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ON THE CONVERGENCE OF A MIXED FINITE ELEMENT METHOD FOR REISSNER-MINDLIN PLATES (*)

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Abstract — *We consider a mixed finite element method, proposed by Weissman and Taylor, to approximate the solution of the Reissner-Mindlin plate problem Only the limit problem of « zero thickness » is studied For this case we provide a convergence result for transverse displacements and rotations, thus showing that the element is locking-free*

INTRODUCTION

The Reissner-Mindlin theory is widely employed by engineers in connection with plate problems. However, it's commonly accepted that finding a good finite element scheme is not at all a trivial task. Indeed, many methods fail the approximation whenever the plate thickness is « too small », because of the well known shear locking phenomenon (*cf.* [8]). Thus, development of general procedures to avoid this problem is still an active area of research. A lot of methods have been proposed so far, but, even if numerical tests show that they work properly, most of them are lacking a rigorous proof of convergence and stability. This is the case of a scheme proposed by Weissman and Taylor (*cf.* [11]).

The aim of this paper is to provide a first analysis for the above method, relatively to a clamped plate.

An outline of the paper is as follows. In section 1 we briefly recall the Reissner-Mindlin model. In agreement with the standard mathematical practice (*cf.* [4], [5]), we introduce a « problem sequence », leading to a well-posed limiting problem. In section 2 we describe the Weissman-Taylor method only for a very simplified geometry. The scheme makes use of

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Wılson's nonconformıng element One could perform a dırect analysıs by means of the usual technıques for nonconformıng elements (*cf* [9]) However, we prefer to statıcally condense the ınternal degrees of freedom, so that we work on a conformıng formulatıon In sectıon 3 we perform our error analysıs for the lımıtıng problem and get an optımal convergence result for transverse dısplacements and rotatıons (proposıtıon 3 1)

Throughout the paper, the letter c wıll denote a constant ındependant of h and t, not necessarly the same at each occurence

1 THE REISSNER MINDLIN MODEL

Let us denote wıth $A = \Omega \times]-t/2, t/2[$ the regıon ın $\mathscr{R}^3$ occupıed by an undeformed elastıc plate of thıckness $t > 0$ The Reıssner-Mındlın model descrıbes the bendıng behavıour of the plate ın terms of the transverse dısplacements and of the fıber rotatıons normal to the mıdplane Ω From a mathematıcal poınt of vıew, the problem consısts ın fındıng the couple $(\underline{\vartheta}(t), w(t))$, mınımızer of the followıng functıonal

$$\Pi_t(\underline{\vartheta}(t), w(t)) = \frac{t^3}{2} a(\underline{\vartheta}(t), \underline{\vartheta}(t)) + + \frac{\lambda t}{2} \| \underline{\vartheta}(t) - \underline{\nabla} w(t) \|^2_{0\ \Omega} - \int_A f_3\, w(t)\, dx\, dy\, dz \quad (1\ 1)$$

over the space $(H^1_0(\Omega))^2 \times H^1_0(\Omega)$ (clamped plate)

In (1 1) we have posed

(ı) $\underline{\vartheta}$ and w are the fıber rotatıons and the transverse dısplacements, respectıvely

(ıı) $a(\cdot, \cdot)$ $(H^1_0(\Omega))^2 \times (H^1_0(\Omega))^2 \to \mathscr{R}$ ıs a bılınear contınuous form defıned by

$$a(\underline{\vartheta}, \underline{\eta}) = = \frac{E}{12(1-\nu^2)} \int_\Omega \left\{ \left(\frac{\partial \vartheta_1}{\partial x} + \nu \frac{\partial \vartheta_2}{\partial y} \right) \frac{\partial \eta_1}{\partial x} + \left(\nu \frac{\partial \vartheta_1}{\partial x} + \frac{\partial \vartheta_2}{\partial y} \right) \frac{\partial \eta_2}{\partial y} + + \frac{1-\nu}{2} \left(\frac{\partial \vartheta_1}{\partial y} + \frac{\partial \vartheta_2}{\partial x} \right) \left(\frac{\partial \eta_1}{\partial y} + \frac{\partial \eta_2}{\partial x} \right) \right\} dx\, dy$$

where $\underline{\vartheta} = (\vartheta_1, \vartheta_2)$, E ıs the Young's modulus and ν ıs the Poısson's ratıo $(0 < \nu \leq 1/2)$

(ııı) $\lambda = \dfrac{Ek}{2(1+\nu)}$ wıth k shear correctıon factor (usually taken as 5/6)

(ıv) $\underline{f} = (0, 0, f_3)$ ıs the transverse load per unıt volume applıed to the plate

Note that, by Korn's inequality, $a(\,.\,,\,.\,)$ is indeed a coercive form over $(H_0^1(\Omega))^2$, i.e. there exists a constant $\alpha > 0$ such that for all $\underline{\vartheta} \in (H_0^1(\Omega))^2$

$$a(\underline{\vartheta}, \underline{\vartheta}) \geq \alpha \|\underline{\vartheta}\|_1^2 .$$

This assures the existence of a unique couple $(\underline{\vartheta}(t), w(t))$ minimizing Π_t.

The standard Euler equations associated with the functional Π_t lead to :

PROBLEM Π_t : *For $t > 0$ fixed, find $(\underline{\vartheta}(t), w(t)) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega)$ such that :*

$$t^3\, a(\underline{\vartheta}(t), \underline{\eta}) + \lambda t(\underline{\nabla} w(t) - \underline{\vartheta}(t), \underline{\nabla} v - \underline{\eta}) = \int_A f_3\, v$$
$$\forall (\underline{\eta}, v) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega) . \quad (1.2)$$

A straightforward discretization by finite elements based on formulation (1.2) typically locks in shear (*cf.* [8]). Therefore, we use here a mixed formulation derived by introducing the scaled shear stress

$$\underline{\gamma} = \lambda t^{-2}(\underline{\nabla} w - \underline{\vartheta})$$

as independent unknown. It turns out that the problem is now changed into a saddle point problem for the new functional

$$\tilde{\Pi}_t(\underline{\vartheta}, w, \underline{\gamma}) = \frac{t^3}{2}\, a(\underline{\vartheta}, \underline{\vartheta}) - \frac{\lambda^{-1} t^5}{2} \|\underline{\gamma}\|_0^2 + t^3(\underline{\gamma}, \underline{\nabla} w - \underline{\vartheta}) - \int_A f_3\, w \quad (1.3)$$

on $(H_0^1(\Omega))^2 \times H_0^1(\Omega) \times (L^2(\Omega))^2$.

