}^n) = \frac{1}{2}\left(\frac{\partial u^n_k}{\partial x_h} + \frac{\partial u^n_h}{\partial x_k}\right) \quad \text{in } \Omega_n\ (n = 1, 2, 3) \tag{2}$$

$\underline{\underline{\sigma}}$ is the stress tensor and $\underline{\underline{e}}$ is the strain tensor. The elastic moduli a^n_{ijkh} meet the usual conditions of symmetry and coercivity which are :

$$a_{ijkh} = a_{ijhk} = a_{khij}\,.$$

And :

$$\exists \alpha \textit{ so that}: \quad \forall \gamma_{ij}\ (\gamma_{ij} = \gamma_{ji}) \quad a_{ijkh}\, \gamma_{kh}\, \gamma_{ij} \geqslant \alpha \gamma_{ij}\, \gamma_{ij}$$

The problem involves three displacement fields $\underline{u}^1, \underline{u}^2, \underline{u}^3$ (defined respectively in Ω_1, Ω_2, Ω_3). As Ω_1 and Ω_2 have a not empty intersection Ω, the deformations of the fabric in Ω are described by two displacement fields $\underline{u}^1$ and $\underline{u}^2$. The interaction between warp and weft is modelled by a linear elastic springback i.e. by the density of surface force equal to $k_{ij}(u^1_j - u^2_j)$. We extend k_{ij} to zero outside Ω. Then, the balance equations are :

$$\frac{\partial \sigma^1_{ij}}{\partial x_j} + f^{1e}_i + k_{ij}(u^2_j - u^1_j) = 0 \quad in\ \Omega_1 \tag{3}$$

$$\frac{\partial \sigma^2_{ij}}{\partial x_j} + f^{2e}_i + k_{ij}(u^1_j - u^2_j) = 0 \quad in\ \Omega_2 \tag{4}$$

$$\frac{\partial \sigma^3_{ij}}{\partial x_j} + f^{3e}_i = 0 \quad in\ \Omega_3\,. \tag{5}$$

The quadratic form associated to the matrix $[k_{ij}]$ is supposed to be positive.

Furthermore, the fields $\underline{u}^n$, $\underline{\underline{\sigma}}^n$ ($n = 1, 2$ ou 3) meet transmission equations and boundary conditions on the interfaces between Ω_1, Ω_2 and Ω_3 :

$$\sigma^1_{ij} n_j = 0 \quad \text{on } \Gamma^{1\,i} \quad \underline{u}^1 = \underline{u}^3 \quad \text{and} \quad \sigma^1_{ij} n_j = \sigma^3_{ij} n_j \quad \text{on } \Gamma^{1\,e} \tag{6}$$

$$\sigma^2_{ij} n_j = 0 \quad \text{on } \Gamma^{2\,i} \quad \underline{u}^2 = \underline{u}^3 \quad \text{and} \quad \sigma^2_{ij} n_j = \sigma^3_{ij} n_j \quad \text{on } \Gamma^{2\,e} . \tag{7}$$

1.2.2. *Uncoated fabrics (U.F.)*

The case of uncoated fabrics (U.F.) is described nearly like coated fabrics (C.F.). The difference is that we consider only two domains Ω_1 and Ω_2 and then two displacement fields $\underline{u}^1$ and $\underline{u}^2$ respectively defined in Ω_1 and Ω_2. Then for U.F., the balance equations are the same as 3 and 4 and the third equation 5 vanishes. On the other hand, the equations 6 and 7 become the boundary conditions :

$$\sigma^1_{ij} n_j = 0 \quad \text{on } \Gamma^{1\,i} \quad \text{and} \quad \Gamma^{1\,e} \tag{8}$$

$$\sigma^2_{ij} n_j = 0 \quad \text{on } \Gamma^{2\,i} \quad \text{and} \quad \Gamma^{2\,e} . \tag{9}$$

2. WEAK FORMULATIONS. EXISTENCE AND UNIQUENESS

2.1. Introduction

The aim of this section is to prove by using the theorem of Lax-Milgram, which is applied to weak formulations, that the problems of this modelling of fabrics with usual boundary conditions (Dirichlet, Neumann, mixed conditions, periodic conditions) are well posed.

We shall see that, in the weak formulations, there are the usual bilinear forms of elasticity for each part of the fabric (warp and weft for U.F., warp, weft and coating for C.F.). For Dirichlet problems, the coercivity of these forms is a consequence of the Poincaré's inequality, and then the term corresponding to the elastic springback between warp and weft is unessential for the existence and uniqueness results.

On the other hand, for U.F. with Neumann boundary conditions, the elastic springback is important because the domain D where the equations are set is not connected (unlike what is usual for problems of elasticity) and the coercivity of the bilinear form is not so obvious.

The problem is similar for U.F. with periodic boundary conditions or for problems with mixed boundary conditions. In this latter case, if on each fiber there are boundary conditions for displacements, then the coercivity is a consequence of the Poincaré's inequality. Otherwise for other conditions, the coercivity depends in a very important way of the elastic springback.

For C.F., in a way, the coating makes the domain D connected and the problem is nearly the same as this of classical linear elasticity.

Then, the « non-connectivity » of the domain D yields many different cases of problems with different boundary conditions. We shall not study here all these cases, and we shall only consider a problem in a simplified domain D which presents however the main characteristics of the problem of existence and uniqueness. This simplified domain is shown on the figure 2. It is composed of two fibers Ω_1 and Ω_2 parallel to X_2 and X_1 axes. For C.F., we add the coating, that is the third domain Ω_3 disjoined from Ω_1 and Ω_2.

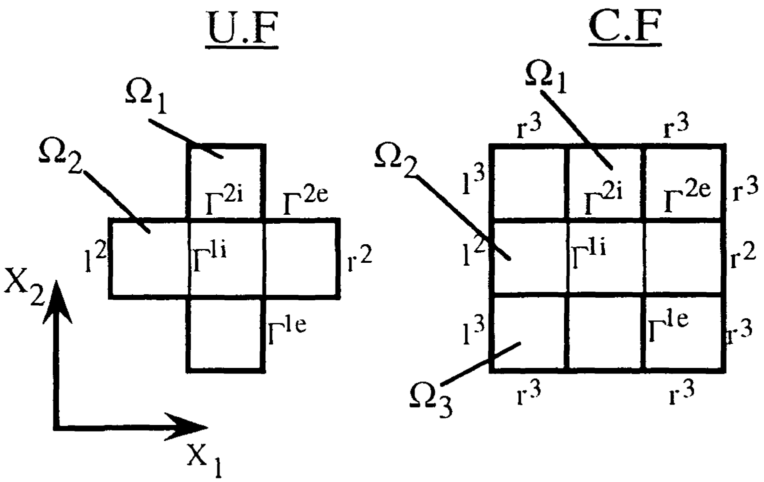

Figure 2. — A simplified model of a U.F. and a C.F.

