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W. B. LIU
JOHN W. BARRETT

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QUASI-NORM ERROR BOUNDS FOR THE FINITE ELEMENT APPROXIMATION OF SOME DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS AND VARIATIONAL INEQUALITIES (*)

by W. B. LIU (1, †) and John W. BARRETT (1)

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Abstract. — *In this paper energy type error bounds are established for the finite element approximation of the following variational inequality problem*

Let K be a closed convex set in the Sobolev space $W_0^{1,p}(\Omega)$ *with* $p \in (1, \infty)$, *where* Ω *is an open set in* R^d $(d = 1$ *or* $2)$ *Given f, find* $u \in K$ *such that for any* $v \in K$

$$\int_\Omega k(x, |\nabla u|)\, \nabla u(x) . \nabla(v(x) - u(x))\, dx \geq \int_\Omega f(x)(v(x) - u(x))\, dx ,$$

where $k \in C(\Omega \times (0, \infty))$ *is a given nonnegative function with* $k(.,t)\, t$ *strictly increasing for* $t \geq 0$, *but possibly degenerate*

In some notable cases these error bounds converge at the optimal approximation rate provided the solution u is sufficiently smooth

Résumé — *Dans cette publication les bornes d'erreur de genre énergétique sont définies pour l'approximation des éléments finis du probleme de variation d'inégalité suivant*

Prenons K, un ensemble convexe fermé dans l'espace de Sobolev $W_0^{1,p}(\Omega)$, $p \in (1, \infty)$, *où* Ω *est un ensemble ouvert dans* R^d $(d = 1$ ou $2)$ *Étant donné f, on cherche* $u \in K$ *de manière a ce que, pour chaque* $v \in K$,

$$\int_\Omega k(x, |\nabla u|)\, \nabla u(x) . \nabla(v(x) - u(x))\, dx \geq \int_\Omega f(x)(v(x) - u(x))\, dx ,$$

où $k \in C(\Omega \times (0, \infty))$ *est une fonction non négative donnée avec* $k(.,t)\, t$ *de progression strictement ascendante pour* $t \geq 0$, *mais qui peut éventuellement dégénérer*

Dans certains cas remarquables, si la solution u est suffisamment régulière, ces bornes d'erreur convergent au taux d'approximation optimal

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(1) Department of Mathematics, Imperial College, London, SW7 2BZ

1. INTRODUCTION

Let Ω be a bounded open set in R^d, $d = 1$ or 2, with a Lipschitz boundary $\partial\Omega$ in the case $d = 2$. Many mathematical models from physical processes have the following form : given f and g, find u such that

$$-\nabla \cdot (k(x, |\nabla u(x)|) \nabla u(x)) = f(x) \quad \forall x \in \Omega \subset R^d, \tag{1.1a}$$

$$u|_{\partial\Omega} = g|_{\partial\Omega} . \tag{1.1b}$$

When k is smooth and satisfies the ellipticity and monotonicity conditions such as those given in [9] and [11], there is much work on its well-posedness. Moreover, optimal error bounds for its finite element approximation have been established in various norms

For many physical models, k does not satisfy these conditions and hence the linearization or deformation procedure used in [9] and [11] cannot be applied at all. A typical example is the p-Laplacian, where $k(t) = t^{p-2}$, $p \in (1, \infty)$ and $p \neq 2$. Throughout we will denote $k(x, t)$ by $k(t)$ for almost every $x \in \Omega$. Such models arise in many physical processes : nonlinear diffusion and filtration, see [19], power-law materials, see [1], and non-Newtonian flows, see [3]. For such cases the monotonicity method plays an essential role in establishing well-posedness and error bounds, see [12]. By this method error estimates for the finite element approximation of (1.1) have been given for a class of k, which includes the p-Laplacian, see [6]. [7] and [12], although the results are only suboptimal in most cases and may be very poor for some important cases (for instance, $k(t) = t^{p-2}$ and $p \neq 2$). On the other hand, numerical computations indicate that the approximation should converge at the optimal rate for such cases at least for sufficiently regular solutions. This has been confirmed recently for the continuous piecewise linear finite element approximation of (1.1) , firstly for the case where $k(t) = t^{p-2}$ in [2] and then for more general cases in [16]. We note that there is little point in considering a higher order approximation due to the lack of regularity of the solutions of (1.1) in general. It has been further shown that the techniques used in dealing with (1 1) can also be applied in a modified form to the case of a quasi-Newtonian flow obeying the power law or the Carreau law, see [3], and the parabolic p-Laplacian, see [4].

In the mathematical literature there has been a huge explosion of work on the p-Laplacian and related degenerate quasilinear elliptic and parabolic equations. This work has naturally led to considering the corresponding variational inequalities (obstacle problems), see [5] for example. It is the purpose of this paper to show that the techniques in [2] can be adapted to study the finite element approximation of the elliptic variational inequality corresponding to (1.1). Error bounds in energy type norms are proved. In some notable cases these error bounds converge at the optimal approximation rate provided the corresponding solution is sufficiently smooth. In addition these results in some cases simplify and improve on those for the equation case presented in [16]. Although there is much work on the finite element approximation of elliptic variational inequalities for the Laplacian, see [10] and [8] for example, and when k is smooth and positive, see [17] and [18], we know of no work for degenerate k except the brief mention of suboptimal bounds in [8].

The outline of this paper is as follows. In the next section we state the precise weak formulation of the problem and then prove some important inequalities. In Section 3 we establish some abstract error bounds for the finite element approximation of the problem. In Section 4 we derive some explicit error bounds from these abstract bounds.

Throughout this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,q}(\Omega)}$. In addition C and M denote two general positive constants independent of h and $x \in \Omega$.

2. THE WEAK FORMULATION AND SOME INEQUALITIES

Let Ω be a bounded open set in R^d with a Lipschitz boundary $\partial\Omega$. Let K be a closed convex set in $W_0^{1,p}(\Omega) \equiv \{v \in W^{1,p}(\Omega) : v = 0$ on $\partial\Omega\}$ and $f \in L^2(\Omega)$. Let k satisfy the following conditions.

ASSUMPTIONS (E) : *We assume that* $k \in C(\Omega \times (0, \infty))$ *and the function* $t \to k(t)\, t$ *is continuous and strictly increasing on* $[0, \infty)$ *and that it vanishes at* $t = 0$, *where throughout the paper we will denote* $k(x, t)$ *by* $k(t)$ *for almost every* $x \in \Omega$. *In addition we assume that there exist constants* $p > 1$ *and* $t_0 > 0$ *such that*

$$M t^{p-1} \leq k(t)\, t \leq C t^{p-1} \quad for \quad t \geq t_0 . \tag{2.1}$$

The problem that we wish to consider is :
(WP) Find $u \in K$ such that

$$\int_\Omega k(x, |\nabla u(x)|)\, \nabla u(x) \,.\, \nabla(v(x) - u(x))\, dx \geq$$

$$\geq \int_\Omega f(x)(v(x) - u(x))\, dx \quad \forall v \in K .$$

Associated with (WP) is the following minimization problem :
(MP) Find $u \in K$ such that

$$J_\Omega(u) = \operatorname*{Min}_{v \in K} J_\Omega(v) , \tag{2.2}$$

where

$$J_{\Omega}(v) = \int_{\Omega} \int_0^{|\nabla v|} k(t)\, t\, dt\, dx - \int_{\Omega} fv\, dx . \tag{2.3}$$

Under Assumptions (E) it is a simple matter to establish the existence of a unique solution u to (MP) by adapting the argument for the equation case given in [6], [7] and [12]. In addition it follows that (WP) and (MP) are equivalent problems.