In order to study a discretization based on formulation (1.3), we make the following choice for f_3 :

$$f_3(t, x, y) = t^2 f(x, y) .$$

Therefore, the associated Euler equations read as follows.

PROBLEM $\tilde{\Pi}_t$: *for $t > 0$ fixed, find $(\underline{\vartheta}, w, \underline{\gamma}) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega) \times (L^2(\Omega))^2$ such that*

$$\begin{cases} a(\underline{\vartheta}, \underline{\eta}) - (\underline{\gamma}, \underline{\eta}) = 0 & \forall \underline{\eta} \in (H_0^1(\Omega))^2 \\ (\underline{\gamma}, \underline{\nabla} v) = (f, v) & \forall v \in H_0^1(\Omega) \\ \lambda^{-1} t^2(\underline{\gamma}, \underline{s}) - (\underline{\nabla} w - \underline{\vartheta}, \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2 . \end{cases} \quad (1.4)$$

For problem $\tilde{\Pi}_t$ it's standard to obtain (*cf.* [5]) the

PROPOSITION 1.1 : *Given $t > 0$, there is a unique triple* $(\underline{\vartheta}, w, \underline{\gamma})$ *in* $(H_0^1(\Omega))^2 \times H_0^1(\Omega) \times (L^2(\Omega))^2$ *solution of the variational system* (1.4) ■

Furthermore, it's also well-known that, due to our choice of loads, the following uniform boundedness result holds (*cf.* [4]).

PROPOSITION 1.2 : *Let* $(\underline{\vartheta}(t), w(t), \underline{\gamma}(t))$ *be the solution of* (1.4), *then there exist two positive constants a and b, independent of t, such that*

$$a \leq \|\underline{\gamma}(t)\|_{\Gamma'} + \|\underline{\vartheta}(t)\|_{1,\Omega} + \|w(t)\|_{1,\Omega} \leq b$$

where $\Gamma' = H^{-1}(\mathrm{div}\,;\Omega)$ *is the dual space of* $\Gamma = H_0(\mathrm{rot}\,,\Omega)$ *and it is supplied with the norm*

$$\|\underline{\gamma}\|^2_{\Gamma'} = \|\underline{\gamma}\|^2_{-1,\Omega} + \|\mathrm{div}\,\underline{\gamma}\|^2_{-1,\Omega} \qquad ■$$

Proposition 1.2 allows us to perform a passage to the weak limit for $t \to 0$ and thus to consider a virtual « limiting problem », never used in the engineering practice but very useful for theoretical purposes, especially as far as an element tendency of suffering from locking is concerned.

It's, in fact, straightforward to obtain (*cf.* [5]) the

PROPOSITION 1.3 (Problem $\tilde{\Pi}_0$) :

Let $(\underline{\vartheta}_0, w_0, \underline{\gamma}_0) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega) \times H^{-1}(\mathrm{div}\,,\Omega)$ be the weak limit of $(\underline{\vartheta}(t), w(t), \underline{\gamma}(t))$, that is

$$\begin{array}{lll} \underline{\gamma}(t) \to \underline{\gamma}_0 & \text{weakly in } H^{-1}(\mathrm{div}\,;\Omega) & \text{for } t \to 0 \\ \underline{\vartheta}(t) \to \underline{\vartheta}_0 & \text{weakly in } (H_0^1(\Omega))^2 & \text{for } t \to 0 \\ w(t) \to w_0 & \text{weakly in } H_0^1(\Omega) & \text{for } t \to 0\,. \end{array}$$

Then $(\underline{\vartheta}_0, w_0, \underline{\gamma}_0)$ solves :

$$\begin{cases} a(\underline{\vartheta}_0, \underline{\eta}) - \langle \underline{\gamma}_0, \underline{\eta} \rangle = 0 & \forall \underline{\eta} \in (H_0^1(\Omega))^2 \\ \langle \underline{\gamma}_0, \underline{\nabla} v \rangle = (f, v) & \forall v \in H_0^1(\Omega) \\ \langle \underline{\nabla} w_0 - \underline{\vartheta}_0, \underline{s} \rangle = 0 & \forall \underline{s} \in H^{-1}(\mathrm{div}\,;\Omega) \end{cases}$$

where $\langle\ ,\ \rangle$ represents the duality pairing between $H^{-1}(\mathrm{div}\,;\Omega)$ and its dual space (which is again the usual $L^2(\Omega)$ inner product when the functions are smooth enough).

Moreover, w_0 solves the Kirchhoff-type problem

$$\begin{cases} E\Delta^2 w_0 = (12 - \nu^2) f & \text{in} \quad \Omega \\ w_0 = 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial w_0}{\partial n} = 0 & \text{on} \quad \partial\Omega . \end{cases}$$

■

2. DISCRETIZATION OF THE PROBLEM

From now on, we will consider only the case of a square plate with edges parallel to the coordinate axes (x, y) and of length L. Moreover a typical mesh T_h will be obtained by partitioning Ω into $n \times n$ equal squares, with $2\,h = L/n$.

Thus, for each square $K \in T_h$ there is an affine map F_K from the standard reference square $\hat{K} = \{(\xi, \eta) : |\xi| \leq 1 \text{ and } |\eta| \leq 1\}$ onto K defined by

$$F_K(\xi, \eta) = (x_K + h\xi, y_K + h\eta) \tag{2.1}$$

where (x_K, y_K) is the barycenter of K. As usual, from an assigned function $\hat{v} : \hat{K} \to \mathcal{R}$ we can get a corresponding function $v_K : K \to \mathcal{R}$ by

$$v_K(x, y) := \hat{v}(F_K^{-1}(x, y)) .$$

In order to perform the analysis of the method proposed by Weissman and Taylor (*cf.* [11]), we first need to introduce the following finite element spaces :

$$\begin{aligned} \Gamma_h &= \left\{\underline{\gamma}_h \in (L^2(\Omega))^2 : \underline{\gamma}_h|_K \in (P_1(K))^2 \quad \forall K \in T_h\right\} \\ \Theta_h &= \left\{\underline{\vartheta}_h \in (H_0^1(\Omega))^2 : \underline{\vartheta}_h|_K \in (Q_1(K))^2 \quad \forall K \in T_h\right\} \\ W_h &= \left\{w_h \in H_0^1(\Omega) : w_h|_K \in Q_1(K) \quad \forall K \in T_h\right\} . \end{aligned}$$

We also consider the space of nonconforming bubbles defined by :

$$B_{NC} = \left\{v \in L^2(\Omega) : \hat{v}|_K \in \text{Span}\,\{1 - \xi^2, 1 - \eta^2\} \quad \forall K \in T_h\right\}$$

and we eventually construct

$$\Theta_h^* = \Theta_h \oplus (B_{NC})^2, \; W_h^* = W_h \oplus B_{NC} .$$

Because of the non conformity of W_h^*, we need to define a differential operator

$$\underline{\nabla}_h : W_h^* \to (L^2(\Omega))^2$$

where $\underline{\nabla}_h v_h$ is the element by element gradient of v_h, thus ignoring possible discontinuities along the element interfaces ; we also define

$$a_h(\cdot,\cdot) : \Theta_h \times \Theta_h \to \mathscr{R} \quad a_h(\cdot,\cdot) = \sum_{K \in T_h} a_K(\cdot,\cdot) .$$

We are now in the position of stating the Weissman-Taylor method.

