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ASYMPTOTICS OF SCATTERING FREQUENCIES WITH SMALL IMAGINARY PARTS FOR AN ACOUSTIC RESONATOR (*)

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Abstract — *In this paper we consider an acoustic resonator with finite thickness of the shell and with the Neumann boundary condition The exterior and the interior of the resonator are connected by a thin tube We construct asymptotics of scattering frequencies with small imaginary parts in power series with respect to a small parameter (radius of the cross-section of the connecting tube) These asymptotics are applied to the scattering problem*

Résumé — *Nous considérons un résonateur acoustique avec paroi d'épaisseur finie et condition aux limites de Neumann L'extérieur et l'intérieur du résonateur sont reliés par un tube étroit Nous construisons le développement asymptotique des fréquences de diffusion de partie imaginaire petite en série de puissances par rapport a un petit paramètre (égal au rayon de la section droite du tube de connexion) Ce comportement asymptotique est appliqué au problème de diffusion*

1. INTRODUCTION

The classical acoustic Helmholtz resonator is an ideal hard sphere with a small hole [1], [2]. The corresponding mathematical model is the Neumann boundary value problem for the Helmholtz equation outside this surface. In 1916 O. M. Rayleigh showed [1] (by non rigorous methods) that, for some low frequencies, the field scattered by a resonator differs essentially from the field scattered by a sphere without the hole. In 1971 J. W. Miles obtained [2] by computational methods that the same situation takes place for some frequencies near any eigenfrequency (a square root of an eigenvalue) of $-\Delta$ in the corresponding ball. In [3], [4] it had been proven that the form of

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the resonator is not relevant for the resonant phenomena and only the poles of the analytic continuation of Green function (scattering frequencies in terms of the Lax-Phillips theory [5]) with small imaginary parts do matter for resonances. These poles converge to a real valued set Σ^{in} of eigenfrequencies of the Neumann boundary value problem in Ω^{in} *(limit interior problem)* as $\varepsilon \to 0$ $(0 < \varepsilon \ll 1$ is the « radius » of the hole). Their asymptotics with respect to ε had been constructed in [6] by J. Sanchez-Hubert, E. Sanchez-Palencia and in [7]-[11]. These asymptotics were obtained by using the method of matched asymptotic expansions [12]-[14].

In this paper we consider an analogue of the Helmholtz resonator with finite thickness of the shell. This analogue Ω_ε has a bounded component Ω^{in} and an unbounded component Ω^{ex} which are connected by a thin tube κ_ε. J. T. Beale showed [15] that there exist complex scattering frequencies τ_ε which converge to an additional real valued set $\Sigma^{tu} = \{\pm \pi m/h\}_{m=1}^{\infty}$ where h is the length of κ_ε. This set is the set of the Dirichlet eigenfrequencies of the interval $[0, h]$. Namely the thin tube κ_ε generates these poles. Beals's results show that Ω_ε is not an exact analogue of the Helmholtz resonator, but this problem is more interesting from the mathematical viewpoint. The one-to-one correspondence (counting multiplicities) of the scattering frequencies and the eigenfrequencies for the perturbed and limit problems are proved in [16] by R. Brown, P. Hislop, A. Martinez and in [17]. In addition, the authors of [16] obtained the following estimates on the rate of convergence of the scattering frequencies with small imaginary parts (we discuss only such scattering frequencies in $\mathbf{R}^3$). If nonzero $k_0 \in \Sigma^{in} \Delta \Sigma^{tu}$ and τ_ε is the scattering frequency converging to k_0 then $\tau_\varepsilon = k_0 + O(\varepsilon^{1/2})$. In [17] the power series asymptotics of two vanishing scattering frequencies τ_ε with respect to the radius of the cross-section of the tube were constructed and the principal terms of τ_ε and Im τ_ε were obtained. In the present paper we construct the power series asymptotics of scattering frequencies τ_ε converging (i) to $k_0 \in \Sigma_1^{in} \setminus \Sigma^{tu}$, where Σ_1^{in} is the set of the square roots of the simple eigenvalues in Ω^{in}, and (ii) to $k_0 \in \Sigma^{tu} \setminus \Sigma^{in}$. In these cases we prove that (i) $\tau_\varepsilon = k_0 + \varepsilon^2 t_1 + O(\varepsilon^3)$, $\operatorname{Im} \tau_\varepsilon = \varepsilon^4 t_2 + O(\varepsilon^5)$ and (ii) $\tau_\varepsilon = k_0 + \varepsilon t_3 + O(\varepsilon^2)$, $\operatorname{Im} \tau_\varepsilon = k_0 + \varepsilon^2 t_4 + O(\varepsilon^3)$, where the constants t_j are expressible in terms of the limit problems. Note that the results of the present paper and [17] are announced in [18]. Note also that similar results for the Dumbbel shaped domain are obtained in [19] by S. Jimbo and in [20].

The paper is organized as follows. In Sections 2, 3 we state the boundary value problem, and the main results, respectively. The asymptotics of the scattering frequencies converging to nonzero $k_0 \in \Sigma_1^{in} \setminus \Sigma^{tu}$ are obtained in Section 4. In Section 5 we consider τ_ε associated with the thin tube. Some remarks on the problem in $\mathbf{R}^n$ for $n > 3$ are given in Section 6.

2. PRELIMINARIES

Let Ω^{in} and Ω be bounded simply connected domains in $\mathbf{R}^3$, $\overline{\Omega^{in}} \subset \Omega$, $\Omega^{ex} = \mathbf{R}^3 \backslash \bar{\Omega}$, and their boundaries $\partial \Omega^{in, ex}$ belong to C^∞. Assume that Ω^{in} coincides with the half-space $x_3 > 0$ at some neighborhood of the origin, Ω^{ex} coincides with the half-space $x_3 < -h$ at some neighborhood of $x_0 = (0, 0, -h)$ where $h > 0$ and the interval $(-h, 0)$ lies on the axis Ox_3 and does not intersect $\Omega^{in} \cup \Omega^{ex}$. The domains $\Omega^{in, ex}$ are the interior and exterior of the following acoustic resonator :

$$\Omega_\varepsilon = \Omega^{in} \cup \Omega^{ex} \cup \kappa_\varepsilon .$$

Now we shall describe the connecting channel κ_ε. It would be most natural to define κ_ε as the narrow cylinder $\tilde{\kappa}_\varepsilon = \omega_\varepsilon \times [0, -h]$ where $\omega(\varepsilon) = \{(x_1, x_2) : (\varepsilon^{-1} x_1, \varepsilon^{-1} x_2) \in \omega\}$ and ω is a two-dimensional simply connected domain with a boundary from C^∞. All propositions of the paper are true in this case, too. But since, we use the results of [15], [17], the notations in this paper are the same, i.e. the connecting tube transits to the exterior and interior of the resonator in a smooth way. Let $\eta(t)$ be a positive function satisfying the following conditions :

$$\eta(t) \in C((-\infty, 0]) \cap C^\infty((-\infty, 0)), \quad \eta^{(n)}(0) = \infty \text{ as } n \geqslant 1 ,$$
$$\eta(t) \equiv 1 \quad \text{as} \quad t \leqslant -1 ,$$
$$\kappa(x, t) = \{x : x \in \omega(\eta(x_3)) \times x_3, t < x_3 \leqslant 0\} ,$$
$$\kappa_\varepsilon = \kappa\left(x\varepsilon^{-1}, -\frac{2}{3} h\varepsilon^{-1}\right) \cup \kappa\left((x_0 - x)\varepsilon^{-1}, -\frac{2}{3} h\varepsilon^{-1}\right) .$$

The boundary value problem for the acoustic resonator Ω_ε reads as following

$$(\Delta + k^2) u_\varepsilon = F , \quad x \in \Omega_\varepsilon , \quad \partial u_\varepsilon / \partial n = 0 , \quad x \in \partial \Omega_\varepsilon , \tag{1}$$

$$\partial u_\varepsilon / \partial r - iku_\varepsilon = o(r^{-1}) , \qquad r \to \infty , \tag{2}$$

where n is the outward normal, $r = |x|$, k is real valued. Let $S(R)$ be the open ball of radius R around the origin, $\bar{\Omega} \subset S(R)$, $F \in L_2(\mathbf{R}^3)$, $\operatorname{supp} F \subset \Omega_{\varepsilon, R} = S(R) \cap \Omega_\varepsilon$. The solutions of (1), (2) (and their analytical continuations) are considered in the class of functions belonging to $W_2^1(\Omega_{\varepsilon, T})$ for any T. Since the belonging of a solution to $W_2^2(\Omega_{\varepsilon, T})$ follows from the belonging of it and F to $W_2^1(\Omega_{\varepsilon, T})$, and $L_2(\mathbf{R}^3)$, respectively, then the boundary conditions are understood in the usual sense. Note that the boundary condition for the resonator with edges (for example, $\kappa_\varepsilon = \tilde{\kappa}_\varepsilon$) has the usual sense outside these edges, too. But, of course, the solution does not belong to W_2^2 at their neighborhood.