In order to prove error bounds for the finite element approximation of $(WP) \equiv (MP)$ one requires stronger assumptions on the function k.

ASSUMPTIONS (A) : *We assume that* $k \in C(\Omega \times (0, \infty))$ *and that there exists a constant* $p > 1$ *such that*

$(A1)$ *There exist constants* $\alpha_1 \in [0, 1]$ *and* $\varepsilon, C > 0$ *such that for all* $t \geq 0$

$$k(t) \leq C\,[t^{\alpha_1}(1+t)^{1-\alpha_1}]^{p-2}$$

and for all $s, t > 0$ *satisfying* $|s/t - 1| \leq \varepsilon$

$$|k(t)\,t - k(s)\,s| \leq C\,|t-s|\,[(t+s)^{\alpha_1}(1+t+s)^{1-\alpha_1}]^{p-2} .$$

$(A2)$ *There exist constants* $\alpha_2 \in [0, 1]$ *and* $M > 0$ *such that for all* $t \geq s \geq 0$

$$k(t)\,t - k(s)\,s \geq M(t-s)[(t+s)^{\alpha_2}(1+t+s)^{1-\alpha_2}]^{p-2} .$$

Assumptions (A) are slightly more general than those in [16], where α_1 and α_2 are either 1 or 0, and can be stated in a more compact way. However, the main advantage to stating the assumptions in this form is that one can obtain some sharper inequalities, see (2.4) and (2.5), than those in [6] and [16] for some k.

Remark 2.1 : We note that if k satisfies Assumptions (A) then it satisfies Assumptions (E). Many functions k met in practical problems satisfy the Assumptions (A); e.g. $k(t) = [t^{\mu}(1+t)^{1-\mu}]^{p-2}$ with $p \in (1, \infty)$ and $\mu \in [0, 1]$ satisfies (A) with $\alpha_1 = \alpha_2 = \mu$; $k(t) = (1+t^2)^{(p-2)/2}$ with $p \in (1, \infty)$ satisfies (A) with $\alpha_1 = \alpha_2 = 0$. Clearly the choice of parameters α_1 and α_2 are not unique. For example if (A) holds for $p \in (1, 2]$ and $\alpha_1, \alpha_2 \in [0, 1]$ then it holds with $\alpha_1 = 1$ and $\alpha_2 = 0$. Similarly if (A) holds for $p \in [2, \infty)$ and $\alpha_1, \alpha_2 \in [0, 1]$ then it holds for $\alpha_1 = 0$ and $\alpha_2 = 1$. In applying our results in this paper we will always choose $\alpha_1 = \alpha_2$ as in the examples above. However, we state our Assumptions (A) in this more general form so as to relate to previous work.

Throughout this section $(. , .)$ denotes the R^d inner product.

LEMMA 2.1 : *Let k satisfy the Assumptions (A1) for $p \in (1, \infty)$ and $\alpha_1 \in [0, 1]$. Then for all $x, y \in R^d$ $(d \geq 1)$ and $\delta \geq 0$ we have that*

$$|k(|x|)x - k(|y|)y| \leq$$
$$\leq C\,|x-y|^{1-\delta}\,[(|x| + |y|)^{\alpha_1}(1 + |x| + |y|)^{1-\alpha_1}]^{p-2+\delta}\,. \quad (2.4)$$

Let k satisfy the Assumptions (A2) for $p \in (1, \infty)$ and $\alpha_2 \in [0, 1]$. Then for all $x, y \in R^d$ $(d \geq 1)$ and $\delta \geq 0$ we have that

$$(k(|x|)x - k(|y|)y, x - y) \geq$$
$$\geq M\,|x-y|^{2+\delta}\,[(|x| + |y|)^{\alpha_2}(1 + |x| + |y|)^{1-\alpha_2}]^{p-2-\delta}\,. \quad (2.5)$$

Proof: We first prove (2.4) with $\delta = 0$. Some ideas similar to those in [2], [6], [7], [12] and [16] will be applied. For any $(x, y) \in R^d \times R^d$ let

$$F(x, y) \equiv |k(|x|)x - k(|y|)y| /$$
$$/\left\{|x-y|\,[(|x| + |y|)^{\alpha_1}(1 + |x| + |y|)^{1-\alpha_1}]^{p-2}\right\}.$$

We wish to prove that F is bounded. For A an $d \times d$ orthogonal matrix it follows that for all $(x, y) \in R^d \times R^d$

$$F(x, y) = F(y, x)\,, \quad F(Ax, Ay) = F(x, y) \quad \text{and} \quad F(0, y) \leq C\,. \quad (2.6)$$

Therefore without loss of generality we can suppose that $x, y \neq 0$. Let $b(x) \equiv [|x|^{\alpha_1}(1 + |x|)^{1-\alpha_1}]^{p-2}$. Then F can be rewritten as

$$F(x, y) \equiv$$
$$\equiv \frac{|k(|x|)x/b(x)|x| - [k(|y|)y/b(y)|y|][b(y)|y|/b(x)|x|]|}{|x/|x| - y/|x||\,[(1 + |y|/|x|)^{\alpha_1}(1 + |y|/(1 + |x|))^{1-\alpha_1}]^{p-2}}\,.$$

We can further assume from (2.6) that $x/|x| = e_1 \equiv (1, 0, \ldots, 0)^T$ and $|y|/|x| \leq 1$. It follows that $F(x, y)$ will be bounded if $y/|x|$ does not tend to e_1 as the function $x \to k(|x|)/b(x)$ is bounded above. It remains to show that $\overline{\lim}_{y/|x| \to e_1} F(x, y) < \infty$.

If $1 - \varepsilon \leq |y|/|x| \leq 1$ for some $\varepsilon \in (0, 1)$ then there exists a constant C such that $|x|\,|y| \leq (|x| + |y|)^2 \leq C\,|x|\,|y|$. Then it follows, since $|x|\,|y| - (x, y) \leq |x-y|^2$, that

$$|k(|x|)x - k(|y|)y|^2 = (k(|x|)|x| - k(|y|)|y|)^2 +$$
$$+ 2\,k(|x|)\,k(|y|)(|x|\,|y| - (x, y))$$

$$\leqslant C\,(|x| - |y|)^2\,[(|x| + |y|)^{\alpha_1}\,(1 + |x| + |y|)^{1-\alpha_1}]^{2(p-2)} +$$
$$+ C\,|x-y|^2\,\{(|x|\,|y|)^{\alpha_1}\,[(1 + |x|)(1 + |y|)]^{1-\alpha_1}\}^{p-2}$$
$$\leqslant C\,|x-y|^2\,[(|x| + |y|)^{\alpha_1}\,(1 + |x| + |y|)^{1-\alpha_1}]^{2(p-2)}$$

