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ERROR ESTIMATES FOR LEAST-SQUARES MIXED FINITE ELEMENTS (*)

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Résumé — *Une méthode éléments finis mixtes des moindres carrés est formulée, et appliquée à une classe de problemes elliptiques du second ordre, pour des domaines bidimensionnels et tridimensionnels La solution primaire u et le flux σ sont approchés en utilisant des espaces éléments finis de polynomes par morceaux, de degrés k et r, respectivement La méthode est non conforme dans la mesure ou l'approximation du flux ne peut pas être satisfaite sur toute la frontière Γ, mais n'est satisfaite qu'aux nœuds de Γ Des estimations d'erreur optimales dans les espaces L^2 et H^1 sont obtenues en faisant l'hypothese habituelle de régularité sur la partition éléments finis (la condition LBB n'est pas requise) Les cas importants où $k = r$ et $k + 1 = r$ sont examinés*

Abstract — *A least-squares mixed finite element method is formulated and applied for a class of second order elliptic problems in two and three dimensional domains The primary solution u and the flux σ are approximated using finite element spaces consisting of piecewise polynomials of degree k and r respectively The method is nonconforming in the sense that the boundary condition for the flux approximation cannot be satisfied exactly on the whole boundary Γ — so it is satisfied only at the nodes on Γ Optimal L^2- and H^1-error estimates are derived under the standard regularity assumption on the finite element partition (the* LBB*-condition is not required) The important cases of $k = r$ and $k + 1 = r$ are considered*

1. INTRODUCTION

Least-squares mixed finite element methods have become a topic of increasing interest since they lead to symmetric algebraic systems and are not subject to the Ladyzhenskaya, Babuška, Brezzi (LBB) consistency requirement. The methods remain, however, relatively little studied compared with the established mixed methods. There are several open theoretical questions related to the formulation and convergence properties as well as numerical behaviour.

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The main idea can be conveniently introduced by means of the representative second-order elliptic boundary-value problem :

$$- \operatorname{div} (A \operatorname{grad} u) = f \quad \text{in} \quad \Omega , \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma , \tag{1.2}$$

where $\Omega \subset R^n$, $n = 2, 3$, is a bounded domain with boundary Γ and A is a positive definite matrix of coefficients. Introducing the flux $\boldsymbol{\sigma} = - A \operatorname{grad} u$, the problem may be recast as the first order system

$$\boldsymbol{\sigma} + A \operatorname{grad} u = 0 \quad \text{in} \quad \Omega , \tag{1.3}$$

$$\operatorname{div} \boldsymbol{\sigma} + cu = f \quad \text{in} \quad \Omega , \tag{1.4}$$

$$u = 0 \quad \text{on} \quad \Gamma . \tag{1.5}$$

The classical mixed method for (1.3)-(1.5) is based on the stationary principle for a saddle-point problem and is subject to the inf-sup condition on the spaces for u and $\boldsymbol{\sigma}$ (see Brezzi [1]). This implies certain restrictions on the polynomial degree k and r for the element bases defining approximations u_h and $\boldsymbol{\sigma}_h$ respectively. In a least-squares mixed formulation the problem is to minimize the L^2-norm of the residuals corresponding to (1.3)-(1.4) and is not subject to the consistency requirement. The following estimates for the least-squares mixed method are proved in [18] : for $k = r$

$$\|u - u_h\|_{1, \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div}, \Omega)} \leq C h^k \tag{1.6}$$

and for $k + 1 = r$

$$\|u - u_h\|_{0, \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div}, \Omega)} \leq C h^r . \tag{1.7}$$

These estimates are optimal in the corresponding norms but it is highly desirable to have a optimal estimate for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0, \Omega}$. This is the aim of this paper. To accomplish this goal we use the fact that curl grad $v = 0$ to introduce the equation

$$\operatorname{curl} (A^{-1} \boldsymbol{\sigma}) = 0$$

which is added to the first order system. Also, a new boundary condition $\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma} = 0$ is imposed on Γ, where $\mathbf{n}$ is the outward normal to Γ and "$\wedge$" denotes the exterior product. This boundary condition cannot be satisfied exactly by the finite element space — so we satisfy it only at the nodes on the boundary. In this sense the method is mildly nonconforming at the boundary. Note that the nonconformity has no negative impact on the stability of the method — the only boundary condition which is necessary for existence and uniqueness is (1.2). We prove the following estimates : for $k = r$

$$\|u - u_h\|_{1, \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1, \Omega} \leq C h^k , \tag{1.8}$$

$$\|u - u_h\|_{0, \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0, \Omega} \leq C h^{k+1} \tag{1.9}$$

and for $k + 1 = r$

$$\|u - u_h\|_{0\ \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1\ \Omega} \leq C h^r , \tag{1 10}$$

$$\|u - u_h\|_{\ 1\ \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0\ \Omega} \leq C h^{r+1},\ k > 1 \tag{1 11}$$

Note that all above estimates are optimal and they depend only on the regularity of the solution and the standard regularity assumption on the finite element partition — there are no other restrictions on the finite element mesh or on the finite element spaces

Some comments conserning several related studies of least-squares methods are warranted to put the current work in perspective, e g see [5, 6, 10, 12, 16] Fix, Gunzburger and Nicolaides [10] presented a mixed method based on the Kelvin principle Optimal L^2-error estimates are proved for a certain class of grids satisfying the so-called Grid Decomposition Property Unfortunately, the latter is a necessary and sufficient condition for stability and optimal accuracy (see also Chen [6]) Chang [5] has proved an estimate similar to (1 9) when A is the identity matrix under the assumption that the boundary condition $\mathbf{n} \wedge \boldsymbol{\sigma} = 0$ is satisfied exactly on Γ This condition is essential for the analysis in [5] But in the present paper we prove that it is not necessary to satisfy this boundary condition in order to have stability (see also [18]) We need it in order to get better estimates and it is sufficient to satisfy such condition approximately The main tool in [5] is the general theory of Agmon, Douglis and Nirenberg for elliptic systems which does not reveal entirely the different nature of u and $\boldsymbol{\sigma}$ It is not clear how to handle the case $k \neq r$ following such approach In the present study we manage to "separate" the consideration of error estimates for u and $\boldsymbol{\sigma}$ Our analysis is closely related to the analysis of finite element approximations for Maxwell equations (see Neittaanmaki and Saranen [17], Saranen [22], Neittaanmaki and Picard [15]) In fact, part of our bilinear form coincides with the bilinear form in these studies The special cases corresponding to the Poisson equation and Helmholtz equation are also considered in Neittaanmaki and Saranen [16], Haslinger and Neittaanmaki [12] For such specific classes of equations it is possible to define a direct approximation to the flux with optimal estimates However, the same approach does not work for the class of problems considered here since these involve a coupled system for u and $\boldsymbol{\sigma}$

