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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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BOUNDARY LAYER RESOLUTION IN HIERARCHICAL MODELS OF LAMINATED COMPOSITES (*)

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Abstract — *The hierarchical modelling of a linear heat conduction problem in an orthotropic sandwich plate of thickness 2 d is analyzed and the asymptotic structure of the solution and the boundary layers are obtained A family of lower dimensional, hierarchical models with increased model order in a* $O(d|\ln d|)$ *vicinity of the lateral edge is constructed by energy projection It is shown that these models converge in the energy norm with optimal order as* $d \to 0$ *to the exact solution regardless of the boundary layers*

Résumé — *On analyse l'équation de la chaleur dans une plaque sandwich tridimensionnelle d'épaisseur 2 d du bord Lipschitzien Pour des données régulieres, qui ne remplissent aucune condition de la compatibilité au bord de la plaque, on obtient la structure asymptotique et les couches limites de la solution tridimensionnelle quand l'épaisseur tend vers zéro On construit une classe hiérarchique des modeles bidimensionnels d'ordre élevé dans le voisinage du bord par la méthode Galerkin On démontre que ces modèles convergent optimalement vers la solution tridimensionnelle en présence des couches limites quand l'épaisseur approche zéro ou, autrement dit, que ces modèles hiérarchiques* résolvent *les couches limites tridimensionnelles*

1. INTRODUCTION

In recent years, structures made of laminated composites have become increasingly important in a number of industries. Often the structural components are in addition thin, *i.e.* we deal with beams, rods, plates and shells. The accurate and effective numerical prediction of their macroscopic as well as of their microscopic responses under external forces has therefore become increasingly important. Typically one exploits the special geometry of the structure by adopting a lower dimensional model, which is obtained by asymptotic techniques as outlined, for example, in [4]. The laminated materials, on the other hand, are dealt with by proposing « averaged »

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effective models with fictitious, homogeneous materials prior to the asymptotic analysis While accurately predicting the macroscopic response of the structures, this class of models does not allow for an accurate assessment of the microscopic features, such as interlamina stresses and boundary effects, which govern the onset of delamination

In the present paper we analyze therefore an alternative approach for a model problem of heat conduction in a thin plate which consists of a stack of orthotropic layers ideally bonded together As in [9] we replace the three-dimensional boundary value problem by a hierarchy of two dimensional problems which approximate the original problem as both thickness tends to zero and the order of the model tends to infinity Moreover, accuracy and complexity of the models are *independent of the number of layers*, and already low order models allow to resolve cross-sectional micro effects accurately, even for a large number of layers, in contrast to the above mentioned « effective » models We view hierarchical modelling as an optimized numerical method for the approximation of three dimensional boundary value problems with special structure — here a stratified material and a thin geometry The effectiveness of this approach depends strongly the boundary layers — solution components that decay exponentially off the lateral boundary of the plate and play an important role in the onset of delamination We show in this paper that a local increase of the model order in a $O(d|\ln d|)$ — neighborhood of the lateral boundary of the plate results in a plate model which resolves the boundary layers of the exact solution, i e the modelling error is of optimal asymptotic order Models of the type investigated here have also been computationally realized and successfully used in engineering applications [2] and are amenable to an adaptive selection of the model order (« d-adaptivity ») [3], [5]

2. PROBLEM FORMULATION

Let $\omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary $\gamma = \partial\omega$ and define, for $0 < d \leqslant 1$, the domain $\Omega = \omega \times (-d, d)$ and its lateral boundary $\Gamma = \gamma \times (-d, d)$ We consider the boundary value problem

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega, \\ \gamma_0 u &= 0 \quad \text{on } \Gamma, \\ \gamma_1 u &= f^{\pm} \quad \text{on } R_{\pm} \end{aligned} \tag{2 1}$$

where $R_{\pm} = \omega \times \{\pm d\}$ The operator L is defined by

$$L = \frac{\partial}{\partial y}\left(a\left(\frac{y}{d}\right)\frac{\partial}{\partial y}\right) + b\left(\frac{y}{d}\right)\nabla_x \cdot (C(x)\nabla_x)$$

where $a, b \in L^{\infty}(-1, 1)$ are independent of d, $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)^{\top}$, and

$$0 < \underline{A} \leq a(z), \quad 0 < \underline{B} \leq b(z), \tag{2.2}$$

γ_0 denotes the trace operator and γ_1 the distributional conormal derivative, defined on $H_L(\Omega) := \overset{\circ}{H}{}^1(\Omega) \cap \{u \mid Lu \in L^2(\Omega)\}$ via Green's formula. The matrix function $C(x)$ is assumed to be symmetric, positive definite, with C^{∞} coefficients which satisfy

$$\underline{C}\, \xi^{\top} \xi \leq \xi^{\top} C(x)\, \xi \leq \bar{C}\, \xi^{\top} \xi \quad \forall x \in \bar{\omega}, \quad \forall \xi \in \mathbb{R}^n \tag{2.3}$$

for some constants $0 < \underline{C} \leq \bar{C}$. We introduce the (strictly positive) differential operator

$$A = -\nabla_x \cdot C(x) \nabla_x, \tag{2.4}$$

with domain $\mathcal{D}(A) \subset \overset{\circ}{H}{}^1(\omega)$ and, obviously, $\mathcal{D}(A^{1/2}) = \overset{\circ}{H}{}^1(\omega)$.

The *weak form* of (2.1) reads : Find $u \in H^1(\Omega, \Gamma)$ such that

$$B(u, v) = F(v) \quad \forall v \in H^1(\Omega, \Gamma), \tag{2.5}$$

where

$$H^1(\Omega, \Gamma) = H^1(\Omega) \cap \{u \mid \gamma_0 u = 0 \text{ on } \Gamma\},$$

$$B(u, v) = \int_{\Omega} \left\{ a\left(\frac{y}{d}\right) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + b\left(\frac{y}{d}\right) \nabla_x u \cdot C(x) \nabla_x v \right\} dy\, dx,$$

$$F(v) = \int_{\omega} (f^+ v(x, d) + f^- v(x, -d))\, dx.$$

THEOREM 2.1 : *For every pair* $f^+, f^- \in L^2(\omega)$ *and every* $0 < d \leq 1$, *there exists a unique weak solution* u *of* (2.5).

We assume in what follows for convenience that

$$f^+ = f^- = f, \quad a(z) = a(-z), \quad b(z) = b(-z), \quad z \in (-1, 1).$$

Then the weak solution $u(x, y)$ of (2.5) satisfies $u(x, y) = u(x, -y)$ for a.e. $x \in \omega$.