Consequently one has that (2 4) holds with $\delta = 0$ On the other hand for all $x, y \in R^d$ and $\delta \geqslant 0$ we have that

$$|x-y|\,[(|x| + |y|)^{\alpha_1}\,(1 + |x| + |y|)^{1-\alpha_1}]^{p-2} \leqslant$$
$$\leqslant |x-y|^{1-\delta}\,[(|x| + |y|)^{\alpha_1}\,(1 + |x| + |y|)^{1-\alpha_1}]^{p-2+\delta}\,,$$

that is, (2 4) holds for any $\delta \geqslant 0$

Similarly (2 5) for the case $\delta = 0$ holds since we have for $M_i > 0$ that

$$(k(|x|)\,x - k(|y|)\,y,\, x-y) = (k(|x|)|x| - k(|y|)|y|)(|x| - |y|) +$$
$$+ (k(|x|) + k(|y|))(|x|\,|y| - (x, y))$$
$$\geqslant M(|x| - |y|)^2\,[(|x| + |y|)^{\alpha_2}\,(1 + |x| + |y|)^{1-\alpha_2}]^{p-2} +$$
$$+ M\left\{[|x|^{\alpha_2}\,(1 + |x|)^{1-\alpha_2}]^{p-2} + [|y|^{\alpha_2}\,(1 + |y|)^{1-\alpha_2}]^{p-2}\right\} \times$$
$$\times\,(|x|\,|y| - (x, y)) \geqslant M_1|x-y|^2\,[(|x| + |y|)^{\alpha_2}\,(1 + |x| + |y|)^{1-\alpha_2}]^{p-2}\,,$$

where we have noted for all $s, t \geqslant 0$ that

$$[s^{\alpha_2}(1 + s)^{1-\alpha_2}]^{p-2} + [t^{\alpha_2}(1 + t)^{1-\alpha_2}]^{p-2} \geqslant$$
$$\geqslant M_2[(s + t)^{\alpha_2}\,(1 + s + t)^{1-\alpha_2}]^{p-2} \qquad (2\ 7)$$

Clearly (2 7) holds for $p \in (1, 2]$ with $M_2 = 1$ For $p \in (2, \infty)$, (2 7) follows from noting that there exist constants M_3 and $M_4(p)$ such that

$$(s + t)^{\alpha_2}\,(1 + s + t)^{1-\alpha_2} \leqslant M_3[s^{\alpha_2}(1 + s)^{1-\alpha_2} + t^{\alpha_2}\,(1 + t)^{1-\alpha_2}]$$

and

$$(s + t)^{p-2} \leqslant M_4[s^{p-2} + t^{p-2}]$$

In addition for all $x, y \in R^d$ and $\delta \geqslant 0$ we have that

$$|x-y|^2\,[(|x| + |y|)^{\alpha_2}\,(1 + |x| + |y|)^{1-\alpha_2}]^{p-2} \geqslant$$
$$\geqslant |x-y|^{2+\delta}\,[(|x| + |y|)^{\alpha_2}\,(1 + |x| + |y|)^{1-\alpha_2}]^{p-2-\delta}$$

and hence (2 5) holds for all $\delta \geqslant 0$ □

Under similar assumptions Chow in [6] has proved (2.4) with $\delta = 2 - p$ for $p \in (1, 2]$, $\alpha_1 = 1$ and $\delta = 0$ for $p \in [2, \infty)$, $\alpha_1 = 0$; (2.5) with $\delta = 0$ for $p \in (1, 2]$, $\alpha_2 = 0$ and $\delta = p - 2$ for $p \in [2, \infty)$, $\alpha_2 = 1$. With these one can establish some error bounds for the finite element approximation of (1.1) (see [6], [7] and [12]). These error bounds, however, are only suboptimal in many important cases. In [2] and [16], sharper inequalities, which can also be viewed as generalizations of those in [6] are established; that is, (2.4) with $\alpha_1 = 1$ and (2.5) with $\alpha_2 = 0$ when $p \in (1, 2]$, (2.4) with $\alpha_1 = 0$ and (2.5) with $\alpha_2 = 1$ when $p \in [2, \infty)$. It is these generalizations that makes the establishment of some optimal error bounds possible for (1.1) by exploiting the associated minimization problem. However, for some k the inequalities (2.4) and (2.5) are even sharper than those in [16]. For example $k(t) = t^{p-2}$, $p \in (1, \infty)$, satisfies the Assumptions (A) with $\alpha_1 = \alpha_2 = 1$. These improved inequalities are absolutely essential in establishing sharp error bounds for the finite element approximation of some degenerate quasilinear problems for which there is no associated minimization problem (for example, the parabolic p-Laplacian, see [4]).

Remark 2.2 : Lemma 2.1 can be generalised so that the inequalities (2.4) and (2.5) remain true if $x \equiv (x_{ij})$ and $y \equiv (y_{ij})$ are $d \times d$ matrices with $(x, y) \equiv \sum_{i,j=1}^{d} x_{i,j}\, y_{i,j}$ and $|x|^2 \equiv (x, x)$. Therefore these crucial inequalities can be applied to systems, such as those studied in [16].

THEOREM 2.1 : *Let k satisfy Assumptions (A). Let $u \equiv u_i \in K$ be the unique solution of (WP) for a given $f_i \in L^2(\Omega)$. Then for $p \in (1, 2]$*

$$\|u_1 - u_2\|_{W^{1,p}(\Omega)} \leq$$
$$\leq C\, \|f_1 - f_2\|_{L^2(\Omega)} [1 + \|u_1\|_{W^{1,p}(\Omega)} + \|u_2\|_{W^{1,p}(\Omega)}]^{2-p}, \quad (2.8)$$

and for $p \in [2, \infty)$

$$(\|u_1 - u_2\|_{W^{1,p}(\Omega)})^{p-1} \leq C\, \|f_1 - f_2\|_{L^2(\Omega)} . \quad (2.9)$$

Proof : The results (2.8) and (2.9) follow by adapting the proofs for the equation case in [6]. □

3. FINITE ELEMENT APPROXIMATION AND ABSTRACT ERROR BOUNDS

In this section we consider the piecewise linear element approximation of $(WP) \equiv (MP)$. Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$ in the case $d = 2$ and $\Omega^h \equiv \Omega$ if $d = 1$. Let T^h be a partitioning of

Ω^h into disjoint open regular d-simplices τ, each of maximum diameter bounded above h, so that $\bar{\Omega}^h \equiv \bigcup_{\tau \in T^h} \bar{\tau}$. We assume in the case $d = 2$ that $P_i \in \partial\Omega^h \Rightarrow P_i \in \partial\Omega$ and dist $(\partial\Omega^h, \partial\Omega) \leq Ch^2$, where P_i, $i = 1 \to J$, is the vertex set associated with the partitioning T^h. We will further assume that $\Omega^h \subseteq \Omega$.

Associated with T^h are the finite dimensional spaces

$$S^h \equiv \{\chi \in C^0(\bar{\Omega}^h) : \chi|_\tau \text{ is linear for all } \tau \in T^h\}$$

and

$$S_0^h \equiv \{\chi \in C^0(\bar{\Omega}) : \chi|_{\bar{\Omega}^h} \in S^h \text{ and } \chi|_{\bar{\Omega}\setminus\Omega^h} = 0\} \subset W_0^{1,\infty}(\Omega^h).$$

Let $\pi_h : C^0(\bar{\Omega}^h) \to S^h$, denote the interpolation operator such that for any $v \in C^0(\bar{\Omega}^h)$, $\pi_h v(P_i) = v(P_i)$ $i = 1 \to J$. We recall the following standard approximation results :

For $m = 0$ or 1, q, $s \in [1, \infty]$ and $v \in W^{2,s}(\tau)$ (so that $v \in C^0(\bar{\tau})$) we have that

$$|v - \pi_h v|_{W^{m,q}(\tau)} \leq C h^{d(1/q - 1/s)} h^{2-m} |v|_{W^{2,s}(\tau)} \quad \forall \tau \in T^h, \tag{3.1a}$$

provided $W^{2,s}(\tau) \subset W^{m,q}(\tau)$. Furthermore if $q > d$ then

$$|v - \pi_h v|_{W^{m,q}(\tau)} \leq C h^{1-m} |v|_{W^{1,q}(\tau)} \quad \forall \tau \in T^h. \tag{3.1b}$$

Let K^h be a closed convex set in S_0^h such that $\pi_h(K \cap C^0(\bar{\Omega})) \subset K^h$ and that $v^h \in K^h \Rightarrow v^h \in K$. We note that for the analysis that follows these assumptions can be relaxed (see [9]).