The outgoing condition for the analytical continuation of u_ε can be expressed by writing u_ε as superposition of the fundamental solution $e(x, y, k) = (4 \pi |x - y|)^{-1} \exp\{ik|x - y|\}$ (see [5], [15]). Elsewhere below we understand as the solution of (1), (2) either the solution of (1),(2) for real valued k, or its analytical continuation for complex k. In such sense, (1), (2) has a unique solution outside the discrete set Σ_ε of the scattering frequencies having no finite limit points, lying below the real axis and coinciding with a set of the poles of the Green function analytical continuation of (1), (2). The residues of the solutions in the pole τ_ε satisfies the equation and the boundary condition in (1) with $k = \tau_\varepsilon$ and $F = 0$. Such residues are defined as eigenfunctions. Of course, they grow exponentially at infinity for fixed ε. It is known [15] that any complex neighborhood K of $k_0 \in \Sigma = \Sigma^{in} \cup \Sigma^{ex} \cap \Sigma^{tu}$, where Σ^{ex} is a set of scattering frequencies of the Neumann boundary value problem *(limit exterior problem)* in Ω^{ex}, lying below the real axis, includes at least one scattering frequency of (1), (2) for a sufficiently narrow tube. On the other hand, if $\bar{K} \cap \Sigma = \emptyset$, then there are no poles of the resonator in $\bar{K}$ for a sufficiently thin tube.

Elsewhere below we shall consider all functions extended to be zero outside the closures of their primary regions of definitions. We define solutions of the united limit problem for (1), (2) as the sum of solutions of the interior, and exterior limit problems. If $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$, then we denote by ψ the corresponding eigenfunction normalized in $L_2(\Omega^{in})$. If $k_0 \in \Sigma^{tu} \backslash \Sigma^{in}$, then assume that

$$\psi(x) = \varepsilon^{-1}(2/h)^{1/2} |\omega|^{-1/2} \sin (k_0 x_3)$$

as $x \in \Pi_\varepsilon = \omega_\varepsilon \times [-h, 0]$.

Denote by $\| \cdot \|_{0, Q}$ the norm in $L_2(Q)$. The following statement, which will foundate our asymptotics, is proved in [17] (see Theorems 1, 3 in the cited work).

PROPOSITION. *Let nonzero* $k_0 \in \Sigma_1 = ((\Sigma_1^{in} \backslash \Sigma^{tu}) \cup (\Sigma^{tu} \backslash \Sigma^{in}))$. *Then there is only one scattering frequency* τ_ε *converging to* k_0 *as* $\varepsilon \to 0$ *and only one eigenfunction (up to a scalar factor) corresponds to it.*

For any k *sufficiently close to* k_0 *and* F *such that* $\operatorname{supp} F \subset S(R)$, *the solution of* (1), (2) *has the following representation*

$$u_\varepsilon(x, k) = (k^2 - \tau_\varepsilon^2)^{-1} \Psi_\varepsilon(x) \int_{\mathbf{R}^3} F(y) \Psi_\varepsilon(y) \, dy + U_\varepsilon(x, k), \tag{3}$$

where $\|U_\varepsilon(x, k)\|_{0, S(R)} \leq C_R \|F\|_{0, S(R)}$ *and the eigenfunction* $\Psi_\varepsilon \to \psi$ *in the same norm as* $\varepsilon \to 0$. *If in addition* $\operatorname{supp} F \cap \kappa_\varepsilon = \emptyset$, *then, for* $\varepsilon \to 0$, *the* *function U_ε converges to the regular part $U_0 = U^{in} \oplus U^{ex}$ of the united limit problem in $L_2(S(R))$ norm uniformly with respect to k.*

Note that this statement can be obtained from [15], [16], too.

Remark : Proposition shows that the first addendum in the right hand side of (3) is the resonant term and that, for real valued $k = k(\varepsilon)$, maximal perturbations are observed in the resonant sector :

$$k = \operatorname{Re} \tau_\varepsilon + O(\operatorname{Im} \tau_\varepsilon) . \tag{4}$$

Of course from (3) we can infer that the resonant term is essential for the radiation problem (*i.e.* $\operatorname{supp} F \cap \Omega^{in} \neq \emptyset$) in the interior Ω^{in}. But even in the resonant sector, (3) does not give any information on an essential influence of the resonant term for the radiation problem outside Ω^{in}, or for the scattering problem ($\operatorname{supp} F \subset \Omega^{ex}$) anywhere. One sees that, for giving conclusion in this problem, we must know the leading terms of asymptotics of Im τ_ε and Ψ_ε outside Ω^{in}. □

3. MAIN RESULTS

In this section we state the main result of the present work. The coefficients of the leading terms are expressed by some values described as follows. Let $G^{in, ex}(x, y, k)$ be the Green functions of the interior and exterior limit problems for (1), (2),

$$\sigma(k) = \lim_{R \to \infty} \int_{\partial S(R)} |G^{ex}(x, x_0, k)|^2 \, ds$$

be the scattering cross-section [2], [21], $\xi = (\xi_1, \xi_2, \xi_3)$,

$$\gamma_\omega = \kappa(\xi, -\infty) \cup \{\xi : \xi_3 > 0\} .$$

Below we shall show the existence of a function $Y(\xi)$ such that this function is harmonic in γ_ω, belongs to $W_2^1(\gamma_\omega \cap S(R))$ for any R, satisfies the homogeneous Neumann condition on $\partial \gamma_\omega$ and has asymptotics as $\rho = |\xi| \to \infty$ in the following form

$$Y(\xi) = O(\rho^{-1}), \quad \xi \geq 0, \quad Y(\xi) = \xi_3 + q_\omega + o(1), \quad \xi < 0$$

where q_ω is some constant. Of course if $\partial \gamma_\omega$ is smooth then Y is smooth, too. Denote by $S^{ex}(R)$ the ball of radius R around x_0. Elsewhere below we assume that if $k_0 \in \Sigma_1^{in}$ then the corresponding eigenfunction does not equal to zero at the origin. The main statement of our work reads as follows.

THEOREM 1 : (i) *The asymptotics of the scattering frequency τ_ε, converging to $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$ as $\varepsilon \to 0$, and of the corresponding eigenfunction have the form*

$$\tau_\varepsilon = k_0 + \sum_{j=\alpha}^{\infty} \varepsilon^j \tau_j , \tag{5}$$

where $\alpha = 2$,

$$\tau_2 = \frac{1}{2} \psi^2(0) \tan^{-1} (k_0 h) |\omega| , \qquad \operatorname{Im} \tau_3 = 0 , \tag{6}$$

$$\operatorname{Im} \tau_4 = -\frac{1}{2} (k_0 |\omega| \psi(0) \sin^{-1} (k_0 h))^2 \sigma(k_0) , \tag{7}$$

$$\Psi_\varepsilon(x) \sim \psi(x) , \quad x \in \Omega^{in} \backslash S(\varepsilon^{1/2}) , \quad \Psi_e(x) \sim \psi(0) , \quad x \in S(2\varepsilon^{1/2}) ,$$

$$\Psi_\varepsilon(x) \sim \psi(0) \sin^{-1} (k_0 h) \sin (k_0(x_3 + h)) , \ x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})) ,$$

$$\Psi_\varepsilon(x) \sim -\varepsilon k_0 \psi(0) \sin^{-1} (k_0 h) Y((x - x_0)/\varepsilon) , \qquad x \in S^{ex}(2\varepsilon^{1/2}) ,$$

$$\Psi_\varepsilon(x) \sim \varepsilon^2 k_0 \psi(0) \sin^{-1} (k_0 h) |\omega| G^{ex}(x, x_0, k_0) , \quad x \in \Omega^{ex} \backslash S^{ex}(\varepsilon^{1/2})$$

in $L_2(S(R))$ for any R.