Remark 2.1 : The assumptions (2.2) and (2.3) imply in particular that on $H^1(\Omega, \Gamma)$ the expressions $\|u\|_{E(\Omega)} := \sqrt{B(u, u)}$ and $|u|_{H^1(\Omega)} = \left(\int_{\Omega} |\nabla u|^2\, dx\, dy\right)^{1/2}$ are equivalent norms :

$$\min\{\underline{A}, \underline{B}\underline{C}\}\, |u|_{H^1(\Omega)} \leq \|u\|_{E(\Omega)} \leq \max\{\bar{A}, \bar{B}\bar{C}\}\, |u|_{H^1(\Omega)}. \tag{2.6}$$

3. HIERARCHICAL MODELLING

Motivated by the special geometry of Ω, we approximate (2.1)-(2.5) by *dimensionally reduced models,* i.e. by elliptic boundary value problems on $\omega \subset \mathbb{R}^n$. Denote by

$$\mathcal{P} := \{\omega_i \mid \omega_i \subseteq \omega,\ 1 \leq i \leq M\}$$

a partition of ω into M domains with Lipschitz boundaries, i.e. $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$ and $\bar{\omega} = \cup\, \bar{\omega}_i$. We associate with each ω_i a non-negative integer q_i, to which we will refer as the *order of the model on* ω_i and we define

$$S(\mathcal{P}, q) := \left\{ u \in H^1(\Omega, \Gamma) \,|\, u|_{\omega_i} = \sum_{j=0}^{q_i} X_j(x)\, \varphi_j\left(\frac{y}{d}\right),\ \omega_i \in \mathcal{P} \right\} \tag{3.1}$$

and $q = \{q_1, \ldots, q_M\}$. Here X_j are unknown coefficient functions to be determined and $\varphi_i(z)$, $z \in (-1, 1)$, are d-independent, a priori selected coefficient functions. For a given selection of $\{\varphi_i\}$, $\mathcal{P}$, q, the *$(\mathcal{P}, q)$-model* is obtained by energy projection : Find $u(\mathcal{P}, q) \in S(\mathcal{P}, q)$ such that

$$B(u(\mathcal{P}, q), v) = F(v) \quad \forall v \in S(\mathcal{P}, q)\,. \tag{3.2}$$

We observe that the *modelling error* $e(\mathcal{P}, q) := u - u(\mathcal{P}, q)$ is optimal, i.e.

$$\|e(\mathcal{P}, q)\|_{E(\Omega)} = \inf_{w \in S(\mathcal{P}, q)} \|u - w\|_{E(\Omega)}\,. \tag{3.3}$$

Another important property of the modelling error is

THEOREM 3.1 : *Assume that φ_0 is constant. Then*

$$\int_{-d}^{d} b\left(\frac{y}{d}\right) e(x, y)\, dy = 0 \quad \text{a.e.} \quad x \in \omega\,. \tag{3.4}$$

Proof: Since $q_i \geq 0$ and $\varphi_0(z) = 1$, we have $\mathring{H}^1(\omega) \otimes 1_y \subset S(\mathcal{P}, q)$, hence, from (3.2),

$$B(e(\mathcal{P}, q), v) = 0 \quad \forall v(x, y) = V(x) \otimes 1_y\,,$$

i.e.

$$0 = \int_\Omega b\left(\frac{y}{d}\right) \nabla_x V \,.\, C(x)\, \nabla_x e\, dx\, dy \quad \forall V \in \mathring{H}^1(\omega)\,.$$

We obtain with Fubini's theorem and integration by parts that

$$A\left(\int_{-d}^{d} b\left(\frac{y}{d}\right) e(x, y)\, dy\right) = 0 \quad \text{in} \quad \omega\,,$$

$$\gamma_0\left(\int_{-d}^{d} b\left(\frac{y}{d}\right) e(x, y)\, dy\right) = 0 \quad \text{on} \quad \gamma\,,$$

with A as in (2.4). Now (3.4) is a consequence of the strict positivity of A. □

For the analysis of the modelling error e in (3.3) and to obtain an insight into a good selection of the basis functions $\varphi_i(z)$ in (3.1) we investigate now the asymptotic structure of the exact solution u.

4. ASYMPTOTIC ANALYSIS OF u

In this section we analyze the structure of the weak solution u of (2.5) as $d \to 0$. We show that it can be separated into a *limiting solution* u_0^N which has product form and into *boundary layers* u_{BL}^N, which have three dimensional character.

THEOREM 4.1 : *Assume that* $f \in H^{2N}(\omega)$, $N \in \mathbb{N}_0$. *Then we have for* $0 < d \leq 1$

$$\|u - u_0^N - u_{BL}^N\|_{E(\Omega)} \leq C_N\, d^{2N + 1/2} \|A^N f\|_{L^2(\omega)}\,. \tag{4.1}$$

The constant C_N *is independent of* d *and given by*

$$C_N = \begin{cases} \|\sqrt{a}\,\psi_{2N+2}'\|_{L^2(-1,1)}\,, & \text{if} \quad N \geq 1 \\ 2\,\Lambda^{1/2} & \text{if} \quad N = 0 \end{cases}$$

with Λ *as in* (4.11) *below. Further,*

$$u_0^N = \sum_{j=0}^{N} d^{-1+2j} (A^{j-1} f)(x)\, \psi_{2j}\left(\frac{y}{d}\right), \tag{4.2}$$

and, for $N \geq 1$,

$$u_{BL}^N = \sum_{j=1}^{N} d^{-1+2j}\, U_j(x, y) \tag{4.3}$$

($u_{BL}^N = 0$ *if* $N = 0$). *The functions* ψ_{2j} *in* (4.2) *are even and defined recursively by*

$$\int_{-1}^{1} a(z)\, \psi_0'\, v'\, dz = 0 \tag{4.4}$$

$$\int_{-1}^{1} a(z)\, \psi_2'\, v'\, dz + \int_{-1}^{1} b(z)\, \psi_0\, v\, dz = v(-1) + v(1)\,, \tag{4.5}$$

$$\int_{-1}^{1} a(z)\, \psi_{2_j}'\, v'\, dz + \int_{-1}^{1} b(z)\, \psi_{2_j - 2}\, v\, dz = 0 \quad j = 2, 3, \dots\,, \tag{4.6}$$

for all $v \in H^1(-1, 1)$ *and* U_j *is the solution of the* Saint-Venant problem

$$\begin{aligned} LU_j &= 0 \quad \text{on} \quad \Omega\,, \\ \gamma_0 U_j &= -\psi_{2_j}\left(\frac{y}{d}\right) \gamma_0(A^{j-1} f)(s) \quad \text{on} \quad \Gamma\,, \\ \gamma_1 U_j &= 0 \quad \text{on} \quad R_{\pm}\,. \end{aligned} \tag{4.7}$$

