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STABILIZED FINITE ELEMENT METHODS FOR MISCIBLE DISPLACEMENT IN POROUS MEDIA (*)

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Abstract — In this paper, we shall derive a new model for the miscible displacement of one incompressible fluid by another in porous media using simple physical conservation laws. For a dilute mixture in which the density can be approximated by a constant, this new model reduces to the standard one used for the last decade. The model is governed by a nonlinear system consisting of pressure and concentration equations. The pressure equation is elliptic, while the concentration equation is parabolic but normally convection-dominated. We then present and analyze some extensions of the stabilized finite element methods that have been developed for steady convection-diffusion problems to the systems of miscible displacement. The analysis is first given to the concentration equation for a given velocity field, and then extended to the general case where the velocity is obtained by solving pressure equations with a mixed finite element method. In both cases, the stabilities and error estimates are given.

Résumé — Dans cet article, nous présentons un nouveau modèle pour le déplacement miscible d'un fluide incompressible par un autre dans les milieux poreux utilisant des lois simples physiques de conservation. Pour un mélange dilué dans lequel la densité peut être approchée par une constante, ce nouveau modèle se réduit à celui utilisé depuis ces dix dernières années. Le modèle est décrit par un système non linéaire composé des équations de la pression et de la concentration. L'équation de la pression est elliptique tandis que l'équation de la concentration est parabolique, mais normalement dominée par la convection. Nous présentons et analysons quelques extensions au système de déplacement miscible des méthodes d'éléments finis stabilisées qui ont été développées pour les problèmes de convection-diffusion stationnaires. On considère d'abord l'équation de la concentration pour un champ de vitesse donné puis le cas général où la vitesse est obtenue par la résolution de l'équation de la pression par une méthode d'éléments finis mixtes. Dans les deux cas, on donne les estimations de la stabilité et de l'erreur.

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1 INTRODUCTION

The numerical simulation of fluid displacement processes in porous media has been widely applied in analyzing petroleum recovery. Miscible displacement of one fluid by another is one such recovery process that has attracted considerable attention in the petroleum industry. It involves injection of a solvent at certain (injection) wells in a petroleum reservoir, with the intention of displacing the resident oil to other (production) wells ([15]).

The standard model for this process derived several years ago has been used in the series of papers written by Douglas, Ewing, Russell, Wheeler, and others [9], [10], [15], [16] and [32]. Recently Douglas *et al* [11] derived a new model using the homogenization theory after finding that the old model is correct only for the special dilute mixture in which density is approximately a constant. Here we shall rederive this new model based only on simple physical conservation laws, which may give us better interpretations for the Darcy's velocity and concentration. As we shall see later, the new model, which consists of one parabolic equation for the concentration and one elliptic equation for the pressure, coincides with the old model for a dilute mixture. The concentration equation is usually convection-dominated and difficult to solve. There have been many numerical methods developed to solve these types of equations.

It is known that for the convection-dominated problems, standard finite element and finite difference methods often exhibit nonphysical oscillation because they are designed for problems with smooth solutions, as occur when diffusion dominates convection. Upwind schemes are first used to stabilize convective flow, especially in reservoir simulation (see references in [25]). These methods suppress the oscillations by incorporating artificial diffusion and often reduce over- and undershooting effects. However, these methods may introduce numerical diffusion of the first order in the spatial grid size and can smear out sharp fronts and produce solutions that strongly depend on the orientation of the difference grid relative to the direction of the streamlines of the flow.

By noting the almost hyperbolic nature of these problems, Douglas *et al* ([13]) proposed and analyzed an approximation method for convective flow based on the characteristics of the hyperbolic part of the differential equation. This method was later applied successfully to the modeling of miscible displacement in porous media when combined with a variety of approximation techniques for the pressure equation, (see [7, 8, 14, 16, 17], and [32] for the formulation and analysis for some of these procedures).

Godunov schemes are often used in the numerical approximations of conservation laws. Recently, Dawson ([6]) used an operator-splitting technique to split the convection-diffusion problem into two parts, convective

and diffusive. He then applied a Godunov scheme to the convective part and a mixed method to the diffusive part, and obtained a maximum principle and L^2 estimates.

Since the convective part possesses hyperbolic features, it is natural to look at some successful schemes for hyperbolic conservation laws. Recently, several high resolution schemes have been invented for scalar conservation laws with nonlinear fluxes; examples include TVD (Total Variation Diminishing) [28], TVB (Total Variation Bounded) [33], and ENO (Essentially NonOscillatory) [21]. The schemes have been shown theoretically and numerically to converge to entropy solutions and are usually stable with respect to total variation; hence, they can capture sharp fronts without introducing oscillation.

Further, Cockburn *et al.* ([5]) combined Discontinuous Galerkin Methods, first introduced by LeSaint and Raviart in 1974, for solving the neutron transport equation, with Runge-Kutta methods for time stepping to approximate conservation laws. These schemes have been shown to be total variation bounded. One advantage of such schemes over TVD, TVB, or ENO schemes is that they can handle more complex boundary data in multidimensional spaces. These schemes are more local in the sense that higher orders are achieved by involving more moments in a single cell, instead of using neighboring cells.

By using splitting techniques as described in [6], Wei ([35]) recently combined a discontinuous Galerkin finite element method with a mixed finite element procedure for a convection-dominated diffusion problem. The combination of these two schemes is natural because both are based on a weak form of the differential equation and utilize discontinuous approximation spaces. The main results were the derivation of a maximum principle, L^2 error estimates, and the TVD property.

Stabilized methods for an advective problem were introduced by Hughes and Brooks ([22]), ([23]), ([4]), who referred to these methods as SUPG (Streamline-Upwind-Petrov-Galerkin) methods. Later, Johnson *et al.* (see references in [26]) gave a convergence analysis for these methods; they referred to them as SD (Streamline-Diffusion) methods. More recently, a canonical form for these methods was given by Hughes *et al.* ([24]), who called them GLS (Galerkin-Least-Square) methods, and further improvements were suggested by Franca *et al.* ([19, 18]), where a new terminology, SFM (Stabilized-Finite-Method), was introduced. The technique to be studied for the miscible displacement problem in this paper is closely related to this collection of stabilized procedures.

This paper is organized as follows. In the next section, we shall derive our model for the incompressible miscible displacement of one fluid by another. Our model, derived by homogenization theory in [11], is slightly different from the one used in [10], [9], and [15].

Our primary concern is the approximation of the concentration. In § 3, as a first step, we apply stabilized methods to the concentration equation with a given velocity field, i.e., the pressure equation is assumed to have been solved independently of the concentration equation. We demonstrate stability and convergence results similar to those obtained in [18] and [19] for linear problems.

Finally, we extend the results above to the coupled miscible displacement system by approximating the concentration equation by stabilized methods and the pressure equation with mixed finite element methods.

We shall use the following notation throughout this paper.

Notation :

$$\begin{aligned} \Omega &\subset R^2 && \text{The domain} \\ I &= [0, T_0] && \text{The time interval} \\ S_m &= \Omega \times I_m, \\ W^{k,p}(\Omega) &&& \text{Standard Sobolev spaces} \\ \|u\|_{k,p,\Omega} &= \|u\|_{W^{k,p}(\Omega)}, \\ \|u\|_{k,m}^2 &= \int_{I_m} \|u\|_{k,2,\Omega}^2 dt, \\ \|u\|_{k,\Omega} &= \|u\|_{k,2,\Omega}, \\ \|u\|_{k,m,T} &= \left(\int_{I_m} \|u\|_{k,2,T}^2 dt \right)^{\frac{1}{2}}, \\ |u|_m &= \|u(\cdot, t_m)\|_{0,\Omega}, \\ |u_{\pm}|_m &= \lim_{t \rightarrow t_m^{\pm}} \|u(\cdot, t)\|_{0,\Omega}, \\ (u, v) &= \int_{\Omega} u \cdot v dx, \\ \langle u, v \rangle_m &= (u(\cdot, t_m), v(\cdot, t_m)), \\ u_{\pm} &= \lim_{t \rightarrow t_m^{\pm}} u(\cdot, t) \end{aligned}$$

2. DERIVATION OF THE MODEL

We shall begin by giving a brief derivation of our miscible displacement model, derived originally in [11] by homogenization. Here, we rederive the

model using only simple physical conservation laws. We use the notation of [1]

2.1. Conservation laws in a fluid continuum

Consider a system composed of a mixture of N chemical species, with each species forming a continuum. Different continua may occupy the same portion of space at the same time, so long as they represent different species. Let dU be a REV (Representative Elementary Volume) of the multispecies system. Let dm_α and dm denote the instantaneous masses of the species α and the fluid system, respectively, in the REV dU . We may then define a mass density ρ_α of the species α as the mass of the species α per unit volume of fluid solution as follows

$$\rho_\alpha = \frac{dm_\alpha}{dU} \quad (2.1)$$

It follows that

$$\sum_{\alpha=1}^N \rho_\alpha = \sum_{\alpha=1}^N \frac{dm_\alpha}{dU} = \left(\sum_{\alpha=1}^N dm_\alpha \right) / dU = \frac{dm}{dU} = \rho, \quad (2.2)$$

where ρ is the density of the system.

In general, the velocity of species α will be different than that of the fluid system. The velocity u_α at a point P (with respect to a fixed coordinate system) is defined as the average velocity within dU of the individual molecules of the species α . Several kinds of averaged velocities can be defined for the system as a whole. The most common are the mass-averaged velocity u^m and volume-averaged velocity u^v , which are defined as follows

$$u^m = \left(\sum_{\alpha=1}^N \rho_\alpha u_\alpha \right) / \sum_{\alpha=1}^N \rho_\alpha = \left(\sum_{\alpha=1}^N \rho_\alpha u_\alpha \right) / \rho = \sum_{\alpha=1}^N \omega_\alpha u_\alpha, \quad (2.3)$$

$$u^v = \sum_{\alpha=1}^N \rho_\alpha v_\alpha u_\alpha, \quad (2.4)$$

where ω_α is called the mass fraction of species α , defined as

$$\omega_\alpha = \frac{\rho_\alpha}{\rho}, \quad \sum_{\alpha=1}^N \omega_\alpha = 1, \quad (2.5)$$

v_α is the partial specific volume. The velocity u^m is often interpreted as momentum per unit mass, since ρu^m represents the momentum per unit volume.

In a homogeneous incompressible single-species fluid, ($N = 1$, and $\rho_\alpha = \rho$), $u_\alpha = u^m = u^v$. In general, these velocities differ both in direction and magnitude. The differences

$$\bar{u}_\alpha^m = u_\alpha - u^m \quad \text{and} \quad \bar{u}_\alpha^v = u_\alpha - u^v \quad (2.6)$$

are diffusive velocities of the species α with respect to the mass-averaged and volume-averaged velocities, respectively. Similarly,

$$\rho_\alpha \bar{u}_\alpha^m = \rho_\alpha (u_\alpha - u^m) \quad \text{and} \quad \rho_\alpha \bar{u}_\alpha^v = \rho_\alpha (u_\alpha - u^v) \quad (2.7)$$

are diffusive mass fluxes of the species α with respect to the mass averaged velocity and volume-averaged velocities, respectively.

In a binary mixture, Fick's law ([1, 2]) relates the diffusive mass fluxes of the species to the diffusivity of the system by

$$\rho_\alpha \bar{u}_\alpha^m = \rho_\alpha (u_\alpha - u^m) = -\rho D_{\alpha\beta} \nabla \omega_\alpha, \quad (2.8)$$

$$\rho_\alpha \bar{u}_\alpha^v = \rho_\alpha (u_\alpha - u^v) = -D_{\alpha\beta} \nabla \rho_\alpha, \quad (2.9)$$

where $D_{\alpha\beta}$ is the binary diffusivity.

For a species α of a multicomponent system, mass conservation gives

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha u_\alpha) = I_\alpha, \quad (2.10)$$

where I_α is the production rate by the chemical reactions of species α per unit volume of the system.

For simplicity, we restrict our analysis below to a binary mixture system of species α and β . Combining (2.8), (2.9), and (2.10), we can write the mass conservation for species α in terms of mass- and volume-averaged velocities

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha u^m - \rho D_{\alpha\beta} \nabla \omega_\alpha) = I_\alpha, \quad (2.11)$$

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha u^v - D_{\alpha\beta} \nabla \rho_\alpha) = I_\alpha \quad (2.12)$$

The similar equations hold for species β . Adding (2.11) and (2.12) to the corresponding equations for β species, respectively, gives the mass conservations of the whole system in terms of the mass and volume-averaged velocities as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u^m) = I, \quad (2.13)$$

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho u^v - D_{\alpha\beta} \nabla \rho) = I, \quad (2.14)$$

where $I = I_\alpha + I_\beta$ is the mass production rate of the system per unit volume. By the equations above, we see that the mass- and volume-averaged velocities are related by

$$u^m - u^v = -\frac{1}{\rho} D_{\alpha\beta} \nabla \rho . \tag{2.15}$$

2.2. Transport equations in porous media

The conservation laws for a fluid system in a continuum described in the last section extend easily to the incompressible miscible displacement of one component, α , by another, β , in a porous medium by using volume-averaging techniques or homogenization ([11]) :

$$\phi \frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha u^m - \rho D \nabla \omega_\alpha) = q_\alpha , \tag{2.16}$$

$$\phi \frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha u^v - D \nabla \rho_\alpha) = q_\alpha , \tag{2.17}$$

with similar equations holding true for β species, and it follows that for the whole system, we have

$$\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u^m) = q , \tag{2.18}$$

$$\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u^v - D \nabla \rho) = q , \tag{2.19}$$

where ϕ is the porosity of the porous medium, $q_\alpha = \phi I_\alpha$ is a source of species α and $q = q_\alpha + q_\beta$, and

$$D = \phi D_{\alpha\beta} , \tag{2.20}$$

where, for the moment, the effect of hydrodynamic dispersion is ignored and only molecular diffusion is considered.

As in (2.15), (2.18) and (2.19) imply that the volume-averaged velocity u^v and mass-averaged velocity u^m are related by

$$u^v = u^m + \frac{1}{\rho} D \nabla \rho . \tag{2.21}$$

The conservation laws (2.16)-(2.17), expressed in terms of densities, are not easy to use. More convenient forms are often written in terms of concentrations, which can be mass concentrations (mass fractions), volumetric concentrations (volume fractions), or molar concentrations (mole fractions) ([2]). In the following, we give a specific definition [31] of concentration and then derive equations.

In addition to limiting ourselves to a binary mixture, we assume that the mixture is incompressible and that there are two « end point » (reference) mixtures 1 and 2, which have known composition $\rho_{i \alpha}$ and $\rho_{i \beta}$, and constant densities ρ_i

$$\rho_1 = \rho_{1 \alpha} + \rho_{1 \beta}, \quad \rho_2 = \rho_{2 \alpha} + \rho_{2 \beta}, \quad (2.22)$$

$$\rho_{1 \alpha} \rho_{2 \beta} - \rho_{1 \beta} \rho_{2 \alpha} \neq 0$$

We further assume that our mixture is combined from these two « end point » mixtures by taking volumes v_1 of mixture 1 and v_2 of mixture 2. Suppose that mixing takes place at constant temperature and pressure and without changing volume. Then, the mixture has a volume $v = v_1 + v_2$. Now, define the concentration c as the volumetric fraction of mixture 1 (31))

$$c = \frac{v_1}{v} = \frac{v_1}{v_1 + v_2} \quad (2.23)$$

Then the mass m_α and density ρ_α of the species α in the mixture can be written in term of reference densities as follows

$$m_\alpha = \rho_{1 \alpha} v_1 + \rho_{2 \alpha} v_2, \quad (2.24)$$

$$\rho_\alpha = \frac{m_\alpha}{v} = \frac{\rho_{1 \alpha} v_1 + \rho_{2 \alpha} v_2}{v} = \rho_{1 \alpha} c + \rho_{2 \alpha} (1 - c) \quad (2.25)$$

Similarly, we can write the equation for species β . It follows that the density ρ of the mixture can be written as

$$\begin{aligned} \rho &= \rho_\alpha + \rho_\beta = \rho_1 c + \rho_2 (1 - c) = \rho_2 (1 + \sigma c), \\ \sigma &= (\rho_1 - \rho_2) (\rho_2)^{-1}, \end{aligned} \quad (2.26)$$

which is the equation of state for the mixture under our assumptions that it is incompressible and mixed without changing volumes. Now, substituting (2.25) and the similar equation for the species β into (2.16)-(2.17), and noting that $\rho_{1 \alpha}$, $\rho_{1 \beta}$, $\rho_{2 \alpha}$, and $\rho_{2 \beta}$ are all constants, we have

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot \left(c u^m - \rho D \nabla \frac{c}{\rho} \right) = \bar{q}_1, \quad (2.27)$$

$$\phi \frac{\partial (1 - c)}{\partial t} + \nabla \cdot \left((1 - c) u^m - \rho D \nabla \frac{1 - c}{\rho} \right) = \bar{q}_2, \quad (2.28)$$