The domains Ω, D and the boundaries Γ^{1e}, Γ^{1i}, Γ^{2e} and Γ^{2i} are defined in a same way as in the beginning of the section 1.2. In this section, we define also l^2 as the left part of the boundary $\partial D \cap \partial\Omega_2$ and r^2 as the right part of the boundary $\partial D \cap \partial\Omega_2$ (*see fig.* 2). For C.F., we define also l^3 as the left part of $\partial\Omega \cap \partial\Omega_3$. The rest of $\partial\Omega \cap \partial\Omega_3$ is called r^3.

In the next paragraph, we give the weak formulations of the problem for periodic boundary conditions and mixed conditions in the case of coated and uncoated fabrics.

2.2. Weak formulations

2.2.1. *Uncoated fabrics with mixed boundary conditions*

The strong formulation of the problem is made up of the balance equations 3, 4, the constitutive law 1 and the transmission equations 8 and 9.

We have to add boundary conditions, for mixed boundary conditions they may be for instance :

$$\underline{u}^2 = 0 \quad \text{on } l^2$$
$$\sigma_{ij}^2 n_j = F_i^2 \text{ on } r^2 \quad \text{and} \quad \sigma_{ij}^1 n_j = F_i^1 \text{ on } \partial D \cap \partial \Omega_1 .$$

With these conditions, the weak formulation of the problem is :

Find $u \in V_{um}$ (u for uncoated, m for mixed boundary conditions) so that : $\forall v \in V_{um}$

$$\sum_{n=1}^{2} \int_{\Omega_n} a_{ijlm}^n \, e_{lm}^x(\underline{u}^n) \, e_{ij}^x(\underline{v}^n) \, dx + \int_{\Omega} k_{ij}(u_j^2 - u_j^1)(v_i^2 - v_i^1) \, dx =$$
$$= \sum_{n=1}^{2} \int_{\Omega_n} f_i^{ne} \, v_i^n \, dx + \int_{r^2} F_i^2 \, v_i^2 \, dx + \int_{\partial D \cap \Omega_1} F_i^1 \, v_i^1 \, dx . \quad (10)$$

With

$$V_{um} = \{ v = (\underline{v}^1, \underline{v}^2) \text{ so that } (v_1^n, v_2^n) \in [H^1(\Omega_n)]^2 \; n = 1, 2 \text{ and } \underline{v}^2 = 0 \text{ on } l^2 \} .$$

2.2.2. *Coated fabrics with mixed boundary conditions*

In this case, the strong formulation is made up of the balance equations 3 to 5, the constitutive law 1 and transmission equations 6 and 7.

For C.F., we consider the mixed boundary conditions :

$$\underline{u}^2 = 0 \text{ on } l^2 \qquad \underline{u}^3 = 0 \text{ on } l^3$$
$$\sigma_{ij}^2 n_j = F_i^2 \text{ on } r^2 \qquad \sigma_{ij}^3 n_j = F_i^3 \text{ on } r^3$$
$$\sigma_{ij}^1 n_j = F_i^1 \text{ on } \partial D \cap \partial \Omega_1$$

With these conditions, the weak formulation of the problem is :

Find $u \in V_{cm}$ (c for coated, m for mixed) so that :

$\forall v \in V_{cm}$

$$\sum_{n=1}^{3} \int_{\Omega_n} a_{ijlm}^n \, e_{lm}^x(\underline{u}^n) \, e_{ij}^x(\underline{v}^n) \, dx + \int_{\Omega} k_{ij}(u_j^2 - u_j^1)(v_i^2 - v_i^1) \, dx =$$
$$= \sum_{n=1}^{3} \int_{\Omega_n} f_i^{ne} \, v_i^n \, dx + \int_{r^2} F_i^2 \, v_i^2 \, dx + \int_{r^3} F_i^3 \, v_i^3 \, dx + \int_{\partial D \cap \Omega_1} F_i^1 \, v_i^1 \, dx . \quad (11)$$

With $V_{cm} = \{v = (\underline{v}^1, \underline{v}^2, \underline{v}^3)$ so that $\underline{v}^n = (v_1^n, v_2^n) \in [H^1(\Omega_n)]^2$ $n = 1, 2, 3$; $\underline{v}^1 = \underline{v}^3$ on Γ^{1e}, $\underline{v}^2 = \underline{v}^3$ on Γ^{2e}; $\underline{v}^2 = 0$ on l^2 and $\underline{v}^3 = 0$ on $l^3\}$.

2.2.3. *Periodic boundary conditions for U.F. and C.F.*

We can also consider periodic boundary conditions so that :

u^m and σ_{ij}^m ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.) have the same values on opposite sides of $\partial D \cap \partial\Omega_m$ ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.). And then $\sigma_{ij}^m n_j$ has opposite values on opposite sides of $\partial D \cap \partial\Omega_m$ ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.). (12)

With these conditions of periodicity, the weak formulation of the problem is :

Find $u \in V_p$ (p for periodic) so that :

$\forall v \in V_p$

$$\sum_{n=1}^{3} \int_{\Omega_n} a_{ijlm}^n \, e_{lm}^x(\underline{u}^n) \, e_{ij}^x(\underline{v}^n) \, dx + \int_{\Omega} k_{ij}(u_j^2 - u_j^1)(v_i^2 - v_i^1) \, dx = \\ = \sum_{n=1}^{p} \int_{\Omega_n} f_i^{ne} \, v_i^n \, dx \quad (13)$$

($p = 2$ for U.F., $p = 3$ for C.F.).

With, for U.F. :

$$V_p = V_{up} = \{v = (\underline{v}^1, \underline{v}^2) \text{ so that } \underline{v}^n = (v_1^n, v_2^n) \in [\dot{H}^1(\Omega_n)]^2,$$

$\underline{v}^n$ has the same value on opposite sides of $\partial D \cap \partial\Omega_n$ $n = 1, 2\}$.

And, for C.F. :

$$V_p = V_{cp} = \{v = (\underline{v}^1, \underline{v}^2, \underline{v}^3) \text{ so that } \underline{v}^n = (v_1^n, v_2^n) \in [H^1(\Omega_n)]^2 \; n = 1, 2, 3;$$

$\underline{v}^1 = \underline{v}^3$ on Γ^{1e}, $\underline{v}^2 = \underline{v}^3$ on Γ^{2e} ; $\underline{v}^n$ has the same value on opposite sides of $\partial D \cap \partial\Omega_n$ $n = 1, 2, 3\}$.

2.3. Existence and uniqueness

The weak problems 10, 11 and 13 can be written in the following way :
Find $u \in V$ so that $\forall v \in V$,

$$a(u, v) = L(v) .$$

The definition of the space V depends on the boundary conditions and on the possible presence of coating (see section 2.2).

To prove the existence and uniqueness of the solution of the problems 10, 11 and 13, we use the theorem of Lax-Milgram. In order to use this theorem, we need to prove that the bilinear and symmetric form $a(v, v)$ involved in these different problems provides the spaces V with a structure of Hilbert space.

In the following, we shall use some theorems and results of functionnal analysis, the terms of which may be found in any treatise of functionnal analysis (see for instance [2] and [5]).