Remark 4.1 : From (4.4) we obtain that ψ_0 is constant. The constant is determined uniquely from the requirement that the Neumann problem (4.5) is solvable. Analogously we see that ψ_{2_j}, $j \geq 1$, are uniquely determined and satisfy

$$\int_{-1}^{1} b(z)\, \psi_{2_j}(z)\, dz = 0\,, \quad j \geq 1\,. \tag{4.8}$$

Proof : i) *Case* $N = 0$: Here $\mathcal{P} = \{\omega\}$, $q = \{0\}$ and $u(\mathcal{P}, q) = X_0(x)\, \psi_0\left(\frac{y}{d}\right)$ where $X_0(x)$ satisfies

$$\begin{aligned} DAX_0 &= d^{-1} f \quad \text{in} \quad \omega\,, \\ X_0 &= 0 \qquad \text{on} \quad \gamma\,, \end{aligned}$$

and $D = \frac{1}{2}\int_{-1}^{1} b(z)\, \psi_0^2\, dz$. Thus $u(\mathcal{P}, q) = d^{-1} D^{-1} A^{-1} f \psi_0\left(\frac{y}{d}\right) = u_0^0(x, y)$, and, since ψ_0 is constant, we have

$$B(e, v) = 0 \quad \text{for all} \quad v \in \overset{\circ}{H}{}^1(\omega) \otimes 1_y\,. \tag{4.9}$$

Now

$$\|e\|_{E(\Omega)} = \sup_{0 \neq v \in H^1(\Omega, \Gamma)} \frac{B(e, v)}{\|v\|_{E(\Omega)}}$$

and, by (4.9), the supremum needs only be taken over all $v \in H^1(\Omega, \Gamma)$ for which

$$\int_{-d}^{d} b\left(\frac{y}{d}\right) v(x, y)\, dy = 0 \quad \text{a.e.} \quad x \in \omega\,. \tag{4.10}$$

Since ψ_0 is constant we have

$$B(e, v) = F(v) - B(u_0^0, v) =$$

$$= F(v) - \psi_0 \int_\omega \left(\nabla_x \left(\int_{-d}^{d} b\left(\frac{y}{d}\right) v(x, y)\, dy \right) \cdot C(x) \nabla_x X_0 \right) dx$$

and the volume terms vanish due to (4.10). Therefore

$$\|e\|_{E(\Omega)}^2 = \sup_{0 \neq v \in H^1(\Omega, \Gamma)} \frac{\left(\int_\omega f(x)(v(x, d) + v(x, -d))\, dx \right)^2}{\|v\|_{E(\Omega)}^2}$$

$$\leqslant \sup_{0 \neq v \in H^1(\Omega, \Gamma)} \frac{\left(\int_\omega f(x)(v(x, d) + v(x, -d))\, dx \right)^2}{\int_{-d}^{d} a\left(\frac{y}{d}\right) \left(\frac{\partial v}{\partial y}\right)^2 dy}$$

$$= 4\, \Lambda d \|f\|_{L^2(\omega)}^2 .$$

Here

$$\Lambda^{-1} = \inf_{H^1(-1, 1)} \frac{\int_{-1}^{1} a(z)(\psi')^2\, dz}{|\psi(1)|^2}$$

and the infimum is taken over all even functions ψ which satisfy $\int_{-1}^{1} b(z)\, \psi(z)\, dz = 0$. Let us calculate Λ. Taking variations, we find that necessarily $(a(z)\, \psi')' = \text{Const.}$, i.e.

$$\psi(z) = \alpha \left(\int_{-1}^{z} \frac{\xi}{a(\xi)}\, d\xi + \beta \right)$$

since ψ is even. Here $\alpha \neq 0$ is arbitrary and, from (4.10),

$$\beta = -\left(\int_{-1}^{1} \int_{-1}^{z} \frac{\xi}{a(\xi)}\, d\xi\, b(z)\, dz \right) \Big/ \int_{-1}^{1} b(z)\, dz .$$

Consequently,

$$\Lambda = \frac{\int_{-1}^{1} b(z) \left(\int_{z}^{1} \frac{\xi}{a(\xi)}\, d\xi \right) dz}{\int_{-1}^{1} b(z)\, dz \int_{-1}^{1} \frac{z^2}{a(z)}\, dz} . \tag{4.11}$$

ii) *Case* $N \geq 1$: Changing variables $z = y/d$, we find that

$$B(u, v) = d^{-1} a(u, v) + db(u, v),$$

where

$$a(u, v) := \int_\omega \int_{-1}^{1} a(z)\, u'\, v'\, dz\, dx,$$

$$b(u, v) := \int_\omega \int_{-1}^{1} b(z)\, \nabla_x v \,.\, C(x)\, \nabla_x u\, dz\, dx$$

are independent of d and a prime denotes $\frac{d}{dz}$. For any $v \in H^1(\Omega, \Gamma)$, we have

$$\begin{aligned} B(u - u_0^N, v) &= F(v) - B(u_0^N, v) \\ &= F(v) - \sum_{j=1}^{N} d^{-2+2j}\, a((A^{j-1} f)\, \psi_{2j}, v) \\ &\quad - \sum_{j=0}^{N} d^{2j}\, b((A^{j-1} f)\, \psi_{2j}, v) \end{aligned}$$

since ψ_0 is constant. We combine terms with equal powers of d and get

$$\begin{aligned} B(u - u_0^N, v) = F(v) - \sum_{j=0}^{N-1} d^{2j} \{ b((A^{j-1} f)\, \psi_{2j}, v) + a((A^j f)\, \psi_{2j+2}, v) \} \\ - d^{2N}\, b((A^{N-1} f)\, \psi_{2N}, v)\,. \end{aligned} \tag{4.12}$$

Integrations by parts with respect to x and (4.5), (4.6) give for $j \geq 1$ that

$$a((A^j f)\, \psi_{2j+2}, v) = -\, b((A^{j-1} f)\, \psi_{2j}, v) + d^{-1} R_j(v)$$

where

$$R_j(v) = \int_{-d}^{d} \int_\gamma (\gamma_0 v)(s, y)\, b\left(\frac{y}{d}\right) \gamma_1(A^{j-1} f)(s)\, ds\, \psi_{2j}\left(\frac{y}{d}\right) dy\,.$$

For $j = 0$ we get analogously

$$a(f \psi_2, v) = -\, b((A^{-1} f)\, \psi_0, v) + F(v) + d^{-1} R_0(v)\,.$$

Inserting into (4.12) yields

$$B(u - u_0^N, v) = -\sum_{j=0}^{N-1} d^{2j-1} R_j(v) - d^{2N}\, b((A^{N-1} f)\, \psi_{2N}, v)\,.$$