The paper is organized as follows in section 2 we give the problem formulation and prove the coercivity of the bilinear form The finite element formulation is described in section 3 Optimal error estimates are derived in section 4

2. PROBLEM FORMULATION

Let Ω be a bounded domain in R^n, $n = 2, 3$, with smooth boundary Γ. Consider the second-order boundary-value problem

$$- \operatorname{div} (A \operatorname{grad} u) + c(x) u = f \quad \text{in} \quad \Omega , \tag{2.1}$$
$$u = 0 \quad \text{on} \quad \Gamma , \tag{2.2}$$

where the matrix of coefficients $A = (a_{ij}(x))_{i,j=1}^{n}$, $x \in \bar{\Omega}$, is positive definite and the coefficients a_{ij} are bounded ; i.e. there exist constants α_1 and α_2 such that

$$\alpha_1 \zeta^T \zeta \leq \zeta^T A \zeta \leq \alpha_2 \zeta^T \zeta . \tag{2.3}$$

for all vectors $\zeta \in R^n$ and all $x \in \bar{\Omega}$.

The standard notations for Sobolev spaces $H^m(\Omega)$ with norm $\| \cdot \|_{m, \Omega}$ and seminorms $| \cdot |_{i, \Omega}$, $0 \leq i \leq m$, are employed throughout. As usual, $L^2(\Omega) = H^0(\Omega)$ and let $H^m(\Omega)^n$ be the corresponding product space. Also, we shall use the spaces $H^s(\Gamma)$ (see Grisvard [11]). Let

$$V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \} .$$

By the Poincaré-Friedrichs inequality

$$\| v \|_{0, \Omega} \leq C_F | v |_{1, \Omega} \quad \text{for all} \quad v \in V . \tag{2.4}$$

Let

$$c_0 = \min \left\{ \inf_{x \in \Omega} c(x), 0 \right\} . \tag{2.5}$$

We make the following assumptions with respect to the coefficients of our equation : there exist constants α_0 and c_1 such that

$$| c(x) | \leq c_1 \quad \text{for all} \quad x \in \bar{\Omega} , \tag{2.6}$$
$$0 < \alpha_0 \leq \alpha_1 + c_0 C_F^2 , \tag{2.7}$$

where C_F is the constant from the Poincaré-Friedrichs inequality above. Hence, the coefficient $c(x)$ may be negative provided that α_1 is sufficiently large.

Now, introducing a new variable $\boldsymbol{\sigma} = - A \operatorname{grad} u$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, we get the following system of first-order differential equations for u and $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} + A \operatorname{grad} u = 0 \quad \text{in} \quad \Omega , \tag{2.8}$$
$$\operatorname{div} \boldsymbol{\sigma} + cu = f \quad \text{in} \quad \Omega , \tag{2.9}$$
$$u = 0 \quad \text{on} \quad \Gamma . \tag{2.10}$$

Let $\mathbf{q} = (q_1, \ldots, q_n)$ be a smooth vector function Denote by curl the following operator

$$\Omega \subset R^2 \quad \operatorname{curl} \mathbf{q} = \partial_1 q_2 - \partial_2 q_1 ,$$
$$\Omega \subset R^3 \quad \operatorname{curl} \mathbf{q} = (\partial_2 q_3 - \partial_3 q_2, \partial_3 q_1 - \partial_1 q_3, \partial_1 q_2 - \partial_2 q_1)$$

Also, when $\Omega \subset R^2$ and $v \in H^1(\Omega)$ we denote $\operatorname{curl} v = (-\partial_2 v, \partial_1 v)$ Since $\operatorname{curl} \operatorname{grad} v = 0$ for smooth v then (2 8) yields

$$\operatorname{curl} A^{-1} \boldsymbol{\sigma} = 0 \tag{2 11}$$

Let $\mathbf{n} = (\nu_1, \ldots, \nu_n)$ be the outward normal to the boundary Γ We introduce the exterior product operator

$$\Omega \subset R^2 \quad \mathbf{n} \wedge \mathbf{q} = \nu_1 q_2 - \nu_2 q_1 ,$$
$$\Omega \subset R^3 \quad \mathbf{n} \wedge \mathbf{q} = (\nu_2 q_3 - \nu_3 q_2, \nu_3 q_1 - \nu_1 q_3, \nu_1 q_2 - \nu_2 q_1)$$

Then, having in mind the boundary condition (2 10), we get $\mathbf{n} \wedge \operatorname{grad} u = 0$ which may be written as

$$\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma} = 0 \quad \text{on} \quad \Gamma \tag{2 12}$$

Next, introduce the following spaces

$$\tilde{\mathbf{W}} = \{\mathbf{q} \in L^2(\Omega)^n \;\; \operatorname{div} \mathbf{q} \in L^2(\Omega)\} , \tag{2 13}$$
$$\mathbf{W} = \{\mathbf{q} \in \tilde{\mathbf{W}} \;\; \operatorname{curl} A^{-1} \mathbf{q} \in L^2(\Omega)^s, \;\; s = 1 \text{ for } n = 2 , \; s = 3 \text{ for } n = 3 , \;\; \mathbf{n} \wedge A^{-1} \mathbf{q} = 0 \text{ on } \Gamma\} \tag{2 14}$$

with norms

$$\|\mathbf{q}\|^2_{H(\operatorname{div})} \equiv \|\mathbf{q}\|^2_{0\,\Omega} + \|\operatorname{div} \mathbf{q}\|^2_{0\,\Omega} ,$$
$$\|\mathbf{q}\|^2_{H(\operatorname{div}\,\operatorname{curl})} \equiv \|\mathbf{q}\|^2_{H(\operatorname{div})} + \|\operatorname{curl} A^{-1} \mathbf{q}\|^2_{0\,\Omega}$$

Let $(\cdot\,,\cdot)_{0\,\Omega}$ be the standard inner product in $L^2(\Omega)$ or $L^2(\Omega)^n$, correspondingly $(\cdot\,,\cdot)_{0\,\Gamma}$ will be the inner product in $L^2(\Gamma)^s$, $s = 1$ for $n = 2$, $s = 3$ for $n = 3$

Now we are ready to formulate the least-squares minimization problem find $u \in V$, $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$J(u, \boldsymbol{\sigma}) = \inf_{v \in V \;\; \mathbf{q} \in \mathbf{W}} J(v, \mathbf{q}) ,$$

where

$$J(v, \mathbf{q}) = (\operatorname{curl} A^{-1} \mathbf{q}, \operatorname{curl} A^{-1} \mathbf{q})_{0\,\Omega} + (\operatorname{div} \mathbf{q} + cv - f, \operatorname{div} \mathbf{q} + cv - f)_{0\,\Omega} + (\mathbf{q} + A \operatorname{grad} v, \mathbf{q} + A \operatorname{grad} v)_{0\,\Omega} \tag{2 15}$$