PROBLEM WT_h For $t > 0$ fixed, find $(\underline{\vartheta}_h, w_h, \underline{\gamma}_h) \in \Theta_h^* \times W_h^* \times \Gamma_h$ such that

$$\begin{cases} a_h(\underline{\vartheta}_h, \underline{\eta}_h) - (\underline{\gamma}_h, \underline{\eta}_h) = 0 & \forall \underline{\eta}_h \in \Theta_h^* \\ (\underline{\gamma}_h, \underline{\nabla}_h v_h) = (f, v_{h,c}) & \forall v_h \in W_h^* \\ \lambda^{-1} t^2 (\underline{\gamma}_h, \underline{s}_h) - (\underline{\nabla}_h w_h - \underline{\vartheta}_h, \underline{s}_h) = 0 & \forall \underline{s}_h \in \Gamma_h \end{cases} \tag{2.2}$$

where $v_{h,c}$ is the conforming part of $v_h \in W_h^*$.

Remark : In their paper, Weissman and Taylor assume the loading term to be L^2-orthogonal to the nonconforming displacements. This is a very restrictive assumption and even a constant load does not meet it. In what follows, we shall show that a convergence result can be obtained whitout any a priori orthogonality condition, but simply dropping out the contribution of nonconforming bubbles to the loading term (*cf.* second equation of (2.2)). □

Formulation of problem WT_{th} is not yet the final formulation on which we will perform our error analysis. Our goal is to reach a conforming formulation, i.e. one in which we do not deal with bubble functions anymore. Even if calculations are rather tedious, the idea is very simple : it's just the well known procedure of static condensation. We will proceed into two steps.

Step 1 - Elimination of nonconforming rotations.

First of all, let's choose a basis for $(B_{NC})^2$ once and for all.

Let's consider $\hat{b}_\iota : \hat{K} \to \mathscr{R}$ defined by

$$\hat{b}_1(\xi,\eta) = \frac{3}{8}(1-\xi^2) \quad \hat{b}_2(\xi,\eta) = \frac{3}{8}(1-\eta^2)$$

(the factor 3/8 has been introduced in order to have $\int_{\hat{K}} b_\iota \, d\xi \, d\eta = 1$).

Set $\mathscr{B} = \{(b_{\iota K}, 0), (0, b_{\iota K})\}_{K \in T_h}^{\iota = 1,2}$ where $b_{\iota K} : K \to \mathscr{R}$ is obtained from $\hat{b}_\iota$ by $F_K : \hat{K} \to K$.

It's easily seen that $\mathscr{B}$ is a basis for the space $(B_{NC})^2$ and that one has

$\int_K b_{iK}(x, y)\,dx\,dy = h^2$. Now, let us consider the first equation of system (2.2):

$$a_h(\underline{\vartheta}_h, \underline{\eta}_h) - (\underline{\gamma}_h, \underline{\eta}_h) = 0 \quad \forall \underline{\eta}_h \in \Theta_h^* . \tag{2.3}$$

From the structure of Θ_h^* it's clear that

(i) There's a unique decomposition

$$\underline{\vartheta}_h = \underline{\vartheta}_C + \underline{\vartheta}_{NC} \quad \text{with} \quad \underline{\vartheta}_C \in \Theta_h \quad \text{and} \quad \underline{\vartheta}_{NC} \in (B_{NC})^2$$

(ii) $\underline{\vartheta}_{NC} = \sum_{K \in T_h} \sum_{i=1}^{4} \alpha_{iK}\, \underline{\eta}^i_{NC,K}$ where

$$\underline{\eta}^1_{NC,K} = (b_{1K}, 0)\,; \quad \underline{\eta}^2_{NC,K} = (b_{2K}, 0)\,;$$

$$\underline{\eta}^3_{NC,K} = (0, b_{1K})\,; \quad \underline{\eta}^4_{NC,K} = (0, b_{2K})$$

and the α_{iK}'s are uniquely determined real numbers.

Henceforth, choosing $\underline{\eta}^i_{NC,K}$ as test functions, equation (2.3) becomes:

$$\sum_{i=1}^{4} \alpha_{iK}\, a_K(\underline{\eta}^i_{NC,K}, \underline{\eta}^j_{NC,K}) = (\underline{\gamma}_h, \underline{\eta}^j_{NC,K})_K - a_K(\underline{\vartheta}_C, \underline{\eta}^j_{NC,K}) \quad j = 1, \ldots, 4 . \tag{2.4}$$

Note that the matrix $c^K_{ij} = [a_K(\underline{\eta}^i_{NC,K}, \underline{\eta}^j_{NC,K})]^4_{i,j=1}$ is a diagonal non-singular one, so that we have

$$\alpha_{iK}\, c^K_{ii} = (\underline{\gamma}_h, \underline{\eta}^i_{NC,K})_K - a_K(\underline{\vartheta}_C, \underline{\eta}^i_{NC,K}) \quad i = 1, \ldots, 4 . \tag{2.4)'$$

Equations (2.4)' allow us to determine the α_{iK}'s (and so $\underline{\vartheta}_{NC}|_K$) in terms of $\underline{\vartheta}_C|_K$ and $\underline{\gamma}_h|_K$. Collecting all these local relations over $K \in T_h$, we can then determine $\underline{\vartheta}_{NC}$ in terms of $\underline{\vartheta}_C$ and $\underline{\gamma}_h$. If we now choose $\underline{\eta}_h \in \Theta_h$ as test functions in (2.3), we get

$$a(\underline{\vartheta}_C, \underline{\eta}_C) + \sum_{K \in T_h} \sum_{i=1}^{4} \alpha_{iK}\, a_K(\underline{\eta}^i_{NC,K}, \underline{\eta}_C) = (\underline{\gamma}_h, \underline{\eta}_C) \quad \forall \underline{\eta}_C \in \Theta_h . \tag{2.4)''$$