Now, since $v \in H^1(\Omega, \Gamma)$, $R_j(v) = 0$ for $0 \leq j \leq N-1$, and

$$b((A^{N-1} f) \psi_{2N}, v) = -a((A^N f) \psi_{2N+2}, v)$$

from integration by parts. We have therefore proved

$$B(u - u_0^N, v) = d^{2N} a((A^N f) \psi_{2N+2}, v).$$

Since $u_0^N \notin H^1(\Omega, \Gamma)$, we correct the nonzero trace of u_0^N on Γ and add

$$u_{BL}^N = \sum_{j=1}^{N} d^{-1+2j} U_j(x, y)$$

where U_j solves (4.7). Then obviously $u_0^N + u_{BL}^N|_\Gamma = 0$, and, since $B(u_{BL}^N, v) = 0 \ \forall v \in H^1(\Omega, \Gamma)$,

$$B(u - u_0^N - u_{BL}^N, v) = d^{2N} a((A^N f) \psi_{2N+2}, v) \leq$$

$$\leq d^{2N} \|A^N f\|_{L^2(\omega)} \|\sqrt{a} \psi'_{2N+2}\|_{L^2(-1,1)} \left\|\sqrt{a} \frac{\partial v}{\partial z}\right\|_{L^2(\omega \times (-1,1))}.$$

Changing variables $y = zd$, we find

$$B(u - u_0^N - u_{BL}^N, v) \leq$$

$$\leq d^{2N+1/2} \|A^N f\|_{L^2(\omega)} \|\sqrt{a} \psi'_{2N+2}\|_{L^2(-1,1)} \left\|\sqrt{a} \frac{\partial v}{\partial y}\right\|_{L^2(\Omega)}$$

from where (4.1), (4.2) follow. □

Remark 4.2 : In the case of Neumann conditions on Γ, the above result is also true ; the only modification consists in that U_j is now a solution of the Saint Venant problem (4.7) with the boundary conditions

$$\gamma_1 U_j = -\gamma_1 \left(\psi_{2j} \left(\frac{y}{d} \right) (A^{j-1} f)(s) \right) \quad \text{on} \quad \Gamma .$$

Remark 4.3 : If $a = b = 1$, i.e. the material is homogeneous, the functions ψ_{2j} are polynomials and it was shown in [7] that $C_N = \sqrt{\frac{2}{3}} \left(\frac{\pi}{2} \right)^{-2N}$.

Remark 4.4 : Based on Theorem 3.1 we select the functions φ_j in the definition 2.1 of $S(\mathcal{P}, q)$ such that u_0^N is well approximated, i.e.

$$\varphi_j = \psi_{2j} \quad j = 0, 1, 2, \ldots \tag{4.13}$$

If $a(z)$, $b(z)$ are piecewise constant, the ψ_{2j} are uniquely determined piecewise polynomials (splines) which can be efficiently computed for any given material by solving the Neumann problems (4 4)-(4 6) with a one dimensional finite element method

This selection ensures also that the models will converge at fixed $d > 0$

PROPOSITION 4 1 [9, II, Theorem 2 1]
The sequence $\{\psi_{2j}\}_{j=0}^{\infty}$ *is dense in* $H^1(-1, 1) \cap \{\psi \mid \psi(z) = \psi(-z)\}$

Remark 4 5 We will also admit $q_i = \infty$ in (3 1) By Proposition 4 1, this corresponds to solving locally, i e on ω_i, a three-dimensional problem

5 BOUNDARY LAYER RESOLUTION

The result (4 1) on the asymptotic structure of the solution (2 5) allows, together with the quasioptimality (3 3) of the modelling error for an estimate of $\|e\|_{E(\Omega)}$ Let us first consider an *uniform model order q*, i e

$$\mathscr{P} = \{\omega\} , \quad q = \{q\} , \quad q \geq 0 \tag{5 1}$$

THEOREM 5 1 *With* $(\mathscr{P}, q)$ *as in* (5 1) *there holds*

$$\|e(\mathscr{P}, q)\|_{E(\Omega)} \leq C(q)\, d^{1/2} \|f\|_{L^2(\omega)}$$

and the rate $d^{1/2}$ *is optimal*

Proof

$$\|e\|_{E(\Omega)} = \sup_{0 \neq v \in H^1(\Omega\ \Gamma)} \frac{B(e, v)}{\|v\|_{E(\Omega)}}$$

and the supremum is taken over all v such that $B(v, w) = 0\ \forall w \in S(\mathscr{P}, q)$ Hence we can estimate as in the proof of Theorem 4 1 (Case $N = 0$) The optimality of the rate $\sqrt{d}$ is seen from Theorem 4 1, too To obtain for example $d^{5/2}$, we would have to include u_{BL}^1 into $u(\mathscr{P}, q)$ However, $u_{BL}^1 \notin S(\mathscr{P}, q)$ for any q □

In the remainder of this section we show that the optimal asymptotic rate $d^{2N+1/2}$ in Theorem 4 1 can be recovered by simply using a more sophisticated model in an $O(d|\ln d|)$-neighborhood of γ and uniform order $q = N$ in the interior of ω We start by proving some technical results used in the subsequent analysis

5.1. Technical Preliminaries

We analyze $B(\,.\,,\,.\,)$ in weighted spaces Let $B(\,.\,,\,.\,)$ be a bilinear form on Hilbert spaces $H_1 \times H_2$, with respective norms $\|\circ\|_1$, $\|\circ\|_2$ Then

$B(.,.)$ is (C, δ)-regular if there exist positive constants C and δ such that

$$|B(u, v)| \leq C \|u\|_1 \|v\|_2 \quad \forall u \in H_1, \forall v \in H_2, \tag{5.2}$$

$$\inf_{\|u\|_1 = 1} \sup_{\|v\|_2 = 1} |B(u, v)| \geq \delta > 0, \tag{5.3}$$

$$\sup_{\|u\|_1 = 1} |B(u, v)| > 0 \quad \forall 0 \neq v \in H_2. \tag{5.4}$$

If $B(.,.)$ is (C, δ)-regular, it is well known (see, for example, [1]) that for every bounded, linear functional $F(.)$ on H_2 there exists exactly one $u \in H_1$ such that

$$B(u, v) = F(v) \quad \forall v \in H_2,$$

and, if

$$\sup_{\|v\|_2 = 1} |F(v)| \leq A, \tag{5.5}$$

we have

$$\|u\|_1 \leq A/\delta. \tag{5.6}$$

Below we will use the space

$$H_\varphi := \left\{ u \in H^1(\Omega, \Gamma) \mid \int_{-d}^{d} b\left(\frac{y}{d}\right) u(x, y)\, dy = 0 \quad \text{a.e.} \quad x \in \omega \right\} \tag{5.7}$$

furnished with the norm $\| \circ \|_\varphi$ given by

$$\|u\|_\varphi^2 := \int_\omega \varphi^2(x) \int_{-d}^{d} \left\{ a\left(\frac{y}{d}\right) \left(\frac{\partial u}{\partial y}\right)^2 + b\left(\frac{y}{d}\right) \nabla_x u \,.\, C(x) \nabla_x u \right\} dy\, dx. \tag{5.8}$$

Then there holds.