2.3.1. *Coercivity of the bilinear forms $a(u, v)$*

In order to prove that $\sqrt{a(v, v)}$ provides the spaces V with a structure of Hilbert spaces, we shall use a classical theorem of functionnal analysis, the terms and the proof of which are recalled hereafter.

THEOREM : Let L and E be two Hilbert spaces for the norms $\| \ \|_L$ and $\| \ \|_E$ such that :

— $E \subset L$ algebrically.

— $\forall v \in E \quad \|v\|_E^2 = \|v\|_L^2 + a(v, v)$ where $a(v, v)$ is a positive quadratic form on E.

— The injection $x \in E \to x \in L$ is compact.

Let W be a closed subspace of E so that $\|v\|_1 = \sqrt{a(v, v)}$ is a prehilbertian norm on W.

Then on W, $\| \ \|_1$ is a norm equivalent to $\| \ \|_E$ and consequently, W equiped with the norm $\| \ \|_1$ is an Hilbert space.

Recall of the proof

It is clear that :

$$\|v\|_1 \leqslant \|v\|_E .$$

We have just to prove that :

$$\exists M \in \mathbb{R} \text{ s.t.} : \forall v \in W \quad \|v\|_L \leqslant M\|v\|_1 .$$

Let assume that this inequality does not hold true. Then :

$$\forall n \, \exists w^n \in W \text{ s.t.} : \|w^n\|_L \geqslant n\|w^n\|_1 .$$

Let set :

$$v^n = \frac{w^n}{\|w^n\|_L} .$$

Then :

$$\forall n\, \exists v^n\ s.t. : \|v^n\|_L = 1 \quad \text{and} \quad \|v^n\|_1 \leqslant \frac{1}{n}.$$

The sequence v^n is bounded in E, indeed :

$$\|v^n\|_E^2 = \|v^n\|_L^2 + \|v^n\|_1^2 \leqslant 1 + \frac{1}{n^2} \leqslant 2 \quad \text{for} \quad n \geqslant 1 .$$

As E is an Hilbert space, there is a subsequence extracted from v^n and v belonging to E so that $v^n \to v$ weakly in E when $n \to \infty$. As W is a closed vector subspace of E, it is closed too for the weak topology and as v_n belongs to W, the limit v belongs to W. As a consequence of the weak convergence of v_n to v, we have :

$$\|v\|_E \leqslant \liminf_{n \to \infty} \|v^n\|_E = \liminf_{n \to \infty} \sqrt{\|v^n\|_L^2 + \|v^n\|_1^2} .$$

Now, the injection $E \to L$ is compact, then v^n converges to v strongly in L and :

$$\|v\|_L = \lim_{n \to \infty} \|v^n\|_L = 1 .$$

Furthermore :

$$\|v^n\|_1 \leqslant \frac{1}{n}.$$

Then :

$$\liminf_{n \to \infty} \sqrt{\|v^n\|_L^2 + \|v^n\|_1^2} = 1 \quad \text{and} \quad \|v\|_E \leqslant 1 .$$

That is :

$$\|v\|_L^2 + \|v\|_1^2 = 1 + \|v\|_1^2 \leqslant 1 .$$

And necessarily :

$$\|v\|_1 = 0 .$$

Now, $v \in W$ and necessarily $v = 0$, which is impossible for $\|v\|_L = 1$.

The theorem is then proved.

We have now to consider the different cases of uncoated and coated fabrics.

Uncoated fabrics :

For this case, we have :

$$L = [L^2(\Omega_1)]^2 X [L^2(\Omega_2)]^2$$

$$E = [H^1(\Omega_1)]^2 X [H^1(\Omega_2)]^2 .$$

And the injection $E \to L$ is compact.

It is clear that :
$V_{um} = \{ v \in E \, s.t. \, \underline{v}^2 = 0 \text{ on } \ell^2 \}$
and $V_{up} = \{ v \in E$ s.t. $\underline{v}^n$ has the same value on opposite sides of $\partial D \cap \partial \Omega_n \; n = 1, 2 \}$ are closed subspaces of E.

The usual norm on E is such that :

$$\| v \|^2 = \sum_{n=1}^{2} \int_{\Omega_n} v_i^n v_i^n \, dx + \sum_{n=1}^{2} \int_{\Omega_n} \frac{\partial v_i^n}{\partial x_j} \frac{\partial v_i^n}{\partial x_j} \, dx .$$

Assuming that the matrix (k_{ij}) is definite positive and bounded and using Korn lemma, it is easy to prove that $\| v \|_E$ defined by :

$$\| v \|_E^2 = \sum_{n=1}^{2} \int_{\Omega_n} v_i^n v_i^n \, dx + \sum_{n=1}^{2} \int_{\Omega_n} e_{ij}(\underline{v}^n) \, e_{ij}(\underline{v}^n) \, dx + $$

$$+ \sum_{n=1}^{2} \int_{\Omega} k_{ij}(v_j^2 - v_j^1)(v_i^2 - v_i^1) \, dx$$

is an Hilbertian norm which is equivalent to the usual one. Let define $a(u, v)$ as :

$$a(u, v) = \sum_{n=1}^{2} \int_{\Omega_n} e_{ij}(\underline{u}^n) \, e_{ij}(\underline{v}^n) \, dx + \sum_{n=1}^{2} \int_{\Omega} k_{ij}(u_j^2 - u_j^1)(v_i^2 - v_i^1) \, dx .$$

We are now almost under the hypothesis of the previous theorem, in order to use it we have just to prove that $\sqrt{a(v, v)}$ is a norm on the spaces V_{um} and V_{up}.

It is clear that $\| \quad \|_1 = \sqrt{a(v, v)}$ is not a norm on V_{up}. Indeed for $v = (\underline{c}, \underline{c})$ (where $\underline{c}$ is a constant vector of $\mathbb{R}^2$), we have :

$$a(v, v) = 0 .$$

This circonstance is usual for boundary value problems with periodic conditions (see [1, 6]) and it is usual then to study the problem in $C_{up}^{\perp}$, the orthogonal space of C_u in E where :

$$C_u = \{ v \in E \; s.t. \; v = (\underline{c}, \underline{c}), \underline{c} \text{ is a constant vector} \} .$$

The space $C_{up}^{\perp}$ is then :

$$C_{up}^{\perp} = \{ v \in E \; s.t. \; \underline{v}^n \text{ has the same value on opposite sides of}$$

$$\partial D \cap \partial \Omega_n \; n = 1, 2 \; and \int_{\Omega_1} \underline{v}^1 \, dx + \int_{\Omega_2} \underline{v}^2 \, dx = 0 \} .$$

But we may state :

LEMMA : *If the matrix k_{ij} is positive definite, then $\sqrt{a(v, v)}$ is a norm on both spaces :*
$V_{um} = \{ v \in E \text{ s.t. } \underline{v}^2 = 0 \text{ on } t^2 \}$
and $C_{up}^{\perp} = \{ v \in E$ s.t. $\underline{v}^n$ has the same value on opposite sides of $\partial D \cap \partial \Omega_n$ $\quad n = 1, 2$ $\quad$ and $\displaystyle\int_{\Omega_1} \underline{v}^1 \, dx + \int_{\Omega_2} \underline{v}^2 \, dx = 0 \}$.