The corresponding variational statement is find $u \in V$, $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$a(u, \boldsymbol{\sigma}, v, \mathbf{q}) = (f, \operatorname{div} \mathbf{q} + cv)_{0, \Omega} \quad \text{for all} \quad v \in V, \quad \mathbf{q} \in \mathbf{W}, \tag{2.16}$$

where

$$a(u, \boldsymbol{\sigma}, v, \mathbf{q}) = \tilde{a}(u, \boldsymbol{\sigma}, v, \mathbf{q}) + (\operatorname{curl} A^{-1} \boldsymbol{\sigma}, \operatorname{curl} A^{-1} \mathbf{q})_{0, \Omega}, \tag{2.17}$$

$$\tilde{a}(u, \boldsymbol{\sigma}, v, \mathbf{q}) = (\operatorname{div} \boldsymbol{\sigma} + cu, \operatorname{div} \mathbf{q} + cv)_{0, \Omega} + (\boldsymbol{\sigma} + A \operatorname{grad} u, \mathbf{q} + A \operatorname{grad} v)_{0, \Omega} \tag{2.18}$$

In order to prove existence and uniqueness of the solution of (2.16) we have to show that the bilinear form $a(.,.)$ is coercive in the space $(V, \mathbf{W})$. First, we shall investigate the coercivity of $\tilde{a}(.,.)$ in the larger space $(V, \tilde{\mathbf{W}})$

THEOREM 2.1 *There exists a constant $C > 0$ such that*

$$C(\|v\|_{1, \Omega}^2 + \|\mathbf{q}\|_{0, \Omega}^2 + \|\operatorname{div} \mathbf{q}\|_{0, \Omega}^2) \leq \tilde{a}(v, \mathbf{q}, v, \mathbf{q}) \tag{2.19}$$

for all $v \in V$, $\mathbf{q} \in \tilde{\mathbf{W}}$

Proof Let β be a positive constant to be specified later and E denote the identity $n \times n$ matrix. Expanding $\tilde{a}(.,.)$,

$$\tilde{a}(v, \mathbf{q}, v, \mathbf{q}) = \int_\Omega [(\operatorname{div} \mathbf{q} + cv)^2 + (\mathbf{q} + A \operatorname{grad} v)^2] \, dx$$

$$= \int_\Omega [(\operatorname{div} \mathbf{q})^2 + 2 cv \operatorname{div} \mathbf{q} + (cv)^2 + \mathbf{q}^2 + 2 \mathbf{q} . A \operatorname{grad} v + (A \operatorname{grad} v)^2 + 2 \beta \mathbf{q} . \operatorname{grad} v - 2 \beta \mathbf{q} . \operatorname{grad} v + (c - \beta)^2 v^2 - (c - \beta)^2 v^2] \, dx$$

Selectively integrating by parts, setting $v = 0$ on Γ and regrouping,

$$\tilde{a}(v, \mathbf{q}, v, \mathbf{q}) = \int_\Omega [(\operatorname{div} \mathbf{q})^2 + 2(c - \beta) v \operatorname{div} \mathbf{q} + (c - \beta)^2 v^2 - (c - \beta)^2 v^2 + (cv)^2 + \mathbf{q}^2 + 2 \mathbf{q} . (A - \beta E) \operatorname{grad} v + (A \operatorname{grad} v)^2] \, dx$$

$$= \int_\Omega [(\operatorname{div} \mathbf{q} + (c - \beta) v)^2 + (2 \beta c - \beta^2) v^2 + \mathbf{q}^2 + 2 \mathbf{q} . (A - \beta E) \operatorname{grad} v + ((A - \beta E) \operatorname{grad} v)^2 - ((A - \beta E) \operatorname{grad} v)^2 + (A \operatorname{grad} v)^2] \, dx$$

$$= \int_\Omega [(\operatorname{div} \mathbf{q} + (c - \beta) v)^2 + (2 \beta c - \beta^2) v^2$$
$$+ (\mathbf{q} + (A - \beta E) \operatorname{grad} v)^2 + 2 \beta A \operatorname{grad} v \cdot \operatorname{grad} v - \beta^2 (\operatorname{grad} v)^2] \, dx$$
$$\geqslant \int_\Omega [(2 \beta c_0 - \beta^2) v^2 + 2 \beta A \operatorname{grad} v \cdot \operatorname{grad} v - \beta^2 (\operatorname{grad} v)^2] \, dx$$
$$\geqslant \int_\Omega [(2 \beta c_0 - \beta^2) C_F^2 + 2 \beta \alpha_1 - \beta^2] (\operatorname{grad} v)^2 \, dx \,, \tag{2.20}$$

where we have used (2.3) and (2.4).

Let $\beta = \dfrac{\alpha_0}{1 + C_F^2}$. Then by (2.7)

$$\beta (2 (c_0 C_F^2 + \alpha_1) - \beta (1 + C_F^2)) = \frac{\alpha_0}{1 + C_F^2} (2 (c_0 C_F^2 + \alpha_1) - \alpha_0)$$
$$\geqslant \frac{\alpha_0^2}{1 + C_F^2} > 0 \,. \tag{2.21}$$

Using (2.21) in (2.20),

$$\tilde{a}(v, \mathbf{q} \,; v, \mathbf{q}) \geqslant C \, |\operatorname{grad} v|_{0, \Omega}^2 \geqslant C \, \|v\|_{1, \Omega}^2 \,. \tag{2.22}$$

Obviously, from (2.18),

$$\tilde{a}(v, \mathbf{q} \,; v, \mathbf{q}) \geqslant \|\mathbf{q} + A \operatorname{grad} v\|_{0, \Omega}^2 \,,$$
$$\tilde{a}(v, \mathbf{q} \,; v, \mathbf{q}) \geqslant \|\operatorname{div} \mathbf{q} + cv\|_{0, \Omega}^2 \,.$$

Hence,

$$\|\mathbf{q}\|_{0, \Omega}^2 \leqslant 2 \|\mathbf{q} + A \operatorname{grad} v\|_{0, \Omega}^2 + 2 \|A \operatorname{grad} v\|_{0, \Omega}^2$$
$$\leqslant C \tilde{a}(v, \mathbf{q} \,; v, \mathbf{q}) \,, \tag{2.23}$$
$$\|\operatorname{div} \mathbf{q}\|_{0, \Omega}^2 \leqslant 2 \|\operatorname{div} \mathbf{q} + cv\|_{0, \Omega}^2 + 2 \|cv\|_{0, \Omega}^2$$
$$\leqslant C \tilde{a}(v, \mathbf{q} \,; v, \mathbf{q}) \,. \tag{2.24}$$