LEMMA 5.1 : *Assume that $u(x, y) \in H_\varphi$. Then, for all open subsets $\sigma \subseteq \omega$,*

$$\int_{\sigma \times (-d, d)} b\left(\frac{y}{d}\right) \varphi^2(x) |u(x, y)|^2\, dy\, dx \leq$$

$$\leq C_3^2 d^2 \int_{\sigma \times (-d, d)} a\left(\frac{y}{d}\right) \varphi^2(x) \left(\frac{\partial u}{\partial y}\right)^2 dy\, dx \tag{5.9}$$

where

$$\frac{1}{C_3^2} = \inf_{\psi \in H^1(-1,1)} \frac{\int_{-1}^{1} a(z)(\psi')^2\, dz}{\int_{-1}^{1} b(z)\psi^2(z)\, dz}$$

and the infimum is taken over all

$$\psi \in H^1(-1, 1) \cap \left\{ \psi \mid \int_{-1}^{1} b(z)\, \psi(z)\, dz = 0 \right\}.$$

Proof : For smooth $u(x, y)$ and all $x \in \omega$, we have

$$\int_{-d}^{d} b\left(\frac{y}{d}\right) u^2(x, y)\, dy \leq C_3^2 d^2 \int_{-d}^{d} a\left(\frac{y}{d}\right)\left(\frac{\partial u}{\partial y}\right)^2 dy$$

by the definition of C_3 and a scaling argument. Multiplying both sides by $\varphi^2(x)$ and integrating over σ completes the proof. □

Now we can prove.

THEOREM 5.2 : *Let* $0 < \varphi \in W^{1,\infty}(\omega)$ *and denote*

$$Q = \max_{1 \leq i \leq n} \left\| \frac{\partial \varphi}{\partial x_i} \Big/ \varphi(x) \right\|_{L^\infty(\omega)} . \tag{5.10}$$

Then the bilinear form $B(.,.)$ *is* $(1, \delta)$*-regular on* $H_\varphi \times H_{\varphi^{-1}}$ *with*

$$\delta = (1 - C_4(\omega, n)\, Qd)\left(1 + 4\sqrt{n\bar{C}}\, Q\, dC_3(1 + \sqrt{n\bar{C}}\, Q\, dC_3)\right)^{-1/2} . \tag{5.11}$$

Proof : It is easily seen from Schwartz' inequality that (5.2) holds with $C = 1$.

Let us prove (5.3). We consider $u \in H_\varphi$ and define $v_u = u\varphi^2$. Then $v_u \in H_{\varphi^{-1}}$ and we have

$$\|v_u\|_{\varphi^{-1}}^2 = \int_\Omega \varphi^{-2}\left\{\varphi^4 a\left(\frac{y}{d}\right)\left(\frac{\partial u}{\partial y}\right)^2 + \right.$$

$$\left. + b\left(\frac{y}{d}\right) \nabla_x(\varphi^2 u) \cdot C(x)\, \nabla_x(\varphi^2 u)\right\} dy\, dx .$$

Since

$$\nabla_x(\varphi^2 u) \cdot C\, \nabla_x(\varphi^2 u) = \varphi^4\, \nabla_x u \cdot C\, \nabla_x u +$$

$$+ 2\, u\varphi^2 \nabla_x u \cdot C\, \nabla_x(\varphi^2) + u^2\, \nabla_x(\varphi^2) \cdot C\, \nabla_x(\varphi^2)$$

we find with Lemma 5.1

$$\|v_u\|^2_{\varphi^{-1}} \leq \left(1 + 4\sqrt{n\bar{C}}\, Q\, dC_3(1 + \sqrt{n\bar{C}}\, Q\, dC_3)\right) \|u\|^2_\varphi .$$

Further,

$$B(u, v_u) = \|u\|^2_\varphi + \int_\Omega \left\{ b\left(\frac{y}{d}\right) u\, \nabla_x(\varphi^2) \cdot C(x)\, \nabla_x u \right\} dx\, dy .$$

Since

$$|\nabla_x(\varphi^2)|^2_2 \leq 4\, nQ^2\, \varphi^4$$

where $| \circ |_2$ denotes the Euclidean norm in $\mathbb{R}^n$, we have

$$\left| \int_\Omega \left\{ b\left(\frac{y}{d}\right) u\, \nabla_x(\varphi^2) \cdot C(x)\, \nabla_x u \right\} dx\, dy \right|^2 \leq$$

$$\leq C_1^2 \int_\Omega \left\{ b\left(\frac{y}{d}\right) |u|^2\, \varphi^{-2} |\nabla_x(\varphi^2)|^2 \right\} dx\, dy \int_\Omega \varphi^2 |\nabla_x u|^2\, dx\, dy$$

$$\leq 4\, nQ^2\, C_1^2\, C_2^2 \int_\Omega \varphi^2\, b\left(\frac{y}{d}\right) |u|^2\, dx\, dy\, \|u\|^2_\varphi$$

$$\leq 4\, nQ^2\, C_1^2\, C_2^2\, C_3^2\, d^2 \int_\Omega \varphi^2\, a\left(\frac{y}{d}\right) \left(\frac{\partial u}{\partial y}\right)^2 dx\, dy\, \|u\|^2_\varphi$$

by Lemma 5.1, so that finally

$$\left| \int_\Omega \left\{ b\left(\frac{y}{d}\right) u\, \nabla_x(\varphi^2) \,.\, C(x)\, \nabla_x u \right\} dx\, dy \right| \leq C_4\, Qd \|u\|_\varphi$$

where $C_4 := 2\sqrt{n} C_1\, C_2\, C_3$. Thus

$$B(u, v_u) \geq (1 - C_4(\omega, n)\, Qd) \|u\|^2_\varphi$$

and we see that (5.3) holds with δ as in (5.11). Condition (5.4) follows readily from the symmetry of B. □

Remark 5 1 Below, we shall use in particular $\varphi(x) = \exp\{\beta \operatorname{dist}(x, \gamma)\}$, $\beta \in \mathbb{R}$. If γ is Lipschitz, we have $\varphi \in W^{1,\infty}(\omega)$ (see, for example, [8, Chap. 6.3]). Then

$$Q = |\beta|\, R(\omega), \quad R(\omega) := \max_{1 \leq i \leq n} \left\| \frac{\partial}{\partial x_i} \operatorname{dist}(x, \gamma) \right\|_{L^\infty(\omega)} \tag{5.12}$$

and $B(\cdot, \cdot)$ is $(1, \delta)$ regular with $\delta > 0$ independent of d, provided

$$|\beta| < 1/(2\, C_4\, R(\omega)\, d) . \tag{5.13}$$

5.2. Decay Estimates for the Boundary Layers

With Theorem 5.2 we can prove that u_{BL}^N in Theorem 4 1 is indeed a boundary effect, i e. that u_{BL}^N decays exponentially off γ.