(ii) *The asymptotics of the scattering frequency τ_ε, converging to $k_0 = \pi m/h \in \Sigma^{tu} \backslash \Sigma^{in}$ as $\varepsilon \to 0$, and of the corresponding eigenfunction have the form* (5), *where* $\alpha = 1$,

$$\tau_1 = 2 q_\omega k_0/h , \tag{8}$$

$$\operatorname{Im} \tau_2 = -k_0^2 h^{-1} |\omega| \sigma(k_0) , \tag{9}$$

$$\Psi_\varepsilon(x) \sim -\varepsilon k_0 (2 h^{-1} |\omega|)^{1/2} G^{in}(x, 0, k_0) , \quad x \in \Omega^{in} \backslash S(\varepsilon^{1/2}) ,$$

$$\Psi_e(x) \sim k_0 \left(\frac{1}{2} h |\omega| \right)^{-1/2} Y(x/\varepsilon) , \qquad x \in S(2\varepsilon^{1/2}) ,$$

$$\Psi_\varepsilon(x) \sim \varepsilon^{-1} \left(\frac{1}{2} h |\omega| \right)^{-1/2} \sin (k_0 x_3) , \quad x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})) ,$$

$$\Psi_\varepsilon(x) \sim (-1)^{m+1} k_0 \left(\frac{1}{2} h |\omega| \right)^{-1/2} Y((x - x_0)/\varepsilon) , \quad x \in S^{ex}(2\varepsilon^{1/2}) ,$$

$$\Psi_\varepsilon(x) \sim \varepsilon (-1)^m k_0 (2 h^{-1} |\omega|)^{1/2} G^{ex}(x, x_0, k_0) , \quad x \in \Omega^{ex} \backslash S^{ex}(\varepsilon^{1/2})$$

in $L_2(S(R))$ for any R.

Due to Theorem 1 the resonant sector (4) in the cases (i), and (ii) has form

$$k = k_0 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_3 + \varepsilon^4 (t + o(1)) ,$$
$$k = k_0 + \varepsilon \tau_1 + \varepsilon^2 (t + o(1)) ,$$

respectively. Here t is any real valued number.

Now, on account of Proposition, Theorem 1 gives

THEOREM 2 *The solution of the scattering problem in the resonant sector corresponding to nonzero* $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$ *has the following form*

$$u_\varepsilon(x, k) \sim \varepsilon^{-2} T\psi(x), \quad x \in \Omega^{in} \backslash S(\varepsilon^{1/2}), \quad u_\varepsilon(x, k) \sim \varepsilon^{-2} T\psi(0), \quad x \in S(2\varepsilon^{1/2}),$$

$$u_\varepsilon(x, k) \sim \varepsilon^{-2} \psi(0) \sin^{-1}(k_0 h) \sin(k_0(x_3 + h)), \quad x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})),$$

$$u_\varepsilon(x, k) \sim - \varepsilon^{-1} T k_0 \psi(0) \sin^{-1}(k_0 h) Y((x - x_0)/\varepsilon), \quad x \in S^{ex}(2\varepsilon^{1/2}),$$

$$u_\varepsilon(x, k) \sim T k_0 \psi(0) \sin^{-1}(k_0 h) | \omega | G^{ex}(x, x_0, k) + U^{ex}(x, k), \quad x \in \Omega^{ex} \backslash S^{ex}(\varepsilon^{1/2})$$

where $T = (2(\tau_4 - t))^{-1} \psi(0) \sin^{-1}(k_0 h) | \omega | U^{ex}(x_0, k_0)$, U^{ex} *is the solution of the limit exterior problem*

The solution of the scattering problem in the resonant sector corresponding to $k_0 = \pi m/h \in \Sigma^{tu} \backslash \Sigma^{in}$ *has the following form*

$$u_\varepsilon(x, k) \sim - T k_0 (2h^{-1} | \omega |)^{1/2} G^{in}(x, 0, k), \quad x \in \Omega^{in} \backslash S(\varepsilon^{1/2}),$$

$$u_\varepsilon(x, k) \sim \varepsilon^{-1} T k_0 \left(\frac{1}{2} h(\omega) \right)^{-1/2} Y(x, \varepsilon), \quad x \in S(2\varepsilon^{1/2}),$$

$$u_\varepsilon(x, k) \sim \varepsilon^{-2} T \left(\frac{1}{2} h | \omega | \right)^{1/2} \sin(k_0 x_3), \quad x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})),$$

$$u_\varepsilon(x) \sim \varepsilon^{-1} T (-1)^{m+1} k_0 \left(\frac{1}{2} h | \omega | \right)^{1/2} Y((x - x_0)/\varepsilon), \quad x \in S^{ex}(2\varepsilon^{1/2}),$$

$$u_\varepsilon(x) \sim (-1)^m T k_0 (2h^{-1} | \omega |)^{1/2} G^{ex}(x, x_0, k), \quad x \in \Omega^{ex} \backslash S^{ex}(\varepsilon^{1/2})$$

where $T = (\tau_2 - t)^{-1} (-1)^m (| \omega | / 2h)^{1/2} U^{ex}(x_0, k_0)$

Theorem 2 shows that the scattering frequencies with small imaginary parts have a resonant nature One can obtain a similar statement for the radiation problem

4. ASYMPTOTICS OF SCATTERING FREQUENCIES ASSOCIATED WITH THE INTERIOR OF THE RESONATOR

In this section we construct the asymptotics of the scattering frequency converging to $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$ by using the method of matched asymptotic expansions However, some auxiliary statements will be proved for the case $k_0 \in \Sigma^{tu} \backslash \Sigma^{in}$, too So, for the sake of brevity, we shall use the following double-valued constants (see also (5), and (ii) in Theorem 1)

(i) If $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$ then $\alpha = 2$, $\beta = 0$, $\alpha_{in} = \beta_{ex} = 1$, $\alpha_{ex} = \beta_{in} = 0$, $b(k) = (k_0^2 - k^2)$.

(ii) If $k_0 \in \Sigma^{tu} \backslash \Sigma^{in}$ then $\alpha = 1$, $\beta = 1$, $\alpha_{in} = \alpha_{ex} = \beta_{in} = \beta_{ex} = 0$, $b(k) \equiv 1$.

In the same way a double index « in, ex » in representations means that there are two formulae there. For instance, the following notations

$$\psi_\varepsilon^{in, ex}(x, k) = \sum_{j=0}^{\infty} \varepsilon^{j+\beta} b(k) R_j^{in, ex}(D_y) G^{in, ex}(x, x_0^{in, ex}, k),$$

$$R_j^{in, ex}(D_y) = \sum_{l=0}^{j} P_l^{in, ex, j}(D_y), \quad P_l^{in, ex, j}(D_y) = \sum_{q=0}^{l} a_{lq}^{in, ex, j} \frac{\partial^l}{\partial^q y_1 \, \partial^{l-q} y_2},$$

where and $a_{lq}^{in, ex, j}$ are some constants, to be read for « in », and for « ex » separately. Hereafter, $x_0^{in} = 0$, $x_0^{ex} = x_0$, $x^{in} = x$, $x^{ex} = x_0 - x$, $r_{in, ex} = |x - x_0^{in, ex}|$, $S^{in}(t) = S(t)$, $Z_j^{in, ex}(\xi)$ are homogeneous harmonic polinomials of order j such that $\partial Z_j^{in, ex}/\partial \xi_3 = 0$ as $\xi_3 = 0$. On the other hand, we shall omit index sometimes (for example, $P_j(D_y)$, or $Z_j(\xi)$). By definition we have :

LEMMA 1 : *Let $k_0 \in \Sigma_1$. Then the coefficients of $\psi_\varepsilon^{in, ex}(x, k)$ have analytic continuations to some neighborhood of k_0, satisfying the homogeneous Neumann boundary condition on $\partial\Omega^{in, ex} \backslash \{x_0^{in, ex}\}$ and are solutions of the Helmholtz equation in $\Omega^{in, ex}$. The coefficients of $\psi_\varepsilon^{ex}(x, k)$ satisfy the outgoing condition at infinity.*

LEMMA 2 : *For k close to $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$, the Green functions and their derivatives have the following form*

$$P_l(D_y) G^{in, ex}(x, x_0^{in, ex}, k) = \alpha_{in, ex}(k_0^2 - k^2)^{-1} \psi(x) P_l(D_y) \psi(0) + \\ + (-1)^l P_l(D_x)((2 \pi r_{in, ex})^{-1} \cos(k r_{in, ex})) + g_l^{in, ex, j}(x, k),$$

where the functions $g_l^{in, ex, j}$ are analitic in k, belong to $C^m(\overline{S^{in, ex}(t)} \cap \overline{\Omega^{in, ex}})$ for any m and satisfy the homogeneous Neumann boundary condition on $\partial\Omega^{in, ex}$ at some neighborhood of $x_0^{in, ex}$.

If k and coefficients of P_l are real valued then $g_l^{in, j}$ is real valued, too.