Proof of this lemma : It is clear that the only point to prove is that :

$$\| v \|_1 = 0 \Rightarrow v = 0 .$$

Now $\quad \| v \|_1 = 0 \Rightarrow$

$$\int_{\Omega_1} e_{ij}(\underline{v}^1) \, e_{ij}(\underline{v}^1) \, dx = 0 \tag{14}$$

$$\int_{\Omega_2} e_{ij}(\underline{v}^2) \, e_{ij}(\underline{v}^2) \, dx = 0 \tag{15}$$

$$\int_{\Omega} k_{ij}(v_j^1 - v_j^2)(v_i^1 - v_i^2) \, dx = 0 . \tag{16}$$

From the equations 14 and 15, we can say that $\underline{v}^1$ and $\underline{v}^2$ are solid body displacements. So :

$$\underline{v}^1 = \underline{a}^1 + \underline{b}^1 \wedge \underline{x}$$

$$\underline{v}^2 = \underline{a}^2 + \underline{b}^2 \wedge \underline{x} .$$

The matrix k_{ij} is supposed to be definite positive. Then, from the relation 16, we get :

$$\underline{v}^2 - \underline{v}^1 = 0 \text{ on } \Omega .$$

And necessarily :

$$\underline{a}^1 = \underline{a}^2 = \underline{a}$$

$$\underline{b}^1 = \underline{b}^2 = \underline{b} .$$

We then study separately the spaces V_{um} and V_{up}. For the space V_{um}, the boundary conditions on l^2 yield :

$$\underline{a} = \underline{b} = 0 .$$

And then :

$$\underline{v}^2 = 0 \; a.e. \text{ in } \Omega_2 \quad \text{and} \quad \underline{v}^1 = 0 \text{ a.e. in } \Omega_1 . \tag{17}$$

For the space $C_{up}^{\perp}$, periodic boundary conditions yield :

$$\underline{b} = 0 .$$

And then :

$$v = (\underline{v}^1, \underline{v}^2) = (\underline{a}, \underline{a}) .$$

So $v \in C_u$. As v is assumed to be in $C_{up}^{\perp}$, it is equal to zero.

For periodic conditions, as usual, in order $L(v)$ (defined as the right hand side of the weak formulation 13) to be a linear continuous form defined on $C_{up}^{\perp}$, the forces $\underline{f}^{ne}$ $(n = 1, 2)$ have to be such that $L(c) = 0$ where $c = (\underline{c}, \underline{c})$, i.e. :

$$\sum_{n=1}^{2} \int_{\Omega_n} \underline{f}^{ne} \, dx = 0 .$$

Then we use the previous theorem and state.

PROPOSITION : *If the matrix k_{ij} is definite positive then :*

1. *The problem* 10 *(mixed boundary conditions) has an unique solution.*

2. *If* $\sum_{n=1}^{2} \int_{\Omega_n} \underline{f}^{ne} \, dx = 0$, *then the problem* 13 *with* $p = 2$ *(periodic boundary conditions) has an unique solution up to an additive constant* $c = (\underline{c}, \underline{c}) (\underline{c} \in \mathbb{R}^2)$.

Coated fabrics

The study for coated fabrics is near the same as for uncoated fabrics. For this case, we set :

$$L = [L^2(\Omega_1)]^2 X[L^2(\Omega_2)]^2 X[L^2(\Omega_3)]^2$$

$$E = [H^1(\Omega_1)]^2 X[H^1(\Omega_2)]^2 X[H^1(\Omega_3)]^2 .$$

And the injection $E \to L$ is compact.

The spaces V_{cm} and V_{cp} have been defined as :

$$V_{cm} = \{v \in E \text{ s.t. } \underline{v}^1 = \underline{v}^3 \text{ on } \Gamma^{1e}, \underline{v}^2 = \underline{v}^3 \text{ on } \Gamma^{2e} ; \underline{v}^2 = 0 \text{ on } l^2, \underline{v}^3 = 0 \text{ on } l^3\}$$

$V_{cp} = \{v \in E$ s.t. $\underline{v}^1 = \underline{v}^3$ on Γ^{1e}, $\underline{v}^2 = \underline{v}^3$ on Γ^{2e} ; $\underline{v}^n$ has the same value on opposite sides of $\partial D \cap \partial\Omega_n$ $n = 1, 2, 3\}$.

As the trace operators on Γ^{1e} and Γ^{2e} are continuous, it is clear that the spaces V_{cm} and V_{cp} are closed subspaces of E.

Contrary to the case of uncoated fabrics, the coerciveness of the bilinear form involved in problems 11 is not a consequence of the positive definiteness of k_{ij} but of the continuity conditions 6 and 7.

Then we define here :

$$a(v, v) = \sum_{n=1}^{3} \int_{\Omega_n} e_{ij}(\underline{v}^n)\, e_{ij}(\underline{v}^n)\, dx .$$

And it is clear from Korn Lemma that :

$$\| v \|_E^2 = \sum_{n=1}^{3} \int_{\Omega_n} v_i^n v_i^n \, dx + a(v, v)$$

defines an hilbertian norm on E equivalent to the « cannonic » one.

As for uncoated fabrics, we may state :

LEMMA : *$\sqrt{a(v, v)}$ is a norm on both spaces :*

$$V_{cm} = \{v \in E \text{ s.t. } \underline{v}^1 = \underline{v}^3 \text{ on } \Gamma^{1e}, \underline{v}^2 = \underline{v}^3 \text{ on } \Gamma^{2e} ; \underline{v}^2 = 0 \text{ on } l^2, \underline{v}^3 = 0 \text{ on } l^3\}$$

and $C_{cp}^{\perp} = \{v \in E$ s.t. $\underline{v}^1 = \underline{v}^3$ on Γ^{1e}, $\underline{v}^2 = \underline{v}^3$ on Γ^{2e} ; $\underline{v}^n$ *has the same value on opposite sides of* $\partial D \cap \partial\Omega_n$ $n = 1, 2, 3$ *and* $\int_{\Omega_1} \underline{v}^1 \, dx + \int_{\Omega_2} \underline{v}^2 \, dx + \int_{\Omega_3} \underline{v}^3 \, dx = 0\}$.