A possible finite element approximation of (MP) is then :
$(MP)^h$: Find $u^h \in K^h$ such that

$$J_{\Omega^h}(u^h) = \operatorname{Min}_{v^h \in K^h} J_{\Omega^h}(v^h).$$

Equivalently, one can define u^h as the unique solution of the following variational inequality :

$(WP)^h$: Find $u^h \in K^h$ such that

$$\int_{\Omega^h} k(|\nabla u^h|)(\nabla u^h, \nabla(v^h - u^h))\, dx \geq \int_{\Omega^h} f(v^h - u^h)\, dx \ \forall v^h \in K^h.$$

The well-posedness of $(WP)^h$ and $(MP)^h$ follows in an analogous way to that of (WP) and (MP).

We now establish some error bounds for such an approximation. Although no error bounds exist in the literature for degenerate elliptic variational inequalities by combining the work on « linear » variational inequalities, i e [10], with the work on degenerate elliptic equations in [6] it is a simple matter to obtain the following error bounds if $u \in W^{2,p}(\Omega)$

If k satisfies the Assumptions (A) with $p \in (1, 2]$, $\alpha_1 = 1$ and $\alpha_2 = 0$ then

$$\|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C h^{p/2}$$

If k satisfies the Assumptions (A) with $p \in [2, \infty)$, $\alpha_1 = 0$ and $\alpha_2 = 1$ then

$$\|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C h^{2/p}$$

Clearly these error bounds are not optimal Below we will establish some improved error bounds by applying the ideas used in [2] and [16] These bounds converge at the optimal rate, provided u is sufficiently smooth, in some notable cases Firstly, we establish some abstract error bounds

Let u be the unique solution of (WP) Then for any $\delta > -2$ and $\alpha \in [0, 1]$ we define for any $v \in W^{1,p}(\Omega^h)$

$$\|v\|^{\rho}_{(p,\delta,\alpha)} \equiv$$

$$\equiv \int_{\Omega^h} |\nabla v|^{2+\delta} \left[(|\nabla u| + |\nabla v|)^{\alpha} (1 + |\nabla u| + |\nabla v|)^{(1-\alpha)}\right]^{(p-2-\delta)} dx ,$$

where $\rho = \max\{p, 2 + \delta\}$ This is well defined for all $v \in W^{1,p}(\Omega^h)$, $p \in (1, \infty)$, since

$$\|v\|^{\rho}_{(p,\delta,\alpha)} \leq \int_{\Omega^h} (1 + |\nabla v| + |\nabla u|)^{\rho} dx$$

It is easy to establish that $\|\,.\,\|_{(p,\delta,\alpha)}$ is a quasi-norm on $W^{1,p}_0(\Omega^h)$, that is, it satisfies all the properties of a norm except the homogeneity property In addition we note for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ that

$$\|v\|^{2+\delta}_{(p,\delta,\alpha_1)} \leq \|v\|^{2+\delta}_{(p,\delta,\alpha_2)} \quad \text{if} \quad p \leq 2 + \delta \qquad (3.2a)$$

and

$$\|v\|^{p}_{(p,\delta,\alpha_2)} \leq \|v\|^{p}_{(p,\delta,\alpha_1)} \quad \text{if} \quad p \geq 2 + \delta \qquad (3.2b)$$

We can now state our abstract error bounds in the following way

THEOREM 3 1 *Let k satisfy Assumptions (A) Let u and u^h be the unique solutions of (WP) and $(WP)^h$ We assume in addition that $u \in L^{\infty}(\Omega)$ and*

$\nabla \cdot (k(|\nabla u|)\nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$. *Then for any* $\delta_1 \in (-2, 0]$, $\delta_2 \geq 0$ *and* $v^h \in K^h$ *we have that*

$$\|u - u^h\|^{\rho_2}_{(p, \delta_2, \alpha_2)} \leq C\left(\|u - v^h\|^{\rho_1}_{(p, \delta_1, \alpha_1)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2\right), \quad (3.3)$$

where $\rho_i = \max\{p, 2 + \delta_i\}$.

Proof · The proof is similar to that of Theorem 2.1 in [2] and Theorem 4.1 in [16]. For any $v^h \in K^h$

$$J_{\Omega^h}(v^h) - J_{\Omega^h}(u) = \int_0^1 J'_{\Omega^h}(u + t(v^h - u))(v^h - u)\,dt =$$

$$\int_0^1 [J'_{\Omega^h}(u + t(v^h - u))(u + t(v^h - u) - u) -$$

$$- J'_{\Omega^h}(u)(u + t(v^h - u) - u)]\, t^{-1}\,dt + J'_{\Omega^h}(u)(v^h - u)$$

$$= A(v^h) + J'_{\Omega^h}(u)(v^h - u),$$

where

$$A(v^h) \equiv \int_0^1 \int_{\Omega^h} (k(|\nabla(u + t(v^h - u))|)\nabla(u + t(v^h - u)) -$$

$$- k(|\nabla u|)\nabla u, \nabla(v^h - u))\,dx\,dt .$$

From $(MP)^h$ we have that

$$A(u^h) + J'_{\Omega^h}(u)(u^h - u) = J_{\Omega^h}(u^h) - J_{\Omega^h}(u) \leq$$

$$\leq J_{\Omega^h}(v^h) - J_{\Omega^h}(u) = A(v^h) + J'_{\Omega^h}(u)(v^h - u) .$$

It follows, since $\Omega^h \subseteq \Omega$ and $(WP) \Rightarrow J'_\Omega(u)(u - u^h) \leq 0$, that

$$A(u^h) \leq A(v^h) + J'_{\Omega^h}(u)(v^h - u^h) = A(v^h) + J'_\Omega(u)(v^h - u^h)$$

$$= A(v^h) + J'_\Omega(u)(v^h - u) + J'_\Omega(u)(u - u^h) \leq A(v^h) + J'_\Omega(u)(v^h - u) .$$

On the other hand, $\nabla \cdot (k(|\nabla u|)\nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$, $u \in L^\infty(\Omega)$, $\Omega^h \subseteq \Omega$ and $\mathrm{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$ and so we have that

$$A(u^h) \leq A(v^h) + C\left(\|u - v^h\|_{L^2(\Omega^h)} + h^2\right) \quad \forall v^h \in K^h . \quad (3.4)$$