We shall construct asymptotics of the eigenfunction corresponding to τ_ε as follows

$$\psi_\varepsilon(x) = \psi_\varepsilon^{in, ex}(x, \tau_\varepsilon), \qquad x \in \Omega^{in, ex} \backslash S^{in, ex}(\varepsilon^{1/2}),$$
$$\psi_\varepsilon(x) = \sum_{j=\beta_{in, ex}}^{\infty} \varepsilon^j v_j^{in, ex}(x^{in, ex}/\varepsilon), \quad x \in S^{in, ex}(2\varepsilon^{1/2}), \tag{10}$$

$$\psi_\varepsilon(x) = \sum_{j=-\beta}^{\infty} \varepsilon^j w_j(x_3), \qquad x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})), \quad (11)$$

Note that ψ_ε equals to Ψ_ε up to a multiplier $1 + o(1)$ as $\varepsilon \to 0$.

The reasons, allowing to determine the orders of the leading terms for these asymptotics are the same as in [6], [7], [22] and that is why they are not explicitly given here. The boundary value problems for the coefficients of series (10) are obtained as follows [7], [14], [22]. In (1) we assume $F = 0$, substitute series (5), (10) in place of k and u_ε and pass to the variable $\xi = x^{in, ex} \varepsilon^{-1}$. Then we extract out equalities with the same degree of ε and take the formal limit as $\varepsilon \to 0$. Finally, we obtain the following system of recurrent boundary value problems :

$$\Delta v_j = - k_0^2 v_{j-2} - \sum_{l=\alpha}^{j-2-\beta_{in\,ex}} \lambda_l v_{j-l-2}, \quad \xi \in \gamma_\omega, \quad \partial v_j / \partial n = 0,$$
$$\xi \in \partial \gamma_\omega \quad (12)$$

where λ_l are the coefficients of the series $\lambda_\varepsilon = \tau_\varepsilon^2 - k_0^2$.

Substituting series (5), (11) into (1), we obtain the ordinary differential equations for w_j :

$$w_j''(x_3) + k_0^2 w_j(x_3) + \sum_{l=\alpha}^{j+\beta} \lambda_l w_{j-l}(x_3) = 0, \quad -h < x_3 < 0 \quad (13)$$

Obviously, the functions

$$w_j(x_3) = - k_0^{-1} \sum_{l=\alpha}^{j+\beta} \lambda_l \int_0^{x_3} \sin(k_0(x_3 - t)) w_{j-l}(t)\, dt +$$
$$+ a_j \sin(k_0 x_3) + b_j \sin(k_0 x_3), \quad (14)$$

where a_j, and b_j are any constants, are the solutions of (13).

Suppose $T_j(\xi)$ are homogeneous functions of order j being either homogeneous polynomials, or homogeneous polynomials multiplied by ρ^{-2q-1} for some integer $q \geqslant 0$, and satisfying the boundary condition $\partial T_j(\xi)/\partial \xi_3 = 0$ as $\xi_3 = 0$, $\xi \neq 0$. Denote by $\tilde{\mathscr{A}}_j$ a set of series $T(\xi) = \sum_{q=-\infty}^{j} T_q(\xi)$.

Let us define the operator $K_q^{in, ex}$ on the sums $U(x, \varepsilon)$ with the form $\psi_\varepsilon^{in, ex}(x, \tau_\varepsilon)$, and (11) as follows [7], [14]. Expand coefficients of $U(x, \varepsilon)$ in series as $\tau_{in, ex} \to 0$ and pass to the variables $\xi = x_{in, ex} \varepsilon^{-1}$. In the double series obtained in such way we extract a sum of the terms $\varepsilon^j \Phi(\xi)$ for $j \leqslant q$ and denote this sum by $K_q^{in, ex}(U(x, \varepsilon))$.

Denote $\psi_j = \partial\psi/\partial x_j$ for $x = 0$. The definition of $K_q^{in, ex}$ and Lemmas 1, 2 give :

LEMMA 3 : *Let $k_0 \in \Sigma_1^{in}\backslash\Sigma^{tu}$, τ_ε be any function having asymptotic* (5) with arbitrary coefficients, *$\psi_\varepsilon^{in, ex}(x, k)$ have arbitrary coefficients $a_{iq}^{in, ex, j}$, and $w_\varepsilon(x)$ have form* (11), (14) *with arbitrary a_j, and b_j.*

(a) *Then, for any integer $N \geq 0$, the following equalities hold*

$$\begin{aligned} K_N^{in, ex}(\psi_\varepsilon^{in, ex}(x, \tau_\varepsilon)) &= \sum_{i = \beta_{in, ex}}^{N} \varepsilon^i V_i^{in, ex}(\xi), \\ K_N^{in, ex}(w_\varepsilon(x)) &= \sum_{i = \beta_{in, ex}}^{N} \varepsilon^i W_i^{in, ex}(\xi). \end{aligned} \tag{15}$$

The series $V_j^{in, ex}$ belong to $\tilde{\mathscr{A}}_j$, are formal asymptotic solutions of (12) *as $\rho \to \infty$, and $\xi_3 \geq 0$, where the functions v_i are replaced by $V_i^{in, ex}$; $W_i^{in, ex}$ are polynomials of order i, having the form*

$$W_0^{in}(\xi) = b_0, \quad W_0^{ex}(\xi) = b_0 \cos(k_0 h) - a_0 \sin(k_0 h),$$

$$W_n^{in}(\xi) = \tilde{W}_n^{in}(\xi) + b_n,$$

$$W_n^{ex}(\xi) = \tilde{W}_n^{ex}(\xi) + b_n \cos(k_0 h) - a_n \sin(k_0 h), \quad n \geq 1,$$

$$\tilde{W}_1^{in}(\xi) = a_0 k_0 \xi_3, \quad \tilde{W}_1^{ex}(\xi) = -k_0(a_0 \cos(k_0 h) + b_0 \sin(k_0 h)) \xi_3,$$

where $\tilde{W}_n^{in, ex}$ are independent of τ_q, a_q, and b_q as $q \geq n$; $W_j^{in, ex}$, and $\tilde{W}_j^{in, ex}$ are formal asymptotic solutions of (12) *as $\rho \to \infty$, and $\xi_3 < 0$, where the functions v_i are replaced by $W_i^{in, ex}(\xi)$.*

(b) *If in addition, $P_0^{in, q} = 0$ as $q \geq 1$ then*

$$V_0^{in}(\xi) = R_0^{in} \psi^2(0),$$

$$\begin{aligned} V_1^{in, ex}(\xi) = \alpha_{in, ex}\, \psi(0) \left(R_0^{in} \sum_{q=1}^{2} \psi_q \xi_q + P_1^{in, 1}(D_x)\, \psi(0) \right) - \\ -\pi^{-1} k_0 \tau_2 \left(R_0^{in, ex} \rho^{-1} + \sum_{i=1}^{\infty} (-1)^i P_i^{in, ex, i}(D_\xi)\, \rho^{-1} \right), \end{aligned}$$

$$\begin{aligned} V_{n+1}^{in, ex}(\xi) = \tilde{V}_{n+1}^{in, ex}(\xi) + \alpha_{in, ex}\, \psi(0)\, P_1^{in, n+1}(D_x)\, \psi(0) - \\ - \pi^{-1} k_0 \tau_{n+2} \left(R_0^{in, ex} \rho^{-1} + \sum_{i=1}^{\infty} (-1)^i P_i^{in, ex, i}(D_\xi)\, \rho^{-1} \right) \\ - \pi^{-1} k_0 \tau_2 \sum_{i = \alpha_{in, ex}}^{\infty} (-1)^i P_i^{in, ex, i+n}(D_\xi)\, \rho^{-1}, \quad n \geq 1, \end{aligned}$$

where $\tilde{V}_{n+1}^{in,ex}(\xi) \in \tilde{\mathscr{A}}_{n+1}$ *are independent of* τ_{q+2}, *and* $P_l^{in,ex,l+q}$ *as* $q \geqslant n$ *and are formal asymptotic solutions of* (12) *as* $\rho \to \infty$, *and* $\xi_3 \geqslant 0$, *where functions* v_l *are replaced by* $V_l^{in,ex}(\xi)$.