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (c u^v - D \nabla c) = \bar{q}_1, \quad (2.29)$$

$$\phi \frac{\partial (1 - c)}{\partial t} + \nabla \cdot \left((1 - c) u^v - D \nabla \frac{1 - c}{\rho} \right) = \bar{q}_2, \quad (2.30)$$

where \bar{q}_1, \bar{q}_2 are the volumetric production rates of mixture 1 and mixture 2, respectively, given by

$$\bar{q}_1 = \frac{\rho_{2,\beta} q_\alpha - \rho_{2,\alpha} q_\beta}{\rho_{1,\alpha} \rho_{2,\beta} - \rho_{2,\alpha} \rho_{1,\beta}}, \quad (2.31)$$

$$\bar{q}_2 = \frac{-\rho_{1,\beta} q_\alpha + \rho_{1,\alpha} q_\beta}{\rho_{1,\alpha} \rho_{2,\beta} - \rho_{2,\alpha} \rho_{1,\beta}}. \quad (2.32)$$

Using (2.26), and adding (2.27) to (2.28) and (2.29) to (2.30), we can rewrite the system above as

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot \left(cu^m - \frac{1}{1 + \sigma c} D \nabla c \right) = \bar{q}_1, \quad (2.33)$$

$$\nabla \cdot \left(u^m + \frac{\sigma}{1 + \sigma c} D \nabla c \right) = \bar{q}, \quad (2.34)$$

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (cu^v - D \nabla c) = \bar{q}_1, \quad (2.35)$$

$$\nabla \cdot u^v = \bar{q}, \quad (2.36)$$

where $\bar{q} = \bar{q}_1 + \bar{q}_2$. It is interesting to see that we have obtained two sets of concentration equations, one in terms of mass-averaged velocity (2.33)-(2.34) and the other in terms of volume-averaged velocity (2.35)-(2.36). These two forms are equivalent through (2.21), which can now be written as

$$u^v = u^m + \frac{1}{\rho} D \nabla \rho = u^m + \frac{\sigma}{1 + \sigma c} D \nabla c. \quad (2.37)$$

The main point in the derivation above is to clarify u^m and u^v . The relation between the velocity of a fluid flow in a porous medium and some potential gradient, is usually called Darcy's law, can be derived from the Navier-Stokes equations by volume averaging or homogenization ([11]). Since the Navier-Stokes equations are momentum equations, the Darcy velocity can be expected to be a mass-averaged velocity. Thus,

$$u^m = -\frac{k}{\mu} (\nabla p - \rho g), \quad (2.38)$$

where $\mu = \mu(c)$ and p are the viscosity and the pressure in the fluid mixture, respectively; k is the permeability of the porous medium, and g is the downward-pointing gravity acceleration vector. By (2.37), the volume-averaged velocity is given by

$$u^v = -\frac{k}{\mu} (\nabla p - \rho g) + \frac{\sigma}{1 + \sigma c} D \nabla c. \quad (2.39)$$

We see that u^m and u^v are different if $\sigma \neq 0$. When $\sigma \approx 0$, a special case corresponding to a dilute mixture whose density is approximately a constant, then $u^m \approx u^v \approx u$. The two sets of concentration equations coincide, and the governing equations (2.33)-(2.39) can be summarized as

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (cu - D \nabla c) = \bar{q}_1, \quad (2.40)$$

$$\nabla \cdot u = \bar{q}, \quad (2.41)$$

$$u = -\frac{k}{\mu} (\nabla p - \rho g) \quad (2.42)$$

The system (2.40)-(2.42) is the miscible displacement model used in reservoir simulations by many authors ([12, 10, 9])

As mentioned earlier, besides the volume fraction defined by (2.23), the concentration can also be defined as the mass fraction of a species

$$c^* = \frac{m_\alpha}{m_\alpha + m_\beta} = \frac{m_\alpha}{m} = \frac{\rho_\alpha}{\rho} = \omega_\alpha \quad (2.43)$$

It follows that

$$\begin{aligned} \rho_\alpha &= \rho c^*, \\ \rho_\beta &= \rho (1 - c^*) \end{aligned}$$

Using the two equations above in conservation laws (2.16)-(2.17) for species α and β yields

$$\phi \frac{(\partial \rho c^*)}{\partial t} + \nabla \cdot (\rho c^* u^m - \rho D \nabla c^*) = q_\alpha, \quad (2.44)$$

$$\phi \frac{(\partial \rho (1 - c^*))}{\partial t} + \nabla \cdot (\rho (1 - c^*) u^m - \rho D \nabla (1 - c^*)) = q_\beta, \quad (2.45)$$

$$\phi \frac{(\partial \rho c^*)}{\partial t} + \nabla \cdot (\rho c^* u^v - D \nabla \rho c^*) = q_\alpha, \quad (2.46)$$

$$\phi \frac{(\partial \rho (1 - c^*))}{\partial t} + \nabla \cdot (\rho (1 - c^*) u^v - D \nabla \rho (1 - c^*)) = q_\beta \quad (2.47)$$

These equations, together with Darcy's law, form the model for miscible displacement in porous media. But this system is not closed without the equation of state, which is a relation between ρ and c for an incompressible mixture like (2.26), or a relation among ρ , c , and p for a compressible mixture. For the special dilute mixture where ρ can be approximated by a constant, Darcy's law, and the fact that $u^m \approx u^v \approx u$, allows us to write the system (2.44)-(2.47) above as

$$\phi \frac{\partial c^*}{\partial t} + \nabla \cdot (c^* u - D \nabla c^*) = q_\alpha^*, \quad (2.48)$$

$$\nabla \cdot u = q^* , \quad (2.49)$$

$$u = - \frac{k}{\mu} (\nabla p - \rho g) , \quad (2.50)$$

where $q_\alpha^* = q_\alpha / \rho$, and $q^* = q_\alpha^* + q_\beta^*$. This system, first derived in [29], is widely used in reservoir simulation ([15, 30]). We note that for a dilute miscible displacement, the two models (2.27)-(2.30) and (2.44)-(2.47) reduce to (2.40)-(2.42) and (2.48)-(2.50), respectively. It is easy to see that the simplified forms (2.40)-(2.42) and (2.48)-(2.50) have the same form except for the source terms on the right hand sides which have different interpretations : the former, \bar{q}_1 , is a volumetric production rate of mixture 1, one of the two « end-point » reference mixtures, while the latter, q_α^* , is a volumetric production rate of species α , one of two components in the mixture.

As previously mentioned, the analysis above is given under the condition that hydrodynamic dispersion is ignored and only molecular diffusion is considered (this is a reasonable assumption only if the fluid velocity is very small). Otherwise, a dispersion term that takes into account the mechanical mixing caused by heterogeneities in the porous medium has to be included in the model. Peaceman [30] suggested a dispersion tensor D in the form

$$D = D(u) = d_m I + |u| \{d_\ell E(u) + d_t E^\perp(u)\} , \quad (2.51)$$

where d_m , d_ℓ , and d_t are, respectively, the molecular, longitudinal, and transverse diffusion constants, I the identity transformation, $E(u)$ the projection in the direction of the flow, and $E^\perp(u)$ the projection on the orthogonal complement of the flow vector ; *i.e.*,

$$\begin{aligned} E_{ij} &= \frac{1}{|u|^2} u_i u_j , \\ E^\perp(u) &= I - E(u) . \end{aligned} \quad (2.52)$$

We remark here that in reality the longitudinal diffusion constant d_ℓ is larger than the transverse diffusion constant d_t , and we shall make this assumption in the following analysis.

The new term $\frac{\sigma}{1 + \sigma c} D \nabla c$ appearing in (2.34) would cause many difficulties in the analysis that follows. In this paper, we shall restrict our study to the special case of a dilute mixture, so that $\sigma \approx 0$ and all models considered above coincide and can be written as

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (uc - D(u) \nabla c) = \tilde{c}q , \quad (2.53)$$

$$\nabla \cdot u = q , \quad (2.54)$$

$$u = - \frac{k}{\mu(c)} (\nabla p - \rho(c) g) . \quad (2.55)$$

The right-hand side term q on (2.53) is the external source which can be written as

$$q = q_i - q_o, \quad (2.56)$$

where $q_i = \max(0, q) \geq 0$ and $q_o = \max(0, -q) \geq 0$ are the flow rates at injection and production wells, respectively, and \tilde{c} is given by

$$\tilde{c} = \begin{cases} c_i \text{ at injection well,} & \text{where } q = q_i \geq 0, \\ c \text{ at production wells,} & \text{where } q = -q_o \leq 0, \end{cases} \quad (2.57)$$

and where c_i is the given concentration of the injected fluid and $D(u)$ is given by (2.51).

For simpler analysis, we assume the following homogeneous boundary conditions :

$$c = 0 \quad \text{on } \partial\Omega \times I, \quad (2.58)$$

$$u \cdot n = 0, \quad \text{on } \partial\Omega \times I. \quad (2.59)$$

Finally, it is necessary to specify the initial concentration,

$$c(x, 0) = c_0(x). \quad (2.60)$$

The purpose of this work is to define and analyze an appropriate discrete approximation method for the problem (2.53)-(2.60). We assume that all data functions, including q , which in reality is nonzero (and nonsmooth) only at wells, are smooth.

3. STABILIZED METHODS FOR CONCENTRATION EQUATIONS

In this section, we combine the time-discontinuous Galerkin methods developed by Johnson *et al.* ([26]) with the stabilized techniques advocated by Hughes and Franca to study a stabilized method for the concentration equation (2.53), which we simplify here as

$$\frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D(u) \nabla c) = f(x), \quad (3.1)$$

with a given smooth velocity field u that satisfies

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \quad (3.2)$$

$$u \cdot n = 0 \quad \text{on } \partial\Omega, \quad (3.3)$$

where, in this section, we take $\phi = 1$ in the concentration equation without loss of generality.

3.1. Finite element spaces and technical lemmas

We shall introduce some finite element spaces to be used in the following sections and then prove some technical lemmas. We shall use the notation of Johnson *et al.* [26].

Let $T_h(\Omega)$ be a quasiregular subdivision of Ω , $0 = t_0 < t_1 < \dots < t_M = T_0$ be a subdivision of the time interval $I = [0, T_0]$, $I_m = (t_{m-1}, t_m)$, $m = 1, 2, \dots, M$,

$$T_h^m(s_m) = \{T_m : T_m = T \times I_m, T \in T_h(\Omega)\} \tag{3.4}$$

be the corresponding subdivision of $s_m = \Omega \times I_m$, with h representing the maximum of the diameters of $T_m \in T_h^m$, and let

$$M_h^m = \{ \phi \in H^1(s_m) : \phi|_{T_m} \in P_k(T_m), T_m = T \times I_m \in T_h^m, \phi|_{\Gamma \times I_m} = 0 \}, \tag{3.5}$$

$$M_h = \prod_{m=1}^M M_h^m, \tag{3.6}$$

where $P_k(T_m)$ denotes the set of polynomials of total degree at most k on T_m . In other words, M_h^m is the set of piecewise polynomial functions on T_h^m of degree at most k that are continuous in x , possibly discontinuous in t across the time levels t_m , $m = 1, 2, 3, \dots, M$, and vanish on $\Gamma = \partial\Omega$. We shall assume that M_h^m is a regular subdivision of s_m ; i.e., for each $T_m \in T_h^m$ there is an inscribed sphere in T_m such that the ratio of the diameter of this sphere and the diameter of T_m is bounded below, independently of T_m and h .

The following standard interpolation error estimate and inverse inequality have been shown in [34] and [3].

LEMMA 3.1: *There are constants c_1 and c_{inv} such that, for any $w \in W^{s,p}(s_m) \cap C(s_m)$ for which $w|_{\Gamma \times T_m} = 0$,*

$$\inf_{w_h \in M_h^m} \|w - w_h\|_{H^r(s_m)} \leq Qh^{\ell+1-r} \|w\|_{H^{\ell+1}(s_m)}. \tag{3.7}$$

$$r = 0, 1, \quad 1 \leq \ell \leq k, \quad p = 2, \tag{3.8}$$

and, for each $v_h \in M_h$, the following inverse inequality holds :

$$c_{inv} \|v_h\|_{W^{r,p}(T)} \leq Qh^{-r} \|v_h\|_{L_p(T)}, \tag{3.9}$$

$$r = 0, \dots, k, \quad 1 \leq p \leq \infty, \tag{3.10}$$

on each $T_m = T \times I_m$, $m = 1, 2, \dots$

By following an idea used in [26], we can show the following result.

LEMMA 3.2 : For $\theta \in \hat{H} = \prod_{m=1}^M H^1(s_m)$, $\theta|_{\partial\Omega} = 0$, and $u \in H^{1,\infty}(s_m)$ and

$\phi(\lambda) \geq \phi_0 > 0$,

$$\int_{I_m} \left\| \phi^{\frac{1}{2}} \theta \right\|_{0,\Omega}^2 dt \leq \left(h_m \left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 + \frac{1}{\phi_0 c_1} h_m \|\phi \theta' + u \cdot \nabla \theta\|_{0,m}^2 \right) \times \\ \times \exp((c_1 + \|\nabla \cdot u\|_{0,\infty,s_m}) h / \phi_0) \quad (3.11)$$

for any $c_1 > 0$, where $h_m = |I_m| = t_m - t_{m-1} \leq h$.

Proof We note that

$$\begin{aligned} \left\| \phi^{\frac{1}{2}} \theta \right\|^2 &= \left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 - \int_t^{t_m} \frac{d}{dt} (\phi \theta(t), \theta(t)) dt \\ &= \left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 - 2 \int_t^{t_m} (\phi \theta'(t), \theta(t)) dt \\ &= \left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 - 2 \int_t^{t_m} (\phi \theta'(t) + u \cdot \nabla \theta, \theta) dt + \\ &\quad + 2 \int_t^{t_m} (u \cdot \nabla \theta, \theta) dt. \end{aligned} \quad (3.12)$$

Since $\theta = 0$ on $\partial\Omega$,

$$\begin{aligned} 2 \int_t^{t_m} (u \cdot \nabla \theta, \theta) dt &= \int_t^{t_m} (\langle u \cdot n, \theta^2 \rangle_{\partial\Omega} - (\nabla \cdot u, \theta^2)) dt \\ &= - \int_t^{t_m} (\nabla \cdot u, \theta^2) dt \\ &\leq 1/\phi_0 \|\nabla \cdot u\|_{0,\infty,s_m} \int_t^{t_m} \left\| \phi^{\frac{1}{2}} \theta \right\|^2 dt. \end{aligned} \quad (3.13)$$

Thus,

$$\begin{aligned} \left\| \phi^{\frac{1}{2}} \theta(t) \right\|^2 &\leq \left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 + 2 \left(\int_{I_m} \|\phi \theta' + u \cdot \nabla \theta\|^2 dt \right)^{\frac{1}{2}} \times \\ &\quad \times \left(\int_t^{t_m} \|\theta\|^2 dt \right)^{\frac{1}{2}} + 1/\phi_0 \|\nabla \cdot u\|_{0,\infty,s_m} \int_t^{t_m} \left\| \phi^{\frac{1}{2}} \theta \right\|^2 dt \leq \\ &\leq \left(\left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 + \frac{1}{\phi_0 c_1} \|\phi \theta' + u \cdot \nabla \theta\|_{0,m}^2 \right) + \\ &\quad + 1/\phi_0 (c_1 + \|\nabla \cdot u\|_{0,\infty,s_m}) \int_t^{t_m} \left\| \phi^{\frac{1}{2}} \theta(t) \right\|^2 dt \\ &\leq \left(\left| \phi^{\frac{1}{2}} \theta_- \right|_m^2 + \frac{1}{\phi_0 c_1} \|\phi \theta' + u \cdot \nabla \theta\|_{0,m}^2 \right) \exp((c_1 + \|\nabla \cdot u\|_{0,\infty,s_m}) h / \phi_0), \end{aligned}$$

where we have used the Gronwall inequality. Integrating over I_m proves the lemma. ■

Next, we shall give some results concerning the dispersion tensor defined by (2.51)-(2.52). It is easy to check the following lemma.

LEMMA 3.3 : Assume that $d_m > 0$. Then, the dispersion tensor $D(u)$ given by (2.51)-(2.52) is symmetric positive definite, and, moreover, for each $\xi, \eta \in R^2$,

$$(d_m + d_t |u|) |\xi|^2 \leq (D(u) \xi, \xi) \leq (d_m + d_\ell |u|) |\xi|^2 \quad (3.14)$$

and

$$(D(u) \xi, \eta) \leq (d_m + d_\ell |u|) |\xi| |\eta|. \quad (3.15)$$

By using the above result, one can show the following lemma.

LEMMA 3.4 : For $\theta \in H^2(T)$, we have

$$\begin{aligned} \|\nabla \cdot (D(u) \nabla \theta)\|_{0,T}^2 &\leq 2(d_m + d_\ell \|u\|_{0,\infty,T})^2 \|\nabla \nabla \theta\|_{0,T}^2 + \\ &\quad + 2(3d_\ell - 2d_t)^2 \|\nabla u\|_{0,\infty,T}^2 \|\nabla \theta\|_{0,T}^2. \end{aligned} \quad (3.16)$$

If, in addition, $\theta \in H^2(T) \cap P_k(T)$, then

$$\|\nabla \cdot (D(u) \nabla \theta)\|_{0,T}^2 \leq (c_{\text{inv}} h_T^2)^{-1} D_T^2 \|\nabla \theta\|_{0,T}^2, \quad (3.17)$$

where

$$D_T = (2(d_m + d_\ell \|u\|_{0,\infty,T})^2 + 2(3d_\ell - 2d_t)^2 \|\nabla u\|_{0,\infty,T}^2 h_T^2 c_{\text{inv}})^{1/2}, \quad (3.18)$$

and c_{inv} is the constant present in the inverse estimate (3.9).