Combining (2.22)-(2.24) we get (2.19). □

From Theorem 2.1 we obtain directly

THEOREM 2.2 : *The bilinear form* $a(\,.\,;\,.\,)$ *is coercive in* $(V, \mathbf{W})$, *i.e. there exists* $C > 0$ *such that*

$$C \left(\|v\|_{1, \Omega}^2 + \|\mathbf{q}\|_{H(\operatorname{div}, \operatorname{curl})}^2 \right) \leqslant a(v, \mathbf{q} \,; v, \mathbf{q}) \tag{2.25}$$

for all $v \in V$, $\mathbf{q} \in \mathbf{W}$. □

Remark The inequality (2 25) does not depend on the boundary condition (2 12) In order to demonstrate existence and uniqueness we need only the boundary condition (2 10) □

THEOREM 2 3 *Let* $f \in L^2(\Omega)$ *Then the problem* (2 16) *has a unique solution* $u \in V$, $\boldsymbol{\sigma} \in \mathbf{W}$

Proof Since $a(.,.)$ is continuous and coercive, the result follows from the Lax-Milgram lemma □

3 FINITE ELEMENT APPROXIMATION

Next, we define finite element spaces corresponding to V and $\mathbf{W}$ Let $\mathcal{T}_h$ be a partition of the domain Ω into finite elements, i e $\Omega = \bigcup_{K \in \mathcal{T}_h} K$ and h be the maximum diameter of the elements We suppose that the same partition is used in the definition of approximation spaces for u and $\boldsymbol{\sigma}$ although this is not necessary

Let $P_k(\Sigma)$, $\Sigma \subset R^n$, be the set of polynomials of degree k on Σ and let $\hat{K}$ denote the master element Suppose that for any $K \in \mathcal{T}_h$ there exists a mapping F_K $\hat{K} \to K$, $F_K(\hat{K}) = K$ with components $(F_K)_i \in P_s(\hat{K})$, $i = 1,$, n As usual, we have the correspondence $v_h(x) = \hat{v}_h(\hat{x})$, $\mathbf{q}_h(x) = \hat{\mathbf{q}}_h(\hat{x})$ for any $x = F_K(\hat{x})$, $\hat{x} \in \hat{K}$, and any functions $\hat{v}_n$, $\hat{\mathbf{q}}_h$ on $\hat{K}$ Define the following approximation spaces (of piecewise polynomials of degree k and r respectively for V_h and $\mathbf{W}_h$)

$$V_h = \left\{ v_h \in C^0(\Omega) \quad v_h\big|_K = \hat{v}_h\big|_{\hat{K}} \in P_k(\hat{K}) \quad \forall K \in \mathcal{T}_h, \quad v_h = 0 \text{ on } \Gamma \right\}, \tag{3 1}$$

$$\mathbf{W}_h = \left\{ \mathbf{q}_h \in C^0(\Omega)^n \quad (\mathbf{q}_h)_i\big|_K = (\hat{\mathbf{q}}_h)_i\big|_{\hat{K}} \in P_r(\hat{K}), \right.$$
$$\left. i = 1, \quad , n, \ \forall K \in \mathcal{T}_h, \ n \wedge A^{-1} \mathbf{q}_h = 0 \text{ at the nodes on } \Gamma \right\} \tag{3 2}$$

In general, we suppose that $1 \leq s \leq \max\{k, r\}$, where s is the degree of polynomials used in the mappings F_K, $K \in \mathcal{T}_h$ This means that for one of the variables (u or $\boldsymbol{\sigma}$) we may have isoparametric elements, while for the other variable the elements may be superparametric (see Carey and Oden [3])

Now, let us comment on the boundary condition Since we can use curved elements and we may have a non-constant matrix A then the boundary condition $\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma} = 0$ on Γ cannot be satisfied on the whole boundary We require this condition to be satisfied only at the nodes on the boundary Hence $\mathbf{W}_h \not\subset \mathbf{W}$ and we have a nonconforming finite element method find

$u_h \in V_h$, $\boldsymbol{\sigma}_h \in \mathbf{W}_h$ such that

$$a(u_h, \boldsymbol{\sigma}_h ; v_h, \mathbf{q}_h) = (f, \operatorname{div} \mathbf{q}_h + c v_h)_{0, \Omega} \quad \text{for all} \quad v_h \in V_h, \quad \mathbf{q}_h \in \mathbf{W}_h . \tag{3.3}$$

Using (2.8)-(2.11) for the exact solution we get the orthogonality property

$$a(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h ; v_h, \mathbf{q}_h) = 0 \quad \text{for all} \quad v_h \in V_h, \quad \mathbf{q}_h \in \mathbf{W}_h . \tag{3.4}$$

Since the inequality (2.25) does not depend on the boundary condition (2.12) we have

$$C (\|v_h\|_{1, \Omega}^2 + \|\mathbf{q}_h\|_{H(\operatorname{div}, \operatorname{curl})}^2) \leq a(v_h, \mathbf{q}_h ; v_h, \mathbf{q}_h) \tag{3.5}$$

for all $v_h \in V_h$, $\mathbf{q}_h \in \mathbf{W}_h$.

Hence the discrete problem (3.3) has a unique solution. Also, it follows in the same manner as in [18] that the condition number of the resulting linear system is $O(h^{-2})$.

In the cases when Ω is a 2D-polygon (3D-polytope) the tangential derivative is not uniquely specified at a corner point and, hence, we have several boundary conditions at the corner points of Ω. The value of σ_h at some corner point can be determined following an approach similar to the one developed in [2] for boundary-flux calculations. This issue and other issues concerning the implementation will be discussed in a forthcoming paper. Note that in the case of affine elements and constant matrix A the boundary condition $\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h = 0$ is satisfied exactly.

4. ERROR ESTIMATES

Let $v_I \in V_h$ and $\mathbf{q}_I \in \mathbf{W}_h$ be the standard finite element interpolants of some function v and some vector function $\mathbf{q}$ respectively, i.e. we have $v(x) = v_I(x)$ and $\mathbf{q}(x) = \mathbf{q}_I(x)$ at any node x (of course, we suppose that v and $\mathbf{q}$ are defined everywhere over $\bar{\Omega}$). From approximation theory we have the estimates (see Ciarlet [7]),

$$\|u - u_I\|_{0, \Omega} + h \|u - u_I\|_{1, \Omega} \leq C h^{k+1} \|u\|_{k+1, \Omega}, \tag{4.1}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{0, \Omega} + h \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{H(\operatorname{div}, \operatorname{curl})} \leq C h^{r+1} \|\boldsymbol{\sigma}\|_{r+1, \Omega} . \tag{4.2}$$

THEOREM 4.1 : *Let $k = r$. Then*

$$\|u - u_h\|_{1, \Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div}, \operatorname{curl})} \leq C h^k (\|u\|_{k+1, \Omega} + \|\boldsymbol{\sigma}\|_{k+1, \Omega}) . \tag{4.3}$$