Using (2.4)', equation (2.4)'' becomes, after some (rather cumbersome) calculations

$$a(\underline{\vartheta}_C, \underline{\eta}_C) - h^2 \beta_2 \sum_K (A\underline{\vartheta}_C, A\underline{\eta}_C)_K - (\underline{\gamma}_h, \underline{\eta}_C) + \\ + h^2 \beta_1 \sum_K (\underline{\gamma}_h, A\underline{\eta}_C)_K = 0 \quad \forall \underline{\eta}_C \in \Theta_h \tag{2.5}$$

where the β_i's are positive constants and A is exactly the linear operator induced by $a(\cdot,\cdot)$ in the natural way Using (2 4)' again, but this time in the third equation of (2 2), we get

$$\lambda^{-1} t^2(\underline{\gamma}_h, \underline{\eta}_h) - (\underline{\nabla}_h w_h - \underline{\vartheta}_C, \underline{s}_h) - h^2 \beta_1 \sum_K (A\underline{\vartheta}_C, \underline{s}_h)_K +$$

$$+ h^2 \beta_3 \sum_K (P_0 \underline{\gamma}_h, \underline{s}_h)_K = 0 \quad \forall \underline{s}_h \in \Gamma_h \quad (2\ 6)$$

where β_3 is constant and P_0 is the $(L^2(\Omega))^2$-orthogonal projection operator onto the piecewise constant functions

Remark In the following, we will explicitely use the value of β_2 For completeness, let us display all the β_i's values

$$\beta_1 = \frac{2}{3} H^{-1} \quad \beta_2 = \frac{2H^{-1}(1 - \nu + 2\nu^2)}{3(1+\nu)^2} \quad \beta_3 = \frac{H^{-1}(3-\nu)}{3 - 3\nu}$$

where $H = \dfrac{E}{12(1-\nu^2)}$ □

Step 2 Elimination of nonconforming transverse displacements

Looking at the second equation of (2 2), we find that

$$(\underline{\gamma}_h, \underline{\nabla}_h v_{NC}) = 0 \quad \forall v_{NC} \in B_{NC}$$

This fact suggests us to split the finite element space Γ_h into $\Gamma_h = \Gamma_h^R \oplus \underline{\nabla}_h B_{NC}$ where Γ_h^R is easily recognized to be

$$\Gamma_h^R = \left\{ \underline{\gamma}_h \in (L^2(\Omega))^2 \quad \underline{\gamma}_h|_K \in Q_{0\,1}(K) \times Q_{1\,0}(K) \quad \forall K \in T_h \right\}$$

($Q_{i\,j}(K)$ is the space of polynomials, defined on K, of degree $\leq i$ in x and of degree $\leq j$ in y)

In order to eliminate the non conforming part of w_h, it's then sufficient to use Γ_h^R both as trial space and as test space for the approximated shear stress

What we have done so far can be summarized in the

PROPOSITION 2 1 *Let* $(\underline{\vartheta}_h, w_h, \underline{\gamma}_h) \in \Theta_h^* \times W_h^* \times \Gamma_h$ *be solution of problem* WT_{th} *If* $\underline{\vartheta}_h = \underline{\vartheta}_C + \underline{\vartheta}_{NC}$ *and* $w_h = w_C + w_{NC}$, *then* $(\underline{\vartheta}_C, w_C, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h^R$ *solves*

PROBLEM CWT_{th}

$$
\begin{cases}
a(\underline{\vartheta}_C, \underline{\eta}_C) - h^2 \beta_2 \sum_K (A\underline{\vartheta}_C, A\underline{\eta}_C)_K - (\underline{\gamma}_h, \underline{\eta}_C) + \\
\qquad + h^2 \beta_1 \sum_K (\underline{\gamma}_h, A\underline{\eta}_C)_K = 0 & \forall\, \underline{\eta}_C \in \Theta_h \\
(\underline{\gamma}_h, \underline{\nabla} v_C) = (f, v_C) & \forall v_C \in W_h \\
\lambda^{-1} t^2 (\underline{\gamma}_h, \underline{s}_h) + h^2 \beta_3 \sum_K (P_0\, \underline{\gamma}_h, \underline{s}_h)_K - (\underline{\nabla} w_C - \underline{\vartheta}_C, \underline{s}_h) - \\
\qquad - h^2 \beta_1 \sum_K (A\underline{\vartheta}_C, \underline{s}_h)_K = 0 & \forall\, \underline{s}_h \in \Gamma_h^R .
\end{cases}
\tag{2.7}
$$
■

Remark : Note that, because $\underline{\eta}^j_{NC} \notin (H^1_0(K))^2$, static condensation of nonconforming bubbles leads to a method in which a consistency error is introduced : it's indeed suddenly seen that the solution of the continuous problem (*cf.* equations (1.4)) does not solve system (2.7), since $\beta_i \neq \beta_j$ for $i \neq j$. □

We haven't yet shown that Weissman-Taylor problem admits a unique solution, at least for the case $t > 0$; to do this, we use the formulation (2.7) above. We need

LEMMA 2.1 : *There exists $\varepsilon > 0$, independent of h, such that*

$$a(\underline{\vartheta}_h, \underline{\vartheta}_h) - h^2 \sum_K \beta_2 (A\underline{\vartheta}_h, A\underline{\vartheta}_h)_K \geq \varepsilon a(\underline{\vartheta}_h, \underline{\vartheta}_h) \quad \forall\, \underline{\vartheta}_h \in \Theta_h .$$

Proof : It's cearly sufficient to prove the lemma *locally*. First of all we have

$$h^2 \beta_2 (A\underline{\vartheta}_h, A\underline{\vartheta}_h)_K = \beta_2 (A\hat{\underline{\vartheta}}_h, A\hat{\underline{\vartheta}}_h)_{\hat{K}}$$

and

$$a_K(\underline{\vartheta}_h, \underline{\vartheta}_h) = a_{\hat{K}}(\hat{\underline{\vartheta}}_h, \hat{\underline{\vartheta}}_h) \quad \forall\, \underline{\vartheta}_h \in \Theta_h$$

so that we just need

$$(1 - \varepsilon)\, a_{\hat{K}}(\hat{\underline{\vartheta}}_h, \hat{\underline{\vartheta}}_h) - \beta_2 (A\hat{\underline{\vartheta}}_h, A\hat{\underline{\vartheta}}_h)_{\hat{K}} \geq 0$$

$$\forall\, \hat{\underline{\vartheta}}_h \in (Q_1(\hat{K}))^2 \text{ for some } \varepsilon \quad 0 < \varepsilon < 1 .$$