Proof of this lemma : The only point to prove is that :

$$\| v \|_1 = 0 \Rightarrow v = 0 .$$

As for U.C., $\|v\|_1 = 0 \Rightarrow$

$$\underline{v}^1 = \underline{a}^1 + \underline{b}^1 \wedge \underline{x}$$
$$\underline{v}^2 = \underline{a}^2 + \underline{b}^2 \wedge \underline{x}$$
$$\underline{v}^3 = \underline{a}^3 + \underline{b}^3 \wedge \underline{x} .$$

As $\underline{v}^1$ and $\underline{v}^3$ satisfy the continuity condition $\underline{v}^1 = \underline{v}^3$ on Γ^{1e}, we get :

$$\forall \underline{x} \in \Gamma^{1e} \quad \underline{a}^1 + \underline{b}^1 \wedge \underline{x} = \underline{a}^3 + \underline{b}^3 \wedge \underline{x} .$$

And then :

$$\underline{a}^1 = \underline{a}^3 = \underline{a}$$
$$\underline{b}^1 = \underline{b}^3 = \underline{b} .$$

In the same way :

$$\underline{a}^2 = \underline{a}^3 = \underline{a}$$
$$\underline{b}^2 = \underline{b}^3 = \underline{b} .$$

Now we study separately the spaces V_{cm} and V_{cp}.

For V_{cm}, the boundary conditions on l^2 and l^3 yield :

$$\underline{a}^2 = \underline{b}^2 = \underline{a}$$
$$\underline{a}^3 = \underline{b}^3 = \underline{a} .$$

And then :

$$v = 0 .$$

For V_{cp}, periodic boundary conditions yield :

$$\underline{b} = 0 .$$

And then :

$$v = (\underline{a}, \underline{a}, \underline{a}) .$$

So $v \in C_c$. As for U.C., v is assumed to be in $C_{cp}^{\perp}$, then it is equal to zero.

For periodic conditions, as usual, in order $L(v)$ (defined as the right hand side of the weak formulation 13 with $p = 3$) to be a linear continuous form defined on $C_{cp}^{\perp}$, the forces $\underline{f}^{ne}$ $(n = 1, 2, 3)$ have to be such that $L(c) = 0$ where $c = (\underline{c}, \underline{c}, \underline{c})$, i.e. :

$$\sum_{n=1}^{3} \int_{\Omega_n} \underline{f}^{ne}\, dx = 0 .$$

Then we use the previous theorem and state :

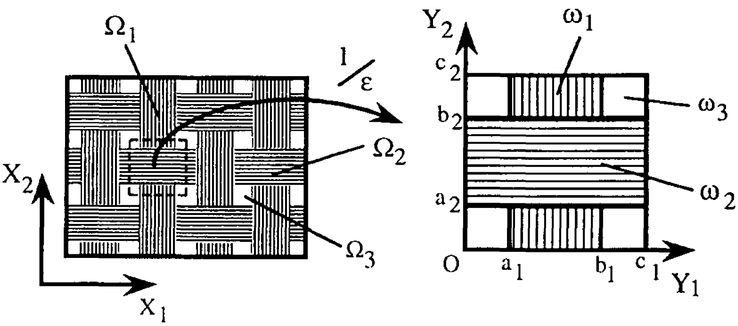

Figure 3. — The basic cell Y.

PROPOSITION : *If the matrix (k_{ij}) is positive (but not necessarily definite) then :*

1. *The problem* 11 *(mixed boundary conditions) has an unique solution.*

2. *If* $\sum_{n=1}^{3} \int_{\Omega_n} \underline{f}^{ne} \, dx = 0$, *then the problem* 13 *with* $p = 3$ *(periodic boundary conditions) has an unique solution up to an additive constant* $c = (\underline{c}, \underline{c}, \underline{c})(\underline{c} \in \mathbb{R}^2)$.

3. HOMOGENIZATION

3.1. Introduction

The aim of the method of homogenization is to replace a finely heterogeneous medium by a macroscopic equivalent medium. As our fabric is periodic, we shall use the method of two scale asymptotic expansions (see [1] and [6]). It assumes the existence of a small parameter ε which measures the size of the period, each of these periods being the homothetic of a basic cell Y of dimensions comparable to the dimensions of the medium (see the figure 3 for C.F. To represent U.C., the coating, i.e. the domains Ω_3 and ω_3, has to be removed).

$\omega_1 =]a_1, b_1[\; X]0, c_2[$

$\omega_2 =]a_2, b_2[\; X]0, c_1[$

$\omega = \omega_1 \cap \omega_2 =]a_1, b_1[\; X]a_2, b_2[$

$\gamma^{1e} = \bar{\omega}_1 \cap \bar{\omega}_3$ (common boundary of ω_1 and ω_3)

$\gamma^{2e} = \bar{\omega}_2 \cap \bar{\omega}_3$

$\gamma^{1i} = \partial\omega_1 \cap \omega_2 = \partial\omega_1 \cap \partial\omega$ (γ^{1i} represents the part of the boundary of ω_1 which is also a part of the boundary of ω)

$\gamma^{2i} = \partial\omega_2 \cap \omega_1 = \partial\omega_2 \cap \partial\omega$

We look for displacements u_i and stresses σ_{ij} as two scale asymptotic expansions :

$$u_i^m = u_i^{m0}(x, x/\varepsilon) + \varepsilon u_i^{m1}(x, x/\varepsilon) + \varepsilon^2 u_i^{m2}(x, x/\varepsilon) + \dots \qquad ((18))$$

$$\sigma_{ij}^m = \frac{1}{\varepsilon}\sigma_{ij}^{m-1}(x, x/\varepsilon) + \sigma_{ij}^{m0}(x, x/\varepsilon) + \dots \qquad ((19))$$

$m = 1, 2$ for U.F. $m = 1, 2, 3$ for C.F. x is the slow or macroscopic variable and y $(y = x/\varepsilon)$ is the quick or microscopic variable.

Then, these expressions 18 and 19 are replaced in the balance and constitutive equations (1 to 4 for U.F., 1 to 5 for C.F.). By identifying equations (which contain the variables x and y) at each order of ε, an equivalent mechanic characteristic of the fabric can be found, i.e. a balance and constitutive equation of the homogenized medium (which depend on the only variable x). It will be seen that the obtained macroscopic models depend on the order of magnitude of k_{ij}^{ε} with respect to ε.

3.2. Asymptotic expansions and macroscopic models

Two types of fabrics (coated and uncoated) are considered as well as three dependances of ε of k_{ij}^{ε} : $k_{ij}^{\varepsilon} = k_{ij}$, $k_{ij}^{\varepsilon} = k_{ij}/\varepsilon$, $k_{ij}^{\varepsilon} = k_{ij}/\varepsilon^2$. It will be seen that the macroscopic models may be different for coated and uncoated fabrics with the same k_{ij}^{ε}.