Proof: Using Theorem 2.2, the orthogonality property (3.4) and the interpolation estimates (4.1) and (4.2),

$$\|u_h - u_I\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{H(\mathrm{div}\,\mathrm{curl})}^2$$
$$\leq C a(u_h - u_I, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\,;\, u_h - u_I, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I)$$
$$= C a(u - u_I, \boldsymbol{\sigma} - \boldsymbol{\sigma}_I\,;\, u_h - u_I, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I)$$
$$\leq C\,(\|u - u_I\|_{1,\Omega}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{H(\mathrm{div}\,\mathrm{curl})}^2)^{1/2} \times (\|u_h - u_I\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{H(\mathrm{div},\,\mathrm{curl})}^2)^{1/2}$$
$$\leq C h^k(\|u\|_{k+1,\Omega} + \|\boldsymbol{\sigma}\|_{k+1,\Omega})\,(\|u_h - u_I\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{H(\mathrm{div},\,\mathrm{curl})}^2)^{1/2}\,.$$

Applying again (4.1), (4.2) and the triangle inequality we get (4.3). □

Now we consider the case of different degree polynomials for u_h and $\boldsymbol{\sigma}_h$.

THEOREM 4.2 : *Let $k + 1 = r$. Then*

$$\|u - u_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,\Omega} \leq C h^r(\|u\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r+1,\Omega})\,. \tag{4.4}$$

Proof For any $v \in V$ let $S_h v \in V_h$ be the following projection :

$$(A \operatorname{grad}(v - S_h v), A \operatorname{grad} v_h)_{0,\Omega} + (c(v - S_h v), c v_h)_{0,\Omega} = 0 \quad \text{for all} \quad v_h \in V_h\,. \tag{4.5}$$

From standard finite element theory, we have the estimate

$$\|v - S_h v\|_{0,\Omega} + h\|v - S_h v\|_{1,\Omega} \leq C h^{k+1}\|v\|_{k+1,\Omega}\,. \tag{4.6}$$

Using Theorem 2.2 and the orthogonality property (3.4) in the same manner as before but with $S_h u$,

$$C\,(\|u_h - S_h u\|_{1,\Omega}^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{H(\mathrm{div},\,\mathrm{curl})}^2)$$
$$\leq a(u_h - S_h u, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\,;\, u_h - S_h u, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I)$$
$$= a(u - S_h u, \boldsymbol{\sigma} - \boldsymbol{\sigma}_I\,;\, u_h - S_h u, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I)$$
$$= (\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I), \operatorname{div}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I))_{0,\Omega} + (c(u - S_h u), \operatorname{div}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I))_{0,\Omega}$$
$$+ (\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I), c\,(u_h - S_h u))_{0,\Omega}$$
$$+ (\operatorname{curl} A^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I), \operatorname{curl} A^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I))_{0,\Omega}$$
$$+ (\boldsymbol{\sigma} - \boldsymbol{\sigma}_I, A \operatorname{grad}(u_h - S_h u))_{0,\Omega} - (u - S_h u, \operatorname{div} A^t(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I))_{0,\Omega}$$
$$+ (\boldsymbol{\sigma} - \boldsymbol{\sigma}_I, \boldsymbol{\sigma}_h - \boldsymbol{\sigma})_{0,\Omega}\,,$$

where we have used (4.5) and integration by parts. Hence from (4.2) and (4.6),

$$(\|u_h - S_h u\|_{1,\Omega} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{H(\mathrm{div},\,\mathrm{curl})})^2$$
$$\leq C h^r(\|\boldsymbol{\sigma}\|_{r+1,\Omega} + \|u\|_{r,\Omega})(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{1,\Omega} + \|u_h - S_h u\|_{1,\Omega})\,. \tag{4.7}$$

Now we shall prove that

$$\|\mathbf{q}_h\|_{1,\Omega} \leq C\,\|\mathbf{q}_h\|_{H(\mathrm{div},\,\mathrm{curl})} \tag{4.8}$$

for all $\mathbf{q}_h \in \mathbf{W}_h$. The following estimate can be found in Saranen [22, Theorem 2.2], see also Neittaanmäki and Saranen [17, Theorem 2.2] :

$$\|\mathbf{q}\|_{1,\Omega} \leq C\,(\|\mathrm{curl}\,A^{-1}\,\mathbf{q}\|_{0,\Omega} + \|\mathrm{div}\,\mathbf{q}\|_{0,\Omega} + \|\mathbf{q}\|_{0,\Omega} + \|\mathbf{n}\wedge A^{-1}\,\mathbf{q}\|_{1/2,\Gamma}) \tag{4.9}$$

for all $\mathbf{q} \in H^1(\Omega)^n$. We set $\mathbf{q} = \mathbf{q}_h$ and in order to get (4.8) it remains to estimate $\|\mathbf{n}\wedge A^{-1}\,\mathbf{q}_h\|_{1/2,\Gamma}$. Let $K \subset \Omega$ be any element that has a side (face) coincident with the boundary Γ, i.e. $K \cap \Gamma = e$, $\dim(e) = n-1$. Let $\hat{K}$ be the corresponding master element and $\hat{e}$ be the side (face) of $\hat{K}$ corresponding to e. As usual, $\mathbf{q}_h(x) = \hat{\mathbf{q}}_h(\hat{x})$, $x = F_K(\hat{x})$, $\hat{x} \in \hat{K}$, where F_K is the mapping from $\hat{K}$ onto K. Similarly, $A^{-1}(x) = \hat{A}^{-1}(\hat{x})$, $\mathbf{n}(x) = \hat{\mathbf{n}}(\hat{x})$. Then

$$\begin{aligned}\|\mathbf{n}\wedge A^{-1}\,\mathbf{q}_h\|_{1/2,e} &\leq C h^{-1/2}(\mathrm{meas}\,(e))^{1/2}\,\|\hat{\mathbf{n}}\wedge \hat{A}^{-1}\,\hat{\mathbf{q}}_h\|_{1/2,\hat{e}} \\ &\leq C h^{-1/2}(\mathrm{meas}\,(e))^{1/2}\,\|\hat{\mathbf{n}}\wedge \hat{A}^{-1}\,\hat{\mathbf{q}}_h\|_{r+1,\hat{e}}\end{aligned}$$

and since $\mathbf{n}\wedge A^{-1}\,\mathbf{q}_h = 0$ at the nodes on e we get by the Bramble-Hilbert lemma

$$\begin{aligned}\|\mathbf{n}\wedge A^{-1}\,\mathbf{q}_h\|_{1/2,e} &\leq C h^{-1/2}(\mathrm{meas}\,(e))^{1/2}\,|\hat{\mathbf{n}}\wedge \hat{A}^{-1}\,\hat{\mathbf{q}}_h|_{r+1,\hat{e}} \\ &\leq C h^{-1/2}(\mathrm{meas}\,(e))^{1/2}\sum_{s=0}^{r+1}|\hat{\mathbf{n}}|_{r+1-s,\infty,\hat{e}}\,|\hat{A}^{-1}\,\hat{\mathbf{q}}_h|_{s,\hat{e}}\,.\end{aligned}$$