If we now split $\hat{\underline{\vartheta}}_h = \hat{\underline{\vartheta}}_1 + \hat{\underline{\vartheta}}_2$ into its linear and pure bilinear part, respectively, it's suddenly seen that

$$(1-\varepsilon)\, a_K(\hat{\underline{\vartheta}}_h, \hat{\underline{\vartheta}}_h) - \beta_2 (A\hat{\underline{\vartheta}}_h, A\hat{\underline{\vartheta}}_h)_K \geqslant$$
$$\geqslant (1-\varepsilon)\, a_K(\hat{\underline{\vartheta}}_2, \hat{\underline{\vartheta}}_2) - \beta_2 (A\hat{\underline{\vartheta}}_2, A\hat{\underline{\vartheta}}_2)_K \quad (2\ 8)$$

Let us put $\hat{\underline{\vartheta}}_2 = \alpha_1(\xi\eta, 0) + \alpha_2(0, \xi\eta) = \alpha_1 \hat{\underline{v}}_1 + \alpha_2 \hat{\underline{v}}_2$

Because we have

$$a(\hat{\underline{v}}_1, \hat{\underline{v}}_1) = a(\hat{\underline{v}}_2, \hat{\underline{v}}_2), \quad (A\hat{\underline{v}}_1, A\hat{\underline{v}}_1) = (A\hat{\underline{v}}_2, A\hat{\underline{v}}_2)$$

and

$$(A\hat{\underline{v}}_1, A\hat{\underline{v}}_2) = a(\hat{\underline{v}}_1, \hat{\underline{v}}_2) = 0,$$

then we get from (2 8) that we only need

$$(1-\varepsilon)\, a(\hat{\underline{v}}_1, \hat{\underline{v}}_1) - \beta_2 (A\hat{\underline{v}}_1, A\hat{\underline{v}}_1) \geqslant 0 \quad (2\ 9)$$

Recalling that $\beta_2 = 2H^{-1}(1-\nu+2\nu^2)/3(1+\nu)^2$, a calculation shows that (2 9) is true whenever $0 < \varepsilon < 4/5$ The proof is complete ■

The following proposition holds

PROPOSITION 2 2 *For $t > 0$ fixed, there is one and only one solution $(\underline{\vartheta}_C, w_C, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h^R$ of system* (2 7)

Proof Assume that $f = 0$ and let $(\underline{\vartheta}_C, w_C, \underline{\gamma}_h)$ be a solution of (2 7) Choosing $\underline{\eta}_C = \underline{\vartheta}_C$, $\underline{v}_C = w_C$, $\underline{s}_h = \underline{\gamma}_h$ in (2 7) and adding the three equations so obtained, we get

$$a(\underline{\vartheta}_C, \underline{\vartheta}_C) - h^2 \beta_1 \sum_K (A\underline{\vartheta}_C, A\underline{\vartheta}_C)_K +$$
$$+ \lambda^{-1} t^2 \left\|\underline{\gamma}_h\right\|_{0\ \Omega}^2 + h^2 \beta_3 \sum_K \left\|P_0\, \underline{\gamma}_h\right\|_{0\ K}^2 = 0 \quad (2\ 10)$$

By lemma 2 1

$$0 \geqslant C \left\|\underline{\vartheta}_C\right\|_{1\ \Omega}^2 + \lambda^{-1} t^2 \left\|\underline{\gamma}_h\right\|_{0\ \Omega}^2$$

so that $\underline{\vartheta}_C = \underline{\gamma}_h = 0$ From the third equation of (2 7) we finally get $\underline{\nabla} w_C = 0$, i e $w_C = 0$ ■

3. ERROR ANALYSIS

We will consider only the « limiting problem » ($t = 0$). Recall that the continuous problem is :

PROBLEM $\tilde{\Pi}_0$. *Find* $(\underline{\vartheta}_0, w_0, \underline{\gamma}_0) \in (H_0^1(\Omega))^2 \times H_0^1(\Omega) \times H^{-1}(\mathrm{div}\,;\Omega)$ *such that*

$$\begin{cases} a(\underline{\vartheta}_0, \underline{\eta}) - \langle \underline{\gamma}_0, \underline{\eta} \rangle = 0 & \forall \underline{\eta} \in (H_0^1(\Omega))^2 \\ \langle \underline{\gamma}_0, \underline{\nabla} v \rangle = (f, v) & \forall v \in H_0^1(\Omega) \\ \langle \underline{\nabla} w_0 - \underline{\vartheta}_0, \underline{s} \rangle = 0 & \forall \underline{s} \in H^{-1}(\mathrm{div}\,, \Omega) \end{cases} \tag{3.1}$$

while the corresponding discretized problem is

PROBLEM CWT_{oh} Find $(\underline{\vartheta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h^R$ such that

$$\begin{cases} a(\underline{\vartheta}_h, \underline{\eta}_h) - h^2 \beta_2 \sum_K (A\underline{\vartheta}_h, A\underline{\eta}_h)_K - (\underline{\gamma}_h, \underline{\eta}_h) + \\ \qquad + h^2 \beta_1 \sum_K (\underline{\gamma}_h, A\underline{\eta}_h)_K = 0 & \forall \underline{\eta}_h \in \Theta_h \\ (\underline{\gamma}_h, \underline{\nabla} v_h) = (f, v_h) & \forall v_h \in W_h \\ h^2 \beta_3 \sum_K (P_0 \underline{\gamma}_h, \underline{s}_h)_K - (\underline{\nabla} w_h - \underline{\vartheta}_h, \underline{s}_h) - \\ \qquad - h^2 \beta_1 \sum_K (A\underline{\vartheta}_h, \underline{s}_h)_K = 0 & \forall \underline{s}_h \in \Gamma_h^R \end{cases}$$

(3 2)

Remark It's easily seen that the couple $(\underline{\vartheta}_h, w_h)$, part of the solution of problem CWT_{oh}, is uniquely determined by equations (3 2). Unfortunately, this is not the case for $\underline{\gamma}_h$ Nevertheless, the L^2-projection of $\underline{\gamma}_h$ over the piecewise constant functions is unique : see, for instance, the third equation of (3.2). □