Proof : Rewrite $\nabla \cdot (D(u) \nabla \theta)$ as

$$\begin{aligned} \nabla \cdot (D(u) \nabla \theta) &= \\ &= d_m \nabla^2 \theta + (d_\ell - d_t) \nabla \cdot (|u| E(u) \nabla \theta) + d_t \nabla \cdot (|u| \nabla \theta). \end{aligned} \quad (3.19)$$

Noting that

$$\begin{aligned} \nabla \cdot (|u| E(u) \nabla \theta) &= (\nabla \cdot u) \frac{u \cdot \nabla \theta}{|u|} - \frac{1}{|u|^3} (u \cdot \nabla u \cdot u) (u \cdot \nabla \theta) \\ &\quad + \frac{u \cdot \nabla u \cdot \nabla \theta}{|u|} + \frac{u \cdot (\nabla \nabla \theta) \cdot u}{|u|}, \end{aligned} \quad (3.20)$$

by using the assumption $d_\ell \geq d_m$, we can estimate (3.19) as follows :

$$\begin{aligned} |\nabla \cdot (D(u) \nabla \theta)| &\leq \\ &\leq d_m |\nabla^2 \theta| + (d_\ell - d_t) |\nabla \cdot (|u| E(u) \nabla \theta)| + d_t |\nabla \cdot (|u| \nabla \theta)| \\ &\leq (d_m + d_\ell |u|) |\nabla \nabla \theta| + (3d_\ell - 2d_t) |\nabla u| |\nabla \theta|. \end{aligned}$$

It follows that

$$\begin{aligned} \|\nabla \cdot (D(u) \nabla \theta)\|_{0,T}^2 &= \int_T |\nabla \cdot (D(u) \nabla \theta)|^2 dx \\ &\leq 2 \int_T (d_m + d_\ell |u|)^2 |\nabla \nabla \theta|^2 + (3 d_\ell - 2 d_t)^2 |\nabla u|^2 |\nabla \theta|^2 dx \\ &\leq 2 (d_m + d_\ell \|u\|_{0,\infty,T})^2 \|\nabla \nabla \theta\|_{0,T}^2 + \\ &\quad + 2 (3 d_\ell - 2 d_t)^2 \|\nabla u\|_{0,\infty,T}^2 \|\nabla \theta\|_{0,T}^2. \end{aligned} \quad (3.21)$$

Thus, (3.16) holds. If, in addition, $\theta \in P_k(T)$, then the inverse inequality (3.9) gives

$$c_{\text{inv}} (h_T^2 \|\nabla \nabla \theta\|_{0,T}^2) \leq \|\nabla \theta\|_{0,T}^2, \quad (3.22)$$

for some constant c_{inv} . It follows that

$$\begin{aligned} \|\nabla \cdot (D(u) \nabla \theta)\|_{0,T}^2 &\leq [2 (d_m + d_\ell \|u\|_{0,\infty,T})^2 (c_{\text{inv}} h_T^2)^{-1} + \\ &\quad + 2 (3 d_\ell - 2 d_t)^2 \|\nabla u\|_{0,\infty,T}^2] \|\nabla \theta\|_{0,T}^2. \end{aligned} \quad (3.23)$$

Thus, (3.17) is proved. \blacksquare

3.2. Formulation of the stabilized methods

In this section, we shall construct our stabilized finite element methods by following the ideas used in [26] and the stabilized method described in the last section for the concentration equation (3.1). Let $\alpha \in \{1, 0, -1\}$. To define our method on the slab $s_m = \Omega \times I_m$, impose the initial values C_-^{m-1} weakly and the boundary values $C = 0$ strongly. Then, the method is given by the following relations :

For $m = 1, 2, \dots, M$, find $C \in M_h^m$ such that

$$B_\alpha^m(u, C, \theta) = F_\alpha^m(u, \theta), \quad \forall \theta \in M_h^m, \quad (3.24)$$

where

$$\begin{aligned} B_\alpha^m(u, C, \theta) &= \int_{I_m} (C_t + u \cdot \nabla C, \theta) dt + \\ &+ \int_{I_m} (D(u) \nabla C, \nabla \theta) dt + \langle [C], \theta_t \rangle_{m-1} \\ &+ \sum_{T \in \mathcal{T}_h} \int_{I_m} \tau_T (C_t + u \cdot \nabla C - \nabla \cdot (D(u) \nabla C), \theta_t + \\ &\quad + u \cdot \nabla \theta + \alpha \nabla \cdot (D(u) \nabla \theta))_T dt, \end{aligned} \quad (3.25)$$

$$F_{\alpha}^m(u, \theta) = \int_{I_m} (f, \theta) dt + \sum_{T \in T_h} \int_{I_m} \tau_T (f, \theta_t + u \cdot \nabla \theta + \alpha \nabla \cdot (D(u) \nabla \theta)) dt, \quad (3.26)$$

where the initial condition $c_-^0 = C_0$ is given by

$$C_0 = \text{the projection of } c_0 \text{ into } M_h^1, \quad (3.27)$$

and τ_T is a stabilizing parameter defined by

$$\tau = \tau_T = \frac{h_T}{2 \|u\|_{0, \infty, T}} \xi(P_{eT}) \quad \text{on } T \quad \text{for } T \in T_h, \quad (3.28)$$

$$\xi(P_{eT}) = \min(P_{eT}, 1). \quad (3.29)$$

P_{eT} is the mesh-dependent Péclet number given by

$$P_{eT}(x) = \frac{m_k \|u\|_{0, \infty, T} h_T}{D_T^2/d_T} \quad \text{on } T \quad \text{for } T \in T_h, \quad (3.30)$$

where D_T is defined by (4.19), and d_T and m_k are given by

$$d_T = d_m + d_t \inf_{x \in T} |u| \quad (3.31)$$

and

$$m_k = \frac{2}{3} \min\left(\frac{1}{2}, c_{\text{inv}}\right). \quad (3.32)$$

By (3.18) and (3.31), we can easily show that

$$D_T^2/d_T^2 \geq \frac{2(d_m + d_t \|u\|_{0, \infty, T})^2}{\left(d_m + d_t \inf_{x \in T} (u)\right)^2} \geq 2. \quad (3.33)$$

As before, (3.24) is referred to as the DW (Douglas-Wang) method for $\alpha = 1$, the SUPG method for $\alpha = 0$, and the GLS method for $\alpha = -1$, respectively, by Franca *et al.* [19, 18].

The existence and uniqueness of a solution of (3.24) for each α is a consequence of the following stability result.

THEOREM 3.1 : Assume that $u \in W^{1, \infty}(\Omega \times (0, T_0))$ satisfies (3.2)-(3.3). Then, for $C \in M_h^m$,

$$B_{\alpha}^m(u, C, C) \geq \frac{1}{2} (|C_-|_m^2 - |C_-|_{m-1}^2) + \frac{1}{2} |[C]|_{m-1}^2 + \frac{1}{2} \left(\int_{I_m} \left\| d_T^{\frac{1}{2}} \nabla C \right\|^2 dt + \sum_{T \in T_h} \int_{I_m} \left\| \tau_T^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|^2 dt \right), \quad (3.34)$$

and it follows that for $C \in M_h$,

$$B_\alpha(u, C, C) = \sum_{m=1}^M B_\alpha^m(u, C, C) \geq \frac{1}{2} |||C|||^2 - |C_-|_0^2, \quad (3.35)$$

where

$$\begin{aligned} |||C|||^2 &= |C_-|_M^2 + |C_-|_0^2 + \sum_{m=1}^M |[C]|_m^2 + \\ &\sum_{m=1}^M \int_{I_m} \left\| d_T^{\frac{1}{2}} \nabla C \right\|^2 dt + \sum_{m=1}^M \sum_{T \in T_h} \int_{I_m} \left\| \tau_T^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|^2 dt, \end{aligned} \quad (3.36)$$

and d_T and D_T are defined by (3.31) and (3.18), respectively.

Proof: Note that

$$\begin{aligned} B_\alpha^m(u, C, C) &= \left[\int_{I_m} (C_t + u \cdot \nabla C, C) dt + \langle [C], C_+ \rangle_{m-1} \right] + \\ &+ \left[\int_{I_m} (D(u) \nabla C, \nabla C) dt \right] + \left[\sum_{T \in T_h} \int_{I_m} \tau_T (C_t + u \cdot \nabla C - \nabla \cdot (D(u) \nabla C), \right. \\ &\left. C_t + u \cdot \nabla C + \alpha \nabla \cdot (D(u) \nabla C))_T dt \right] \equiv T_1 + T_2 + T_3. \end{aligned} \quad (3.37)$$

Since u satisfies (3.2)-(3.3), we have

$$\begin{aligned} T_1 &= \int_{I_m} (C_t + u \cdot \nabla C, C) dt + \langle [C], C_+ \rangle_{m-1} \\ &= \frac{1}{2} \int_{I_m} \frac{d}{dt} (C, C) dt + \int_{I_m} (u \cdot \nabla C, \nabla C) dt + \langle [C], C_+ \rangle_{m-1} \\ &= \frac{1}{2} \langle C_-, C_- \rangle_m - \frac{1}{2} \langle C_+, C_+ \rangle_{m-1} + \frac{1}{2} \int_{I_m} (u \cdot n, C^2)_{\partial\Omega} dt - \\ &\quad - \frac{1}{2} \int_{I_m} (\nabla \cdot u, C^2) dt + \langle [C], C_+ \rangle_{m-1} \\ &= \frac{1}{2} [\langle C_-, C_- \rangle_m - \langle C_-, C_- \rangle_{m-1}] + \frac{1}{2} \langle [C], [C] \rangle_{m-1}. \end{aligned} \quad (3.38)$$

By Lemma 3.3, we have

$$T_2 = \int_{I_m} (D(u) \nabla C, \nabla C) dt \geq \sum_{T \in T_h} \int_{I_m} d_T \|\nabla C\|_{0,T}^2 dt, \quad (3.39)$$

$$\begin{aligned}
T_3 &= \sum_{T \in T_h} \int_{I_m} \tau_T (C_t + u \cdot \nabla C - \nabla \cdot (D(u) \nabla C), \\
&C_t + u \cdot \nabla C + \alpha \nabla \cdot (D(u) \nabla C))_T dt \\
&\geq \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - (1 - \alpha) \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T} \cdot \right. \\
&\quad \left. \cdot \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T} - \alpha \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 \right) dt, \quad (3.40)
\end{aligned}$$

By Lemma 3.4 and (3.28)-(3.32), we have

$$\begin{aligned}
\left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 &= \tau_T \|\nabla(D(u) \nabla C)\|_{0,T}^2 \leq \tau_T (c_{\text{inv}} h_T^2)^{-1} D_T^2 \|\nabla C\|_{0,T}^2 \\
&= \frac{h_T}{2 \|u\|_{0,\infty,T}} \frac{\xi(P_{eT})}{P_{eT}} P_{eT} \frac{1}{c_{\text{inv}} h_T^2} D_T^2 \|\nabla C\|_{0,T}^2 \\
&\leq \frac{h_T}{2 \|u\|_{0,\infty,T}} \frac{m_k \|u\|_{0,\infty,T} h_T}{D_T^2/d_T} \frac{1}{c_{\text{inv}} h_T^2} D_T^2 \|\nabla C\|_{0,T}^2 \\
&= \frac{m_k}{2 c_{\text{inv}}} d_T \|\nabla C\|_{0,T}^2 \leq \frac{1}{3} d_T \|\nabla C\|_{0,T}^2.
\end{aligned}$$

That is,

$$\left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 \leq \frac{1}{3} d_T \|\nabla C\|_{0,T}^2. \quad (3.41)$$

To simplify the analysis of (3.40), we treat the three cases for α separately.

For $\alpha = 1$ by (3.41), equation (3.40) can be simplified into

$$\begin{aligned}
T_3 &= \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 \right) dt \geq \\
&\geq \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 dt - \frac{1}{3} \sum_{T \in T_h} \int_{I_m} d_T \|\nabla C\|_{0,T}^2 dt. \quad (3.42)
\end{aligned}$$

Similarly, for $\alpha = 0$,

$$\begin{aligned}
T_3 &\geq \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - \right. \\
&\quad \left. - \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\| \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\| \right) dt
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{T \in T_h} \int_{I_m} \left(\frac{3}{4} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 \right) dt \\
&\geq \frac{3}{4} \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 dt - \frac{1}{3} \sum_{T \in T_h} \int_{I_m} d_T \|\nabla C\|_{0,T}^2 dt. \quad (3.43)
\end{aligned}$$

Finally, for $\alpha = -1$, taking β such that $0 < \beta < 1$, we have

$$\begin{aligned}
T_3 &\geq \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - 2 \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T} \times \right. \\
&\quad \left. \times \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T} + \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 \right) dt \\
&\geq \sum_{T \in T_h} \int_{I_m} (1 - \beta) \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 + \\
&\quad + \left(1 - \frac{1}{\beta} \right) \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla C) \right\|_{0,T}^2 dt \\
&\geq \sum_{T \in T_h} \int_{I_m} \left((1 - \beta) \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 + \frac{1}{3} \left(1 - \frac{1}{\beta} \right) d_T \|\nabla C\|_{0,T}^2 \right) dt \\
&= \sum_{T \in T_h} \int_{I_m} \left(\frac{1}{2} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - \frac{1}{3} d_T \|\nabla C\|_{0,T}^2 \right) dt, \quad (3.44)
\end{aligned}$$

where β has been set to $1/2$ to obtain the last equality. In all three cases of α , we have shown that

$$T_3 \geq \sum_{T \in T_h} \int_{I_m} \left(\frac{1}{2} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 - \frac{1}{3} d_T \|\nabla C\|_{0,T}^2 \right) dt. \quad (3.45)$$

Substituting (3.38), (3.39), and (3.45) into (3.37) yields

$$\begin{aligned}
B_\alpha^m(u, C, C) &\geq \frac{1}{2} [\langle C_-, C_- \rangle_m - \langle C_-, C_- \rangle_{m-1}] + \frac{1}{2} \langle [C], [C] \rangle_{m-1} + \\
&\quad + \frac{2}{3} \sum_{T \in T_h} \int_{I_m} d_T \|\nabla C\|_{0,T}^2 dt + \frac{1}{2} \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (C_t + u \cdot \nabla C) \right\|_{0,T}^2 dt.
\end{aligned}$$

Thus, (3.34) is proved. Taking the sum over m in the above inequality yields (3.35). The lemma is proved. \blacksquare

3.3. Error estimates

In this section, we study the convergence features of the methods given by (3.24)-(3.26).