(c) *If in addition,* τ_2, τ_3, $P_l^{in,ex,l}$, $P_l^{in,ex,l+1}$, a_0, a_1, b_0, b_1 *are real valued then*

$$\operatorname{Im} V_1^{in,ex} = \operatorname{Im} V_2^{in} = \operatorname{Im} \tilde{V}_3^{in} = 0 ,$$

$$\operatorname{Im} \tilde{V}_2^{ex} \equiv -2\, k_0\, \tau_2 R_0^{ex} \operatorname{Im} G^{ex}(x_0, x_0, k_0) ,$$

$$\operatorname{Im} W_0^{in,ex} = \operatorname{Im} W_1^{in,ex} = 0 , \quad \operatorname{Im} W_2^{in} = \operatorname{Im} b_2 ,$$

$$\operatorname{Im} W_2^{ex} = \operatorname{Im} b_2 \cos (k_0 h) - \operatorname{Im} a_2 \sin (k_0 h) ,$$

$$\operatorname{Im} W_3^{in} = k_0 \operatorname{Im} a_2\, \xi_3 + \operatorname{Im} b_3 .$$

Remark : Note that, if τ_ε has a power series asymptotics of the form $\tau_\varepsilon = k_0 + \varepsilon \tau_1 + O(\varepsilon^2)$ then (a) hold, too, but with $\alpha = 1$, and $\beta_{ex} = 0$ in (12), and (15). Namely because of this reason we distinguish (a). □

Lemma 3 shows that the problem of $\psi_\varepsilon^{in,ex}(x, \tau_\varepsilon)$, and (10), (11) matching in power orders has been reduced to the existence of the solutions $v_l^{in,ex}(\xi)$ of (12) such that their asymptotics coincide with $V_l^{in,ex}(\xi)$ as $\rho \to \infty$, $\xi_3 \geqslant 0$, and with $W_l^{in,ex}(\xi)$ as $\rho \to \infty$, $\xi_3 < 0$ up to exponentially small terms. Suppose that $0 = \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots$ are the eigenfrequencies of the Neumann problem for $-\Delta$ in the domain ω on the plane $\xi' = (\xi_1, \xi_2)$, $\phi_q(\xi')$ are the corresponding eigenfunctions normalized in $L_2(\omega)$. Denote by $\tilde{\mathscr{B}}_m$ a set of series

$$H(\xi) = Q_0(\xi_3) + \sum_{j=1}^{\infty} Q_j(\xi_3)\, \phi_j(\xi') \exp \{\mu_j\, \xi_3\} ,$$

where $Q_j(\xi_3)$ are arbitrary polynomials of order m. Denote by $\mathscr{A}_{q,m}$ a set of functions $v(\xi)$ belonging to $C^\infty(\overline{\gamma_\omega \cap S(R)})$ for any R, satisfying the homogeneous Neumann condition on $\partial \gamma_\omega$, and having differentiable asymptotics from $\tilde{\mathscr{A}}_q$, and $\tilde{\mathscr{B}}_m$ at infinity for $\xi_3 \geqslant 0$, and $\xi_3 < 0$, respectively.

LEMMA 4 : *Let the function* $f(\xi) \in \mathscr{A}_{q,m}$, *the series* $V(\xi) \in \tilde{\mathscr{A}}_{q+2}$, *and the polynomials* $W(\xi_3)$ *of order* $m+1$ *be formal asymptotic solutions of the equation* $\Delta U = F$ *as* $\rho \to \infty$, $\xi_3 \geqslant 0$, *and* $\xi_3 < 0$, *respectively, up to exponentially small terms. Then there exists a function* $v \in \mathscr{A}_{q+2,m+1}$ *being the solution of the boundary value problem*

$$\Delta v = f , \quad \xi \in \gamma_\omega , \quad \partial v / \partial n = 0 , \quad \xi \in \partial \gamma_\omega ,$$

and having the following asymptotics as $\rho \to \infty$

$$v(\xi) = V(\xi) + \sum_{j=0}^{\infty} Z_j(\xi)\,\rho^{-2j-1}, \qquad \xi_3 \geq 0, \tag{16}$$

$$v(\xi) = W(\xi_3) + q_0 + O(\xi_3^{m+1}\exp\{\mu_1 \xi_3\}), \quad \xi_3 < 0, \tag{17}$$

where q_0 is some constant If V, and W are real valued then v is real valued too

Proof Let $v_+ \in C^\infty(\{\xi \ \xi_3 \geq 0\})$, $v_- \in C^\infty(\omega \times (-1, \infty)) \cap C(\omega \times [-1, \infty))$ be solutions of the boundary value problems

$$\Delta v_+ = f, \quad \xi_3 > 0, \quad \partial v_+/\partial \xi_3 = 0, \quad \xi_3 = 0,$$

$$\Delta v_- = f, \quad \xi \in \omega \times (-1, \infty), \quad \partial v_-/\partial n = 0, \quad \xi \in \partial\omega \times (-1, \infty),$$

$$v_- = 0, \quad \xi_3 = -1,$$

which have differentiable asymptotics from $\tilde{\mathscr{A}}_{q+2}$ and $\tilde{\mathscr{B}}_{m+1}$ at infinity of the form (16), (17), respectively The existence of the such functions is obvious On the other hand, if $\tilde{f} \in C^\infty(\gamma_\omega)$ has a bounded support then the boundary value problem

$$\Delta u = \tilde{f}, \quad \xi \in \gamma_\omega, \quad \partial u/\partial n = 0, \quad \xi \in \partial\gamma_\omega \tag{18}$$

has a solution in $C^\infty(\overline{\gamma_\omega})$ with differentiable asymptotics (16), (17) where $V(\xi) \equiv 0$, $W(\xi) \equiv 0$ Let $\chi(t) \in C^\infty((-\infty, \infty))$ be a cut-off function such that $\chi \equiv 0$ for $t \leq 1$, and $\chi \equiv 1$ for $t \geq 2$ Then $v = \chi(\rho)(v_+ + v_-) - u$, where u is the solution of (18) with $\tilde{f} = \Delta(\chi(\rho)(v_+ + v_-)) - f$ satisfies all statements of the lemma □

COROLLARY *There are harmonic functions $X_1, X_2 \in \mathscr{A}_{1\,0}$, $Y \in \mathscr{A}_{0\,0}$ such that their asymptotics at infinity have the following form*

$$Y(\xi) = c_\omega \rho^{-1} + \sum_{j=1}^{\infty} Z_j(\xi)\,\rho^{-2j-1}, \quad X_i(\xi) = \xi_i + \sum_{j=1}^{\infty} Z_j^{(i)}(\xi)\,\rho^{-2j-1}, \qquad \xi_3 \geq 0,$$

$$Y(\xi) = \xi_3 + q_\omega + O(\exp\{\mu_1 \xi_3\}) \quad X_i(\xi) = q_\omega^{(i)} + O(\exp\{\mu_1 \xi_3\}) \qquad \xi_3 < 0,$$

where $c_\omega = -(2\pi)^{-1}\,|\omega|$

Lemma 4 and integration by parts of

$$\int_{S(R) \cap \gamma_\omega} \Delta X_i(\xi)\,d\xi = 0, \quad \int_{S(R) \cap \gamma_\omega} \Delta Y(\xi)\,d\xi = 0$$

as $R \to \infty$, prove this statement □

Our construction will require the following obvious statement

LEMMA 5 : *For any* Z_j *there is* P_j *such that* $Z_j(\xi)\,\rho^{-2j-1} = P_j(D_\xi)\,\rho^{-1}$. *If* Z_j *is real valued then the coefficients of* P_j *are real valued, too.*

Now we can prove the main statement of the method of matched asymptotic expansions. Denote by $v_{\varepsilon,N}^{in,ex}(\xi)$ the partial sum of (10).