Our error bound is

PROPOSITION 3.1 . *Let* $(\underline{\vartheta}_0, w_0, \underline{\gamma}_0)$ *be the solution of problem* $\tilde{\Pi}_0$ *and* $(\underline{\vartheta}_h, w_h, \underline{\gamma}_h)$ *a solution of problem* CWT_{oh}, *then there exists a constanc c independent of h, such that one has*

$$\|\underline{\vartheta}_h - \underline{\vartheta}_0\|_{1,\Omega} + \|w_h - w_0\|_{1\ \Omega} + h\|P_0\,\underline{\gamma}_h - P_0\,\underline{\gamma}_0\|_{0\ \Omega} \le$$

$$\le \mathrm{ch}\ (\|\underline{\vartheta}_0\|_{3\ \Omega} + \|\underline{\gamma}_0\|_{0\ \Omega} + h\,\|\underline{\gamma}_0\|_{1\ \Omega})\,. \tag{3 3}$$

■

Before turning to the proof of proposition 3.1, let us recall some known facts about the method of Bathe-Dvorkin (*cf.* [3]), which we will use to perform our error analysis. In that method the space

$$Q_h = \{\underline{\mu}_h \in H_0(\mathrm{rot}\,;\,\Omega) : \underline{\mu}_h|_K \in Q_{0,1}(K) \times Q_{1,0}(K) \quad \forall K \in T_h\}$$

is used to approximate shear stress. Note that $Q_h \subseteq \Gamma_h^R$ and they are *locally* built up by the same functions. Because of the conformity $Q_h \subseteq H_0(\mathrm{rot}\,;\,\Omega)$, it's then possible to define an interpolation operator

$$R_h : W = (H^1(\Omega))^2 \cap H_0(\mathrm{rot}\,;\,\Omega) \to Q_h$$

locally determined by

$$\int_{e_i} (\underline{\eta} - R_h\,\underline{\eta}) \cdot \underline{t}\, ds = 0 \quad \forall e_i \text{ edge of } K \in T_h \tag{3.4}$$

satisfying

(i) $\|\underline{\eta} - R_h\,\underline{\eta}\|_0 \leq \mathrm{ch}\ \|\underline{\eta}\|_1 \quad \forall \underline{\eta} \in W,$ (3.5)

(ii) if $\underline{\eta} \in W$ with rot $\underline{\eta} = 0$ is smooth enough, then there is $(\underline{\eta}_I, v_I) \in \Theta_h \times W_h$ such that

$$\begin{cases} \|\underline{\eta} - \underline{\eta}_I\|_1 \leq \mathrm{Ch}\ \|\underline{\eta}\|_3 & (3.6)\\ R_h\,\underline{\eta}_I = \underline{\nabla} v_I\,. & (3.7)\end{cases}$$

For a detailed discussion of Bathe-Dvorkin element, see for instance [3], [6] and [7].

We need the following

LEMMA 3.1 : *Under our hypotheses on* $\{T_h\}_{h>0}$ *and* Ω,

$$R_h\,\underline{\vartheta}_h = \Pi_h\,\underline{\vartheta}_h \quad \forall \underline{\vartheta}_h \in \Theta_h$$

where $\Pi_h : (H_0^1(\Omega))^2 \to \Gamma_h$ is the usual $L^2(\Omega)$-projection.

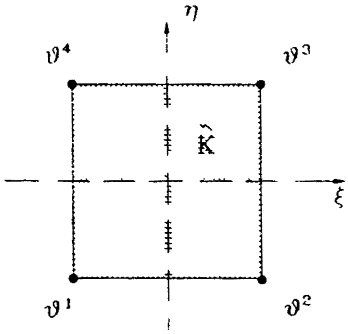
Proof : It's sufficient to prove the lemma on the reference element, because in our case we have

$$\hat{\Pi}_h\,\hat{\underline{\vartheta}}_h = \Pi_h\,\underline{\vartheta}_h\big|_K \quad \text{and} \quad \hat{R}_h\,\hat{\underline{\vartheta}}_h = R_h\,\underline{\vartheta}_h|_K\,.$$

Let $(\hat{R}_h\, \hat{\underline{\vartheta}}_h)_1$ and $(\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1$ be the first components of $\hat{R}_h\, \hat{\underline{\vartheta}}_h$ and $\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h$, respectively. If $\hat{\vartheta}_h = (\hat{\vartheta}_1, \hat{\underline{\vartheta}}_2) \in (Q_1(\hat{K}))^2$, from (3.5) we get

$$\begin{cases} (\hat{R}_h\, \hat{\underline{\vartheta}}_h)_1\,(0, 1) = \dfrac{\vartheta^3 + \vartheta^4}{2} \\ (\hat{R}_h\, \hat{\underline{\vartheta}}_h)_1\,(0, -1) = \dfrac{\vartheta^1 + \vartheta^2}{2} \end{cases} \tag{3.8}$$

where ϑ^i is the value of $\hat{\vartheta}_i$ at the i-th node



On the other hand

$$(\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1\,(\xi, \eta) = \frac{1}{4}\,[(\vartheta^1 + \vartheta^2 + \vartheta^3 + \vartheta^4) + (-\vartheta^1 - \vartheta^2 + \vartheta^3 + \vartheta^4)\,\eta]$$

so that

$$\begin{cases} (\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1\,(0, 1) = \dfrac{\vartheta^3 + \vartheta^4}{2} \\ (\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1\,(0, -1) = \dfrac{\vartheta^1 + \vartheta^2}{2}\,. \end{cases} \tag{3.9}$$

But both $(\hat{R}_h\, \hat{\underline{\vartheta}}_h)_1$ and $(\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1$ belong to $Q_{0,1}(\hat{K})$, so that (3.8) and (3.9) imply $(\hat{R}_h\, \hat{\underline{\vartheta}}_h)_1 = (\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_1$. A similar argument shows that $(\hat{R}_h\, \hat{\underline{\vartheta}}_h)_2 = (\hat{\Pi}_h\, \hat{\underline{\vartheta}}_h)_2$. The lemma is proved. ■

If we now introduce the notation

$$\mathscr{A}_h(\underline{\vartheta}_h, w_h, \underline{\gamma}_h\,;\, \underline{\eta}_h, v_h, \underline{s}_h) = a(\underline{\vartheta}_h, \underline{\eta}_h) - h^2\,\beta_2 \sum_K (A\underline{\vartheta}_h, A\underline{\eta}_h)_k +$$

$$+ (\underline{\gamma}_h, \underline{\nabla} v_h - \underline{\eta}_h) + h^2\,\beta_1 \sum_K (\underline{\gamma}_h, A\underline{\eta}_h)_K + h^2\,\beta_3 \sum_K (P_0\,\underline{\gamma}_h, \underline{s}_h)_K -$$

$$- (\underline{\nabla} w_h - \underline{\vartheta}_h, \underline{s}_h) - h^2\,\beta_1 \sum_K (A\underline{\vartheta}_h, \underline{s}_h)_K$$

then our finite element method is

$$\mathcal{A}_h(\underline{\vartheta}_h, w_h, \underline{\gamma}_h\, ;\, \underline{\eta}_h, v_h, \underline{s}_h) = (f, v_h) \quad \forall (\underline{\eta}_h, v_h, \underline{s}_h)\,.$$