Noting for all v_1, v_2 and $t \in [0, 1]$ the inequality

$$t(|\nabla v_1| + |\nabla v_2|)/2 \leq (|\nabla(v_1)| + |\nabla(v_1 + tv_2)|) \leq 2(|\nabla v_1| + |\nabla v_2|),$$

it follows from (2.4) that for all $\delta_1 \in (-2, 0]$

$$A(v^h) \leqslant C \int_0^1 \int_{\Omega^h} t^{1+\delta_1} |\nabla(u - v^h)|^{2+\delta_1} \times$$
$$\times [(|\nabla u| + |\nabla u + t\,\nabla(v^h - u)|)^{\alpha_1}$$
$$\times (1 + |\nabla u| + |\nabla u + t\,\nabla(v^h - u)|)^{1-\alpha_1}]^{(p-2-\delta_1)}\, dx\, dt \leqslant$$
$$\leqslant C\, \|u - v^h\|^{\rho_1}_{(p,\,\delta_1,\,\alpha_1)}\,. \tag{3.5}$$

Similarly it follows from (2.5) that for all $\delta_2 \geqslant 0$

$$A(u^h) \geqslant M\, \|u - u^h\|^{\rho_2}_{(p,\,\delta_2,\,\alpha_2)}\,. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) yields the desired result (3.3). □

COROLLARY 3.1a : *Under the assumptions of Theorem* 3.1 *we have with* $\rho_i = \max\{p, 2 + \delta_i\}$ *that* .

If $p \in (1, 2]$ *then for all* $\delta_1 \in (-2, 0]$ *and* $v^h \in K^h$

$$|u - u^h|^2_{W^{1,p}(\Omega^h)} \leqslant C\,(\|u - v^h\|^{\rho_1}_{(p,\,\delta_1,\,\alpha_1)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2)\,. \tag{3.7}$$

If $p \in [2, \infty)$ *then for all* $\delta_2 \geqslant 0$ *and* $v^h \in K^h$

$$\|u - u^h\|^{\rho_2}_{(p,\,\delta_2,\,\alpha_2)} \leqslant$$
$$\leqslant C\,(\|v^h\|_{W^{1,p}(\Omega^h)})(\|u - v^h\|^2_{W^{1,p}(\Omega^h)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2)\,; \tag{3.8}$$

and if in addition $u \in W^{1,\infty}(\Omega)$ *and* $\|v^h\|_{W^{1,\infty}(\Omega)} \leqslant C$, *then for any* $s \in [1, 2]$

$$\|u - u^h\|^{\rho_2}_{(p,\,\delta_2,\,\alpha_2)} \leqslant C\,(\|u - v^h\|^s_{W^{1,s}(\Omega^h)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2)\,. \tag{3.9}$$

Proof If $p \in (1, 2]$ then from a Holder inequality and (3.2*a*) we have that

$$|u - u^h|^2_{W^{1,p}(\Omega^h)} \leqslant$$
$$\leqslant \left\{\int_{\Omega^h} [1 + |\nabla u| + |\nabla(u - u^h)|]^p\, dx\right\}^{(2-p)/p} \|u - u^h\|^2_{(p,\,0,\,0)}$$
$$\leqslant C\,\|u - u^h\|^2_{(p\,0,\,0)} \leqslant C\,\|u - u^h\|^2_{(p,\,0,\,\alpha_2)}\,.$$

Hence the desired result (3.7) follows from (3.3) with $\delta_2 = 0$.

Similarly, if $p \in [2, \infty)$ we have from a Holder inequality and (3.2b) that

$$\|u - v^h\|^p_{(p, 0, \alpha_1)} \leq \|u - v^h\|^p_{(p, 0, 0)}$$

$$\leq \left\{ \int_{\Omega^h} [1 + |\nabla u| + |\nabla (u - v^h)|]^p \, dx \right\}^{(p-2)/p} \|u - v^h\|^2_{W^{1,p}(\Omega^h)}$$

$$\leq C(\|v^h\|_{W^{1,p}(\Omega^h)}) \|u - v^h\|^2_{W^{1,p}(\Omega^h)} .$$

Hence the desired result (3.8) follows from (3.3) with $\delta_1 = 0$.

Finally choosing $\delta_1 = s - 2$ in (3.3) and noting (3.2b) yields that

$$\|u - v^h\|^p_{(p, s-2, \alpha_1)} \leq$$

$$\leq \|u - v^h\|^p_{(p, s-2, 0)} \leq C(\|v^h\|_{W^{1,\infty}(\Omega^h)}) \|u - v^h\|^s_{W^{1,s}(\Omega^h)}$$

and hence the desired result (3.9). □

COROLLARY 3.1b : *Under the assumptions of Theorem* 3.1 *we have*

If $p \in (1, 2]$ *and* $u \in W^{1, 2 + \alpha_1(p-2)}(\Omega)$ *then for all* $v^h \in K^h$

$$|u - u^h|^2_{W^{1,p}(\Omega^h)} \leq$$

$$\leq C([\|u - v^h\|_{W^{1, 2+\alpha_1(p-2)}(\Omega^h)}]^{2 + \alpha_1(p-2)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2) . \tag{3.10}$$

If $p \in [2, \infty)$ *then for all* $v^h \in K^h$

$$[|u - u^h|_{W^{1, 2+\alpha_2(p-2)}(\Omega^h)}]^{2 + \alpha_2(p-2)} \leq$$

$$\leq C(\|v^h\|_{W^{1,p}(\Omega^h)})(\|u - v^h\|^2_{W^{1,p}(\Omega^h)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2) ; \tag{3.11}$$

and if in addition $u \in W^{1,\infty}(\Omega)$ *and* $\|v^h\|_{W^{1,\infty}(\Omega)} \leq C$, *then for any* $s \in [1, 2]$

$$[|u - u^h|_{W^{1, 2+\alpha_2(p-2)}(\Omega^h)}]^{2 + \alpha_2(p-2)} \leq$$

$$\leq C(\|u - v^h\|^s_{W^{1,s}(\Omega^h)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2) . \tag{3.12}$$

Proof · The result (3.10) follows directly from (3.7) by noting that $\|v\|^2_{(p, 0, \alpha_1)} \leq [|v|_{W^{1, 2+\alpha_1(p-2)}(\Omega^h)}]^{2 + \alpha_1(p-2)}$ for all $v \in W^{1, 2+\alpha_1(p-2)}(\Omega^h)$ and

$\alpha_1 \in [0, 1]$. The results (3.11) and (3.12) follow directly from (3.8) and (3.9) by noting that $\|v\|^p_{(p, 0, \alpha_2)} \geqslant [|v|_{W^{1, 2+\alpha_2(p-2)}}(\Omega^h)]^{2+\alpha_2(p-2)}$ for all $v \in W^{1, 2+\alpha_2(p-2)}(\Omega^h)$ and $\alpha_2 \in [0, 1]$. □

If $\alpha_1 = \alpha_2 = \alpha$, which is usually the case, then one has from (3.3) on choosing $\delta_1 = \delta_2 = 0$ that for any $v^h \in K^h$

$$\|u - u^h\|^\rho_{(p, 0, \alpha)} \leqslant C\left(\|u - v^h\|^\rho_{(p, 0, \alpha)} + \|u - v^h\|_{L^2(\Omega^h)} + h^2\right), \quad (3.13)$$

where $\rho = \max\{p, 2\}$. Hence u^h converges to u in the quasi-norm $\|\cdot\|_{(p, 0, \alpha)}$ as $h \to 0$ at the optimal rate; that is the same rate as the approximation error $\inf_{v^h} \|u - v^h\|_{(p, 0, \alpha)}$; since in general $\inf_{v^h} \|u - v^h\|^\rho_{(p, 0, \alpha)}$ is the dominant term on the right hand side of (3.13) as $h \to 0$. Therefore one needs to determine at what rate $\inf_{v^h} \|u - v^h\|_{(p, 0, \alpha)} \to 0$ as $h \to 0$ and the relations between these quasi-norms and standard Sobolev norms. These questions are addressed in the next section.