3.2.1. *Assumption* $k_{ij}^{\varepsilon} = k_{ij}$

The problem defined with the following equations is considered :

1. Equations 1 to 5 and conditions 6, 7 for C.F.
2. Equations 1 to 4 and conditions 8, 9 for U.F.

With the boundary conditions (for example) :

$$\underline{u}^m = 0 \quad \text{on } \partial D \cap \partial \Omega_n^{\varepsilon} \ (n = 1, 2, 3 \text{ for C.F. } n = 1, 2 \text{ for U.F.}) .$$

Carrying the asymptotie expansions 18 and 19 in the expressions of $\frac{\partial \sigma_{ij}^m}{\partial x_j}$ ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.) and $e_{kh}^x(\underline{u}^m)$ (see equation 2) yields :

$$\frac{\partial \sigma_{ij}^m}{\partial x_j} = \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}\frac{\partial \sigma_{ij}^{m,-1}}{\partial y_j} + \frac{\partial \sigma_{ij}^{m,-1}}{\partial x_j}\right) + \left(\frac{1}{\varepsilon}\frac{\partial \sigma_{ij}^{m0}}{\partial y_j} + \frac{\partial \sigma_{ij}^{m0}}{\partial x_j}\right)$$
$$+ \varepsilon\left(\frac{1}{\varepsilon}\frac{\partial \sigma_{ij}^{m1}}{\partial y_j} + \frac{\partial \sigma_{ij}^{m1}}{\partial x_j}\right) + \dots$$

$$e_{kh}^x(\underline{u}^m) = \left[\frac{1}{\varepsilon} e_{kh}^y(\underline{u}^{m0}) + e_{kh}^x(\underline{u}^{m0})\right] + \varepsilon\left[\frac{1}{\varepsilon} e_{kh}^y(\underline{u}^{m1}) + e_{kh}^x(\underline{u}^{m1})\right]$$
$$+ \varepsilon^2\left[\frac{1}{\varepsilon} e_{kh}^y(\underline{u}^{m2}) + e_{kh}^x(\underline{u}^{m2})\right] + \dots$$

Using these expansions in the balance and constitutive equations 1 to 4 for U.F., 1 to 5 for C.F., and identifying them at each order of ε yields the following equations in the cell Y:

$$\sigma_{ij}^{n,-1} = a_{ijkh}^{n} e_{kh}^{y}(\underline{u}^{n0}) \tag{20}$$

$$\sigma_{ij}^{n0} = a_{ijkh}^{n}[e_{kh}^{x}(\underline{u}^{n0}) + e_{kh}^{y}(\underline{u}^{n1})] \tag{21}$$

$$\frac{\partial \sigma_{ij}^{n,-1}}{\partial y_j} = 0 \tag{22}$$

$$\frac{\partial \sigma_{ij}^{n0}}{\partial y_j} + \frac{\partial \sigma_{ij}^{n,-1}}{\partial x_j} = 0 \,. \tag{23}$$

In ω_n ($n = 1, 2$ for U.F. and $n = 1, 2, 3$ for C.F.). And :

$$\frac{\partial \sigma_{ij}^{11}}{\partial y_j} + \frac{\partial \sigma_{ij}^{10}}{\partial x_j} + k_{ij}(u_j^{20} - u_j^{10}) + f_i^{1e} = 0 \quad \text{in } \omega_1 \tag{24}$$

$$\frac{\partial \sigma_{ij}^{21}}{\partial y_j} + \frac{\partial \sigma_{ij}^{20}}{\partial x_j} + k_{ij}(u_j^{10} - u_j^{20}) + f_i^{2e} = 0 \quad \text{in } \omega_2 \tag{25}$$

$$\frac{\partial \sigma_{ij}^{31}}{\partial y_j} + \frac{\partial \sigma_{ij}^{30}}{\partial x_j} + f_i^{3e} = 0 \quad \text{in } \omega_3 \text{ (only for C.F.)} \,. \tag{26}$$

With transmission equations :

- For C.F. :

$$\sigma_{ij}^{1a} n_j = 0 \text{ on } \gamma^{1i}, \quad \underline{u}^{1a} = \underline{u}^{3a} \quad \text{and} \quad \sigma_{ij}^{1a} n_j = \sigma_{ij}^{3a} n_j \text{ on } \gamma^{1e} \tag{27}$$

$$\sigma_{ij}^{2a} n_j = 0 \text{ on } \gamma^{2i}, \quad \underline{u}^{2a} = \underline{u}^{3a} \quad \text{and} \quad \sigma_{ij}^{2a} n_j = \sigma_{ij}^{3a} n_j \text{ on } \gamma^{2e} \tag{28}$$

$a = -1, 0, 1.$

- For U.F. :

$$\sigma_{ij}^{ma} n_j = 0 \text{ on } \gamma^{me} \quad \textit{and} \quad \gamma^{mi} \; (m = 1, 2 \; a = -1, 0, 1) \,. \tag{29}$$

And periodic boundary conditions on the boundary of Y:

u_i^{mb} and σ_{ij}^{ma} ($a = -1, 0, 1$ $b = 0, 1, 2$) have the same values on opposite sides of $\partial Y \cap \partial \omega_m$ ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.). And then $\sigma_{ij}^{ma} n_j$ has opposite values on opposite sides on $\partial Y \cap \partial \omega_m$ ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.). (30)

The study of the section 2 for C.F. may be applied here with $k_{ij} = 0$ and the solution of the problem (20, 22) with transmission equations ((27, 28) for C.F., 29 for U.F.) and periodic boundary conditions 30 is :

$$\sigma_{ij}^{n,-1} = 0 \tag{31}$$

$$u_i^{n0} = u_i^{n0}(x) \quad (n = 1, 2 \text{ for U.F.}, n = 1, 2, 3 \text{ for C.F.}) . \tag{32}$$

For C.F., using the transmission equations allows to conclude that :

$$\underline{u}^{10} = \underline{u}^{20} = \underline{u}^{30} = \underline{u}^{0}(x) .$$

For U.F., nothing else can be said about $\underline{u}^{10}$ and $\underline{u}^{20}$.

From 32 and the linearity of the equations 20, 21, 22 and 23, the displacement $\underline{u}^{k1}$ ($k = 1, 2$ for U.F., $k = 1, 2, 3$ for C.F.) can be looked under the form :

$$u_i^{k1} = \chi_i^{klm} e_{lm}^x(\underline{u}^{k0}) + \tilde{u}_i^1(x) . \tag{33}$$

It may easily be proved (see section 2) that the vectors $\underline{\chi}^{nkh}$ ($n = 1, 2$ for U.F., $n = 1, 2, 3$ for C.F.) are solutions of the problems :

$$\tau_{ij}^{nkh} = a_{ijlm}^n e_{lm}^y(\underline{\chi}^{nkh})$$

$$\frac{\partial \tau_{ij}^{nkh}}{\partial y_j} + \frac{\partial a_{ijkh}^n}{\partial y_j} = 0 \quad (n = 1, 2 \text{ for U.F.}, n = 1, 2, 3 \text{ for C.F.}) .$$

With transmission equations (adapted to the vectors $\underline{\chi}^{nkh}$) 6, 7 for C.F., 8, 9 for U.F. and periodic boundary conditions 30.

The weak formulation of this problem is :

$$\chi^{kh} = (\underline{\chi}^{1\,kh}, \underline{\chi}^{2\,kh}) \text{ for U.F.} \quad \text{and} \quad \chi^{kh} = (\underline{\chi}^{1\,kh}, \underline{\chi}^{2\,kh}, \underline{\chi}^{3\,kh}) \text{ for C.F.}$$

Find $\chi^{kh} \in W$ so that $\forall v \in W$,

$$\sum_{n=1}^{p} \int_{\omega_n} a_{ijlm}^n e_{lm}^y(\underline{\chi}^{nkh}) e_{ij}^y(\underline{v}^n)\, dy = -\sum_{n=1}^{p} \int_{\omega_n} a_{ijkh}^n \left(\frac{\partial v_j^n}{\partial y_i}\right) dy \tag{34}$$

($p = 2$ for U.F., $p = 3$ for C.F.).