LEMMA 3.5 : Let $u \in W^{1, \infty}(\Omega \times I)$ satisfy (3.2) and (3.3). Then, for each $\alpha = 1, 0, -1$,

$$B_\alpha(u, c, \theta) \leq Q |||c||| \cdot |||\theta||| \quad \text{for } c \in \hat{H}, \quad \theta \in M_h, \quad (3.46)$$

where Q is a constant depending only on Ω and norms of u ; \hat{H} is defined by

$$\hat{H} = \prod_{m=1}^M H^1(s_m) \quad (3.47)$$

and

$$\begin{aligned} |||c||| &= \sum_{m=1}^M |c_-|_m^2 + \sum_{m=1}^M \int_{I_m} \left\| \tau^{-\frac{1}{2}} c \right\|^2 dt + \sum_{m=1}^M \int_{I_m} \left\| d^{\frac{1}{2}} \nabla c \right\|^2 dt + \\ &+ \sum_{m=1}^M \int_{I_m} \left\| \tau^{\frac{1}{2}} (c_t + u \cdot \nabla c) \right\|^2 dt + \sum_{m=1}^M \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla c) \right\|_{0,T}^2 dt. \end{aligned} \quad (3.48)$$

Proof: Note that

$$\begin{aligned} B_\alpha^m(u, c, \theta) &= \left[\int_{I_m} (c_t + u \cdot \nabla c, \theta) dt + \langle [c], \theta \rangle_{m-1} \right] + \\ &+ \left[\int_{I_m} (D(u) \nabla c, \nabla \theta) dt \right] \\ &+ \left[\sum_{T \in T_h} \int_{I_m} \tau_T (c_t + u \cdot \nabla c - \nabla \cdot (D(u) \nabla c), \right. \\ &\quad \left. \theta_t + u \cdot \nabla \theta + \alpha \nabla \cdot (D(u) \nabla \theta))_T dt \right] \equiv \\ &\equiv T_1 + T_2 + T_3. \end{aligned} \quad (3.49)$$

By using (3.2) and (3.3) and integration by parts, it is easy to check that

$$\begin{aligned} T_1 &= \int_{I_m} (c_t + u \cdot \nabla c, \theta) dt + \langle [c], \theta \rangle_{m-1} \\ &= \langle c_-, \theta_- \rangle_m - \langle c_+, \theta_+ \rangle_{m-1} - \int_{I_m} (c, \theta_t + u \cdot \nabla \theta) dt + \langle [c], \theta_+ \rangle_{m-1} \end{aligned}$$

$$\begin{aligned}
 &= \langle c_-, \theta_- \rangle_m - \langle c_-, \theta_- \rangle_{m-1} - \langle c_-, [\theta] \rangle_{m-1} - \int_{I_m} (c, \theta_t + u \cdot \nabla \theta) dt \\
 &\leq \langle c_-, \theta_- \rangle_m - \langle c_-, \theta_- \rangle_{m-1} + |c_-|_{m-1} |[\theta]|_{m-1} \\
 &+ \int_{I_m} \left\| \tau^{-\frac{1}{2}} c \right\| \left\| \tau^{\frac{1}{2}} (\theta_t + u \cdot \nabla \theta) \right\| dt.
 \end{aligned} \tag{3.50}$$

With (3.14) in Lemma 3.3, we can get

$$\begin{aligned}
 T_2 &= \int_{I_m} (D(u) \nabla c, \nabla \theta) dt \leq \int_{I_m} \int_{\Omega} (d_m + d_\ell |u|) |\nabla c| |\nabla \theta| dx dt \\
 &\leq \int_{I_m} \sum_{T \in T_h} \frac{d_m + d_\ell \sup_T |u|}{d_T} \int_T d_T |\nabla c| |\nabla \theta| dx dt \\
 &\leq Q \int_{I_m} \left\| d^{\frac{1}{2}} \nabla c \right\| \left\| d^{\frac{1}{2}} \nabla \theta \right\| dt,
 \end{aligned} \tag{3.51}$$

where Q is a constant depending on $\|u\|_{0, \infty, \Omega}$. Using (3.41), we have

$$\begin{aligned}
 T_3 &= \sum_{T \in T_h} \int_{I_m} \tau_T (c_t + u \cdot \nabla c - \nabla \cdot (D(u) \nabla c), \theta_t \\
 &+ u \cdot \nabla \theta + \alpha \nabla \cdot (D(u) \nabla \theta))_T dt \\
 &\leq \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (c_t + u \cdot \nabla c) \right\|_{0, T} + \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla c) \right\|_{0, T} \right) \\
 &\cdot \left(\left\| \tau^{\frac{1}{2}} (\theta_t + u \cdot \nabla \theta) \right\|_{0, T} + \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla \theta) \right\|_{0, T} \right) dt \\
 &\leq \sum_{T \in T_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (c_t + u \cdot \nabla c) \right\|_{0, T} + \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla c) \right\|_{0, T} \right) \\
 &\cdot \left(\left\| \tau^{\frac{1}{2}} (\theta_t + u \cdot \nabla \theta) \right\|_{0, T} + \left\| d^{\frac{1}{2}} \nabla \theta \right\|_{0, T} \right) dt.
 \end{aligned} \tag{3.52}$$

Substituting (3.50)-(3.52) into (3.49) and taking the sum over m gives (3.46). Thus, the lemma is proved. ■

To obtain an error estimate, we first study approximability by M_h .

LEMMA 3.6 : Let $u \in W^{1, \infty}(\Omega \times I)$ and $c \in H^1(I, H^k(\Omega)) \cap L^2(I, H^{k+1}(\Omega))$. Then,

$$\inf_{C \in M_h} |||c - C||| \leq Q \sum_{m=1}^M \sum_{T \in T_h} h_T^{2k} \left(\int_{I_m} \|c\|_{k+1, T}^2 + \|c_t\|_{k, T}^2 dt \right) \cdot (h_T H(P_{eT}(x) - 1) + H(1 - P_{eT})), \quad (3.53)$$

where $||| \cdot |||$ is defined by (3.48), and H is the Heaviside function such that $H(x) = 1$ if $x > 0$ and $H(x) = 0$, otherwise.

Proof : Let $\eta = c - C$. By (3.48),

$$\begin{aligned} & ||| \eta |||^2 = \\ &= \sum_{m=1}^M \sum_{T \in T_h} \left[\|\eta\|_{m, T}^2 + \int_{I_m} \left\| \tau^{-\frac{1}{2}} \eta \right\|_{0, T}^2 dt + \int_{I_m} \left\| d^{\frac{1}{2}} \nabla \eta \right\|_{0, T}^2 dt + \right. \\ & \quad \left. + \int_{I_m} \left\| \tau^{\frac{1}{2}} (\eta_t + u \cdot \nabla \eta) \right\|_{0, T}^2 dt + \int_{I_m} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla \eta) \right\|_{0, T}^2 dt \right] \\ & \equiv \sum_{m=1}^M \sum_{T \in T_h} ||| \eta |||^2_{m, T}. \end{aligned} \quad (3.54)$$

To estimate each term in the equation above, consider the two cases for P_{eT} separately.

First, let T be such an element that $P_{eT}(x) \geq 1$. By (3.28)-(3.30),

$$\begin{aligned} \left\| \tau^{-\frac{1}{2}} \eta \right\|_{0, T}^2 &= \int_T \frac{1}{\tau} \eta^2 dx = \left(\frac{h_T}{2 \|u\|_{0, \infty, T}} \right)^{-1} \|\eta\|_{0, T}^2 \\ &= \frac{2 \|u\|_{0, \infty, T}}{h_T} \|\eta\|_{0, T}^2. \end{aligned} \quad (3.55)$$

To estimate the next term, $\left\| d^{\frac{1}{2}} \nabla \eta \right\|_{m, T}^2$, noting that $P_{eT}(x) = \frac{m_k \|u\|_{0, \infty, T} h_T}{D_T^2/d_T} \geq 1$, we have

$$d_T \leq \frac{m_k \|u\|_{0, \infty, T} h_T}{D_T^2/d_T^2}, \quad \|u\|_{0, \infty, T} \geq \frac{(D_T^2/d_T)}{m_k h_T}. \quad (3.56)$$

Thus, by (3.56) and (3.33), we see that

$$\begin{aligned} \left\| d_T^{\frac{1}{2}} \nabla \eta \right\|^2 &\leq \frac{m_k \|u\|_{0, \infty, T} h_T}{D_T^2/d_T^2} \|\nabla \eta\|_{0, T}^2 \\ &\leq \int_T Q \|u\|_{0, \infty, T} h_T |\nabla \eta| dx \leq Q h_T \|\nabla \eta\|_{0, T}^2, \end{aligned} \tag{3.57}$$

and similarly, we can estimate the third term as

$$\begin{aligned} \left\| \tau^{\frac{1}{2}} (\eta_t + u \cdot \nabla \eta) \right\|_{0, T}^2 &= \tau_T \|\eta_t + u \cdot \nabla \eta\|_{0, T}^2 \\ &= \frac{h_T}{2 \|u\|_{0, \infty, T}} (\|\eta_t\|_{0, T}^2 + \|u\|_{0, \infty, T}^2 \|\nabla \eta\|_{0, T}^2) \\ &\leq Q h_T (h_T \|\eta_t\|_{0, T}^2 + \|\nabla \eta\|_{0, T}^2). \end{aligned} \tag{3.58}$$

It is easy to check that, by (3.16) in Lemma 3.4 we have

$$\begin{aligned} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla \eta) \right\|_{0, T}^2 &\leq 2 \tau_T (d_m + d_\ell \|u\|_{0, T})^2 \|\nabla \nabla \eta\|_{0, T}^2 + \\ &\quad + 2 \tau_T (3 d_\ell - 2 d_t)^2 \|\nabla u\|_{0, \infty, T}^2 \|\nabla \eta\|_{0, T}^2. \end{aligned} \tag{3.59}$$

To simplify (3.59), use (3.18) and (3.33) to show that

$$\begin{aligned} \tau_T 2 (d_m + d_\ell \|u\|_{0, \infty, T})^2 &\leq \tau_T D_T^2 \leq \frac{h_T}{2 \|u\|_{\infty, T}} D_T^2 \leq \frac{h_T}{2 \|u\|_{\infty, T}} D_T^2 P_{eT}^2(x) = \\ &= \frac{h_T}{2 \|u\|_{\infty, T}} D_T^2 \frac{m_k^2 \|u\|_{0, \infty, T}^2 h_T^2}{D_T^4 d_T^2} \leq \frac{m_k^2}{4} \|u\|_{0, \infty, T} h_T^3 \end{aligned} \tag{3.60}$$

and

$$\begin{aligned} 2 \tau_T (3 d_\ell - d_t)^2 \|\nabla u\|_{0, \infty, T}^2 &\leq \frac{D_T^2}{c_{\text{inv}} h_T^2} \tau_T \leq \frac{1}{c_{\text{inv}} h_T^2} D_T^2 \tau_T \leq \\ &\leq \frac{1}{c_{\text{inv}} h_T^2} \frac{m_k^2}{4} \|u\|_{0, \infty, T} h_T^3 \leq \frac{m_k^2}{4 c_{\text{inv}}} \|u\|_{0, \infty, T} h_T. \end{aligned} \tag{3.61}$$

Then,

$$\left\| \tau^{\frac{1}{2}} \nabla \cdot (D(u) \nabla \eta) \right\|_{0, T}^2 \leq Q [h_T^3 \|\nabla \nabla \eta\|^2 + h_T \|\nabla \eta\|^2]. \tag{3.62}$$

Substituting (3.55), (3.57), (3.58), and (3.62) into (3.54) yields

$$\begin{aligned} |||\eta|||_{m,T}^2 &\leq Q \frac{1}{h_T} \left[|\eta_-|_{m,T}^2 h_T + \int_{I_m} \|\eta\|_{0,T}^2 dt + \right. \\ &\quad \left. + \int_{I_m} (h_T^2 \|\nabla \eta\|_{0,T}^2 + h_T^3 \|\eta_t\|_{0,T}^2 + h_T^4 \|\nabla \nabla \eta\|_{0,T}^2) dt \right]. \end{aligned} \quad (3.63)$$

Next, let T be such that $0 \leq P_{eT}(x) \leq 1$. We can similarly show that

$$\begin{aligned} |||\eta|||_{T,m}^2 &\leq Q \left[|\eta_-|_{m-1}^2 + \frac{1}{h_T^2} \int_{I_m} \|\eta\|_{0,T}^2 dt + \right. \\ &\quad \left. + \int_{I_m} \|\nabla \eta\|_{0,T}^2 dt + h_T^2 \int_I \|\eta_t\|_{0,T}^2 dt + \int_{I_m} \|\nabla \nabla \eta\|_{0,T}^2 dt \right] \\ &\leq Q \frac{1}{h_T^2} \left[h_T^2 |\eta_-|_{m-1}^2 + \int_{I_m} (\|\eta\|_{0,T}^2 + h_T^2 \|\nabla \eta\|_{0,T}^2 + h_T^4 \|\nabla \nabla \eta\|_{0,T}^2) dt \right. \\ &\quad \left. + h_T^4 \int_{I_m} \|\eta_t\|_{0,T}^2 dt \right]. \end{aligned} \quad (3.64)$$

Combining (3.63) and (3.64), we have shown for any element T :

$$\begin{aligned} |||\eta|||_{m,T}^2 &\leq \\ &\leq Q \left[h_T |\eta_-|_{m-1}^2 + \int_{I_m} \|\eta\|_{0,T}^2 dt + h_T^2 \|\nabla \eta\|_{0,T}^2 + h_T^4 \|\nabla \nabla \eta\|_{0,T}^2 dt + \right. \\ &\quad \left. + h_T^3 \int_{I_m} \|\eta_t\|^2 dt \right] \cdot \left[\frac{1}{h_T^2} H(1 - P_{eT}) + \frac{1}{h_T} H(P_{eT} - 1) \right], \end{aligned} \quad (3.65)$$

where H is the Heaviside function. By Lemma 3.1 and the standard interpolation theory, we have

$$\begin{aligned} \inf_{c \in M_h} \left[h_T |\eta_-|_{m-1}^2 + \int_{I_m} \|\eta\|_{0,T}^2 dt + \int_{I_m} h_T^2 (\|\nabla \eta\|_{0,T}^2 + h_T \|\eta_t\|_{0,T}^2) dt + \right. \\ \left. + \int_{I_m} h_T^4 \|\nabla \nabla \eta\|^2 dt \right] &\leq Q h_T^{2k+2} \left[\int_{I_m} (\|c\|_{k+1,T}^2 + \|c_t\|_{k,T}^2) dt \right]. \end{aligned} \quad (3.66)$$

Combining (3.65), (3.66), and (3.54) proves the lemma. ■

We can now demonstrate our main convergence result.

THEOREM 3.2 : *Let $c \in H^1(I, H^k(\Omega)) \cap L^2(I, H^{k+1}(\Omega))$ be the exact solution to (3.1). Let C be the numerical solution to (3.24) for $\alpha \in \{1, 0, -1\}$. Then,*

$$|||c - C|||^2 \leq Q \sum_{m=1}^M \sum_{T \in T_h} h_T^{2k} \left[\int_{I_m} \|c\|_{k+1, T}^2 + \|c_t\|_{k, T}^2 dt \right] \cdot [H(P_{eT} - 1)h_T + H(1 - P_{eT})], \quad (3.67)$$

where $|||\cdot|||$ is defined by (3.36).

Proof : Since the stabilized method (3.24) is consistent, then the exact solution c satisfies

$$B_\alpha^m(u, c, \theta) = F_\alpha^m(u, \theta), \quad \theta \in M_h^m. \quad (3.68)$$

Let

$$e = c - C = (c - \phi) + (\phi - C) \equiv \eta + \xi, \quad (3.69)$$

where $\phi \in M_h^m$. By (3.68) and (3.24),

$$B_\alpha^m(u, \xi, \xi) = B_\alpha^m(u, e - \eta, \xi) = B_\alpha^m(u, e, \xi) + B_\alpha^m(u, -\eta, \xi) = B_\alpha^m(u, -\eta, \xi)$$

for each m . By Theorem 3.1, Lemma 3.5, and the equation above

$$\begin{aligned} |||\xi|||^2 &\leq 2|\xi_-|_0^2 + 2B_\alpha(u, \xi, \xi) = 2|\xi_-|_0^2 + 2B_\alpha(u, -\eta, \xi) \\ &\leq Q(|\xi_-|_0^2 + |||\eta||| |||\xi|||) \\ &\leq Q(|\xi_-|_0^2 + |||\eta|||^2) + \frac{1}{2}|||\xi|||^2. \end{aligned} \quad (3.70)$$

Thus,

$$|||\xi|||^2 \leq Q[|\xi_-|_0^2 + |||\eta|||^2]. \quad (3.71)$$

It is easy to verify that

$$|||\eta|||^2 \leq Q|||\eta|||^2. \quad (3.72)$$

Thus, by (3.69)-(3.72)

$$|||e|||^2 \leq 2[|||\eta|||^2 + |||\xi|||^2] \leq Q[|\xi_-|_0^2 + |||\eta|||^2]. \quad (3.73)$$

Recall that the numerical method (3.24) began with the initial values $C_-(0) = C^0$, the projection of c_0 ; i.e.,

$$(C_0, \theta) = (c_0, \theta), \quad \forall \theta \in M_h^0. \quad (3.74)$$

It follows that

$$(\xi_-, \xi_-) = (\eta_-, \xi_-) \leq |\eta_-| |\xi_-|,$$

so,

$$|\xi_-|_0^2 \leq |\eta_-|_0^2. \quad (3.75)$$

Therefore, by (3.73),

$$\|e\|^2 \leq Q[|\eta_-|_0^2 + \|\eta\|^2] \leq Q\|\eta\|^2. \quad (3.76)$$

The theorem follows from Lemma 3.6. ■

4. STABILIZED METHODS FOR MISCIBLE DISPLACEMENT

We study a stabilized method for the miscible displacement model (2.53)-(2.55), which, when combined with (2.56)-(2.57), can be rewritten as

$$\phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (D(u) \nabla c) + q_i c = c_i q_i, \quad (4.1)$$

where $q_i \geq 0$ and $q_0 \geq 0$ are the rates at injection and production wells, respectively. The pressure equation and boundary and initial conditions do not change :

$$\nabla \cdot u = q, \quad (4.2)$$

$$u = -\frac{k}{\mu(c)} (\nabla p - \rho g) \equiv -a(c) (\nabla p - g(c)), \quad (4.3)$$

where

$$a(c) = \frac{k}{\mu(c)}, \quad (4.4)$$

$$g(c) = \rho(c) g; \quad (4.5)$$

and

$$c = 0 \quad \text{on} \quad \partial\Omega \times I, \quad (4.6)$$

$$u \cdot n = 0 \quad \text{on} \quad \partial\Omega \times I; \quad (4.7)$$

and

$$c(x) = c_0(x) \quad \text{on} \quad \Omega. \quad (4.8)$$

We first review some results regarding the pressure equations (4.2)-(4.3) and then study stabilized methods for (4.1)-(4.8). The stability and error estimates are given next and the existence and uniqueness results are given last.

4.1. Approximation of the pressure equation

We shall follow the ideas Douglas *et al*, used in [10] and [9] to approximate the pressure equation using mixed finite element methods. We shall not give proofs in this section (for details, see [10] and [9]).