Since the boundary is assumed smooth, we have

$$|\hat{\mathbf{n}}|_{r+1-s,\infty,\hat{e}} \leq C h^{r+1-s}, \quad s = 0, \dots, r+1\,.$$

Also, from the smoothness of coefficients,

$$|\hat{A}^{-1}|_{s,\infty,\hat{e}} \leq C h^{s}, \quad s = 0, \dots, r+1\,.$$

Now, using equivalence of the norms in finite dimensional spaces,

$$\|\mathbf{n}\wedge A^{-1}\,\mathbf{q}_h\|_{1/2,e} \leq C h^{-1/2}(\mathrm{meas}\,(e))^{1/2}\sum_{s=0}^{r} h^{r+1-s}\,|\hat{\mathbf{q}}_h|_{s,\hat{e}}$$

$$\leq Ch^{-1/2}(\text{meas }(e))^{1/2}\,(h^2|\hat{\mathbf{q}}_h|_{0,\hat{e}} + h|\hat{\mathbf{q}}_h|_{1,\hat{e}})$$

$$\leq Ch^{-1/2}(\text{meas }(e))^{1/2}\,(h^2|\hat{\mathbf{q}}_h|_{0,\hat{K}} + h|\hat{\mathbf{q}}_h|_{1,\hat{K}})$$

$$\leq Ch^{-1/2}(\text{meas }(e))^{1/2}\,h^2(\text{meas }(K))^{-1/2}\,\|\mathbf{q}_h\|_{1,K}$$

$$\leq Ch\|\mathbf{q}_h\|_{1,K}\,.$$

Hence

$$\|\mathbf{n}\wedge A^{-1}\mathbf{q}_h\|_{1/2,\Gamma} \leq Ch\|\mathbf{q}_h\|_{1,\Omega_b}, \tag{4.10}$$

where Ω_b is the set of elements which have a common side (face) with the boundary Γ. Then (4.10) and (4.9) with $\mathbf{q} = \mathbf{q}_h$ imply

$$\|\mathbf{q}_h\|_{1,\Omega} \leq C\,(\|\mathbf{q}_h\|_{H(\text{div, curl})} + \|\mathbf{n}\wedge A^{-1}\mathbf{q}_h\|_{1/2,\Gamma})$$
$$\leq C\,\|\mathbf{q}_h\|_{H(\text{div, curl})} + Ch\|\mathbf{q}_h\|_{1,\Omega_b}\,.$$

Hence, for sufficiently small h, the term $Ch\|\mathbf{q}_h\|_{1,\Omega_b}$ is absorbed by $\|\mathbf{q}_h\|_{1,\Omega}$ and we get (4.8). The inequality (4.10) explains why the assumption "h is sufficiently small" is not very restrictive.

Now (4.7) becomes

$$\|u_h - S_h u\|_{1,\Omega} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I\|_{1,\Omega} \leq Ch^r(\|u\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r+1,\Omega})\,. \tag{4.11}$$

Applying the estimate

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{1,\Omega} \leq Ch^r\|\boldsymbol{\sigma}\|_{r+1,\Omega}\,,$$

(4.6), (4.11) and the triangle inequality we get the desired result. □

Remark : Obviously, the validity of (4.8) does not depend on k. Hence, using (4.8) we get (in the case of $k = r$)

$$\|u - u_h\|_{1,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,\Omega} \leq Ch^k(\|u\|_{k+1,\Omega} + \|\boldsymbol{\sigma}\|_{k+1,\Omega}), \tag{4.12}$$

which improves the estimate in Theorem 4.1. □

As an intermediate step toward the final optimal estimates, we introduce the following auxiliary problem : find $\xi \in V$, $\boldsymbol{\eta} \in \mathbf{W}$ such that

$$a(\xi, \boldsymbol{\eta}\,; v, \mathbf{q}) = (G, v)_{0,\Omega} + (\mathbf{F}, \mathbf{q})_{0,\Omega} \quad \text{for all} \quad v \in V,\ \mathbf{q} \in \mathbf{W}\,, \tag{4.13}$$

where $G \in H^1(\Omega)$ and $\mathbf{F} \in H^1(\Omega)^n$ will be specified later.

THEOREM 4 3 *The following a priori estimates hold*

$$\|\xi\|_{2\ \Omega} + \|\boldsymbol{\eta}\|_{2\ \Omega} \leq C(\|G\|_{0\ \Omega} + \|\mathbf{F}\|_{0\ \Omega}), \qquad (4\ 14)$$

$$\|\xi\|_{3\ \Omega} + \|\boldsymbol{\eta}\|_{2\ \Omega} \leq C(\|G\|_{1\ \Omega} + \|\mathbf{F}\|_{0\ \Omega}) \qquad (4\ 15)$$

Proof Using the first Friedrichs' inequality (Saranen [21], Křížek, Neittaanmaki [13]),

$$C\|\mathbf{q}\|_{1\ \Omega} \leq \|\operatorname{curl} A^{-1}\mathbf{q}\|_{0\ \Omega} + \|\operatorname{div}\mathbf{q}\|_{0\ \Omega} + \|\mathbf{q}\|_{0\ \Omega} \qquad (4\ 16)$$

Then from Theorem 2 2 and (4 16),

$$\begin{aligned}\|\xi\|_{1\ \Omega}^2 + \|\boldsymbol{\eta}\|_{1\ \Omega}^2 &\leq C(\|\xi\|_{1\ \Omega}^2 + \|\boldsymbol{\eta}\|_{H(\operatorname{div}\ \operatorname{curl})}^2) \\ &\leq C a(\xi, \boldsymbol{\eta}, \xi, \boldsymbol{\eta}) \\ &\leq C((G, \xi)_{0\ \Omega} + (\mathbf{F}, \boldsymbol{\eta})_{0\ \Omega})\end{aligned}$$

Hence

$$\|\xi\|_{1\ \Omega} + \|\boldsymbol{\eta}\|_{1\ \Omega} \leq C(\|G\|_{0\ \Omega} + \|\mathbf{F}\|_{0\ \Omega}) \qquad (4\ 17)$$