LEMMA 5.2 : *Consider the Saint Venant problem*

$$\begin{aligned} LU &= 0 \quad \text{in} \quad \Omega , \\ \gamma_0 U &= g(s)\, \psi\left(\frac{y}{d}\right) \quad \text{on} \quad \Gamma , \\ \gamma_1 u &= 0 \quad \text{on} \quad R_{\pm} \end{aligned} \tag{5 14}$$

where $g \in H^{1/2}(\gamma)$ *and* $\psi \in H^1(-1, 1)$ *satisfies*

$$\int_{-1}^{1} b(z)\, \psi(z)\, dz = 0 \tag{5.15}$$

Then the solution U *satisfies* (4 10) *and, if* β *satisfies* (5.13),

$$\| U \|_{\varphi} \leq C_8(\omega, n)\, d^{\,1/2} \| \psi \|_{H^1(-1\,1)} \| g \|_{H^{1/2}(\gamma)} .$$

Proof We cast (5.14) into the variational form . Find $U \in H^1(\Omega)$ such that $U = g(s)\, \psi\left(\frac{y}{d}\right)$ on Γ and

$$B(U, V) = 0 \quad \forall V \in H^1(\Omega) . \tag{5.16}$$

To construct a particular solution $\tilde{G}(x, y)$ of (5.16), we observe that since γ is Lipschitz, there exists an extension $g(x)$ such that (see, for example, [6])

$$\| g \|_{H^1(\omega)} \leq C_5(\omega) \| g \|_{H^{1/2}(\gamma)}$$

Hence $\tilde{G}(x, y) := g(x)\, \psi\left(\frac{y}{d}\right)$ is an extension of $g\psi$ to $H^1(\Omega)$ and, by Lemma 5.1 with $\varphi = 1$, we have the estimate

$$\begin{aligned} \| \tilde{G} \|^2_{H^1(\Omega)} &= | \tilde{G} |^2_{H^1(\Omega)} + \underline{B}^{-1} \| \sqrt{b}\tilde{G} \|^2_{L^2(\Omega)} \leq \\ &\leq (1 + \underline{B}^{-1} \bar{A} C_3^2\, d^2) | \tilde{G} |^2_{H^1(\Omega)} \end{aligned} \tag{5 17}$$

Further,

$$\| \tilde{G} \|^2_{H^1(\Omega)} \leq C_6\, d^{-1} \| \psi \|^2_{H^1(-1\,1)} \| g \|^2_{H^{1/2}(\gamma)} , \tag{5.18}$$

where C_6 depends only on ω. Now define $G(x, y) := \tilde{G}/\varphi(x)$, with $\varphi(x)$ as in (5.12). Then

$$\|G\|_{\varphi}^2 \leq \max\{\bar{A}, \bar{B}\bar{C}\} \int_{\Omega} \left\{ \left(\frac{\partial \tilde{G}}{\partial y} \right)^2 + \varphi^2 \left| \nabla_x (\tilde{G}\varphi^{-1}) \right|^2 \right\} dx\, dy .$$

Setting $\rho(x) := \text{dist}(x, \gamma)$, we find

$$\left| \nabla_x (\varphi^{-1} \tilde{G}) \right|^2 \leq 2\, \varphi^{-2} \left\{ \left| \nabla_x \tilde{G} \right|^2 + \beta^2 |\nabla_x \rho|^2 |\tilde{G}|^2 \right\}$$

and, since $\tilde{G}$ satisfies (4.10), we can use Lemma 5.1 with $\varphi = 1$ to get

$$\int_{\Omega} |\nabla_x \rho|^2 |\tilde{G}|^2\, dx \leq nR^2(\omega)\, \underline{B}^{-1} \int_{\Omega} b\left(\frac{y}{d}\right) |\tilde{G}|^2\, dx\, dy$$
$$\leq nR^2(\omega)\, \underline{B}^{-1}\, C_3^2\, d^2 \int_{\Omega} a\left(\frac{y}{d}\right) \left| \frac{\partial \tilde{G}}{\partial y} \right|^2 dx\, dy .$$

Hence

$$\|G\|_{\varphi}^2 \leq C_7(\omega, n) \left| \tilde{G} \right|_{H^1(\Omega)}^2 \tag{5.19}$$

provided that β satisfies (5.13). Combining (5.19) with (5.18), we arrive at

$$\|G\|_{\varphi} \leq C_6\, C_7\, d^{-1/2} \|\psi\|_{H^1(-1,1)} \|g\|_{H^{1/2}(\gamma)} . \tag{5.20}$$

Now we split the solution $U = W + G$, where W solves the problem : Find $W \in H^1(\Omega, \Gamma) \cap H_{\varphi}$, such that

$$B(W, V) = -B(G, V) =: G(V) \quad \forall V \in H^1(\Omega, \Gamma) .$$

By Theorem 5.1, $B(.,.)$ is $(1, \delta)$ regular on $H_{\varphi} \times H_{\varphi^{-1}}$, hence

$$|G(V)| = |B(G, V)| \leq \|G\|_{\varphi}\, \|V\|_{\varphi^{-1}},$$

and from (5.6) we find $\|W\|_{\varphi} \leq \|G\|_{\varphi}/\delta$, i.e.

$$\|U\|_{\varphi} \leq \|W\|_{\varphi} + \|G\|_{\varphi} \leq (1 + \delta^{-1}) \|G\|_{\varphi}$$

and referring to (5.20) completes the proof. □

The previous Lemma implies the desired decay estimate of u_{BL}^N.

THEOREM 5.3 : *Assume that* $f \in H^{2N}(\omega)$, $N \geq 1$, *and that* $\varphi(x) = \exp\{\beta \text{ dist}(x, \gamma)\}$ *with* β *satisfying* (5.13). Then

$$\left\| u_{BL}^N \right\|_{\varphi} \leq C_8(\omega, n)\, d^{1/2}\, \Phi(N, f, d) \tag{5.21}$$

where

$$\Phi(N, f, d) = \sum_{j=1}^{N} d^{2j-2} \| \psi_{2j} \|_{H^1(-1,1)} \| \gamma_0(A^{j-1} f) \|_{H^{1/2}(\gamma)}$$

remains bounded as $d \to 0$.

Proof: Recall that for $N \geq 1$ (see (4.3))

$$u_{BL}^N = \sum_{j=1}^{N} d^{-1+2j} U_j(x, y)$$

where U_j solves (5.14) with

$$g(s) = -\gamma_0(A^{j-1} f), \quad \psi = \psi_{2j}.$$

Due to $f \in H^{2N}(\omega)$ and (4.8), the assumptions of Lemma 5.2 are satisfied for $1 \leq j \leq N$ and we get

$$\| U_j \|_{\varphi} \leq C_8(\omega, n) d^{-1/2} \| \psi_{2j} \|_{H^1(-1,1)} \| \gamma_0(A^{j-1} f) \|_{H^{1/2}(\gamma)}.$$

Hence we have (5.21) □

5.3. Boundary Layer Resolution

We will now prove that the optimal asymptotic rate of convergence of $d^{2N+1/2}$ can be recovered, if we use instead of the uniform model order N in (5.1)

$$\mathcal{P} = \{\omega_t, \sigma_t\}, \quad q = \{N, M\}, \tag{5.22}$$

where $\omega_t := \{x \in \omega \mid \text{dist}(x, \gamma) > t\}$, $\sigma_t = \omega \backslash \bar{\omega}_t$, $t > 0$ is a parameter at our disposal and $M \geq N$ is an elevated model order near the edge γ of the plate.