Let $H(\text{div}, \Omega)$ be the set of vector functions $v \in L^2(\Omega)^2$ such that $\nabla \cdot v \in L^2(\Omega)$, and let

$$V = H(\text{div}, \Omega) \cap \{v \mid v \cdot n = 0 \text{ on } \partial\Omega\} \tag{4.9}$$

Clearly, the solution p of equation (4.2)–(4.3) is determined only up to an additive constant, and we shall avoid this trivial difficulty by considering

$$W = L^2(\Omega) / \{\varphi = \text{constant on } \Omega\} \tag{4.10}$$

For α and β in V , $\varphi \in W$, and $\theta \in L^\infty(\Omega)$, define bilinear forms in the variables (α, β) and (α, φ) , respectively, by

$$A(\theta, \alpha, \beta) = \left(\frac{1}{a(\theta)} \alpha, \beta \right) = \sum_{i=1}^2 \left(\frac{1}{a(\theta)} \alpha_i, \beta_i \right), \tag{4.11}$$

$$B(\alpha, \varphi) = -(\nabla \cdot \alpha, \varphi) \tag{4.12}$$

Then, the pressure equation is equivalent to solving the family of saddle-point problems given by

$$A(c, u, v) + B(v, p) = (g(c), v), \quad \forall v \in V, \tag{4.13}$$

$$B(u, w) = -(q, w), \quad \forall w \in W, \tag{4.14}$$

for a map $\{u, p\} : I \mapsto V \times W$

Let h be the maximum diameters of elements in quasiregular partitions of Ω for the concentration equation and the pressure equation, respectively. Let $\tilde{V}_h \times \tilde{W}_h$ be the RT space of index k associated with the triangulation or quadrilateralization of Ω for the pressure. Let

$$V_h = \{v \in \tilde{V}_h \mid v \cdot n = 0 \text{ on } \partial\Omega\}, \tag{4.15}$$

$$W_h = \tilde{W}_h / \{\varphi = \text{constant on } \Omega\} \tag{4.16}$$

It is not hard to see that $\{V_h, W_h\}$ preserves the relation

$$\text{div } V_h = W_h \tag{4.17}$$

that holds for $\{\tilde{V}_h, \tilde{W}_h\}$. The approximation of $V \times W$ by $V_h \times W_h$ satisfies

the relations for $v \in V$ and $w \in W$ such that

$$\begin{aligned} \inf_{v_h \in V_h} \|v - v_h\|_{L^2(\Omega)} &\leq Q \|v\|_{H^{k+1}(\Omega)} h^{k+1}, \\ \inf_{v_h \in V_h} \|v - v_h\|_{H(\text{div})} &\leq Q \{ \|v\|_{H^{k+1}(\Omega)^2} + \|\nabla \cdot v\|_{H^{k+1}(\Omega)} \} h^{k+1}, \quad (4.18) \\ \inf_{w_h \in W_h} \|w - w_h\|_W &\leq Q \|w\|_{H^{k+1}(\Omega)} h^{k+1}, \end{aligned}$$

whenever the norms on the right-hand side are finite ([10, 9]). Let P_h be the orthogonal L_2 projection from W into W_h . By standard interpolation theory, for $w \in W \cap W^{s,p}(\Omega)$,

$$\|P_h w - w\|_{L^p(\Omega)} \leq Q h^s \|w\|_{W^{s,p}(\Omega)}, \quad 0 \leq s \leq k, \quad 1 \leq p < \infty. \quad (4.19)$$

Furthermore, Johnson *et al.*, ([27]) proved that the projector P_h satisfies the following property :

$$\|P_h w - w\|_{L^\infty(\Omega)} \leq Q h^s \left(\log \frac{1}{h} \right)^{1/2} \|w\|_{s+1}, \quad 0 \leq s \leq k, \quad (4.20)$$

for $w \in W \cap W^{s,\infty}(\Omega)$. Thus, it follows from (4.20) and (4.17) that

$$\|P_h(\text{div } v) - \text{div } v\|_{L^\infty(\Omega)} \leq Q h^s \left(\log \frac{1}{h} \right)^{1/2} \|\text{div } v\|_{s+1}, \quad 0 \leq s \leq k, \quad (4.21)$$

for any $v \in V$ and $\text{div } v \in H^{k+1}(\Omega)$.

Assume the concentration $C \in M_h$ to be given. Our continuous-time approximation procedure for the pressure is defined by the following mixed method :

Find $U = U(C) \in V_h$, $P = P(C) \in W_h$ such that

$$A(C, U, v) + B(v, P) = (g(C), v), \quad V \in V_h, \quad t \in I, \quad (4.22)$$

$$B(U, \varphi) = -(q, \varphi), \quad \varphi \in W_h. \quad (4.23)$$

It is frequently valuable to decompose the analysis of convergence of finite element methods by passing through an elliptic projection of the solution of the differential problem into the finite element space. Consider first the map $\{\tilde{U}, \tilde{P}\} : I \rightarrow V_h \times W_h$ given by

$$A(c, \tilde{U}, v) + B(v, \tilde{P}) = (g(c), v), \quad v \in V_h, \quad (4.24)$$

$$B(\tilde{U}, \varphi) = -(q, \varphi), \quad \varphi \in W_h. \quad (4.25)$$

It is well known that

$$\|u - \tilde{U}\|_V + \|p - \tilde{P}\|_W \leq Q [\|p\|_{L^\infty(I; H^{k+3}(\Omega))} h^{k+1}], \quad (4.26)$$

if the solution p is smooth enough ([10, 9])

The difference between the numerical solution (U, P) and the elliptic projection (\tilde{U}, \tilde{P}) of the exact solution (u, p) is given by the following lemma proven by Douglas *et al*, ([10, 9])

LEMMA 4.1 *Let (U, P) and (\tilde{U}, \tilde{P}) be the solutions to (4.22)-(4.23) and (4.24)-(4.25), respectively. Then,*

$$\|U - \tilde{U}\|_V + \|P - \tilde{P}\|_W \leq Q \left\{ 1 + \|\tilde{U}\|_{L^\infty(\Omega)} \right\} \|c - C\|_{L^2(\Omega)} \quad (4.27)$$

If, in addition, $p \in L^\infty(I, H^{k+3}(\Omega))$ and $k > 0$, then the quasiregularity of the grid implies that \tilde{U} is bounded in $L^\infty(I, L^\infty(\Omega))$. Thus,

$$\|U - \tilde{U}\|_V + \|P - \tilde{P}\|_W \leq Q \|p\|_{L^\infty(I; H^{k+3}(\Omega))} \|c - C\|_{L^2(\Omega)} \quad (4.28)$$

4.2. Stabilized methods and their stabilities

We retain the notation of § 3.2 in the formulation of the stabilized finite element methods for (4.1)-(4.8) which are given as follows

For $\alpha \in \{1, 0, -1\}$ and $m = 1, 2, \dots, M$, find $C \in M_h^m$ such that

$$B_\alpha^m(U, C, U, \theta) = F_\alpha^m(U, \theta), \quad \forall \theta \in M_h^m, \quad (4.29)$$

where

$$\begin{aligned} B_\alpha^m(U, C, V, \theta) &= \int_{I_m} (\phi C_t + U \cdot \nabla C, \theta) dt + \int_{I_n} (D(U) \nabla C, \nabla \theta) dt \\ &+ \langle \phi [C], \theta_+ \rangle_{m-1} + \int_{I_m} (q_i C, \theta) dt \\ &+ \sum_{T \in \mathcal{T}_h} \int_{I_m} \tau_T (\phi C_t + U \cdot \nabla C - \nabla \cdot (D(U) \nabla C) + q_i C, \\ &\phi \theta_t + V \cdot \nabla \theta + \alpha \nabla \cdot (D(V) \nabla \theta))_T dt, \end{aligned} \quad (4.30)$$

$$\begin{aligned} F_\alpha^m(V, \theta) &= \int_{I_m} (c_i q_i, \theta) dt + \sum_{T \in \mathcal{T}_h} \int_{I_m} (c_i q_i, \\ &\tau_T (\phi \theta_t + V \cdot \nabla \theta + \alpha \nabla \cdot (D(V) \nabla \theta)))_T dt, \end{aligned} \quad (4.31)$$

where $U = U(C)$ is the solution of (4.22)-(4.23) and the stabilizing parameter τ_T is defined similarly to (3.28)-(3.32) by

$$\tau = \tau_T(t) = \frac{h_T}{2 \|U\|_{0, \infty, T}} \xi(P_{eT}) \quad \text{on } T \text{ for } T \in T_h, \quad (4.32)$$

$$\xi(P_{eT}) = \min(P_{eT}, 1), \quad (4.33)$$

where P_{eT} is the mesh-dependent Péclet number

$$P_{eT}(x) = \frac{m_k \|U\|_{0, \infty, T} h_T}{D_T^2/dT} \quad \text{on } T \text{ for } T \in T_h, \quad (4.34)$$

with D_T , d_T , and m_k given by

$$D_T^2 = 2(d_m + d_\ell \|U\|_{0, \infty, T})^2 + 2(3d_\ell - 2d_t)^2 \|\nabla U\|_{0, \infty, T}^2 h_T^2 c_{\text{inv}}, \quad (4.35)$$

$$d_T = d_m + d_t \inf_{x \in T} |U|, \quad (4.36)$$

$$m_k = \frac{2}{3} \min\left(\frac{1}{2}, c_{\text{inv}}\right), \quad (4.37)$$

where c_{inv} is defined by (3.9).

We remark here that Lemma 4.1, Theorem 3.2, and the quasiregularity of the mesh imply that $\|U\|_{0, \infty, T}$ is bounded independently of h ; therefore, $\|\nabla U\|_{0, \infty, T}^2 h_T^2$ is also bounded above, by the inverse inequality. Thus, the parameter τ_T is well defined. Noting the similarity between the definitions of the stability parameter τ_T defined here and the one found in (3.28)-(3.32) in the last section, it is easy to verify, by following the proofs in (3.41) and (3.33), that

$$\left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla C) \right\|_{0, T}^2 \leq \frac{1}{3} d_T \|\nabla C\|_{0, T}^2, \quad (4.38)$$

$$D_T^2/d_T^2 \geq \frac{2(d_m + d_\ell \|U\|_{0, \infty, T})^2}{\left(d_m + d_t \inf_{x \in T} (U)\right)^2} \geq 2. \quad (4.39)$$

Before we establish the existence and uniqueness of (4.29), we want to demonstrate the following stability result.

THEOREM 4.1 : *Let $U \in V_h$ be a solution to (4.22), and let u be the exact solution and $u \in H^1(I, H^k(\Omega)) \cap L^2(I, H^{k+1}(\Omega))$. Then, for sufficiently*

small h ,

$$B_\alpha(U, \theta, U, \theta) \equiv \sum_{m=1}^M B_\alpha^m(U, \theta, U, \theta) \geq \frac{1}{4} ||| \theta |||_\phi^2 - \langle \phi \theta_-, \theta_- \rangle_0$$

for $\alpha \in \{1, 0, -1\}$ and any $\theta \in M_h$, where

$$\begin{aligned} ||| \theta |||_\phi^2 &= \langle \phi \theta_-, \theta_- \rangle_M + \langle \phi \theta_-, \theta \rangle_0 + \sum_{m=1}^M \left\| \phi^{\frac{1}{2}} [\theta] \right\|_{m-1}^2 + \\ &+ \sum_{m=1}^M \int_{I_M} \left\| |q|^{\frac{1}{2}} \theta \right\|^2 dt + \sum_{m=1}^M \sum_{T \in T_h} \int_{I_m} d_T \|\nabla \theta\|_{0,T}^2 dt + \\ &+ \sum_{m=1}^M \sum_{T \in T_h} \int_{I_M} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 dt. \end{aligned} \tag{4.41}$$

Moreover, if C is a solution to our scheme (4.29), then

$$||| C |||_{\phi, m}^2 \leq Q \int_{I_m} \left(\left\| c_i q_i^{\frac{1}{2}} \right\|^2 + \sum_{T \in T_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 \right) dt, \tag{4.42}$$

where

$$\begin{aligned} ||| C |||_{\phi, m}^2 &= \langle \phi C_-, C_- \rangle_m - \langle \phi C_-, C_- \rangle_{m-1} + \langle \phi [C], [C] \rangle_{m-1} \\ &\quad + \int_{I_m} (|q|, C^2) dt + \sum_{T \in T_h} \int_{I_m} d_T \|\nabla C\|^2 dt \\ &\quad + \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi C_t + U \cdot \nabla C) \right\|_{0,T}^2 dt. \end{aligned} \tag{4.43}$$

Proof For fixed $U \in V_h$ and $\theta \in M_h$, it is easy to show that

$$\begin{aligned} B_\alpha^m(U, \theta, U, \theta) &= \\ &= \left[\int_{I_m} (\phi(x) \theta_t, \theta) dt + \langle \phi [\theta], \theta_+ \rangle_{m-1} \right] + \left[\int_{I_m} (U \cdot \nabla \theta, \theta) dt \right] \\ &\quad + \left[\int_{I_m} (q_i \theta, \theta) dt + \int_{I_m} (D(U) \nabla \theta, \nabla \theta) dt \right] \\ &\quad + \left[\sum_{T \in T_h} \int_{I_M} \tau_T (\phi \theta_t + U \cdot \nabla \theta - \nabla \cdot (D(U) \nabla \theta) + q_i \theta, \right. \\ &\quad \left. \phi \theta_t + U \cdot \nabla \theta + \alpha \nabla \cdot (D(U) \nabla \theta)) dt \right] \\ &\equiv T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{4.44}$$

We shall estimate each term T_i in the modestly long argument that follows. First, T_1 and T_2 can be rewritten in the forms by integrating by parts

$$\begin{aligned} T_1 &\equiv \int_{I_m} (\phi(x) \theta_t, \theta) dt + \langle \phi(x) [\theta], \theta_+ \rangle_{m-1} \\ &= \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1}, \end{aligned} \quad (4.45)$$

$$\begin{aligned} T_2 &\equiv \int_{I_m} (U \cdot \nabla \theta, \theta) dt = \int_{I_m} \left(\frac{1}{2} \langle U \cdot n, \theta^2 \rangle_{\partial \Omega} - \frac{1}{2} (\nabla \cdot U, \theta^2) \right) dt \\ &= -\frac{1}{2} \int_{I_m} (\nabla \cdot U, \theta^2) dt \\ &= \frac{1}{2} \int_{I_m} (\nabla \cdot (u - U), \theta^2) dt - \frac{1}{2} \int_{I_m} (q, \theta^2) dt, \end{aligned} \quad (4.46)$$

where u is the exact solution to (4.2); we have used the definition (3.5).

Let U be the solution of the pressure equation (4.22)-(4.23). Then,

$$(\operatorname{div} U, \varphi) = (q, \varphi) = (\operatorname{div} u, \varphi), \quad \forall \varphi \in W_h,$$

so that

$$(\nabla \cdot U - \nabla \cdot u, \varphi) = 0, \quad \forall \varphi \in W_h. \quad (4.47)$$

By (4.17), we know that $\nabla \cdot U \in W_h$. Thus,

$$\nabla \cdot U = P_h(\nabla \cdot u), \quad (4.48)$$

where P_h is the L_2 projection into W_h . By (4.21), we have for $s \geq 0$

$$\|\nabla \cdot U - \nabla \cdot u\|_{L^\infty(\Omega)} \leq Qh^s \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \|\operatorname{div} u\|_{s+1}. \quad (4.49)$$

Note that there exists a function $\varphi \in W_h$,

$$\begin{aligned} (\nabla \cdot (u - U), \theta^2) &= (\nabla \cdot (u - U), \theta^2 - \varphi) \\ &\leq \|\nabla \cdot u - \nabla \cdot U\|_{L^\infty(\Omega)} \|\theta^2 - \varphi\|_{L^1(\Omega)} \\ &\leq Qh \|\nabla \cdot u - \nabla \cdot U\|_{L^\infty(\Omega)} \|\nabla \theta^2\|_{L^1(\Omega)}, \end{aligned} \quad (4.50)$$

where we have used the property that the space W_h possesses optimal approximation properties in $L^1(\Omega)$, as well as in $L^2(\Omega)$, for functions that are orthogonal to constants, as is $\nabla \cdot (u - U)$ ([10, 9]). Substituting (4.49)

with $s = 0$ into the inequality (4.50) yields

$$|(\nabla \cdot (u - U), \theta^2)| \leq Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \|\nabla \cdot u\|_1 \|\nabla \theta\| \|\theta\|. \tag{4.51}$$

Substituting (4.51) into (4.46), we have

$$T_2 \geq -\frac{1}{2} \int_{I_m} (q, \theta^2) dt - Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \int_{I_m} \|\nabla \cdot u\|_1 \|\nabla \theta\| \|\theta\| dt. \tag{4.52}$$