THEOREM 3 : *Let* $k_0 \in \Sigma_1^{in}\backslash\Sigma^{iu}$. *Then there exist a function* τ_ε *having an asymptotics of the form* (5), *a series* $\psi_\varepsilon^{in,ex}(x,k)$, (10), (11) *such that* $P_0^{in,j} = 0$ *as* $j \geqslant 1$, *the* $v_j^{in,ex}(\xi) \in \mathcal{A}_{j,j}$ *are solutions of* (12), *and* w_j *have form* (14),

$$P_0^{in} = \psi^{-1}(0)\,, \quad v_0^{in} \equiv b_0 = R_0^{in}\,\psi^2(0)\,, \quad a_0 = b_0 \tan^{-1}(k_0 h)\,,$$

$$v_1^{ex} = -\,k_0(b_0 \sin(k_0 h) + a_0 \cos(k_0 h))\,Y\,,$$

$$\tau_2 = -\,\pi a_0\, c_\omega (R_0^{in})^{-1}\,,\ R_0^{ex} = \pi \tau_2^{-1}(b_0 \sin(k_0 h) + a_0 \cos(k_0 h))\, c_\omega\,,$$

$$\operatorname{Im} \tau_4 = -\,2\, k_0^2\, \tau_2\, \pi^{-1}\, c_\omega \sin^{-1}(k_0 h)\, R_0^{ex} (R_0^{in})^{-1}\, \sigma(k_0)\,, \tag{19}$$

and for any integer $N \geqslant 1$ *and* $\rho \to \infty$, *the following differentiable equalities hold*

$$K_N^{in,ex}(\psi_\varepsilon^{in,ex}(x,\tau_\varepsilon)) = v_{\varepsilon,N}^{in,ex}(\xi)\,, \quad \xi_3 \geqslant 0\,, \tag{20}$$

$$K_N^{in,ex}(w_\varepsilon(x)) = v_{\varepsilon,N}^{in,ex}(\xi) + O(\xi_3^N \exp\{\mu_1 \xi_3\})\,, \quad \xi_3 < 0\,, \tag{21}$$

Proof : Let τ_ε be any function with the asymptotics (5) where τ_l are arbitrary coefficients. Then due to Lemma 2 $\psi_\varepsilon^{in}(x,\tau_\varepsilon) \to R_0^{in}\,\psi(0)\,\psi(x)$, and $R_0^{ex}\,\psi_\varepsilon^{ex}(x,\tau_\varepsilon) \to 0$ as $\varepsilon \to 0$ formally. On the other hand, due to Proposition, $\Psi_\varepsilon \to \psi$ as $\varepsilon \to 0$. So, we put $P_0^{in} = \psi^{-1}(0)$. Note that our construction given below is independent of the concrete value of P_0^{in} and this value is choosen only because of the above mentioned reason.

Putting v_0^{in}, a_0, and b_0 by (19), due to Lemma 3 we obtain (20), (21) for $N = 0$. Obviously, v_0^{in} satisfies (12).

Choose v_1^{ex} by (19) and define

$$v_1^{in}(\xi) = \psi(0)\left(R_0^{in} \sum_{i=1}^{2} \psi_i\, X_i(\xi) + P_1^{in,1}(D_x)\,\psi(0)\right) + a_0\, k_0\, Y(\xi)\,,$$

where $P_1^{in,1}$ is not determined. Then due to Lemma 3 and Corollary we have at infinity :

$$v_1^{in}(\xi) - V_1^{in}(\xi) = (a_0\, k_0\, a_\omega + \pi^{-1} k_0\, \tau_2 R_0^{in})\,\rho^{-1} +$$
$$+ \sum_{i=1}^{\infty} (a_0\, k_0\, Z_i(\xi)\,\rho^{-2i-1} + (-1)^i\, \pi^{-1} k_0\, \tau_2\, P_i^{in,i}(D_\varepsilon)\,\rho^{-1})\,,$$

$$v_1^{ex}(\xi) - V_1^{ex}(\xi) =$$
$$= (-k_0(b_0 \sin(k_0 h) + a_0 \cos(k_0 h))\, a_\omega + \pi^{-1} k_0 \tau_2 R_0^{ex})\, \rho^{-1} +$$
$$+ \sum_{\iota=1}^{\infty} (-k_0(b_0 \sin(k_0 h) + a_0 \cos(k_0 h))\, Z_\iota(\xi)\, \rho^{-2\iota-1}$$
$$+ (-1)^{\iota}\, \pi^{-1} k_0 \tau_2 P_\iota^{ex,\iota}(D_\xi)\, \rho^{-1}), \quad \xi_3 \geqslant 0, \tag{22}$$

$$v_1^{in}(\xi) - W_1^{in}(\xi) = a_0 k_0 q_\omega + \psi(0)\left(R_0^{in} \sum_{\iota=1}^{2} \psi_\iota q_\omega^{(\iota)} + P_1^{in,1}(D_x)\, \psi(0)\right) -$$
$$- b_1 \cos(k_0 h) + O(\exp\{\mu_1 \xi_3\}),$$

$$v_1^{ex}(\xi) - W_1^{ex}(\xi) = -k_0(b_0 \sin(k_0 h) + a_0 \cos(k_0 h))\, q_\omega -$$
$$- b_1 \cos(k_0 h) + a_1 \sin(k_0 h) + O(\exp\{\mu_1 \xi_3\}), \quad \xi_3 < 0. \tag{23}$$

Putting the right hand sides of (22) to be zero, we obtain (19) for τ_2, and R_0^{ex}, due to Lemma 5 determine $P_\iota^{in,ex,\iota}$ (hence, v_1^{in}) and obtain (20) for $N = 1$. Now we consider τ_ε as any function with the asymptotics (5) where τ_2 is above defined and the other coefficients are arbitrary. Note that $\operatorname{Im} P_\iota^{in,ex,\iota} = 0$.

Putting the right hand sides of (23) to be zero up to exponentially small terms, we determine b_1, a_1 and obtain (21) for $N = 1$. Note that $\operatorname{Im} b_1 = \operatorname{Im} a_1 = 0$.

The proof is then obtained by induction using Lemmas 3-5 and Corollary. At the n-th step we determine the solution $\tilde{v}_n^{in,ex} \in \mathscr{A}_{n,n}$ of (12) such that at infinity $\tilde{v}_n^{in,ex} = \tilde{V}_n^{in,ex} + O(\rho^{-1})$ as $\xi_3 \geqslant 0$ and $\tilde{v}_n^{in,ex} = \tilde{W}_n^{in,ex} + O(1)$ as $\xi_3 < 0$. Equating $V_n^{in,ex} - \alpha_{in,ex}\, \psi(0)\, P_1^{in,n}(D_x)\, \psi(0)$ and the asymptotics of $\tilde{v}_n^{in,ex}$ we obtain $\tau_{n+1} n$ and $P_{\iota+1}^{in,ex,n+\iota}$. Then putting $v_n^{in} = \tilde{v}_n^{in} + \psi(0)\, P_1^{in,n}(D_x)\, \psi(0)$, and $v_n^{ex} = \tilde{v}_n^{ex}$ we get (20) for $N = n$. Now equating $W_n^{in,ex}$ and the asymptotics of $v_n^{in,ex}$ up to exponential terms we obtain b_n, a_n and (21) for $N = n$. Of course, after this step we consider τ_ε as any function with asymptotics (5) where, for $j \leqslant n + 1$, the coefficients τ_j are above determined and other coefficients are arbitrary.

In this way we see that the condition of (c) of Lemma 3 holds true. Then the boundary value problems for $\operatorname{Im} v_2^{ex}$ and $\operatorname{Im} v_3^{in,ex}$ read as follows

$$\Delta \operatorname{Im} v_j = 0, \quad \xi \in \gamma_\omega, \quad \partial \operatorname{Im} v_j / \partial n = 0, \quad \xi \in \partial \gamma_\omega.$$

Using the equality $k_0\, \sigma(k_0) = \operatorname{Im} G^{ex}(x_0, x_0, k_0)$ (see, for instance, [7]) we obtain the following chains for the imaginary parts :

$$\operatorname{Im} V_2^{in} = \operatorname{Im} \tilde{W}_2^{in} = 0, \quad \Rightarrow \operatorname{Im} W_2^{in} = 0 \Rightarrow \operatorname{Im} b_2 = 0 \Rightarrow \operatorname{Im} W_2^{ex} =$$
$$= -a_2 \sin(k_0 h);$$

$$\operatorname{Im} \tilde{V}_2^{ex}(\xi) \equiv -2 k_0 \tau_2 R_0^{ex} \operatorname{Im} G^{ex}(x_0, x_0, k_0),$$

$$\operatorname{Im} W_2^{ex} = -\operatorname{Im} a_2 \sin(k_0 h) \Rightarrow$$

$$\operatorname{Im} v_2^{ex} = -2 k_0^2 \tau_2 R_0^{ex} \sigma(k_0) = -a_2 \sin(k_0 h) \Rightarrow \operatorname{Im} a_2 =$$

$$= 2 k_0^2 \tau_2 \sin^{-1}(k_0 h) R_0^{ex} \sigma(k_0) \Rightarrow$$

$$\Rightarrow \operatorname{Im} W_3^{in}(\xi) = 2 k_0^3 \tau_2 \sin^{-1}(k_0 h) R_0^{ex} \sigma(k_0) \xi_3 + \operatorname{Im} b_3 ;$$

$$\operatorname{Im} W_3^{in}(\xi) = 2 k_0^3 \tau_2 \sin^{-1}(k_0 h) R_0^{ex} \sigma(k_0) \xi_3 + \operatorname{Im} b_3 ,$$

$$\operatorname{Im} \tilde{V}_3^{in}(\xi) \equiv 0 \Rightarrow \operatorname{Im} \tilde{v}_3^{in}(\xi) = 2 k_0^3 \tau_2 \sin^{-1}(k_0 h) R_0^{ex} \sigma(k_0) Y(\xi) \Rightarrow$$

$$\Rightarrow 2 k_0^3 \tau_2 \sin^{-1}(k_0 h) R_0^{ex} \sigma(k_0) c_\omega = -\operatorname{Im} \tau_4 k_0 \pi R_0^{in} .$$

The latter equality gives (19) for Im τ_4. The theorem is proved. □

Let us denote by $\psi_{\varepsilon, N}^{in, ex}(x, k)$ the partial sums of the series $\psi_\varepsilon^{in, ex}(x, k)$ and by $w_{\varepsilon, N}(x)$ the partial sums of expansion (11). Theorem 3 implies (see, for instance [4], [7]).