Finally, all the above results in this paper simplify slightly when applied to the equation (1.1) with $g = 0$, as opposed to the variational inequality. We have the following simplification of Theorem 3.1.

THEOREM 3.2 : *Let k satisfy Assumptions (A). Let u and u^h be the unique solutions of (WP) with $K \equiv W_0^{1,p}(\Omega)$ and $(WP)^h$ with $K^h \equiv S_0^h$. Then for any $\delta_1 \in (-2, 0]$, $\delta_2 \geqslant 0$ and $v^h \in S_0^h$ we have with $\rho_i = \max\{p, 2 + \delta_i\}$ that*

$$\|u - u^h\|^{\rho_2}_{(p, \delta_2, \alpha_2)} \leqslant C \|u - v^h\|^{\rho_1}_{(p, \delta_1, \alpha_1)}. \quad (3.14)$$

Proof: The proof follows that of Theorem 3.1 except that $J'_\Omega(u)(v^h - u^h) = 0$ and so in place of (3.4) we have that $A(u^h) \leqslant A(v^h)$ $\forall v^h \in S_0^h$. The desired result (3.14) then follows. □

COROLLARY 3.2 : *Under the Assumptions of Theorem 3.2 we have that the results (3.7)-(3.13) with « $\|u - v^h\|_{L^2(\Omega^h)} + h^2$ » removed from their right hand sides.*

Proof: The results follow immediately from the proofs of (3.7)-(3.13) with (3.3) replaced by (3.14). □

4. EXPLICIT ERROR BOUNDS

In this section we apply our abstract quasi-norm error bounds established in the last section to some problems and obtain more explicit results in familiar norms.

THEOREM 4.1 : *Let k satisfy Assumptions (A) with $p \in (1, 2]$. Let u and u^h be the unique solutions of (WP) and $(WP)^h$. If $\nabla . (k(|\nabla u|) \nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$ and $u \in W^{2, 2+\alpha_1(p-2)}(\Omega)$, then*

$$\|u - u^h\|_{W^{1,p}(\Omega^h)} \leqslant C h^{(2+\alpha_1(p-2))/2} . \tag{4.1}$$

Furthermore if $u \in W^{2,\nu/(\nu-1)}(\Omega)$, where $\nu = 2 + \alpha_1(p-2) \in [p, 2]$, and if there exists an open set $D \subset \Omega$ with a Lipschitz boundary Γ (or whose number of elements is finite in the case $d = 1$) such that $u \in C^{2, \alpha_1(2-p)/\nu}(\bar{D}) \cap W^{3,1}(D)$ and $u \in C^{2, \alpha_1(2-p)/\nu}(\bar{\Omega} \backslash D) \cap W^{3,1}(\Omega \backslash D)$, then

$$\|u - u^h\|_{W^{1,p}(\Omega^h)} \leqslant C h . \tag{4.2}$$

Proof. Firstly, we note for all $r \in [1, \infty)$, $w \in W^{1,r}(\Omega)$ and w_1^h, $w_2^h \in S_0^h$ that

$$\|w - w_1^h\|_{W^{1,r}(\Omega^h)} \leqslant C [|w - w_1^h|_{W^{1,r}(\Omega^h)} + \|w - w_2^h\|_{W^{1,r}(\Omega^h)}] . \tag{4.3}$$

The error bound (4.1) then follows directly from (3.10) by choosing $v^h = \pi_h u$, applying (4.3) with $r = p$, $w \equiv u$, $w_1^h \equiv u^h$, $w_2^h \equiv \pi_h u$ and the interpolation result (3.1a).

We now prove (4.2). Let $T_\Gamma^h \equiv \{\tau \in T^h : \tau \cap \Gamma \neq \varnothing\}$ and we set $\bar{G}^h \equiv \bigcup_{\tau \in T_\Gamma^h} \bar{\tau}$. From our assumptions it follows that mes $(G^h) \leqslant C h$. We now consider $\bar{\Omega}^h \equiv \bar{\Omega}_1^h \cup \bar{\Omega}_2^h \cup \bar{\Omega}_3^h$, where $\Omega_1^h \subseteq D$, $\Omega_2^h \subseteq \Omega \backslash \bar{D}$ and $\Omega_3^h \equiv G^h$. From (3.7) with $\delta_1 = 0$ and $v^h = \pi_h u$, (4.3), (3.1a) and noting that $\nu/(\nu - 1) \geqslant 2$ we have that

$$\begin{aligned}
&\|u - u^h\|_{W^{1,p}(\Omega^h)}^2 \leqslant \\
&\leqslant C h^2 + C \sum_{i=1}^3 \int_{\Omega_i^h} |\nabla(u - \pi_h u)|^2 \, [(|\nabla u| + |\nabla(u - \pi_h u)|)^{\alpha_1} \times \\
&\quad \times (1 + |\nabla u| + |\nabla(u - \pi_h u)|)^{(1-\alpha_1)}]^{(p-2)} \, dx \leqslant \\
&\leqslant C h^2 + C \sum_{i=1}^3 \int_{\Omega_i^h} |\nabla(u - \pi_h u)|^2 \, (|\nabla u| + |\nabla(u - \pi_h u)|)^{\alpha_1(p-2)} \, dx .
\end{aligned} \tag{4.4}$$

Let $H[u]$ be the Euclidean norm of the Hessian matrix of u. Then for any $\tau \subseteq \Omega_i^h$, $i = 1$ or 2, we have for all $x \in \bar{\tau}$ that

$$|\nabla(u - \pi_h u)(x)| \leq Ch \sup_{y \in \tau} H[u](y)$$
$$\leq Ch\, H[u](x) + Ch \sup_{y \in \tau} |H[u](x) - H[u](y)|$$
$$\leq Ch\, H[u](x) + Ch^{2/\nu} \|u\|_{C^{2,\alpha_1(2-p)/\nu}(\bar{\Omega}_i^h)} .$$

In addition it is easy to check that the function $\Psi(t) \equiv t^2(a+t)^\beta$ with $a \geq 0$ and $\beta \in (-1, 0]$ is increasing on R^+ and $\Psi(|t_1 + t_2|) \leq 2[\Psi(|t_1|) + \Psi(|t_2|)]$ for all $t_1, t_2 \in R$. Therefore we have for $i = 1$ and 2 that

$$\int_{\Omega_i^h} |\nabla(u - \pi_h u)|^2 (|\nabla u| + |\nabla(u - \pi_h u)|)^{\alpha_1(p-2)} dx \leq$$
$$\leq Ch^2 \int_{\Omega_i^h} |H[u]|^2 |\nabla u|^{\alpha_1(p-2)} dx + C(\|u\|_{C^{2,\alpha_1(2-p)/\nu}(\Omega_i^h)})\, h^2 . \tag{4.5}$$