At this point, we can finally give the

Proof of proposition 3.1 : Let us set

$$\underline{\varepsilon}_\vartheta = \underline{\vartheta}_h - \underline{\vartheta}_I\, ; \quad \varepsilon_w = w_h - w_I\, ; \quad \underline{\varepsilon}_\gamma = \underline{\gamma}_h - \underline{\gamma}_I \tag{3.10}$$

where $(\underline{\vartheta}_I, w_I)$ are the interpolations of $(\underline{\vartheta}_0, w_0)$ as in Bathe-Dvorkin method, and $\underline{\gamma}_I = P_0\, \underline{\gamma}_0$. It's clear that (*cf.* lemma 2.1)

$$\alpha\,(\|\underline{\varepsilon}_\vartheta\|_{1,\Omega}^1 + h^2\, \|P_0\, \underline{\varepsilon}_\gamma\|_{0,\Omega}^2) \leq \mathcal{A}_h(\underline{\varepsilon}_\vartheta, \varepsilon_w, \underline{\varepsilon}_\gamma\, ;\, \underline{\varepsilon}_\vartheta, \varepsilon_w, \underline{\varepsilon}_\gamma) \tag{3.11}$$

for some $\alpha > 0$ independent of h.

Add and substract $\mathcal{A}_h(\underline{\vartheta}_0, w_0, \underline{\gamma}_0\, ;\, \underline{\varepsilon}_\vartheta, \varepsilon_w, \underline{\varepsilon}_\gamma)$ to the right hand-side of (3.11) ; taking into account (3.1) and (3.2) we then get :

$$\mathcal{A}_h(\underline{\varepsilon}_\vartheta, \varepsilon_w, \underline{\varepsilon}_\gamma\, ;\, \underline{\varepsilon}_\vartheta, \varepsilon_w, \underline{\varepsilon}_\gamma) = A_1 + A_2 - A_3 - A_4 + A_5 + A_6 - A_7$$

where

$$A_1 = h^2 \sum_K \beta_2(A\underline{\vartheta}_I, A\underline{\varepsilon}_\vartheta)_K\, ; \quad A_2 = h^2 \sum_K \beta_1(\underline{\gamma}_I, A\underline{\varepsilon}_\vartheta)_K\, ;$$

$$A_3 = h^2 \sum_K \beta_1(\underline{\varepsilon}_\gamma, A\underline{\vartheta}_I)_K\, ; \quad A_4 = h^2 \sum_K \beta_3(\underline{\gamma}_I, \underline{\varepsilon}_\gamma)_K\, ;$$

$$A_5 = a(\underline{\vartheta}_0 - \underline{\vartheta}_I, \underline{\varepsilon}_\vartheta)\, ; \quad A_6 = (\underline{\gamma}_0 - \underline{\gamma}_I, \underline{\nabla}\varepsilon_w - \underline{\varepsilon}_\vartheta)\, ;$$

$$A_7 = (\underline{\varepsilon}_\gamma, \underline{\nabla}(w_0 - w_I) - (\underline{\vartheta}_0 - \underline{\vartheta}_I))\,.$$

Let us estimate the terms above

(A_1) On each $K \in T_h$ we have

$$h^2(A\underline{\vartheta}_I, A\underline{\varepsilon}_\vartheta)_K = h^2(A(\underline{\vartheta}_I - \underline{\vartheta}_0), A\underline{\varepsilon}_\vartheta)_K + \\ + h^2(A\underline{\vartheta}_0, A\underline{\varepsilon}_\vartheta)_K \leq \mathrm{ch}\, \|\underline{\vartheta}_0\|_{2,K}\, \|\underline{\varepsilon}_\vartheta\|_{1,K}\,.$$

Thus

$$|A_1| \leq \mathrm{ch}^2\, \|\underline{\vartheta}_0\|_{2,\Omega}^2 + \frac{a_1}{2}\, \|\underline{\varepsilon}_\vartheta\|_{1,\Omega}^2 \quad \text{with} \quad a_1 > 0 \quad \text{to be chosen}\,. \tag{3.12}$$

(A_2) We have $(\underline{\gamma}_I, A\underline{\varepsilon}_\vartheta)_K = (\underline{\gamma}_0, A\underline{\varepsilon}_\vartheta)_K$ because

$$P_0\, \underline{\gamma}_0 = \underline{\gamma}_I \quad \text{and} \quad A\underline{\varepsilon}_\vartheta|_K \in (P_0(K))^2\,.$$

Then, by a simple scaling argument,

$$|A_2| \leqslant ch \left\| \underline{\gamma}_0 \right\|_{0,\Omega} \| \underline{\varepsilon}_\vartheta \|_{1,\Omega} \leqslant ch^2 \left\| \underline{\gamma}_0 \right\|_{0,\Omega}^2 + \frac{a_2}{2} \| \underline{\varepsilon}_\vartheta \|_{1,\Omega}^2 \quad \text{with} \quad a_2 > 0 \text{ to be chosen} . \tag{3.13}$$

(A_3) One has

$$(\underline{\varepsilon}_\gamma, A \underline{\vartheta}_I)_K = (P_0 \, \underline{\varepsilon}_\gamma, A \underline{\vartheta}_I)_K = (P_0 \, \underline{\varepsilon}_\gamma, A(\underline{\vartheta}_I - \underline{\vartheta}_0))_K + (P_0 \, \underline{\varepsilon}_\gamma, A \underline{\vartheta}_0)_K \leqslant c \, \|P_0 \, \underline{\varepsilon}_\gamma\|_{0,K} \, \|\underline{\vartheta}_0\|_{2,K}$$

so that

$$|A_3| \leqslant ch^2 \, \|P_0 \, \underline{\varepsilon}_\gamma\|_{0,\Omega} \, \|\underline{\vartheta}_0\|_{2,\Omega} \leqslant ch^2 \, \|\underline{\vartheta}_0\|_{2,\Omega}^2 + \frac{a_3}{2} h^2 \, \|P_0 \, \underline{\varepsilon}_\gamma\|_{0,\Omega}^2 \quad \text{with} \quad a_3 > 0 \text{ to be chosen} . \tag{3.14}$$