With :

$$W = \{v = (\underline{v}^1, \underline{v}^2, \underline{v}^3) \text{ so that } \underline{v}^n = (v_1^n, v_2^n) \in [H^1(\omega_n)]^2 ;$$

$\underline{v}^n$ has the same value on opposite sides of $\partial Y \cap \partial \omega_n$ $n = 1, 2, 3$ and $\underline{v}^1 = \underline{v}^3$ on γ^{1e}, $\underline{v}^2 = \underline{v}^3$ on $\gamma^{2e}\}$ for C.F.

And $W = \{v = (\underline{v}^1, \underline{v}^2)$ so that $\underline{v}^n = (v_1^n, v_2^n) \in [H^1(\omega_n)]^2$; $\underline{v}^n$ has the same value on opposite sides of $\partial Y \cap \partial \omega_n$ $n = 1, 2\}$ for U.F.

Taking the average on ω_n ($n = 1, 2$ for U.F., $n = 1, 2, 3$ for C.F.) of the balance equations (24 and 25 for U.F., 24, 25 and 26 for C.F.), the macroscopic equations of the medium are obtained for the assumption $k_{ij}^{\varepsilon} = k_{ij}$. Here, the case of C.F. and U.F. will be different because the results are not the same if the fabric is coated or not.

- *Coated fabrics*

The model takes only one displacement field $\underline{u}^0$ into account. It is a model of linear two-dimensional elasticity which is governed by the equations :

$$\frac{\partial \tilde{\sigma}_{ij}^0}{\partial x_j} + \tilde{f}_i^e = 0$$

$$\tilde{\sigma}_{ij}^0 = \frac{1}{|Y|} \left[\sum_{n=1}^{3} \int_{\omega_n} (a_{ijkh}^n + a_{ijlm}^n \, e_{lm}^y(\underline{\chi}^{nkh}))\, dy \right] e_{kh}^x(\underline{u}^0) . \tag{35}$$

Where $\tilde{f}_i^e$ and $\tilde{\sigma}_{ij}^0$ are the means over Y of f_i^e and σ_{ij}^0, that is (for $\tilde{\sigma}_{ij}^0$) :

$$\tilde{\sigma}_{ij}^0(x) = \frac{1}{|Y|} \left[\sum_{n=1}^{3} \int_{\omega_n} \sigma_{ij}^{n0}(x, y)\, dy \right] .$$

- *Uncoated fabrics*

The model is a model of linear two-dimensional elasticity. It includes two displacement fields $\underline{u}^{10}$ and $\underline{u}^{20}$ both defined in D which satisfy the following coupled equations :

— Problem 1

$$\frac{\partial \tilde{\sigma}_{ij}^{10}}{\partial x_j} + \tilde{f}_i^{1e} + k_{ij}(u_j^{20} - u_j^{10}) \frac{|\omega|}{|\omega_1|} = 0$$

$$\tilde{\sigma}_{ij}^{10} = \left[\frac{1}{|\omega_1|} \int_{\omega_1} (a_{ijkh}^1 + a_{ijlm}^1 \, e_{lm}^y(\underline{\chi}^{1kh}))\, dy \right] e_{kh}^x(\underline{u}^{10}) .$$

— Problem 2

$$\frac{\partial \tilde{\sigma}_{ij}^{20}}{\partial x_j} + \tilde{f}_i^{2e} + k_{ij}(u_j^{10} - u_j^{20}) \frac{|\omega|}{|\omega_2|} = 0$$

$$\tilde{\sigma}_{ij}^{20} = \left[\frac{1}{|\omega_2|} \int_{\omega_2} (a_{ijkh}^2 + a_{ijlm}^2 \, e_{lm}^y(\underline{\chi}^{2kh}))\, dy \right] e_{kh}^x(\underline{u}^{20}) .$$

Where $\tilde{f}_i^{ne}$ and $\tilde{\sigma}_{ij}^{n0}$ are the means over ω_n $(n = 1, 2)$ of f_i^{ne} and σ_{ij}^{n0}, that is (for $\tilde{\sigma}_{ij}^{n0}$) :

$$\tilde{\sigma}_{ij}^{n0}(x) = \frac{1}{|\omega_n|} \int_{\omega_n} \sigma_{ij}^{n0}(x, y)\, dy .$$

So, in this case, the fabric behaves like two elastic slicks occupying all the surface of the sample and being in interaction with coupling forces which are proportional to the difference of the displacements of each slick.

3.2.2. *Assumption* $k_{ij}^{\varepsilon} = k_{ij}/\varepsilon$

The balance equations 23, 24, 25, 26 defined for $k_{ij}^{\varepsilon} = k_{ij}$ become in this case :

- In ω_1 :

$$\frac{\partial \sigma_{ij}^{10}}{\partial y_j} + \frac{\partial \sigma_{ij}^{1,-1}}{\partial x_j} + k_{ij}(u_j^{20} - u_j^{10}) = 0 \tag{36}$$

$$\frac{\partial \sigma_{ij}^{11}}{\partial y_j} + \frac{\partial \sigma_{ij}^{10}}{\partial x_j} + k_{ij}(u_j^{21} - u_j^{11}) + f_i^{1e} = 0 . \tag{37}$$

- In ω_2 :

$$\frac{\partial \sigma_{ij}^{20}}{\partial y_j} + \frac{\partial \sigma_{ij}^{2,-1}}{\partial x_j} + k_{ij}(u_j^{10} - u_j^{20}) = 0 \tag{38}$$

$$\frac{\partial \sigma_{ij}^{21}}{\partial y_j} + \frac{\partial \sigma_{ij}^{20}}{\partial x_j} + k_{ij}(u_j^{11} - u_j^{21}) + f_i^{2e} = 0 . \tag{39}$$

- In ω_3, for C.F. only :

$$\frac{\partial \sigma_{ij}^{30}}{\partial y_j} + \frac{\partial \sigma_{ij}^{3,-1}}{\partial x_j} = 0 \tag{40}$$

$$\frac{\partial \sigma_{ij}^{31}}{\partial y_j} + \frac{\partial \sigma_{ij}^{30}}{\partial x_j} + f_i^{3e} = 0 . \tag{41}$$

The equations 20, 21, 22, 27, 28, 29 and 30 remain the same.