Applying Green's formula it is easy to deduce for all $w \in W^{2,1}(\Omega)$ and $\beta \in (-1, 0)$ that

$$\int_\Omega |w|^\beta |\nabla w|^2 dx \leq C [\|w\|_{C^0(\bar{\Omega})}]^{\beta+1} \|w\|_{W^{2,1}(\Omega)},$$

see Lemma 3.1 in [2]. Hence it follows that

$$\int_{\Omega_i^h} |H[u]|^2 |\nabla u|^{\alpha_1(p-2)} dx \leq C [\|u\|_{C^1(\Omega_i^h)}]^{\alpha_1(p-2)+1} \|u\|_{W^{3,1}(\Omega_i^h)} . \tag{4.6}$$

Next we note from (3.1a) that

$$\int_{\Omega_3^h} |\nabla(u - \pi_h u)|^2 (|\nabla u| + |\nabla(u - \pi_h u)|)^{\alpha_1(p-2)} dx \leq$$
$$\leq \int_{\Omega_3^h} |\nabla(u - \pi_h u)|^\nu dx$$
$$\leq C [\text{mes}(\Omega_3^h)]^{2-\nu} [|\nabla(u - \pi_h u)|_{W^{1,\nu/(\nu-1)}(\Omega_3^h)}]^\nu \leq Ch^2 [\|u\|_{W^{2,\nu/(\nu-1)}(\Omega_3^h)}] .$$

Combining the above with (4.4)-(4.6) and noting the assumed regularity on u yields the desired result (4.2). □

We have the following simplification of (4.2) in the equation case

COROLLARY 4.1 : *Let k satisfy Assumptions (A) with $p \in (1, 2]$. Let u and u^h be the unique solutions of (WP) with $K \equiv W_0^{1,p}(\Omega)$ and $(WP)^h$ with*

$K^h \equiv S_0^h$ *If* $u \in W^{2\ 2+\alpha_1(p-2)}(\Omega)$ *then* (4 1) *holds Moreover if* $u \in C^{2\ \alpha_1(2-p)/\nu}(\bar{\Omega}) \cap W^{3\ 1}(\Omega)$, *where* $\nu = 2 + \alpha_1(p-2) \in [p, 2]$, *then* (4 2) *holds*

Proof The proof follows directly from the proof of Theorem 4 1 with $\Omega_1^h \equiv \Omega^h$, $\Omega_2^h \equiv \Omega_3^h \equiv \emptyset$ and noting Corollary 3 2 □

Remark 4 1 We note the following points concerning Theorem 4 1 and Corollary 4 1

(i) If $t \to k(t)\,t$ is globally Lipschitz for $t \geq 0$, then we can set $\alpha_1 = 0$ in Assumptions (A) and so from (4 1) u^h converges to u at the optimal rate in $W^{1\ p}(\Omega)$ if $u \in H^2(\Omega)$ This regularity requirement on u is certainly achievable in the equation case for a wide class of data, see [13] For example $k(t) = (1 + t^2)^{(p-2)/2}$, $p \in (1, 2]$, satisfies Assumptions (A) with $\alpha_1 = \alpha_2 = 0$

(ii) The regularity requirements on u do not seem minimal In many cases it can be further weakened For example for a quasi-uniform partition T^h and under some additional assumptions on f, one can weaken the regularity requirement for (4 2) to hold in the equation case to $u \in W^{1+2/p\ p}(\Omega)$, which seems the weakest possible, see [14] Similarly, one can weaken the regularity requirement for (4 2) to hold for the variational inequality to $u \in W^{2\ \nu/(\nu-1)}(\Omega) \cap W^{1+2/p\ p}(D) \cap W^{1+2/p\ p}(\Omega \backslash D)$

(iii) For the variational inequality it is not realistic to assume that $u \in C^{2\ \alpha_1(2-p)/\nu}(\bar{\Omega}) \cap W^{3\ 1}(\Omega)$ or $u \in W^{1+2/p\ p}(\Omega)$ as in the case of the equation, see [14] and [15] where conditions on the data are given for this to be achieved A more realistic assumption for global regularity is that $u \in W^{2\ q}(\Omega)$ for any $q \geq 2$, although we have not yet proved this In general one can only expect u to have higher regularity either side of the free boundary Hence the introduction of Γ in Theorem 4 1

We now apply Theorem 4 1 to the following obstacle problem Let k satisfy Assumptions (A) with $p \in (1, 2]$ let $K \equiv \{v \in W_0^{1\ p}(\Omega) \quad v \geq \varphi\}$, where we assume that the obstacle $\varphi \in W_0^{1\ p}(\Omega) \cap C^0(\bar{\Omega})$ and is convex It then follows that $K^h \equiv \{v^h \in S_0^h \quad v^h \geq \pi_h \varphi\}$ is well-defined and that $\pi_h[K \cap C^0(\bar{\Omega})] \subset K^h \subset K$ (We note that if φ is not convex, then K^h is not a subset of K However, one can adapt the methods used in [10] to obtain similar results to the above) In addition for u to achieve the required regularity assumptions for (4 1) and (4 2) to hold it is necessary for the obstacle φ to satisfy these conditions, since $u \equiv \varphi$ in the contact set For example in the case of (4 2) we require that $\varphi \in C^{2\ \alpha_1(2-p)/\nu}(\bar{\Omega}) \cap W^{3\ 1}(\Omega)$ Furthermore, we would choose $D \equiv \{x \in \Omega \quad u(x) > \varphi(x)\}$ so that Γ is the free boundary of the contact set Then provided u satisfies the regularity requirements and the free boundary is regular one can apply Theorem 4 1 Unfortunately, such results are not available in the literature at present

Let us now consider the case $p \in [2, \infty)$ We have the following result

THEOREM 4 2 *Let k satisfy Assumptions (A) with* $p \in [2, \infty)$ *Let u and* u^h *be the unique solutions of (WP) and* $(WP)^h$ *If* $\nabla . (k(|\nabla u|)\nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$ *and* $u \in W^{1\ \infty}(\Omega) \cap W^{2\ s}(\Omega)$, $s \in [1, 2]$, then

$$\|u - u^h\|_{W^{1\ 2+\alpha_2(p\ \ 2)}(\Omega^h)} \leq C h^{s/[2+\alpha_2(p\ \ 2)]} \qquad (4\ 7)$$

Proof The result (4 7) with the seminorm on the left-hand side follows from (3 12) with $v^h \equiv \pi_h u$ and (3 1) by noting that $2 + d(2^{-1} - s^{-1}) \geq s$ The result for the norm follows from (4 3) with $r = 2 + \alpha_2(p - 2)$, $w \equiv u$, $w_1^h \equiv u^h$, $w_2^h \equiv \pi_h u$ and noting for $u \in W^{1\ \infty}(\Omega) \cap W^{2\ s}(\Omega)$, $s \in [1, 2]$, and $r \geq 2$ that

$$\|u - \pi_h u\|_{W^{1\ r}(\Omega')} \leq C\,[\|u - \pi_h u\|_{W^{1\ s}(\Omega^h)}]^{s/r} \qquad (4\ 8)$$

□

If the assumptions of Theorem 4 2 hold with $\alpha_2 = 0$ and $s = 2$ then (4 7) yields an optimal $H^1(\Omega^h)$ error bound For example $k(t) = (1 + t^2)^{(p\ \ 2)/2}$ for $p \in [2, \infty)$ satisfies $(A2)$ with $\alpha_2 = 0$ However, for degenerate problems $\alpha_2 \in (0, 1]$ and then the error bound (4 7) degenerates as $p \to \infty$