LEMMA 6 : *Let $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$, τ_ε, and the series $\psi_\varepsilon^{in, ex}$,* (10), (11) *satisfy the statements of Theorem* 3. *Then, for any k sufficiently close to k_0, the function*

$$\psi_{\varepsilon, N}(x, k) = \chi(r\varepsilon^{-1/2}) \psi_{\varepsilon, N}^{in}(x, k) + \chi(|x - x_0| \varepsilon^{-1/2}) \psi_{\varepsilon, N}^{ex}(x, k) +$$
$$+ \chi(r\varepsilon^{-1/2}) \chi(|x - x_0| \varepsilon^{-1/2}) w_{\varepsilon, N}(x) + (1 - \chi(r\varepsilon^{-1/2})) v_{\varepsilon, N}^{in}(x)$$
$$+ (1 - \chi(|x - x_0| \varepsilon^{-1/2})) v_{\varepsilon, N}^{ex}(x)$$

belongs to $W_2^1(\Omega_\varepsilon \cap S(R))$, for any R, converges to ψ in $L_2(S(R))$ as $k \to k_0$ and satisfies (1), (2) *for $F(x) = F_{\varepsilon, N}(x, k) \in L_2(\mathbf{R}^3)$ such that* $\operatorname{supp} F_{\varepsilon, N} \subset \kappa_\varepsilon \cup S(2\varepsilon^{1/2}) \cup S^{ex}(2\varepsilon^{1/2})$ *and*

$$\|F_{\varepsilon, N}(\,.\,, \tau_\varepsilon)\|_{0, \mathbf{R}^3} < C_N \varepsilon^{N/2} .$$

The functions $\psi_{\varepsilon, N}(x, k)$, and $F_{\varepsilon, N}(x, k)$ are analytical at some neighborhood of k_0.

Now we can justify the expansions constructed in Theorem 3.

Proof of statement (i) of Theorem 1 : Let us denote τ_ε from Theorem by $\tilde{\tau}_\varepsilon$. Let τ_ε be the scattering frequency converging to k_0. Suppose that there exists such a d that $|\tilde{\tau}_\varepsilon - \tilde{\tau}_\varepsilon| > C\varepsilon^d$ where $c > 0$. Then due to Proposition, and Lemma 6 $\|\psi_{\varepsilon, N}(\,.\,, \tilde{\tau}_\varepsilon)\|_{0, (S(R))} < C | \varepsilon^{N/2 - d}$ and, hence, for $N > 2d$ we have $\psi_{\varepsilon, N}(x, \tilde{\tau}_\varepsilon) \to 0$ as $\varepsilon \to 0$ in $L_2(S(R))$. On the other hand, due to Lemma 6 $\psi_{\varepsilon, N}(x, k) \to \psi$ for any $k \to k_0$ in the same norm. This contradiction proves that $|\tilde{\tau}_\varepsilon - \tilde{\tau}_\varepsilon| = O(\varepsilon^m)$ for any m.

Using (3) for $\psi_{\varepsilon, N}(x, k)$, and k close to τ_ε and taking the limit as $k \to \tau_\varepsilon$, we obtain that

$$\psi_{\varepsilon, N}(x, \tau_\varepsilon) \doteq c(\varepsilon, N)\, \Psi_\varepsilon(x) + \tilde{\psi}_{\varepsilon, N}(x) .$$

Due to Lemma 6, and Proposition $\|\psi_{\varepsilon, N}\|_{0, S(R)} < C_{R, N}\, \varepsilon^{N/2}$ and then $c(\varepsilon, N) = 1 + o(1)$ as $\varepsilon \to 0$ for any fixed N. The equalities (19) give (5)-(7), and the principal terms of Ψ_ε outside Ω^{in}. Thus, statement (i) of Theorem 1 is proved. □

5. ASYMPTOTICS OF THE SCATTERING FREQUENCIES ASSOCIATED WITH TUBE

In this section we construct the asymptotics of a scattering frequency converging to $k_0 \in \Sigma^{tu} \setminus \Sigma^{in}$. We use the following consequence of Lemmas 1, 2 (the analogue of Lemma 3).

LEMMA 7 : *Let $k_0 = \pi m/h \in \Sigma^{tu} \setminus \Sigma^{in}$, τ_ε be any function having the asymptotics* (5) *with arbitrary coefficients, $\psi_\varepsilon^{in, ex}(x, k)$ have arbitrary coefficients $a_{iq}^{in, ex, j}$, $w_\varepsilon(x)$ have form* (11), (14) *where $b_{-1} = a_j = 0$ for $j \geq 0$ and a_{-1}, and the other b_j are arbitrary.*

(a) *Then* (15) *holds for any integer $N \geq 0$, the series $V_j^{in, ex} \in \tilde{\mathscr{A}}_{j-1}$, and the polynomials W_j of order $j+1$ have the following form*

$$W_i^{in}(\xi) = \tilde{W}_i^{in}(\xi) + b_i ,$$

$$W_i^{ex}(\xi) = \tilde{W}_i^{ex}(\xi) + (-1)^m (b_i - a_{-1}\, \tau_{i+1}\, h) ,$$

$$\tilde{W}_0^{in}(\xi) = a_{-1}\, k_0\, \xi_3 , \quad \tilde{W}_0^{ex}(\xi) = (-1)^{m+1}\, a_{-1}\, k_0\, \xi_3 ,$$

$$V_i^{in, ex}(\xi) = \tilde{V}_i^{in, ex}(\xi) + (2\pi)^{-1} \sum_{j=0}^{\infty} (-1)^j P_j^{in, ex, j+i}(D_\xi)\, \rho^{-1} ,$$

$$V_0^{in, ex}(\xi) = 0$$

where the series $\tilde{V}_i^{in, ex} \in \tilde{\mathscr{A}}_{i-1}$, and the polynomials $\tilde{W}_i^{in, ex}$ of order $i+1$ are independent of τ_{q+1}, b_q, $P_j^{in, ex, j+q}$ for $q \geq i$. $\tilde{V}_j^{in, ex}$, and $V_j^{in, ex}$ are formal asymptotic solutions of (12) *as $\rho \to \infty$, and $\xi_3 \geq 0$, where functions v_i are replaced by $V_i^{in, ex}$; $\tilde{W}_j^{in, ex}$, and $W_j^{in, ex}$ are formal asymptotic solutions of* (12) *as $\rho \to \infty$, and $\xi_3 < 0$, where the functions v_i are replaced by $W_i^{in, ex}$.*

(b) *If in addition, τ_1, b_0, and $R_0^{in, ex}$ are real valued then*

$$\operatorname{Im} \tilde{V}_1^{in} = \operatorname{Im} \tilde{W}_1^{in, ex} = 0 , \quad \operatorname{Im} \tilde{V}_1^{ex} = R_0^{ex}\, k_0\, \sigma(k_0) .$$

The following statement is an analogue of Theorem 3.

THEOREM 4 : *Let* $k_0 = \pi m/h \in \Sigma^{in}\setminus\Sigma^{in}$. *Then there exist a function* τ_ε *having an asymptotics of the form* (5), *the series* $\psi_\varepsilon^{in, ex}(x, k)$ (10), (11) *such that* $v_j^{in, ex}(\xi) \in \mathscr{A}_{j-1, j}$ *are solutions of* (12), ω_j *have form* (14) *where* $b_{-1} = a_j = 0$ *for* $j \geqslant 0$,

$$a_{-1} = (2/h)^{1/2}\,|\,\omega\,|^{-1/2}\,,\ v_0^{in} = a_{-1}\,k_0\,Y\,,\ v_0^{ex} = (-1)^{m+1}\,a_{-1}\,k_0\,Y\,,$$

$$b_0 = a_{-1}\,k_0\,q_\omega\,,\ \tau_1 = k_0\,h^{-1}\,q_\omega + (a_{-1}\,h)^{-1}\,b_0\,,\ R_0^{in} = 2\,a_{-1}\,\pi k_0\,c_\omega\,,$$

$$R_0^{ex} = (-1)^{m+1}\,2\,a_{-1}\,\pi k_0\,c_\omega\,,\quad \operatorname{Im}\tau_2 = (-1)^{m+1}\,(a_{-1}\,h)^{-1}\,R_0^{ex}\,k_0\,\sigma(k_0) \tag{24}$$

and (20), (21) *hold for any integer* $N \geqslant 0$.