Setting $v = 0$ in (4 13) we obtain the variational problem find $\boldsymbol{\eta} \in \mathbf{W}$ such that

$$\begin{aligned}(\operatorname{curl} A^{-1}\boldsymbol{\eta}, \operatorname{curl} A^{-1}\mathbf{q})_{0\ \Omega} &+ (\operatorname{div}\boldsymbol{\eta}, \operatorname{div}\mathbf{q})_{0\ \Omega} + (\boldsymbol{\eta}, \mathbf{q})_{0\ \Omega} \\ &= (\mathbf{F} - A \operatorname{grad}\xi + \operatorname{grad} c\xi, \mathbf{q})_{0\ \Omega} \qquad (4\ 18)\end{aligned}$$

holds for all $\mathbf{q} \in \mathbf{W}$ We have the regularity estimate (Mehra [14], cited in Saranen [22] , Neittaanmaki and Saranen [16])

$$\|\boldsymbol{\eta}\|_{2\ \Omega} \leq C\|\mathbf{F} - A \operatorname{grad}\xi + \operatorname{grad} c\xi\|_{0\ \Omega} \qquad (4\ 19)$$

Similarly, letting $\mathbf{q} = 0$ in (4 13) and using integration by parts we get the problem find $\xi \in V$ such that

$$(A \operatorname{grad}\xi, A \operatorname{grad} v)_{0\ \Omega} + (c\xi, cv)_{0\ \Omega} = (G + \operatorname{div} A^{t}\boldsymbol{\eta} - c \operatorname{div}\boldsymbol{\eta}, v)_{0\ \Omega} \qquad (4\ 20)$$

for all $v \in V$ The following *a priori* estimates for this problem hold (see e g Grisvard [11])

(i) if the domain is convex or the boundary Γ is of class $C^{1\ 1}$

$$\|\xi\|_{2\ \Omega} \leq C\|G + \operatorname{div} A^T\boldsymbol{\eta} - c \operatorname{div}\boldsymbol{\eta}\|_{0\ \Omega}, \qquad (4\ 21)$$

(ii) if the boundary Γ is of class $C^{2,1}$

$$\|\xi\|_{3,\Omega} \leq C\,\|G + \operatorname{div} A^T \boldsymbol{\eta} - c \operatorname{div} \boldsymbol{\eta}\|_{1,\Omega} \tag{4.22}$$

Then

$$\begin{aligned}\|\xi\|_{2,\Omega} + \|\boldsymbol{\eta}\|_{2,\Omega} &\leq C\,\|G + \operatorname{div} A^T \boldsymbol{\eta} - c \operatorname{div} \boldsymbol{\eta}\|_{0,\Omega} \\ &\quad + C\,\|\mathbf{F} - A \operatorname{grad} \xi + \operatorname{grad} c\xi\,\|_{0,\Omega} \\ &\leq C\,(\|G\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Omega}) + C\,(\|\xi\|_{1,\Omega} + \|\boldsymbol{\eta}\|_{1,\Omega}) \\ &\leq C\,(\|G\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Omega}),\end{aligned} \tag{4.23}$$

where (4.17), (4.19) and (4.21) have been used. Similarly, applying (4.17), (4.19), (4.22) and (4.23),

$$\begin{aligned}\|\xi\|_{3,\Omega} + \|\boldsymbol{\eta}\|_{2,\Omega} &\leq C\,\|G + \operatorname{div} A^T \boldsymbol{\eta} - c \operatorname{div} \boldsymbol{\eta}\|_{1,\Omega} \\ &\quad + C\,\|\mathbf{F} - A \operatorname{grad} \xi + \operatorname{grad} c\xi\,\|_{0,\Omega} \\ &\leq C\,(\|G\|_{1,\Omega} + \|\mathbf{F}\|_{0,\Omega}) + C\,(\|\xi\|_{1,\Omega} + \|\boldsymbol{\eta}\|_{2,\Omega}) \\ &\leq C\,(\|G\|_{1,\Omega} + \|\mathbf{F}\|_{0,\Omega}),\end{aligned} \tag{4.24}$$

which is the desired result. □

Now, we are able to prove the final estimates.

THEOREM 4.4. *If $k = r$ then*

$$\|u - u_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \leq C h^{k+1}(\|u\|_{k+1,\Omega} + \|\boldsymbol{\sigma}\|_{k+1,\Omega}) \tag{4.25}$$

If $k + 1 = r$, $k > 1$, then

$$\|u - u_h\|_{-1,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \leq C h^{r+1}(\|u\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r+1,\Omega}) \tag{4.26}$$

Proof. Let us consider the variational problem (4.18). Obviously, $\boldsymbol{\eta}$ is a weak solution of the following problem

$$\begin{aligned}-(A^{-1})^T \operatorname{curl}(\operatorname{curl} A^{-1} \boldsymbol{\eta}) - \operatorname{grad} \operatorname{div} \boldsymbol{\eta} + \boldsymbol{\eta} &= \mathbf{F} - A \operatorname{grad} \xi + \operatorname{grad} c\xi \quad \text{in } \Omega \\ \mathbf{n} \wedge A^{-1} \boldsymbol{\eta} &= 0 \quad \text{on } \Gamma, \\ \operatorname{div} \boldsymbol{\eta} &= 0 \quad \text{on } \Gamma,\end{aligned}$$

see Saranen [22]. Then for $\mathbf{p} \in H^1(\Omega)^n$ we get

$$\begin{aligned}(\operatorname{curl} A^{-1} \boldsymbol{\eta}, \operatorname{curl} A^{-1} \mathbf{p})_{0,\Omega} &+ (\operatorname{div} \boldsymbol{\eta}, \operatorname{div} \mathbf{p})_{0,\Omega} + (\boldsymbol{\eta}, \mathbf{p})_{0,\Omega} \\ &= (-(A^{-1})^T \operatorname{curl}(\operatorname{curl} A^{-1} \boldsymbol{\eta}) - \operatorname{grad} \operatorname{div} \boldsymbol{\eta} + \boldsymbol{\eta}, \mathbf{p})_{0,\Omega} \\ &\quad + (\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1} \mathbf{p})_{0,\Gamma} \\ &= (\mathbf{F} - A \operatorname{grad} \xi + \operatorname{grad} c\xi, \mathbf{p})_{0,\Omega} + (\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1} \mathbf{p})_{0,\Gamma}\end{aligned}$$

Hence

$$(\mathbf{F}, \mathbf{p})_{0, \Omega} = a(\xi, \boldsymbol{\eta} ; 0, \mathbf{p}) - (\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1} \mathbf{p})_{0, \Gamma} \qquad (4.27)$$

for all $\mathbf{p} \in H^1(\Omega)^n$. On the other hand, from (4.13) with $\mathbf{q} = \mathbf{0}$,

$$(G, v)_{0, \Omega} = a(\xi, \boldsymbol{\eta} ; v, \mathbf{0}) \quad \text{for all} \quad v \in V . \qquad (4.28)$$

Setting $\mathbf{p} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $v = u - u_h$ in (4.27) and (4.28) respectively, and using (3.4) and (2.12)

$$\begin{aligned}(\mathbf{F}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)_{0, \Omega} &+ (G, u - u_h)_{0, \Omega} \\ &= a(\xi, \boldsymbol{\eta} ; u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - (\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))_{0, \Gamma} \\ &= a(\xi - \xi_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I ; u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\ &\quad + (\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h)_{0, \Gamma} , \qquad (4.29)\end{aligned}$$

where ξ_I and $\boldsymbol{\eta}_I$ are the interpolants of ξ and $\boldsymbol{\eta}$.