THEOREM 4 3 *Let k satisfy Assumptions (A) with* $p \in [2, \infty)$ *Let u and* u^h *be the unique solutions of (WP) and* $(WP)^h$ *Let* $\nabla . (k(|\nabla u|)\nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$ *and* $u \in W^{1\ \infty}(\Omega) \cap W^{2\ s}(\Omega)$, $s \in [1, 2]$ *It follows that for any* $q \in [2, p]$ *and* $r \in [1, q)$ *that if* $\int_D |\nabla u|^{\alpha_2 r(p\ \ q)/(q\ \ r)}\,dx < \infty$, *where* $D \subseteq \Omega^h$, *then*

$$|u - u^h|_{W^{1\ r}(D)} \leq C h^{s/q} \qquad (4\ 9)$$

Proof It follows from (3 9) with $v^h \equiv \pi_h u$, (3 1) by noting that $2 + d(2^{-1} - s^{-1}) \geq s$ and Holder's inequality that

$$[|u - u^h|_{W^{1\ r}(D)}]^q \leq$$

$$\leq \int_D |\nabla(u - u^h)|^q\,|\nabla u|^{\alpha_2(p-q)}\,dx \times \left(\int_D |\nabla u|^{-\alpha_2 r(p\ \ q)/(q-r)}\,dx\right)^{(q-r)/r} \leq$$

$$\leq C\,\|u - u^h\|^2_{(p\ q\ \ 2\ \alpha_2)} \leq C h^s \qquad (4\ 10)$$

Hence the desired result (4 9) □

We now apply Theorem 4 3 to the obstacle problem discussed above

THEOREM 4.4 : *Let k satisfy Assumptions (A) for $p \in (2, \infty)$ and $\alpha_1 = \alpha_2 = \alpha \in (0, 1]$, and be differentiable on $(0, \infty)$. Let $\varphi \in W_0^{1,p}(\Omega)$ be convex, $K \equiv \{v \in W_0^{1,p}(\Omega) : v \geq \varphi\}$ and $K^h \equiv \{v^h \in S_0^h : v^h \geq \pi_h \varphi\}$. Let u and u^h be the unique solutions of (WP) and $(WP)^h$. We further assume that $\nabla . (k(|\nabla u|) \nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$, φ, $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, 2]$, and $|f|^{-\gamma} \in L^1(D)$, where $\gamma > 0$ and $D \equiv \{x \in \Omega^h : u(x) > \varphi(x)\}$.*

Then for any $r \in [1, p)$, we have that

$$|u - u^h|_{W^{1,r}(D)} \leq$$
$$\leq C h^{\min \{s/2,\, s[r(\gamma + s) + (p-2)\gamma s]/r[p(\gamma + s) + (p-2)\gamma s]\}} . \quad (4.11)$$

Consequently, we have that

$$|u - u^h|_{W^{1,1}(D)} \leq C h^{\min \{s/2,\, s/[1 + \gamma^{-1} + s^{-1}]\}} \quad (4.12)$$

and hence if $s = 2$ and $\gamma = 2$ that

$$|u - u^h|_{W^{1,1}(D)} \leq C h . \quad (4.13)$$

Proof : Setting $\underline{w} = \nabla u$, we have that $|\underline{w}| \in W^{1,s}(\Omega)$ since $\nabla |\underline{w}| \equiv \left(\sum_{i=1}^{d} w_i . \nabla w_i \right) / |\underline{w}|$. As $-\nabla . (k(|\nabla u|) \nabla u) = f$ a.e. in D, it follows that

$$k(|\underline{w}|) \nabla . \underline{w} + k'(|\underline{w}|)|\underline{w}|(\underline{w} . \nabla |\underline{w}|)/|\underline{w}| = -f \quad \text{a.e. in } D . \quad (4.14)$$

It follows from Assumptions (A) and the differentiability of k on $(0, \infty)$ that there exist two continuous functions N_1 and N_2 on $[0, \infty)$ such that

$$k(t) \leq t^{\alpha(p-2)} N_1(t)$$

and

$$k'(t)\, t \equiv (k(t)\, t)' - k(t) \leq (k(t)\, t)' \leq$$
$$\leq t^{\alpha(p-2)} N_2(t) \quad \text{on} \quad (0, \infty) . \quad (4.15)$$

Combining (4.14) and (4.15) yields that there exists $\eta \in L^s(D)$ such that

$$|f| \leq \eta\, |\nabla u|^{\alpha(p-2)} \quad \text{a.e. in} \quad D . \quad (4.16)$$

As $\alpha > 0$ and $p > 2$ it follows from (4.16) that for $q \in [2, p)$ and $r \in [1, q)$

$$\int_D |\nabla u|^{-\alpha r(p-q)/(q-r)}\, dx \leq \int_D [\eta\, |f|^{-1}]^\sigma\, dx , \quad (4.17)$$

where $\sigma \equiv r(p-q)/[(p-2)(q-r)] > 0$. Hence if $\sigma < s$ on setting $\rho = s/\sigma \in (1, \infty)$ and $\gamma = \sigma\rho/(\rho - 1) = \sigma s/(s-\sigma)$ we have from a Holder inequality that

$$\int_D |\nabla u|^{-\alpha r(p-q)/(q-r)}\, dx \leqslant$$

$$\leqslant \left[\int_D \eta^s\, dx\right]^{\sigma/s} \left[\int_D |f|^{-\gamma}\, dx\right]^{(\rho-1)/\rho} \leqslant C\,. \quad (4.18)$$

Noting that $\gamma = \sigma s/(s-\sigma) \Rightarrow \sigma = \gamma s/(\gamma + s) > s \Rightarrow q = r\,[p(\gamma + s) + (p-2)\,\gamma s]/[r(\gamma + s) + (p-2)\,\gamma s]$. As $r \in [1, p)$ and $p > 2$ it follows that $q \in (r, p)$. However, we also require $q \geqslant 2$ for (4.17), (4.18) and hence (4.9) to hold. It follows for $r \in [2\,\gamma s/(\gamma + s + \gamma s), p)$ that $q = r[p(\gamma + s) + (p-2)\,\gamma s]/[r(\gamma + s) + (p-2)\,\gamma s] \in [2, p)$ and hence (4.9) holds. On the other hand for $r \in [1, 2\,\gamma s/(\gamma + s + \gamma s))$ we have that $q < 2$ and $|u - u^h|_{W^{1,r}(D)} \leqslant C\,|u - u^h|_{W^{1,2\gamma s/(\gamma + s + \gamma s)}(D)} \leqslant C h^{s/2}$. Hence the desired result (4.11) holds. Finally, the results (4.12) and (4.13) follow directly from (4.11). □

Remark 4.2 : Clearly, unlike (4.7), the error bounds (4.12) and (4.13) do not degenerate as $p \to \infty$. If $\bar{D} \cap \partial\Omega^h$ has positive measure then by applying (4.3) we can replace the semi-norm by the norm in (4.11)-(4.13).

Remark 4.3 : In an analogous manner to the proof of Corollary 4.1 the assumption that $\nabla\,.\,(k(|\nabla u|)\,\nabla u) + f \in L^2(\Omega) \cap L^\infty(\Omega \backslash \Omega^h)$ in Theorems 4.2-4.4 above can be dropped in the case of the equation, $K \equiv W_0^{1,p}(\Omega)$ and $K^h \equiv S_0^h$.

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