Similarly, by Lemma 3.4, we have

$$\begin{aligned} T_3 &= \int_{I_m} (q_t, \theta, \theta) dt + \int_{I_m} (D(U) \nabla \theta, \nabla \theta) dt \\ &\geq \int_{I_m} (q_t, \theta^2) dt + \sum_{T \in T_h} \int_{I_m} \left\| d_T^{\frac{1}{2}} \nabla \theta \right\|_{0,T}^2 dt, \end{aligned} \tag{4.53}$$

$$\begin{aligned} T_4 &= \sum_{T \in T_h} \int_{I_m} \tau_T (\phi \theta_t + U \cdot \nabla \theta - \nabla \cdot (D(U) \nabla \theta) + q_t, \theta) \\ &\quad \phi \theta_t + U \cdot \nabla \theta + \alpha \nabla \cdot (D(U) \nabla \theta))_T dt \\ &\geq \sum_{T \in T_h} \left[\int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 dt \right. \\ &\quad \left. - (1 - \alpha) \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T} \right. \right. \\ &\quad \left. \left. \cdot \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla \theta) \right\|_{0,T} \right) dt + \alpha \int_{I_m} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla \theta) \right\|_{0,T}^2 dt \right] \\ &\quad - \sum_{T \in T_h} \left[\int_{I_m} \left\| \tau^{\frac{1}{2}} q_t \right\|_{0,T} \left(\left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T} + \right. \right. \\ &\quad \left. \left. + |\alpha| \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla \theta) \right\|_{0,T} \right) dt \right] \\ &\equiv T_{41} + T_{42}, \end{aligned} \tag{4.54}$$

where T_{4_i} denotes the two sums in the equation above. To further simplify (4.54), we shall estimate T_{41} and T_{42} as follows. By using (4.38) and

following the proof of (3.45), we have

$$T_{41} \cong \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(\frac{1}{2} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 - \frac{1}{3} d_T \|\nabla \theta\|_{0,T}^2 \right) dt, \quad (4.55)$$

$$\begin{aligned} T_{42} &\cong - \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 + \frac{1}{4} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 \right. \\ &\quad \left. + \frac{1}{2} |\alpha| \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 + \frac{1}{2} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla \theta) \right\|_{0,T}^2 \right) dt \\ &\cong - \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(\frac{1}{4} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 + \left(1 + \frac{1}{2} |\alpha| \right) \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 \right. \\ &\quad \left. + \frac{1}{2} \left\| \tau^{\frac{1}{2}} \nabla \cdot (D(U) \nabla \theta) \right\|_{0,T}^2 \right) dt \\ &\cong \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(-\frac{1}{4} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 - \frac{1}{6} d_T \|\nabla \theta\|_{0,T}^2 \right) \\ &\quad - \left(1 + \frac{1}{2} |\alpha| \right) \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 \right) dt. \quad (4.56) \end{aligned}$$

Substituting (4.55) and (4.56) into (4.54) gives

$$\begin{aligned} T_4 &\cong \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(\frac{1}{4} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 - \frac{1}{2} d_T \|\nabla \theta\|_{0,T}^2 \right. \\ &\quad \left. - \left(1 + \frac{1}{2} |\alpha| \right) \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 \right) dt. \quad (4.57) \end{aligned}$$

Substituting (4.45), (4.52), (4.53), and (4.57) into (4.44) gives

$$\begin{aligned} B_\alpha^m(U, \theta, U, \theta) &\cong \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \\ &\quad + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1} \\ &\quad - \frac{1}{2} \int_{I_m} (q, \theta^2) dt - Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \int_{I_m} \|\nabla \cdot u\|_1 \|\nabla \theta\| \|\theta\| dt \\ &\quad + \int_{I_m} (q_t, \theta^2) dt + \int_{I_m} \sum_{T \in \mathcal{T}_h} d_T \|\nabla \theta\|_{0,T}^2 dt \end{aligned}$$

$$\begin{aligned}
 & + \sum_{T \in \mathcal{T}_h} \int_{I_m} \left(\frac{1}{4} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 - \frac{1}{2} d_T \|\nabla \theta\|_{0,T}^2 \right) dt \\
 & - \frac{3}{2} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 dt \\
 & = \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1} \\
 & + \frac{1}{2} \int_{I_m} (q_t + q_0, \theta^2) dt + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt \\
 & + \frac{1}{4} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 \\
 & - Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \cdot \|\nabla \cdot u\|_{0, \infty, s_m} \int_{I_m} \|\nabla \theta\| \|\theta\| dt \\
 & - \frac{3}{2} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 dt, \tag{4.58}
 \end{aligned}$$

where we have used (2.56).

Since $\theta \in M_h^m \in H_0^1(\Omega)$ for each $t \in I$, the Poincaré inequality implies that

$$\begin{aligned}
 Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \|\nabla \cdot u\|_{L^\infty(I, H^1)} \int_{I_m} \|\nabla \theta\| \|\theta\| dt \\
 \leq Qh \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \|\nabla \cdot u\|_{0, \infty, s_m} \int_{I_m} \sum_{T \in \mathcal{T}_h} \|\nabla \theta\|^2 dt \\
 \leq \frac{1}{4} \int_{I_m} \sum_{T \in \mathcal{T}_h} d_T \|\nabla \theta\|^2 dt \tag{4.59}
 \end{aligned}$$

for small h , since $h \left(\log \frac{1}{h} \right)^{\frac{1}{2}} \rightarrow 0$ as $h \rightarrow 0$ and $u \in W^{1, \infty}$. Thus, substituting (4.59) into (4.58) yields

$$\begin{aligned}
 B_\alpha^m(U, \theta, U, \theta) \geq \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \\
 + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{I_m} (|q|, \theta^2) + \frac{1}{4} \sum_{T \in T_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt \\
 & + \frac{1}{4} \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 dt \\
 & - \frac{3}{2} \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 dt
 \end{aligned} \tag{4.60}$$

for small h .

Now, by (4.32) and (4.39), we see that

$$\begin{aligned}
 \tau & = \frac{h_T}{2 \|U\|_{0,\infty,T}} \xi(P_{eT}) \leq \frac{h_T}{2 \|U\|_{0,\infty,T}} P_{eT} \\
 & = \frac{h_T}{2 \|U\|_{0,\infty,T}} \frac{m_k \|U\|_{0,\infty,T}}{D_T^2/d_T} \leq \frac{m_k h_T^2}{4 d_T}.
 \end{aligned} \tag{4.61}$$

Thus,

$$\left\| (\tau q_t) \right\|_{0,\infty,T} \leq \tau_T \|q_t\|_{0,\infty,T} \leq \frac{m_k h^2}{4 d_T} \|q_t\|_{0,\infty,T} \leq \frac{1}{8},$$

if h is sufficiently small. Therefore,

$$\begin{aligned}
 \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} q_t \theta \right\|_{0,T}^2 dt & \leq \\
 & \leq \sum_{T \in T_h} \left\| (\tau q_t) \right\|_{0,\infty,T} \int_{I_m} (q_t, \theta^2)_T dt \\
 & \leq \frac{1}{8} \sum_{T \in T_h} \int_{I_m} (q_t, \theta^2)_T dt \leq \frac{1}{8} \sum_{T \in T_h} \int_{I_m} (|q|, \theta^2)_T dt,
 \end{aligned} \tag{4.62}$$

where we have used the fact that $|q| \geq q_t \geq 0$. Substituting (4.62) into (4.60) yields

$$\begin{aligned}
 B_\alpha^m(U, \theta, U, \theta) & \geq \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \\
 & \qquad \qquad \qquad + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1} \\
 & + \frac{1}{4} \int_{I_m} (|q|, \theta^2) dt + \frac{1}{4} \sum_{T \in T_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt \\
 & + \frac{1}{4} \sum_{T \in T_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 dt
 \end{aligned} \tag{4.63}$$

for small h .

Next, by (4.38) we can similarly show that

$$\begin{aligned}
 F_\alpha^m(U, \theta) &= \int_{I_m} (c_i q_i, \theta) dt + \\
 &\quad + \sum_{T \in \mathcal{T}_h} \int_{I_m} (c_i q_i, \tau_T(\phi \theta_t + U \cdot \nabla \theta + \alpha \nabla \cdot (D(U) \nabla \theta)))_T dt \\
 &\leq Q \int_{I_m} \left(\left\| c_i q_i^{\frac{1}{2}} \right\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 \right) dt \\
 &\quad + \frac{1}{8} \int_{I_m} \left(\left\| \tau_T^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|^2 + \frac{1}{3} \left\| d_T^{\frac{1}{2}} \nabla \theta \right\|^2 + \left\| |q| \frac{1}{2} \theta \right\|^2 \right) dt. \tag{4.64}
 \end{aligned}$$

Thus, combining with (4.63), (4.64), and (4.29) we have, for any solution θ to (4.29),

$$\begin{aligned}
 &\frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \\
 &\quad + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1} + \frac{1}{4} \int_{I_m} (|q|, \theta^2) dt + \\
 &\quad + \frac{1}{8} \sum_{T \in \mathcal{T}_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt + \frac{1}{8} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau_T^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0,T}^2 dt \\
 &\leq Q \int_{I_m} \left(\left\| c_i q_i^{\frac{1}{2}} \right\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 \right) dt. \tag{4.65}
 \end{aligned}$$

Taking the sum over m proves the theorem. ■

4.3. Error estimates

In this section, we shall prove the following convergence result :

THEOREM 4.2 : *Let the exact solution c be in $H^1(I, H^k(\Omega))$, and let (u, p) , the exact solutions to (4.2), be smooth enough such that $u \in L^\infty(I, H^{k+1}(\Omega))$. Further, we assume that either u satisfies the condition*

$$\left\| \frac{|\nabla u|}{|u|} \right\|_{0, \infty}^2 \leq Q, \tag{4.66}$$

or the dispersion tensor D is independent of u as :

$$d_\varphi = 0 \quad \text{and} \quad d_t = 0. \tag{4.67}$$

Let C be the numerical solution of (4.29). Then,

$$|||c - C|||_{\phi}^2 \leq Q \sum_{m=1}^M \sum_{T \in T_h} h_T^{2k} \left[\int_{I_m} \|c\|_{k+1, T}^2 + \|c_t\|_{k, T}^2 dt \right] \cdot [H(P_{eT} - 1)h_T + H(1 - P_{eT})], \quad (4.68)$$

where $|||\cdot|||_{\phi}$ is defined by (4.41).

Proof: Noting that c_t, q is independent of c and (4.29) is a residual method, we have

$$B_{\alpha}^m(u, c, U, \theta) = F_{\alpha}^m(U, \theta), \quad \theta \in M_h^m. \quad (4.69)$$

Thus, for any $\tilde{C} \in M_h^m$, by subtracting (4.69) from (4.29), we have for any $\theta \in M_h^m$

$$\begin{aligned} 0 &= F_{\alpha}^m(U, \theta) - F_{\alpha}^m(U, \theta) = B_{\alpha}^m(U, C, U, \theta) - B_{\alpha}^m(u, c, U, \theta) \\ &= B_{\alpha}^m(U, \xi, U, \theta) + B_{\alpha}^m(U, -\eta, U, \theta) + B_{\alpha}^m(U, c, U, \theta) - \\ &\quad - B_{\alpha}^m(u, c, U, \theta), \end{aligned} \quad (4.70)$$

where

$$\xi = C - \tilde{C} \quad \text{and} \quad \eta = c - \tilde{C}. \quad (4.71)$$

Taking $\theta = \xi \in M_h^m$ in (4.70) gives

$$\begin{aligned} B_{\alpha}^m(U, \xi, U, \xi) &= \\ &= B_{\alpha}^m(U, \eta, U, \xi) + [B_{\alpha}^m(u, c, U, \xi) - B_{\alpha}^m(U, c, U, \xi)]. \end{aligned} \quad (4.72)$$

Assume that c is sufficiently smooth. By checking the proof of Lemma 3.5, we can prove likewise that

$$|B_{\alpha}(U, \eta, U, \xi)| = \left| \sum_{m=1}^M B_{\alpha}^m(U, \eta, U, \xi) \right| \leq Q |||\eta|||_{\phi} |||\xi|||_{\phi}, \quad (4.73)$$

where $|||\eta|||_{\phi}$ is defined by (4.41), and $|||\eta|||_{\phi}$ is defined similarly to (3.48) by

$$\begin{aligned} |||\eta|||_{\phi}^2 &= \sum_{m=1}^M \left| \phi^{\frac{1}{2}} \eta \right|_m^2 + \sum_{m=1}^M \int_{I_m} \left\| \tau^{-\frac{1}{2}} \eta \right\|^2 dt + \sum_{m=1}^M \int_{I_m} \left\| d_T^{\frac{1}{2}} \nabla \eta \right\|^2 dt \\ &+ \sum_{m=1}^M \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \eta_t + U \cdot \nabla \eta) \right\|^2 dt + \sum_{m=1}^M \int_{I_m} \left\| \tau^{\frac{1}{2}} (\nabla \cdot (D(U) \nabla \eta)) \right\|_{0, T}^2 dt. \end{aligned} \quad (4.74)$$

It remains to show that $B_\alpha^m(u, c, U, \xi) - B_\alpha^m(U, c, U, \xi)$ can be bounded by (4.94). First, we note that

$$\begin{aligned}
 B_\alpha^m(u, c, U, \xi) - B_\alpha^m(U, c, U, \xi) &= \left[\int_{I_m} ((u - U) \nabla c, \xi) dt \right] + \\
 &+ \left[\int_{I_m} ((D(u) - D(U)) \nabla c, \nabla \xi) dt \right] \\
 &+ \left[\sum_{\tau \in \mathcal{T}_h} \int_{I_m} ((u - U) \nabla c - \nabla \cdot ((D(u) - D(U)) \nabla c), \right. \\
 &\quad \left. \tau_\tau (\phi \xi_t + U \cdot \nabla \xi - \alpha \nabla \cdot (D(U) \nabla \xi)))_{\tau} dt \right] \\
 &\equiv T_1 + T_2 + T_3.
 \end{aligned} \tag{4.75}$$

We estimate each term T_i as follows :

$$T_1 \leq Q \int_{I_m} \|u - U\| \|\xi\| dt, \tag{4.76}$$

$$T_2 \leq Q \int_{I_m} \left\| d^{-\frac{1}{2}} (D(u) - D(U)) \right\| \left\| d^{\frac{1}{2}} \nabla \xi \right\| dt, \tag{4.77}$$

$$\begin{aligned}
 T_3 &\leq Q \sum_{\tau \in \mathcal{T}_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (u - U) \right\|_{0, \tau} + \left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0, \tau} \right) \\
 &\cdot \left(\left\| \tau^{\frac{1}{2}} (\phi \xi_t + U \nabla \xi) \right\|_{0, \tau} + \left\| \tau^{\frac{1}{2}} (\nabla \cdot D(U) \nabla \xi) \right\|_{0, \tau} \right) \\
 &\leq Q \sum_{\tau \in \mathcal{T}_h} \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (u - U) \right\|_{0, \tau} + \left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0, \tau} \right) \\
 &\cdot \left(\left\| \tau^{\frac{1}{2}} (\phi \xi_t + U \nabla \xi) \right\|_{0, \tau} + \left\| d_T^{\frac{1}{2}} \nabla \xi \right\|_{0, \tau} \right) dt.
 \end{aligned} \tag{4.78}$$

By substituting each T_i into (4.75) and using (4.61), we have

$$\begin{aligned}
 |B_\alpha^m(u, c, U, \xi) - B_\alpha^m(U, c, U, \xi)| &\leq \\
 &\leq Q \int_{I_m} (\|\xi\|^2 + \|u - U\|^2 + \|D(u) - D(U)\|^2 + \\
 &\quad + \sum_{\tau \in \mathcal{T}_h} \left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0, \tau}^2) dt \\
 &\quad + \varepsilon \left(\int_{I_m} \left\| d^{\frac{1}{2}} \nabla \xi \right\|^2 dt + \int_{I_m} \sum_{\tau \in \mathcal{T}_h} \left\| \tau^{\frac{1}{2}} (\phi \xi_t + U \cdot \nabla \xi) \right\|_{0, \tau}^2 dt \right).
 \end{aligned} \tag{4.79}$$

To further simplify the equation above, we need to estimate $\|u - U\|$, $\|D(u) - D(U)\|$, and $\left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0,T}$. All that remains is to check each of these terms to be bounded by (4.80), (4.82), and (4.93) below, respectively. First, by Lemma 4.1 and (4.20), we can easily show that

$$\|u - U\|^2 \leq Q [\|c - C\|^2 + h^{2(k+1)}] \leq Q [\|\xi\|^2 + \|\eta\|^2 + h^{2(k+1)}] \quad (4.80)$$

Next, to estimate $D(u) - D(U)$, by following Douglas *et al* [10, 9], we can easily check the relation

$$|D(u) - D(U)| \leq Q |u - U| \quad (4.81)$$

It follows from (4.80) that

$$\|D(u) - D(U)\|^2 \leq Q \|u - U\|^2 \leq Q [\|\xi\|^2 + \|\eta\|^2 + h^{2(k+1)}] \quad (4.82)$$

Finally, it remains to estimate $\left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0,T}$. By (3.19), we have on each element T

$$\begin{aligned} \tau_T^{\frac{1}{2}} (\nabla \cdot (D(u) \nabla c) - \nabla \cdot (D(U) \nabla c)) &= \\ &= d_\ell \tau_T^{\frac{1}{2}} (\nabla \cdot (|u| E(u) \nabla c) - \nabla \cdot (|U| E(U) \nabla c)) + \\ &+ d_t \tau_T^{\frac{1}{2}} (\nabla \cdot (|u| E^\perp(u) \nabla c) - \nabla \cdot (|U| E^\perp(U) \nabla c)) \\ &\equiv d_\ell \tau_T^{\frac{1}{2}} T_1 + d_t \tau_T^{\frac{1}{2}} T_2 \end{aligned} \quad (4.83)$$