As in the previous section, it has been shown that :

$$\underline{u}^{n0} = \underline{u}^{n0}(x) \quad \sigma_{ij}^{n,-1} = 0 \quad (n = 1, 2 \text{ for U.F.}, n = 1, 2, 3 \text{ for C.F.}) .$$

For C.F., there is also :

$$\underline{u}^{10} = \underline{u}^{20} = \underline{u}^{30} = \underline{u}^{0}(x)$$

σ_{ij}^{10} is a periodic function of y (see the conditions 30). Furthermore, $\sigma_{ij}^{1,-1}$ is zero (see 31) and the quadratic form which is associated to the matrix $[k_{ij}]$ is supposed to be positive. Because of these three reasons, we can say that for U.F., integrating on y the equation 36 yields :

$$\underline{u}^{10} = \underline{u}^{20} = \underline{u}^{0}(x) .$$

Looking for $\underline{u}^{k1}$ in a similar way as in the equation 33, the same problem as 34 has been obtained for the set of vectors χ^{kh}, wether the fabric is coated or not. The macroscopic equations have exactly the same form as 35 for C.F. For U.F., these equations become :

$$\frac{\partial \tilde{\sigma}_{ij}^{0}}{\partial x_j} + \tilde{f}_i^{e} = 0$$

$$\tilde{\sigma}_{ij}^{0} = \frac{1}{|Y|}\left[\sum_{n=1}^{2} \int_{\omega_n} (a_{ijkh}^{n} + a_{ijlm}^{n}\, e_{lm}^{y}(\underline{\chi}^{nkh}))\, dy\right] e_{kh}^{x}(\underline{u}^{0}) .$$

Where $\tilde{f}_i^{e}$ and $\tilde{\sigma}_{ij}^{0}$ are the mean over Y of f_i^{e} and σ_{ij}^{0}, that is (for $\tilde{\sigma}_{ij}^{0}$) :

$$\tilde{\sigma}_{ij}^{0}(x) = \frac{1}{|Y|}\left[\sum_{n=1}^{2} \int_{\omega_n} \sigma_{ij}^{n0}(x, y)\, dy\right].$$

Here, as in the case $k_{ij}^{\varepsilon} = k_{ij}$ for C.F., the vector fields χ^{mkh} ($m = 1, 2, 3$ for C.F., $m = 1, 2$ for U.F.) do not depend on the coupling term k_{ij} and so do the homogeneous elastic moduli. For C.F., the reason is that the coating compels the displacements in warp and weft to be near, and as the coupling matrix $[k_{ij}]$ is rather weak, the interaction between warp and weft becomes negligible.

3.2.3. *Assumption* $k_{ij}^{\varepsilon} = k_{ij}/\varepsilon^2$

The balance equations 22, 23, 24, 25, 26 defined for $k_{ij}^{\varepsilon} = k_{ij}$ become in this case :

- In ω_1 :

$$\frac{\partial \sigma_{ij}^{1,-1}}{\partial y_j} + k_{ij}(u_j^{20} - u_j^{10}) = 0 \tag{42}$$

$$\frac{\partial \sigma_{ij}^{10}}{\partial y_j} + \frac{\partial \sigma_{ij}^{1,-1}}{\partial x_j} + k_{ij}(u_j^{21} - u_j^{11}) = 0 \tag{43}$$

$$\frac{\partial \sigma_{ij}^{11}}{\partial y_j} + \frac{\partial \sigma_{ij}^{10}}{\partial x_j} + k_{ij}(u_j^{22} - u_j^{12}) + f_i^{1\,e} = 0 . \tag{44}$$

- In ω_2 :

$$\frac{\partial \sigma_{ij}^{2,-1}}{\partial y_j} + k_{ij}(u_j^{10} - u_j^{20}) = 0 \tag{45}$$

$$\frac{\partial \sigma_{ij}^{20}}{\partial y_j} + \frac{\partial \sigma_{ij}^{2,-1}}{\partial x_j} + k_{ij}(u_j^{11} - u_j^{21}) = 0 \tag{46}$$

$$\frac{\partial \sigma_{ij}^{21}}{\partial y_j} + \frac{\partial \sigma_{ij}^{20}}{\partial x_j} + k_{ij}(u_j^{12} - u_j^{22}) + f_i^{2\,e} = 0 . \tag{47}$$

- In ω_3, for C.F. only :

$$\frac{\partial \sigma_{ij}^{3,-1}}{\partial y_j} = 0 \tag{48}$$

$$\frac{\partial \sigma_{ij}^{30}}{\partial y_j} + \frac{\partial \sigma_{ij}^{3,-1}}{\partial x_j} = 0 \tag{49}$$

$$\frac{\partial \sigma_{ij}^{31}}{\partial y_j} + \frac{\partial \sigma_{ij}^{30}}{\partial x_j} + f_i^{3\,e} = 0 . \tag{50}$$

It may be proved that the solutions of the problem made of the equations 42, 45, 48, 20 and conditions 27, 28, 30 for C.F., 42, 45, 20 and conditions 29, 30 for U.F. are so that u^{10}, u^{20}, u^{30} depend only on x and are equal :

$$(u^{10}, u^{20}, u^{30}) = (u^0(x), u^0(x), u^0(x)) \quad \text{for C.F.}$$
$$(u^{10}, u^{20}) = (u^0(x), u^0(x)) \quad \text{for U.F.}$$

On the other hand, the weak problem for χ^{kh} is :
Find $\chi^{kh} \in W$ so that $\forall v \in W$,

$$\sum_{n=1}^{p} \int_{\omega_n} a_{ijlm}^n\, e_{lm}^y(\underline{\chi}^{nkh})\, e_{ij}^y(\underline{v}^n)\, dy + \int_{\omega} k_{ij}(\chi_j^{2\,kh} - \chi_j^{1\,kh})(v_i^2 - v_i^1)\, dy =$$

$$= - \sum_{n=1}^{p} \int_{\omega_n} a_{ijkh}^n \left(\frac{\partial v_j^n}{\partial y_i} \right) dy \quad (p = 2 \text{ for U.F.}, p = 3 \text{ for C.F.}) \tag{51}$$

W is defined in the same way as in the case $k_{ij}^{\varepsilon} = k_{ij}$.

Finally, the macroscopic equations can be written in a similar way as in the equation 35. Thus, it may be seen that, in this case, the elastic moduli depend on the coupling term k_{ij} through the vectors χ^{mkh} ($m = 1, 2$ for U.F., $m = 1, 2, 3$ for C.F.) which are solutions of the problem 51.

4. CONCLUSION AND PERSPECTIVES

This study gives macroscopic models of fabrics with flat fibers under different assumptions about the coupling coefficient k_{ij}^{ε} which characterizes the linear elastic springback between warp and weft.

For a coated or uncoated fabric with a high coupling term, the coating or the high coupling term between warp and weft make that the fabric can be modelled with only one displacement field which satisfy two-dimensional linear elasticity equations. The elastic moduli can depend on the coupling term. For an uncoated fabric with a small coupling matrix, the fabric is modelled with two coupled displacement fields. So it's equivalent to two superimposed slicks being in interaction with a linear elastic springback.

The elastic moduli of warp, weft, coating and the coupling matrix being known, a numerical computation using the finite element developped for this modelling may give the macroscopic equivalent coefficients.

In the present paper, only elastic behaviours have been studied ; it is possible to consider other behaviours as viscoelasticity, as well for the different materials as for the coupling between warp and weft, or Coulomb friction between warp and weft.

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