THEOREM 5.4 : *Let $(\mathcal{P}, q)$ be as in* (5.22) *with $M = \infty$ (see* Remark 4.5 *for the meaning of infinite model order) and $f \in H^{2N}(\omega)$. Then there exist constants $C_{11} = 8NC_4(\omega, n) R(\omega)$ and C_{12} which are independent of d so that*

$$\| u - u(\mathcal{P}, q) \|_{E(\Omega)} \leq C_{12} d^{2N+1/2}$$

provided that

$$t \geq C_{11} d |\ln d| . \tag{5.23}$$

Proof: Due to (3.3), we estimate for any $\omega \in S(\mathcal{P}, q)$

$$\| e(\mathcal{P}, q) \|_{E(\Omega)} \leq \| u - w \|_{E(\Omega)} \leq$$
$$\leq \| u - u_0^N - u_{BL}^N \|_{E(\Omega)} + \| u_0^N + u_{BL}^N - w \|_{E(\Omega)}$$

and by Theorem 4.1 it remains to estimate the last term. Let $\tilde{\chi}(\xi)$ be a nonnegative C^∞ cut-off function satisfying

$$\tilde{\chi}(\xi) = \begin{cases} 1 & 0 \leq \xi \leq 1/2 , \\ 0 & 1 \leq \xi , \end{cases} \tag{5.24}$$

and define $\chi(x) := \tilde{\chi}(\operatorname{dist}(x, \gamma)/t)$ for $t > 0$. Then with Remark 5.1 $\chi \in W^{1,\infty}(\omega)$ and $\operatorname{supp}(1 - \chi(x)) \subseteq \bar{\Omega}_{t/2}$. We select $w = u_0^N + \chi u_{BL}^B \in S(\mathcal{P}, q)$ and have

$$\| u_0^N + u_{BL}^N - w \|_{E(\Omega)}^2 = \| (1 - \chi) u_{BL}^N \|_{E(\Omega_{t/2} \setminus \Omega_t)}^2 + \| u_{BL}^N \|_{E(\Omega_t)}^2 ,$$

where $\Omega_t = \omega_t \times (-d, d)$. We estimate

$$\| (1 - \chi) u_{BL}^N \|_{E(\Omega_{t/2} \setminus \Omega_t)}^2 \leq$$

$$\leq \int_{\Omega_{t/2} \setminus \Omega_t} \left\{ a\left(\frac{y}{d}\right) \left(\frac{\partial u_{BL}^N}{\partial y} \right)^2 + \bar{C} b\left(\frac{y}{d}\right) |\nabla_x ((1 - \chi) u_{BL}^N)|^2 \right\} dx\, dy$$

$$\leq \int_{\Omega_{t/2} \setminus \Omega_t} \left\{ a\left(\frac{y}{d}\right) \left(\frac{\partial u_{BL}^N}{\partial y} \right)^2 + 2\bar{C} b\left(\frac{y}{d}\right) |\nabla_x u_{BL}^N|^2 + \frac{C_9^2(\omega)}{t^2} b\left(\frac{y}{d}\right) (u_{BL}^N) \right\} dx\, dy$$

where $C_9^2 = 2 n\bar{C} \| \tilde{\chi}' \|_{L^\infty} R^2(\omega)$. Using Lemma 5.1 on the last term, we find that

$$\| (1 - \chi) u_{BL}^N \|_{E(\Omega_{t/2} \setminus \Omega_t)}^2 \leq C_{10}^2 \| u_{BL}^N \|_{E(\Omega_{t/2} \setminus \Omega_t)}^2$$

where

$$C_{10}^2(\omega) = \max \left\{ 1 + C_9^2 C_3^2 d^2 t^{-2}, 2\bar{C}/\underline{C} \right\} .$$

On $\omega_{t/2}$ obviously $\exp(\beta t/2) \leq \exp(\beta \operatorname{dist}(x, \gamma))$ for $\beta \geq 0$, hence

$$e^{\beta t} \| u_{BL}^N \|_{E(\Omega_{t/2} \setminus \Omega_t)}^2 \leq \| u_{BL}^N \|_\varphi^2 \leq C_8^2 d \Phi^2(N, f, d)$$

by Theorem 5.3. Analogously

$$\| u_{BL}^N \|_{E(\Omega_t)}^2 \leq e^{-2\beta t} C_8^2 d \Phi^2(N, f, d) .$$

Hence we find

$$\| u_0^N + u_{BL}^N - w \|_{E(\Omega)} \leq d^{1/2} e^{-\beta t/2} C_8 (1 + C_{10}^2)^{1/2} \Phi(N, f, d) , \tag{5.25}$$

and we observe that (5.23) implies the boundedness of C_{10} as $d \to 0$. Now we require the bounds (5.25) and (4.1) to be of the same order in d, i.e.

$$C_N d^{2N} \|A^N f\|_{L^2(\omega)} \sim e^{-\beta t/2} C_8 (1 + C_{10}^2)^{1/2} \Phi(N, f, d) . \quad (5.26)$$

Selecting $\beta = 1/(2 C_4 R(\omega) d)$ as in (5.13), we get for t satisfying (5.23) and for $0 < d \leqslant 1$ that

$$\begin{aligned} e^{-\beta t/2} C_8 (1 + C_{10}^2)^{1/2} \Phi(N, f, d) &\leqslant e^{-\frac{C_{11} |\ln d|}{4 C_4 R}} C_8 (1 + C_{10}^2)^{1/2} \Phi \\ &= d^{\frac{C_{11}}{4 C_4 R}} C_8 (1 + C_{10}^2)^{1/2} \Phi . \end{aligned}$$

Selecting $C_{11} = 8 N C_4 R(\omega)$, we see that (5.26) is satisfied for all sufficiently small d. Adding (5.25) and the upper bound in Theorem 4.1, we obtain the assertion of the theorem with $C_{12} := C_N \|A^N f\|_{L^2(\omega)} + C_8 (1 + C_{10}^2)^{1/2} \Phi(N, f, d)$ where Φ is as in (5.21) and remains uniformly bounded as $d \to 0$. □

We have actually proved the following stronger assertion.