We shall estimate T_1 here, T_2 can be bounded similarly. First, (3.20) implies that

$$\begin{aligned} T_1 &\equiv \nabla \cdot (|u| E(u) \nabla c) - \nabla \cdot (|U| E(U) \nabla c) \\ &= \left((\nabla \cdot u) \frac{u \cdot \nabla c}{|u|} - (\nabla \cdot U) \frac{U \cdot \nabla c}{|U|} \right) \\ &+ \left(-\frac{1}{|u|^3} (u \cdot \nabla u \cdot u) (u \cdot \nabla c) + \frac{1}{|U|^3} (U \cdot \nabla U \cdot U) (U \cdot \nabla c) \right) \\ &+ \left(\frac{u \cdot \nabla u \cdot \nabla c}{|u|} - \frac{U \cdot \nabla U \cdot \nabla c}{|U|} \right) \\ &+ \left(\frac{u \cdot (\nabla \nabla c) \cdot u}{|u|} - \frac{U \cdot (\nabla \nabla c) \cdot U}{|U|} \right) \\ &\equiv T_{11} + T_{12} + T_{13} + T_{14} \end{aligned} \quad (4.84)$$

If $|u| > 0$, then

$$\left| \frac{u}{|u|} - \frac{U}{|U|} \right| = \left| \frac{(u-U)|U| + U(|U| - |u|)}{|u||U|} \right| \leq \frac{2}{|u|} |u - U| \quad (4.85)$$

The above equality is derived under the assumption that $|U| > 0$, but the final result clearly holds for any U . By (4.85),

$$\begin{aligned} T_{11} &= (\nabla \cdot u - \nabla \cdot U) \frac{U}{|U|} \cdot \nabla c + (\nabla \cdot u) \left(\frac{u}{|u|} - \frac{U}{|U|} \right) \cdot \nabla c \\ &\leq |\nabla \cdot (u - U)| |\nabla c| + 2 \frac{|\nabla \cdot u|}{|u|} |\nabla c| |u - U|, \end{aligned} \quad (4.86)$$

$$\begin{aligned} T_{12} &= - \left(\frac{u}{|u|} \cdot \nabla u \cdot \frac{u}{|u|} \left(\frac{u}{|u|} - \frac{U}{|U|} \right) \cdot \nabla c \right) \\ &\quad - \left(\frac{u}{|u|} \cdot \nabla u \cdot \frac{u}{|u|} - \frac{U}{|U|} \cdot \nabla U \cdot \frac{U}{|U|} \right) \left(\frac{U}{|U|} \right) \cdot (\nabla c) \\ &\leq 6 \frac{|\nabla u|}{|u|} |\nabla c| |u - U| + |\nabla u - \nabla U| |\nabla c|, \end{aligned} \quad (4.87)$$

$$T_{13} \leq 2 \frac{|\nabla u|}{|u|} |\nabla c| |u - U| + |\nabla u - \nabla U| |\nabla c|, \quad (4.88)$$

$$\begin{aligned} T_{14} &= \left(\frac{u}{|u|} - \frac{U}{|U|} \right) (\nabla \nabla c) u + \frac{U}{|U|} (\nabla \nabla c) (u - U) \\ &\leq 3 |\nabla \nabla c| |u - U| \end{aligned} \quad (4.89)$$

Substituting (4.86)-(4.89) into (4.84) yields

$$\begin{aligned} T_1 \leq |\nabla \cdot (u - U)| + \frac{2|\nabla \cdot u| + 8|\nabla u|}{|u|} |\nabla c| |u - U| + \\ + 2|\nabla u - \nabla U| |\nabla c| + 3|\nabla \nabla c| |u - U| \end{aligned} \quad (4.90)$$

By (4.61),

$$\begin{aligned} \left\| \tau^{\frac{1}{2}} T_1 \right\|_{0, T} \leq Q \tau_T^{\frac{1}{2}} \|T_1\| \leq Q h \|T_1\|_{0, T} \leq Q \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0, \infty} \right) \cdot \\ \cdot (h \|\nabla \cdot (u - U)\|_{0, T} + h \|(u - U)\|_{0, T} + h \|\nabla(u - U)\|_{0, T}) \end{aligned} \quad (4.91)$$

Similarly,

$$\begin{aligned} \left\| \tau^{\frac{1}{2}} T_2 \right\|_{0, T} \leq Q \tau_T^{\frac{1}{2}} \|T_2\| \leq Q h \|T_2\|_{0, T} \leq Q \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0, \infty} \right) \cdot \\ \cdot (h \|\nabla \cdot (u - U)\|_{0, T} + h \|(u - U)\|_{0, T} + h \|\nabla(u - U)\|_{0, T}) \end{aligned} \quad (4.92)$$

Substituting (4.91) and (4.92) into (4.83) yields

$$\begin{aligned}
 & \sum_{T \in T_h} \left\| \tau^{\frac{1}{2}} \nabla \cdot ((D(u) - D(U)) \nabla c) \right\|_{0,T} \leq \\
 & \leq \sum_{T \in T_h} \left(d_\ell \left\| \tau^{\frac{1}{2}} T_1 \right\|_{0,T} + d_t \left\| \tau^{\frac{1}{2}} T_2 \right\|_{0,T} \right) \cdot \\
 & \leq Q \sum_{T \in T_h} \left(\left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty} \right) (d_\ell + d_t) \right) \cdot \\
 & \quad \cdot (h \|\nabla \cdot (u - U)\|_{0,T} + h \| (u - U) \|_{0,T} + h \|\nabla(u - U)\|_{0,T}) \\
 & \leq Q \sum_{T \in T_h} \left(\left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty} \right) (d_\ell + d_t) (h^{k+1} + \|c - C\|) \right) \\
 & \leq Q \sum_{T \in T_h} \left(\left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty} \right) (d_\ell + d_t) (h^{k+1} + \|\eta\| + \|\xi\|) \right), \quad (4.93)
 \end{aligned}$$

where we have used the inequalities (4.18), (4.26), (4.28), and the inverse inequality (3.9). Substituting (4.80), (4.82), and (4.93) into equation (4.79) gives

$$\begin{aligned}
 & |B_\alpha^m(u, c, U, \xi) - B_\alpha^m(U, c, U, \xi)| \leq \\
 & \leq Q \left(1 + \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty}^2 \right) (d_\ell^2 + d_t^2) \right) \int_{I_m} (\|\xi\|^2 + \|\eta\|^2 + h^{2k+2}) dt \\
 & \quad + \varepsilon \left(\int_{I_m} \left\| d^{\frac{1}{2}} \nabla \xi \right\|^2 dt + \int_{I_m} \sum_{T \in T_h} \left\| \tau^{\frac{1}{2}} (\phi \xi_t + U \cdot \nabla \xi) \right\|_{0,T}^2 dt \right) \\
 & \leq Q \left(1 + \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty}^2 \right) (d_\ell^2 + d_t^2) \right) \int_{I_m} (\|\xi\|^2 + \|\eta\|^2 + h^{2k+2}) dt \\
 & \quad + \varepsilon \|\xi\|_\phi^2. \quad (4.94)
 \end{aligned}$$

Substituting the inequalities (4.94) and (4.73) into (4.72) gives

$$\begin{aligned}
 & B_\alpha(U, \xi, U, \xi) = \left| \sum_{m=1}^M B_\alpha^m(U, \xi, U, \xi) \right| \\
 & \leq \sum_{m=1}^M |B_\alpha^m(U, \eta, U, \xi)| + \sum_{m=1}^M |B_\alpha^m(u, c, U, \xi) - B_\alpha^m(U, c, U, \xi)| \\
 & \leq Q \left(1 + \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0,\infty}^2 \right) (d_\ell^2 + d_t^2) \right) \cdot \\
 & \quad \cdot \int_{I_m} (\|\xi\|^2 + \|\eta\|^2 + h^{2k+2}) dt + \varepsilon \|\varepsilon\|_\phi^2 + Q \|\eta\|_\phi^2. \quad (4.95)
 \end{aligned}$$

On the other hand, by Theorem 4.1,

$$||| \xi |||_{\phi}^2 \leq 4 B_{\alpha}(U, \xi, U, \xi) + 4 \langle \phi \xi_-, \xi_- \rangle_0. \tag{4.96}$$

Combining the two inequalities above gives

$$\begin{aligned} ||| \xi |||_{\phi}^2 &\leq Q \left(||| \eta |||_{\phi}^2 + \left| \phi^{\frac{1}{2}} \xi_- \right|_0^2 \right) + \\ &+ Q \left(1 + \left(1 + \left\| \frac{|\nabla u|}{|u|} \right\|_{0, \infty}^2 \right) (d_{\ell}^2 + d_t^2) \right) \cdot \\ &\cdot \int_{I_m} (\|\xi\|^2 + \|\eta\|^2 + h^{2k+2}) dt. \end{aligned} \tag{4.97}$$

By Lemma 3.2, we see that, for sufficiently small h ,

$$\begin{aligned} \int_{I_m} \|\xi\|^2 dt &\leq \frac{1}{\phi_0} \int_{I_m} \left\| \phi^{\frac{1}{2}} \xi \right\|^2 dt \\ &\leq \frac{1}{\phi_0} \left(h \left| \phi^{\frac{1}{2}} \xi_- \right|_m^2 + \frac{h}{\phi_0 c_1} \int_{I_m} \|\phi(\theta' + U \cdot \nabla \theta)\|_0^2 dt \right) \cdot \\ &\quad \cdot \exp((c_1 + \|\nabla \cdot U\|_{0, \infty, s_m}) h / \phi_0) \\ &\leq \varepsilon \left[\left| \phi^{\frac{1}{2}} \xi_- \right|_m^2 + \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \xi' + U \cdot \nabla \xi) \right\|_0^2 dt \right] \\ &\leq \varepsilon ||| \xi |||_{\phi}, \end{aligned} \tag{4.98}$$

where $\phi \geq \phi_0 > 0$; we also used (4.18) and (4.26) to show the boundedness of ∇U .

Under the assumption that (4.66) or (4.97) holds, combining (4.97) and (4.98) yields

$$\begin{aligned} ||| \xi |||_{\phi}^2 &\leq Q \left(||| \eta |||_{\phi}^2 + \left| \phi^{\frac{1}{2}} \xi_- \right|_0^2 + h^{2k+2} \right) \\ &\leq Q (||| \eta |||_{\phi}^2 + h^{2k+2}), \end{aligned} \tag{4.99}$$

where (3.75) has been used in the last inequality.

The theorem now follows from the above inequality, Lemma 3.6, and standard interpolation theory. ■

It is not easy to see from the convergence result (4.68) what the order of approximation is in general. But, if $d_{\ell} = d_t = 0$ by the theorem, we can say

that $\|c - C\|_{H^1} = O\left(h^k \sqrt{\frac{h}{d_m}}\right)$ roughly in the convection-dominated region $P_{eT} > 1$, and $\|c - C\|_{H^1} = O\left(h^k \sqrt{\frac{1}{d_m}}\right)$ in the diffusion-dominated region $P_{eT} \leq 1$

4.4. Existence and uniqueness

Under the assumption that (4 29) is uniquely solvable, we have derived stability and error estimates In this section, we shall follow the idea given in [26] and apply a variant of Brouwer’s fixed point theorem to prove that, given $C(\cdot, t_{m-1})$, (4 29) has a solution on slab S_m First, define a scalar product in M_h^m by

$$[c, \theta]_m = \langle c_-, \theta_- \rangle_m + \langle c_+, \theta_+ \rangle_m + \int_{I_m} (c, \theta) dt \tag{4 100}$$

Clearly, $(M_h^m, [\cdot, \cdot])$ is a Hilbert space Let $\|\cdot\|_m$ be the norm induced by $[\cdot, \cdot]$, i e ,

$$\|c\|_m = [c, c]_m^{\frac{1}{2}} \tag{4 101}$$

Define a mapping P^m from $(M_h^m, \|\cdot\|_m) \rightarrow (M_h^m, \|\cdot\|_m)$ by

$$[P^m C, \theta]_m = B_\alpha^m(U, C, U, \theta) - F_\alpha^m(U, \theta), \tag{4 102}$$

where $U = U(C)$ is the solution to (4 22)-(4 23) Clearly, the equation (4 29) has a solution if and only if P^m has a zero point It is easy to see that P^m is well-defined To show that P^m is a continuous mapping from $(M_h^m, \|\cdot\|_m)$ into itself, assume that C_n and C belong to M_h^m and are such that $\|C_n - C\|_m \rightarrow 0$ as $n \rightarrow \infty$ We want to show that $\|P^m C_n - P^m C\|_m \rightarrow 0$ as $n \rightarrow \infty$ Since M_h^m is a finite-dimensional Hilbert space, it suffices to show that, for any $\theta \in M_h^m$,

$$[P^m C_n - P^m C, \theta] \rightarrow 0 \text{ as } n \rightarrow \infty ,$$

i e ,

$$B_\alpha^m(U_n, C_n, \theta) - F_\alpha^m(U_n, \theta) \rightarrow B_\alpha^m(U, C, U, \theta) - F_\alpha^m(U, \theta) \tag{4 103}$$

By Lemma 4 1 and quasiregularity of the meshes, $\|C_n - C\|_m \rightarrow 0$ implies that,

for any fixed h ,

$$\int_{I_m} \|U_n - U\|_{1, \infty, \Omega}^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{4.104}$$

$$\int_{I_m} \|U_n - U\|_{1, \Omega}^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.105}$$

Then, (4.103) follows the observations above.

By (4.63) and (4.64) we see that, for small h ,

$$\begin{aligned} [P^m \theta, \theta]_m &= B_\alpha^m(U, \theta, U, \theta) - F_\alpha^m(U, \theta) \\ &\cong \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m - \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_{m-1} + \frac{1}{2} \langle \phi [\theta], [\theta] \rangle_{m-1} \\ &\quad + \frac{1}{4} \int_{I_m} (|q|, \theta^2) dt + \frac{1}{4} \sum_{T \in \mathcal{T}_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt \\ &\quad + \frac{1}{4} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0, T}^2 dt \\ &\quad - \mathcal{Q} \int_{I_m} \left(\left\| c_i q_i^{\frac{1}{2}} \right\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 \right) dt \\ &\quad - \frac{1}{8} \int_{I_m} \left(\left\| \tau_T^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|^2 + \frac{1}{3} \left\| d_T^{\frac{1}{2}} \nabla \theta \right\|^2 \right) dt \\ &\cong \frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m + \frac{1}{4} \langle \phi \theta_+, \theta_+ \rangle_{m-1} + \frac{1}{4} \int_{I_m} (|q| \theta, \theta) dt \\ &\quad + \frac{1}{8} \sum_{T \in \mathcal{T}_h} \int_{I_m} d_T \|\nabla \theta\|^2 dt + \frac{1}{8} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0, T}^2 dt \\ &\quad - \mathcal{Q} \int_{I_m} \left(\left\| c_i q_i^{\frac{1}{2}} \right\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 + \langle \phi \theta_-, \theta_- \rangle_{m-1} \right) dt. \tag{4.106} \end{aligned}$$

By Lemma 3.2, there exist $\gamma > 0$ such that, for small h ,

$$\begin{aligned} &\frac{1}{2} \langle \phi \theta_-, \theta_- \rangle_m + \frac{1}{4} \langle \phi \theta_+, \theta_+ \rangle_{m-1} + \\ &\quad + \frac{1}{8} \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}} (\phi \theta_t + U \cdot \nabla \theta) \right\|_{0, T}^2 dt \\ &\cong \gamma \left[\int_{I_m} (\theta, \theta) dt + \langle \theta_-, \theta_- \rangle_m + \langle \theta_+, \theta_+ \rangle_{m-1} \right] = \gamma \|\theta\|_m^2, \tag{4.107} \end{aligned}$$

where $\| \cdot \|_m$ is defined by (4.101). Therefore, (4.106) can be further simplified to

$$[P^m \theta, \theta] \geq \gamma \|\theta\|_m^2 - Q \int_{I_m} \left(\|c_i q_i^{\frac{1}{2}}\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 + \langle \phi \theta_-, \theta_- \rangle_{m-1} \right) dt \geq 0 \quad (4.108)$$

if

$$\|\theta\|_m^2 \geq \frac{Q}{\gamma} \left[\int_{I_m} \left(\|c_i q_i^{\frac{1}{2}}\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 + \langle \phi \theta_-, \theta_- \rangle_{m-1} \right) dt \right]. \quad (4.109)$$

By the fixed-point theorem given in Corollary 1.1 of ([20], p. 279), (4.109) implies that there exists $\theta \in M_h^m$ such that

$$P^m \theta = 0 \quad \text{and} \quad \|\theta\|^2 \leq \frac{Q}{\gamma} \left[\int_{I_m} \left(\|c_i q_i^{\frac{1}{2}}\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 + \langle \phi \theta_-, \theta_- \rangle_{m-1} \right) dt \right], \quad (4.110)$$

which implies the following existence theorem.