(A_4) From $(\underline{\gamma}_I, \underline{\varepsilon}_\gamma)_K = (\underline{\gamma}_I, P_0 \, \underline{\varepsilon}_\gamma)_K$ we get

$$|A_4| \leqslant ch^2 \left\| \underline{\gamma}_0 \right\|_{0,\Omega} \|P_0 \, \underline{\varepsilon}_\gamma\|_{0,\Omega} \leqslant ch^2 \left\| \underline{\gamma}_0 \right\|_{0,\Omega}^2 + \frac{a_4}{2} h^2 \, \|P_0 \, \underline{\varepsilon}_\gamma\|_{0,\Omega}^2 \quad \text{with} \quad a_4 > 0 \text{ to be chosen} . \tag{3.15}$$

(A_5) It's straightforward to obtain

$$|A_5| \leqslant \frac{c}{2 \, a_5} \, \|\underline{\vartheta}_0 - \underline{\vartheta}_I\|_{I,\Omega}^2 + \frac{ca_5}{2} \, \|\underline{\varepsilon}_\vartheta\|_{I,\Omega}^2 \quad \text{with} \quad a_5 > 0 \text{ to be chosen} . \tag{3.16}$$

(A_6) We have $(\underline{\gamma}_0 - \underline{\gamma}_I, \underline{\nabla}(w_h - w_I) - (\underline{\vartheta}_h - \underline{\vartheta}_I))$. Now, from the third equation of (3.2) we see that $\underline{\nabla} w_h = \Pi_h \, \underline{\vartheta}_h + , \underline{\psi}$, where $\underline{\psi}$ is piecewise constant. By lemma 3.1 we obtain

$$\begin{aligned} |A_6| &= (\underline{\gamma}_0 - \underline{\gamma}_I, R_h(\underline{\vartheta}_h - \underline{\vartheta}_I) - (\underline{\vartheta}_h - \underline{\vartheta}_I)) \leqslant \\ &\leqslant \left\| \underline{\gamma}_0 - \underline{\gamma}_I \right\|_{0,\Omega} \| \underline{\varepsilon}_\vartheta - R_h \, \underline{\varepsilon}_\vartheta \|_{0,\Omega} \leqslant ch^2 \left\| \underline{\gamma}_0 \right\|_{1,\Omega} \| \underline{\varepsilon}_\vartheta \|_{1,\Omega} \leqslant \\ &\leqslant ch^4 \left\| \underline{\gamma}_0 \right\|_{1,\Omega}^2 + \frac{a_6}{2} \| \underline{\varepsilon}_\vartheta \|_{1,\Omega}^2 \quad \text{with } a_6 > 0 \text{ to be chosen} . \end{aligned} \tag{3.17}$$

(A_7) We have $A_7 = (\underline{\varepsilon}_\gamma, \underline{\nabla} w_I - \Pi_h \, \underline{\vartheta}_I)$ and so, by lemma 3.1

$A_7 = (\underline{\varepsilon}_\gamma, R_h \underline{\vartheta}_I - R_h \underline{\vartheta}_I) = 0$. If we now take $a_1, \ldots, a_6$ small enough, estimates (3.12)–(3.17) together with (3.11) and (3.6) give

$$\| \underline{\vartheta}_h - \underline{\vartheta}_I \|_{1, \Omega} + h \left\| P_0(\underline{\gamma}_h - \underline{\gamma}_I) \right\|_{0, \Omega} \leqslant$$
$$\leqslant \text{ch} \left(\| \underline{\vartheta}_0 \|_{3, \Omega} + \left\| \underline{\gamma}_0 \right\|_{0, \Omega} + h \left\| \underline{\gamma}_0 \right\|_{1, \Omega} \right). \quad (3.18)$$

Therefore, by the triangle inequality

$$\| \underline{\vartheta}_h - \underline{\vartheta}_0 \|_{1, \Omega} + h \left\| P_0(\underline{\gamma}_h - \underline{\gamma}_0) \right\|_{0, \Omega} \leqslant$$
$$\leqslant \text{ch} \left(\| \underline{\vartheta}_0 \|_{3, \Omega} + \left\| \underline{\gamma}_0 \right\|_{0, \Omega} + h \left\| \underline{\gamma}_0 \right\|_{1, \Omega} \right). \quad (3.19)$$

As far as an estimate for transverse displacements is concerned, let us consider the third equation of (3.2). We have *locally* $\underline{\nabla} w_h = \Pi_h \underline{\vartheta}_h - h^2 \beta_1 A \underline{\vartheta}_h + h^2 \beta_3 P_0 \underline{\gamma}_h$ on each $K \in T_h$, so that

$$\| \underline{\nabla} w_h - \underline{\nabla} w_I \|_{0, K} = \left\| \Pi_h \underline{\vartheta}_h - R_h \underline{\vartheta}_I - h^2 \beta_1 A \underline{\vartheta}_h + h^2 \beta_3 P_0 \underline{\gamma}_h \right\|_{0, K} \leqslant$$
$$\leqslant \text{(lemma 3.1 and scaling arguments)} \leqslant$$
$$\leqslant \| \underline{\vartheta}_h - \underline{\vartheta}_I \|_{1, K} + \text{ch} \, \| \underline{\vartheta}_h \|_{1, K} + h^2 \beta_3 \left\| P_0 \underline{\gamma}_h \right\|_{0, K}.$$

Squaring and summing up over $K \in T_h$, we get

$$\| \underline{\nabla} w_h - \underline{\nabla} w_I \|_{0, \Omega}^2 \leqslant c \left(\| \underline{\vartheta}_h - \underline{\vartheta}_I \|_{I, \Omega}^2 + h^2 \, \| \underline{\vartheta}_h \|_{1, \Omega}^2 + h^4 \left\| P_0 \underline{\gamma}_h \right\|_{0, \Omega}^2 \right)$$

and finally, by (3.18) and the triangle inequality,

$$\| w_h - w_0 \|_{1, \Omega} \leqslant \text{ch} \left(\| \underline{\vartheta}_0 \|_{3, \Omega} + \left\| \underline{\gamma}_0 \right\|_{0, \Omega} + h \left\| \underline{\gamma}_0 \right\|_{1, \Omega} \right) \quad (3.20)$$

(here we have also used Korn's inequality).

Taken together, estimates (3.19) and (3.20) complete the proof of proposition 3.1. ■

CONCLUSIONS

We have considered a mixed method for the Reissner-Mindlin plate problem, proposed by Weissman and Taylor. For this scheme, we have developed our analysis only for the limit case $t = 0$, proving that the transverse displacements and the rotations converge with optimal rate. Even if we haven't dealt with the problem of uniform convergence in the plate thickness t, we believe that the behaviour for $t = 0$ is indeed very indicative of an element performance.

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