Remark 5.2. Let $\mathscr{P}$ be as in (5.22) and t as in (5.23). Then, if $S(\{\omega\}, N) \subset S(\mathscr{P}, q)$,

$$\|e(\mathscr{P}, q)\|_{E(\Omega)} \leqslant C_N d^{2N + 1/2} + \\ + \inf_{w \in S(\mathscr{P}, q)} \|\chi (u_0^N + u_{BL}^N) - w\|_{E(\Sigma_t)} \quad (5.27)$$

where

$$\Sigma_t := \sigma_t \times (-d, d) .$$

This, together with Proposition 4.1, shows that Theorem 5.4 also holds if a finite, sufficiently large model order $M(N, d)$ is selected in σ_t.

6. CONCLUDING REMARKS

In this final section we briefly address the design of $S(\mathscr{P}, q)$ in the vicinity of γ. To this end we assume that for a mesh $\Delta_0 = \{z_j \mid -1 := z_0 < z_1 < \cdots < z_K := 1\} \subset [-1, 1]$ the functions $a(z)$, $b(z)$ are piecewise constant on (z_j, z_{j+1}) and that γ is smooth. Then it is well known that u and u_{BL}^N become singular on the sets $\gamma \times \{z_j d\}$, due to edge — and interface singularities (see *fig.* 1 for $K = 3$ layers). The functions ψ_{2j} in (4.4)-(4.6) are in this case piecewise polynomials of degree $2j$ and the size of M necessary in (5.27) is governed by the regularity of $u_0^N + u_{BL}^N$ in

Σ_t. Although the subspace $S(\mathcal{P}, q)$ in (3.1) has a simple structure, it is not very well suited for the approximation of singular solutions. Remark 5.2 allows to alter $S(\mathcal{P}, q)$ in σ_t to obtain better approximation properties, as long as $S(\{\omega\}, N) \subset S(\mathcal{P}, q)$. Let us indicate how to obtain a suitable modification of $S(\mathcal{P}, q)$.

Figure 1. — 3-layer laminate with edge ($e_\pm$) and interface ($i_\pm$) singularities

For a mesh $\Delta := \{z_j \mid -1 =: z_0 < z_1 < \cdots < z_L := 1\}$ in $[-1, 1]$ and a polynomial degree vector $\underline{p} := \{p_1, \ldots, p_L\}$, $S^{\underline{p}}(\Delta)$ denotes the space of continuous, piecewise polynomial functions of degrees p_i on $(-1, 1)$, and with basis $\varphi_k(z)$. Partition $(0, t)$ with t as in (5.23) into M subintervals : $t =: t_0 > t_1 > \cdots > t_M := 0$ and define

$$\sigma_j := \omega_{t_j} \backslash \omega_{t_{j-1}},\ \Sigma_j := \sigma_j \times (-d, d),\ j = 1, \ldots, M,\ \sigma_0 = \omega_t,\ \Sigma_0 = \Omega_t\,,$$

where $\omega_t := \{x \in \omega \mid \text{dist}\,(x, \gamma) > t\}$. Then associate with each σ_j a mesh Δ_j on $(-1, 1)$ and a polynomial degree distribution $\underline{p}_j$, satisfying

$$\Delta_{j-1} \subseteq \Delta_j\,,\ \underline{p}_{j-1} \leq \underline{p}_j \text{ componentwise } j = 1, \ldots, M\,, \tag{6.1}$$

i.e. all meshes Δ_j are refinements of Δ_0 and the interfaces between layers are mesh-points. If we set $\underline{p}_0 = \{2N, \ldots, 2N\}$, we have

$$\psi_{2i} \in S^{\underline{p}_0}(\Delta_0) \subseteq S^{\underline{p}_j}(\Delta_j)\,, \quad i = 1, \ldots, N, j = 1, \ldots, M\,. \tag{6.2}$$

Now the subspace $S(\mathcal{P}, q)$ is defined as follows :

$$S(\mathcal{P}, q) := \Big\{ u \,|\, u|_{\Sigma_j} = \sum_\ell X_\ell^{(j)}(x)\, \varphi_\ell^{(j)}\Big(\frac{y}{d}\Big),\ X_\ell^{(j)} \in H^1(\sigma_j),\ j = 1, \ldots, M \tag{6.3}$$
$$\text{and } \quad u|_{\Omega_t} = \sum_\ell X_\ell^{(0)}(x)\, \psi_{2\ell}\Big(\frac{y}{d}\Big),\ X_\ell^{(0)} \in H^1(\omega_t)\Big\} \cap H^1(\Omega, \Gamma)\,.$$

Then (6.2) implies that $S(\{\omega\}, N) \subset S(\mathcal{P}, q)$ and a suitable selection of the meshes Δ_j and the sequence $\{t_j\}$ amounts to $h-p$ refinement towards the singularities Figure 2 shows a possible subdivision of the domain Ω in lateral normal direction with $M = 3$

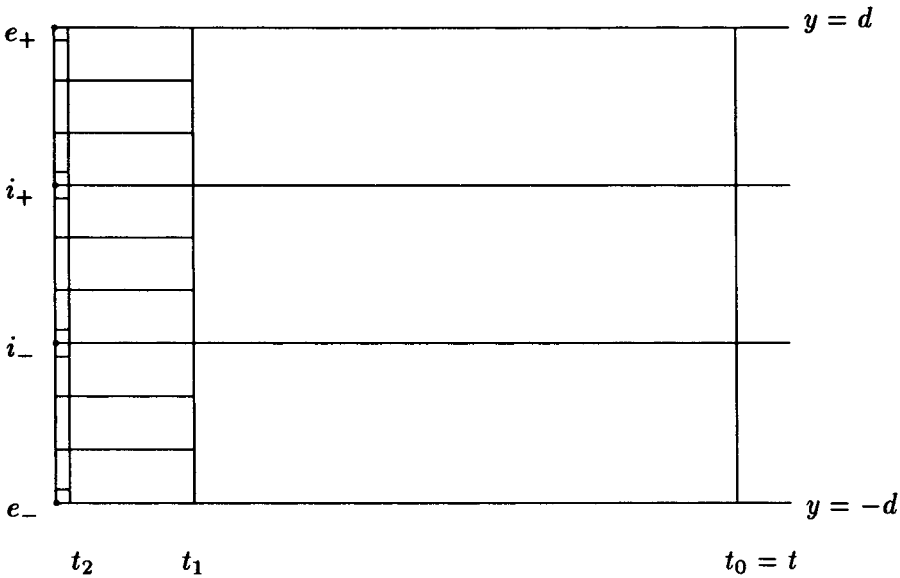

Figure 2. — Effective domain partitioning using $M = 3$ *layers*

So far we only analyzed the semidiscretization error under the assumption that the coupled, elliptic system for the unknown coefficient functions $X_\ell^{(i)}$ in (6 3) can be solved exactly Our conclusions remain valid, however, if this system is also discretized with a sufficiently accurate finite element method The resulting scheme is a conforming discretization of the three dimensional problem (2 1) with $h-p$ refinement in the boundary layer and a single layer of « brick » elements with the nonpolynomial shape functions ψ_{2j} in the cross section for the interior of the plate

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