THEOREM 4.3 : *For sufficiently small h , (4.29) has at least one solution C in M_h^m for each m . Moreover,*

$$\|C\|_m^2 \leq \frac{Q}{\gamma} \left[\int_{I_m} \left(\|c_i q_i^{\frac{1}{2}}\|^2 + \sum_{T \in \mathcal{T}_h} \left\| \tau_T^{\frac{1}{2}} c_i q_i \right\|^2 + \langle \phi C_-, C_- \rangle_{m-1} \right) dt \right], \quad (4.111)$$

where $\| \cdot \|_m$ is defined by (4.101).

To establish uniqueness, let $C_-(\cdot, t_{m-1})$ be given and assume that (C_i, U_i) $i = 1, 2$ are two solutions to (4.29); i.e.,

$$B_\alpha^m(U_i, C_i, U_i, \theta) = F_\alpha^m(U_i, \theta) \quad \text{for} \quad \theta \in M_h^m. \quad (4.112)$$

We want to show that $C_1 = C_2$, with the consequence that $U_1 = U_2$, gives uniqueness. Let $\xi = C_1 - C_2$. By (4.112),

$$\begin{aligned} B_\alpha^m(U_1, \xi, U_1, \xi) &= [B_\alpha^m(U_2, C_2, U_1, \xi) - B_\alpha^m(U_1, C_2, U_1, \xi)] + \\ &+ [B_\alpha^m(U_2, C_2, U_2, \xi) - B_\alpha^m(U_2, C_2, U_1, \xi)] \\ &+ [F_\alpha^m(U_1, \xi) - F_\alpha^m(U_2, \xi)] \equiv T_1 + T_2 + T_3. \end{aligned} \quad (4.113)$$

Before we estimate each term T_i , we would like to remark that allowing the dispersion tensor $D(u)$ to depend on u causes many difficulties in the analysis, just as it did in the proof of Theorem 4.3, which required one of the two conditions (4.66) or (4.67) to be held. With u replaced by U in (4.66), similar conditions will have to hold here in order to prove uniqueness. Unfortunately, we do not have any estimates to guarantee that $\|\nabla U/U\|_{0,\infty}$ will be bounded. Therefore, we shall assume in this section that (4.67) holds, i.e., we ignore the dependence of D on u .

$$D(u) = d_m I \tag{4.114}$$

As a consequence of quasiregularity of the meshes, and the error estimate (4.68), and the discussion at end of the proof of Theorem 4.2, we can show the following a priori estimates

$$\beta_k^{-1} \|C_i\|_{1,\infty,T} + \|U_i\|_{0,\infty,T} \leq Q \tag{4.115}$$

where

$$\beta_k = \max_{P_{eT}(T) > 1} \left(h^{k-1} (h/d_m)^{\frac{1}{2}} + 1 \right) + \max_{P_{eT}(T) \leq 1} \left(h^{k-1} d_m^{-\frac{1}{2}} + 1 \right) \tag{4.116}$$

Now, by Theorem 4.1, (4.61), and (4.28), we can show that

$$\begin{aligned} T_1 &= B_\alpha^m(U_2, C_2, U_1, \xi) - B_\alpha^m(U_1, C_2, U_1, \xi) \\ &= \int_{I_n} ((U_2 - U_1) \cdot \nabla C_2, \xi) dt + \sum_{T \in \mathcal{T}_I} \int_{I_m} ((U_2 - U_1) \cdot \nabla C_2, \\ &\quad \tau_T(U_1)(\phi \xi + U_1 \cdot \nabla \xi - \alpha \nabla \cdot (D(U_1) \cdot \nabla \xi)))_T dt \\ &\leq Q \sum_{T \in \mathcal{T}_I} \int_{I_n} \|\nabla C_2\|_{0,\infty,T} \|U_2 - U_1\|_{0,T} \|\xi\|_{0,T} dt \\ &\quad + Q \sum_{T \in \mathcal{T}_I} \int_{I_n} \tau_T^{\frac{1}{2}} \|\nabla C_2\|_{0,\infty} \|U_2 - U_1\|_{0,T} \\ &\quad \times \left(\left\| \tau^{\frac{1}{2}} (\phi \xi + U_1 \cdot \nabla \xi) \right\|_{0,T} + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|_{0,T} \right) dt \\ &\leq Q \beta_k \int_{I_n} \|U_2 - U_1\| \|\xi\| dt \\ &\quad + Q h \beta_k \int_I \|(U_2 - U_1)\| \left(\left\| \tau^{\frac{1}{2}} (\phi \xi + U_1 \cdot \nabla \xi) \right\| + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\| \right) dt \end{aligned}$$

$$\begin{aligned}
&\leq Q\beta_k \int_{I_m} \|\xi\|^2 dt + Q(h\beta_k)^2 \int_{I_m} \|\xi\|^2 dt \\
&\quad + \varepsilon \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (\phi \xi + U_1 \cdot \nabla \xi) \right\|^2 + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|^2 \right) dt \\
&\leq Q(\beta_k + (h\beta_k)^2) \int_{I_m} \|\xi\|^2 dt \\
&\quad + \varepsilon \int_{I_m} \left(\left\| \tau^{\frac{1}{2}} (\phi \xi + U_1 \cdot \nabla \xi) \right\|^2 + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|^2 \right) dt, \tag{4.117}
\end{aligned}$$

$$\begin{aligned}
T_2 &= B_\alpha^m(U_2, C_2, U_2, \xi) - B_\alpha^m(U_2, C_2, U_1, \xi) \\
&= \sum_{T \in T_h} \int_{I_m} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2 + q_t C_2, \tau_1(\phi \xi_t + U_1 \cdot \nabla \xi \\
&\quad - \alpha \nabla \cdot D \nabla \xi)) - (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2 + q_t C_2, \\
&\quad \tau_2(\phi \xi_t + U_2 \cdot \nabla \xi - \alpha \nabla \cdot D \nabla \xi))_T dt, \tag{4.118}
\end{aligned}$$

where

$$\tau_1 = \tau(U_1) = \frac{h_T}{2\|U_1\|_{0,\infty,T}} \xi(P_{eT}), \tag{4.119}$$

$$\tau_2 = \tau(U_2) = \frac{h_T}{2\|U_2\|_{0,\infty,T}} \xi(P_{eT}), \tag{4.120}$$

and $\xi(P_{eT})$ is defined by (4.33)-(4.34). Since

$$\begin{aligned}
&\tau_1(\phi \xi_t + U_1 \cdot \nabla \xi - \alpha \nabla \cdot (D \nabla \xi)) - \tau_2(\phi \xi_t + U_2 \cdot \nabla \xi - \alpha \nabla \cdot (D \nabla \xi)) = \\
&= (\tau_1 - \tau_2)(\phi \xi_t + U_1 \cdot \nabla \xi + \alpha \nabla \cdot (D \nabla \xi)) + \tau_2(U_1 - U_2) \cdot \nabla \xi, \tag{4.121}
\end{aligned}$$

(4.118) can be written as

$$\begin{aligned}
T_2 &= \sum_{T \in T_h} \int_{I_m} \frac{(\tau_1 - \tau_2)}{(\tau_2 \tau_1)^{\frac{1}{2}}} \left(\tau_2^{\frac{1}{2}} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2 + c_t q_t), \right. \\
&\quad \left. \tau_1^{\frac{1}{2}} (\phi \xi_t + U_1 \cdot \nabla \xi + \alpha \nabla \cdot (D \nabla \xi)) \right)_T
\end{aligned}$$

$$\begin{aligned}
 & + \tau_2^{\frac{1}{2}} \left(\tau_2^{\frac{1}{2}} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2 + c_i q_i), (U_1 - U_2) \cdot \nabla \xi \right)_T dt \\
 & \leq \sum_{T \in T_h} \int_{I_m} \left(\left| \frac{(\tau_1 - \tau_2)}{(\tau_2 \tau_1)^{\frac{1}{2}}} \right| \left(\left\| \tau_2^{\frac{1}{2}} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2) \right\|_{0,T} \right. \right. \\
 & \quad \left. \left. + \|c_i q_i\|_{0,T} \right) \times \left\| \tau_1^{\frac{1}{2}} (\phi \xi_t + U_1 \cdot \nabla \xi + \alpha \nabla \cdot (D \nabla \xi)) \right\|_{0,T} \right. \\
 & \quad \left. + \tau_2^{\frac{1}{2}} \left(\left\| \tau_2^{\frac{1}{2}} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2) \right\|_{0,T} \right. \right. \\
 & \quad \left. \left. + \|c_i q_i\|_{0,T} \right) \left\| (U_1 - U_2) \cdot \nabla \xi \right\|_{0,T} \right) dt \\
 & \leq Q \sum_{T \in T_h} \int_{I_m} \left(\left| \frac{(\tau_1 - \tau_2)}{(\tau_2 \tau_1)^{\frac{1}{2}}} \right|^2 + \tau_2^{\frac{1}{2}} \left\| (U_1 - U_2) \cdot \nabla \xi \right\|_{0,T} \right) dt \\
 & \quad \varepsilon \left\| \tau_1^{\frac{1}{2}} (\phi \xi_t + U_1 \cdot \nabla \xi + \alpha \nabla \cdot (D \nabla \xi)) \right\|_{0,T}^2, \tag{4.122}
 \end{aligned}$$

where we have used the estimate

$$\left\| \tau_2^{\frac{1}{2}} (\phi C_{2t} + U_2 \cdot \nabla C_2 - \nabla \cdot D \nabla C_2) \right\|_{0,T} \leq Q, \tag{4.123}$$

which can be proven from the stability. We now estimate the first two terms in (4.122). By (4.61), the second term can be bounded as follows :

$$\begin{aligned}
 \tau_2^{\frac{1}{2}} \left\| (U_1 - U_2) \cdot \nabla \xi \right\|_{0,T} & \leq Q h_T \|U_1 - U_2\|_{0,\infty,T} \|\nabla \xi\|_{0,T} \\
 & \leq Q \|U_1 - U_2\|_{0,T} \|\nabla \xi\|_{0,T} \\
 & \leq Q \|\xi\|_{0,T} \|\nabla \xi\|_{0,T} \\
 & \leq Q \|\xi\|_{0,T}^2 d_m^{-1} + \varepsilon \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|_{0,T}^2. \tag{4.124}
 \end{aligned}$$

To estimate the first term in (4.122), we consider cases for $P_{eT}(U_1)$ and $P_{eT}(U_2)$.

Case 1 : $P_{eT}(U_i) \geq 1$ for $i = 1, 2$.

From (4.119), we can see that

$$\begin{aligned} \left| \frac{\tau_1 - \tau_2}{\sqrt{\tau_1 \tau_2}} \right| &= \frac{\left| \|U_2\|_{0, \infty, T} - \|U_1\|_{0, \infty, T} \right|}{\sqrt{\|U_2\|_{0, \infty, T} \|U_1\|_{0, \infty, T}}} \\ &\leq \frac{m_k h_T}{d_m} (\|U_1 - U_2\|_{0, \infty, T}) \\ &\leq Q \|\xi\|_{0, T} d_m^{-1}, \end{aligned} \quad (4.125)$$

where we have used the inverse inequality $h \|U_1 - U_2\|_{0, \infty, T} \leq Q \|U_1 - U_2\|_{0, T}$ and the fact that $P_{eT}(U_i) \geq 1$ implies that $\|U_i\|_{0, T}^{-1} \leq (m_k h_T)/d_m$.

Case 2 : $P_{eT}(U_i) \leq 1$ for $i = 1, 2$:

$$\left| \frac{\tau_1 - \tau_2}{\sqrt{\tau_1 \tau_2}} \right| = 0, \quad (4.126)$$

since $\tau_1 = \tau_2 = m_k h_T^2/d_m$.

Case 3 : $P_{eT}(U_1) \leq 1$ and $P_{eT}(U_2) \geq 1$.

It is easy to show that

$$\tau_1 = \frac{m_k h_T^2}{d_m} \geq \frac{h_T}{2 \|U_2\|_{0, \infty, T}} = \tau_2 > 0, \quad (4.127)$$

$$\begin{aligned} \left| \frac{\tau_1 - \tau_2}{\sqrt{\tau_1 \tau_2}} \right| &\leq \frac{\tau_1}{\sqrt{\tau_1 \tau_2}} \\ &= (\tau_1/\tau_2)^{\frac{1}{2}} = \left(\frac{m_k h_T^2}{d_m} \cdot \frac{2 \|U_2\|_{0, \infty, T}}{h_T} \right)^{\frac{1}{2}} \\ &\leq \left(\frac{m_k}{d_m} \|U_2\|_{0, \infty, T} \right)^{\frac{1}{2}} h_T^{\frac{1}{2}} \leq Q h_T^{\frac{1}{2}} d_m^{-\frac{1}{2}}. \end{aligned} \quad (4.128)$$

It follows from (4.124)-(4.128) and (4.122) that

$$\begin{aligned} T_2 &\leq Q \sum_{T \in \mathcal{T}_m} \int_{I_m} (\|\xi\|_{0, T}^2 (d_m^{-2} + d_m^{-1}) + h_T d_m^{-1}) dt + \\ &\quad + \varepsilon \int_{I_m} \left(\left\| \tau_1^{\frac{1}{2}} (\phi \xi_t + U_1 \cdot \nabla \xi) \right\|^2 + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|^2 \right) dt. \end{aligned} \quad (4.129)$$

Similarly,

$$\begin{aligned}
 T_3 &= F_\alpha^m(U_1, \xi) - F_\alpha^m(U_2, \xi) \\
 &= \sum_{T \in \mathcal{T}_m} \int_{I_m} (c_t q, \tau_1(\phi \xi_t + U_1 \cdot \nabla \xi - \alpha \nabla \cdot (D \nabla \xi)))_{0,T} \\
 &\quad - (c_t q, \tau_1(\phi \xi_t + U_2 \nabla \xi - \alpha \nabla \cdot (D \nabla \xi)))_{0,T} \\
 &\leq \sum_{T \in \mathcal{T}_m} \int_{I_m} (c_t q, (\tau_1 - \tau_2)(\phi \xi_t + U_1 \cdot \nabla \xi - \alpha \nabla \cdot (D \nabla \xi)))_{0,T} \\
 &\quad - (c_t q, \tau_2(U_1 - U_2) \nabla \xi) dt \\
 &\leq Q \sum_{T \in \mathcal{T}_m} \int_{I_m} (\|\xi\|_{0,T}^2 (d_m^{-2} + d_m^{-1}) + h_T d_m^{-1}) dt \\
 &\quad + \varepsilon \int_{I_m} \left(\left\| \tau_1^{\frac{1}{2}}(\phi \xi_t + U_1 \cdot \nabla \theta) \right\|_{0,T}^2 + \left\| d_m^{\frac{1}{2}} \nabla \xi \right\|^2 \right) dt. \tag{4.130}
 \end{aligned}$$

Substituting (4.117), (4.129), and (4.130) into (4.113) gives

$$\begin{aligned}
 |||\xi|||_{\phi, m}^2 &= \langle \phi \xi_-, \xi_- \rangle_m - \langle \phi \xi_-, \xi_- \rangle_{m-1} + \langle \phi [\xi], [\xi] \rangle_{m-1} \\
 &\quad + \int_{I_m} (|q|, \xi^2) dt + \sum_{T \in \mathcal{T}_h} \int_{I_m} d_T \|\nabla \xi\|^2 dt \\
 &\quad + \sum_{T \in \mathcal{T}_h} \int_{I_m} \left\| \tau^{\frac{1}{2}}(\phi \xi_t + U \cdot \nabla \xi) \right\|_{0,T}^2 dt \\
 &\leq Q \left[(\beta_k + (h\beta_k)^2 + d_m^{-2} + d_m^{-1}) \int_{I_m} \|\xi\|^2 dt + h d_m^{-1} \right] \\
 &\leq Q [h d_m^{-1} + (\beta_k + (h\beta_k)^2 + d_m^{-2} + d_m^{-1}) \\
 &\quad \times \left(h \left| \phi^{\frac{1}{2}} \xi_- \right|_m^2 + \frac{1}{c_1 \phi_0} h \|\phi \xi' + u \cdot \nabla \xi\|_{0,m}^2 \right) \exp^{(c_1 \phi_0 + \|\nabla u\|_{\infty, \Omega_m}) h / \phi_0}] \\
 &\leq Q (h d_m^{-1} + (\beta_k + (h\beta_k)^2 + d_m^{-2} + d_m^{-1})) \\
 &\quad \times \left(h \left| \phi^{\frac{1}{2}} \xi_- \right|_m^2 + h^2 \int_{I_m} \|(\phi \xi_t + U \cdot \nabla \xi)\|^2 dt \right) \\
 &\leq Q (\beta_k + (h\beta_k)^2 + (d_m^{-2} + d_m^{-1})) h |||\xi|||_{\phi, m}^2, \tag{4.131}
 \end{aligned}$$

where we have used Lemma 3.2. Thus,

$$||| \xi |||_{\phi, m} \leq Q(\beta_k + (h\beta_k)^2 + (d_m^{-2} + d_m^{-1})) h. \quad (4.132)$$

For fixed d_m , as $h \rightarrow 0$, (4.132) and (4.116) imply that

$$||| \xi |||_{\phi, m} = 0, \quad (4.133)$$

so that we have the following uniqueness result.

THEOREM 4.4 : *Assume that the dispersion tensor $D(u)$ is independent of u . Then, the stabilized method (4.29) is uniquely solvable for sufficiently small h .*

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