Proof : Since $\Psi_\varepsilon - \varepsilon^{-1}(2/h)^{1/2}\,|\,\omega\,|^{-1/2}\sin(k_0\,x_3) \to 0$ as $\varepsilon \to 0$ in $L_2(\kappa_\varepsilon)$ then a_{-1} is choosen by (24).

Putting $v_0^{in, ex}$ by (24), due to Lemma 7 we obtain that at infinity

$$v_0^{in}(\xi) - W_0^{in}(\xi) = (a_{-1}\,k_0\,q_\omega - b_0) + O(\exp\{\mu_1\,\xi_3\})\,,$$

$$v_0^{ex}(\xi) - W_0^{ex}(\xi) = (-1)^m\,(-a_{-1}\,k_0\,q_\omega - b_0 + a_{-1}\,\tau_1\,h)\,,\quad \xi_3 < 0\,,$$

$$\begin{aligned} v_0^{in}(\xi) - V_0^{in}(\xi) = {} & (a_{-1}\,k_0\,c_\omega - (2\,\pi)^{-1}\,R_0^{in})\,\rho^{-1} + \\ & + \sum_{l=1}^{\infty} (a_{-1}\,k_0\,Z_l(\xi)\,\rho^{-2l-1} - (2\,\pi)^{-1}\,(-1)^l\,P_l^{in, l}(D_\xi)\,\rho^{-1})\,, \end{aligned} \tag{25}$$

$$\begin{aligned} v_0^{ex}(\xi) - V_0^{ex}(\xi) = {} & ((-1)^{m+1}\,a_{-1}\,k_0\,c_\omega - (2\,\pi)^{-1}\,R_0^{ex})\,\rho^{-1} + \\ & + \sum_{l=1}^{\infty} ((-1)^{m+1}\,a_{-1}\,k_0\,Z_l(\xi)\,\rho^{-2l-1} - (2\,\pi)^{-1}\,(-1)^l\,P_l^{ex, l}(D_\xi)\,\rho^{-1})\,, \\ & \xi_3 \geqslant 0\,. \end{aligned} \tag{26}$$

Equating the right hand sides of (25) to zero up to exponentially small terms, we determine b_0, and τ_1 in the form (24) and obtain (21) for $N = 0$. Putting the right sides of (26) to be zero, we get (24) for $P_l^{in, ex, l}$ and obtain (20) for $N = 0$.

In the same way, using Lemmas 4, 5, 7 one can construct all τ_j, b_j, and $R_j^{in, ex}$. The imaginary part of τ_ε is determined by the following short chain (see (b) in Lemma 7)

$$\operatorname{Im}\tilde V_1^{ex} = R_0^{ex}\,k_0\,\sigma(k_0)\,,\quad \operatorname{Im}\tilde V_1^{in} = 0 \Rightarrow \operatorname{Im} v_1^{ex} \equiv R_0^{ex}\,k_0\,\sigma(k_0)\,,$$

$$\operatorname{Im} v_1^{in} \equiv 0 \Rightarrow \operatorname{Im} W_1^{ex} \equiv R_0^{ex}\,k_0\,\sigma(k_0)\,,$$

$$\operatorname{Im} W_1^{in} \equiv 0 \Rightarrow (-1)^m\,(\operatorname{Im} b_1 - \operatorname{Im}\tau_2\,a_{-1}\,h) = R_0^{ex}\,k_0\,\sigma(k_0)\,,$$

$$\operatorname{Im} b_1 = 0 \Rightarrow (-1)^{m+1}\operatorname{Im}\tau_2\,a_{-1}\,h = R_0^{ex}\,k_0\,\sigma(k_0)\,.$$

The latter equality gives (24) for $\operatorname{Im}\tau_2$. The theorem is proved. □

The justification of the constructed expansions follows in the same way as in the previous section. Equalities (24) give (8), (9), and the principal terms of Ψ_ε. Thus, the statement (ii) of Theorem 1 is proved, too.

6. REMARKS

The asymptotics for a resonator in $\mathbf{R}^n$, $n > 3$ can be constructed in the same way. We consider that $\Omega^{in, ex} \subset \mathbf{R}^n$ coincide with the half-spaces $x_n > 0$, and $x_n < -h$ at some neighborhood of the origin, and $(0, \dots, 0, -h)$, respectively, that ω is some $(n-1)$-dimensional domain in the hyperplane $x_n = 0$, in the definition of κ_ε the variable x_3 must be replaced by x_n, and etc.

Note, that in this case the main term of $G^{in, ex}(x, x_0^{in, ex}, k)$ by x has the form $2(|S_n|\, r_{in, ex})^{-1}$, where S_n is the unite sphere in $\mathbf{R}^n$, and the analogue of Y has the following asymptotics at infinity

$$Y(\xi) = c_\omega \rho^{-n+2} + O(\rho^{-n+1}), \quad \xi_n \geqslant 0, \quad Y(\xi) = \xi_n + q_\omega + o(1), \quad \xi_n < 0$$

where $c_\omega = -2|S_n|^{-1}|\omega|$.

Taking into consideration these remarks, assuming the scattering frequency in the form $k_0 + \varepsilon^{n-1}\tau_{n-1} + o(\varepsilon^{n-1})$ as $k_0 \in \Sigma_1^{in} \backslash \Sigma^{tu}$ and following our construction one obtains that

$$\tau_{n-1} = \frac{1}{2}|S_n|\, a_0\, c_\omega (R_0^{in})^{-1},$$

$$\operatorname{Im} \tau_\varepsilon = 2(k_0 \tau_2)^2 (a_0 \sin(k_0 h))^{-1} R_0^{ex} \sigma(k_0)\, \varepsilon^{2n-2} + o(\varepsilon^{n-2}),$$

the leading terms of Ψ_ε coincide with leading terms of $\psi_\varepsilon^{in, ex}(x, \tau_\varepsilon)$ (10), and (11) and (19) holds.

In the case $\tau_\varepsilon \to k_0 \in \Sigma^{tu} \backslash \Sigma^{in}$ only the orders of $\operatorname{Im} \tau_\varepsilon$ and Ψ_ε are changed. Namely,

$$\tau_\varepsilon \sim k_0 + \varepsilon\tau_1, \qquad \operatorname{Im} \tau_\varepsilon \sim \varepsilon^{n-1} \operatorname{Im} \tau_2,$$

$$\Psi_\varepsilon(x) \sim \varepsilon^{(n-1)/2} R_0^{in, ex} G^{in, ex}(x, x_0^{in, ex}, \tau_\varepsilon), \quad x \in \Omega^{in, ex} \backslash S^{in, ex}(\varepsilon^{1/2}),$$
$$\Psi_\varepsilon(x) \sim \varepsilon^{(3-n)/2} v_0^{in, ex}(x^{in, ex}/\varepsilon), \qquad x \in S^{in, ex}(2\varepsilon^{1/2}),$$
$$\Psi_\varepsilon(x) \sim \varepsilon^{-(n-1)/2} a_{-1} \sin(k_0 x_n), \quad x \in \kappa_\varepsilon \backslash (S^{ex}(\varepsilon^{1/2}) \cup S(\varepsilon^{1/2})),$$

where the coefficients satisfy (24).

In the considered cases our results define more precisely the estimates $|\tau_\varepsilon - k_0| = O(\varepsilon^{(n-2)/2})$ for nonzero $k_0 \in \Sigma^{in} \backslash \Sigma^{tu}$ and $|\tau_\varepsilon - k_0| = O(\varepsilon^{1/2})$ as $k_0 \in \Sigma^{tu} \backslash \Sigma^{in}$ obtained in [16]. Note that, the estimate $|\tau_\varepsilon - k_0| = O(\varepsilon^{(n-2)})$ holds for $k_0 \in \Sigma^{in} \backslash \Sigma^{tu}$ without our restrictions on the simplicity of nonzero

k_0 and $\psi(0) \neq 0$. Similar results were obtained [8], [9] for the Helmholtz resonator with an infinitely thin shell. Note also, that in $\mathbf{R}^3$ the vanishing τ_ε has the asymptotic $\tau_\varepsilon \sim \varepsilon t$ [17], [18].

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