First, we estimate the boundary term in (4.29). Using the trace theorem and (4.14),

$$\begin{aligned}(\operatorname{curl} A^{-1} \boldsymbol{\eta}, \mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h)_{0, \Gamma} &\leq \|\operatorname{curl} A^{-1} \boldsymbol{\eta}\|_{0, \Gamma} \|\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h\|_{0, \Gamma} \\ &\leq C \|\boldsymbol{\eta}\|_{2, \Omega} \|\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h\|_{0, \Gamma} \\ &\leq C (\|G\|_{0, \Omega} + \|\mathbf{F}\|_{0, \Omega}) \|\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h\|_{0, \Gamma} . \end{aligned} \qquad (4.30)$$

In order to estimate $\|\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h\|_{0, \Gamma}$ the technique from the proof of (4.8) in Theorem 4.2 will be used. As before, let $K \subset \Omega$ be an element which has a side (face) coincident with the boundary Γ, $K \cap \Gamma = e$. Then

$$\begin{aligned}\|\mathbf{n} \wedge A^{-1} \boldsymbol{\sigma}_h\|_{0, e} &\leq C (\text{meas } (e))^{1/2} \sum_{s=0}^{l} h^{r+1-s} |\hat{\boldsymbol{\sigma}}_h|_{s, \hat{e}} \\ &\leq C (\text{meas } (e))^{1/2} \left(\sum_{s=0}^{r} h^{r+1-s} |\hat{\boldsymbol{\sigma}}_h - \hat{\boldsymbol{\sigma}}_I|_{s, \hat{e}} \right. \\ &\quad \left. + \sum_{s=0}^{l} h^{r+1-s} (|\hat{\boldsymbol{\sigma}}_I - \hat{\boldsymbol{\sigma}}|_{s, \hat{e}} + |\hat{\boldsymbol{\sigma}}|_{s, \hat{e}}) \right) . \end{aligned} \qquad (4.31)$$

Now, from the equivalence of the norms in finite dimensional spaces,

$$\begin{aligned}\sum_{s=0}^{l} h^{r+1-s} |\hat{\boldsymbol{\sigma}}_h - \hat{\boldsymbol{\sigma}}_I|_{s, \hat{e}} &\leq C h |\hat{\boldsymbol{\sigma}}_h - \hat{\boldsymbol{\sigma}}_I|_{0, \hat{e}} \\ &\leq C h (\text{meas } (e))^{-1/2} |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I|_{0, e} \\ &\leq C h (\text{meas } (e))^{-1/2} (|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|_{0, e} + |\boldsymbol{\sigma} - \boldsymbol{\sigma}_I|_{0, e}) . \end{aligned} \qquad (4.32)$$

Also,

$$\sum_{s=0}^{r} h^{r+1-s}(|\hat{\boldsymbol\sigma}_I - \hat{\boldsymbol\sigma}|_{s,e} + |\hat{\boldsymbol\sigma}|_{s,e}) \leq C \sum_{s-0}^{r} h^{r+1-s}(|\hat{\boldsymbol\sigma}|_{r,e} + |\hat{\boldsymbol\sigma}|_{s,e})$$
$$\leq C h^{r+1}(\text{meas } (e))^{-1/2} \, \|\boldsymbol\sigma\|_{r,e} \, . \quad (4.33)$$

Hence (4.31)-(4.33) lead to

$$\|\mathbf{n} \wedge A^{-1} \boldsymbol\sigma_h\|_{0,\Gamma} \leq Ch(|\boldsymbol\sigma_h - \boldsymbol\sigma|_{0,\Gamma} + |\boldsymbol\sigma - \boldsymbol\sigma_I|_{0,\Gamma}) + Ch^{r+1}\|\boldsymbol\sigma\|_{r,\Gamma}$$
$$\leq Ch\|\boldsymbol\sigma_h - \boldsymbol\sigma\|_{1,\Omega} + Ch^{r+1}\|\boldsymbol\sigma\|_{r+1,\Omega} \, , \quad (4.34)$$

where the trace theorem has been used. This completes the estimate for the boundary term in (4.29).

Now we proceed with the first term in (4.29). Using the Cauchy-Schwarz inequality,

$$a(\xi - \xi_I, \boldsymbol\eta - \boldsymbol\eta_I \, ; u - u_h, \boldsymbol\sigma - \boldsymbol\sigma_h)$$
$$\leq C\,(\|\boldsymbol\eta - \boldsymbol\eta_I\|_{1,\Omega} + \|\xi - \xi_I\|_{1,\Omega})\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{1,\Omega}$$
$$+ C\,(\|\boldsymbol\eta - \boldsymbol\eta_I\|_{1,\Omega} + \|\xi - \xi_I\|_{0,\Omega})\|u - u_h\|_{0,\Omega}$$
$$+ C\,(\|\boldsymbol\eta - \boldsymbol\eta_I\|_{0,\Omega} + \|\xi - \xi_I\|_{1,\Omega})\|u - u_h\|_{1,\Omega} \, .$$

Consider the case $k = r$ and select $G = u - u_h$ and $\mathbf{F} = \boldsymbol\sigma - \boldsymbol\sigma_h$. Then

$$a(\xi - \xi_I, \boldsymbol\eta - \boldsymbol\eta_I \, ; u - u_h, \boldsymbol\sigma - \boldsymbol\sigma_h)$$
$$\leq Ch(\|\boldsymbol\eta\|_{2,\Omega} + \|\xi\|_{2,\Omega})(\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{1,\Omega} + \|u - u_h\|_{1,\Omega})$$
$$\leq Ch^{k+1}(\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{0,\Omega} + \|u - u_h\|_{0,\Omega})(\|\boldsymbol\sigma\|_{k+1,\Omega} + \|u\|_{k+1,\Omega}) \, , \quad (4.35)$$

where we have used (4.3) and (4.12). We get (4.25) from (4.29), (4.35), (4.34) and (4.12).

Let $k + 1 = r$, $k > 1$, $\mathbf{F} = \boldsymbol\sigma - \boldsymbol\sigma_h$ and $G = 0$. Then

$$a(\xi - \xi_I, \boldsymbol\eta - \boldsymbol\eta_I \, ; u - u_h, \boldsymbol\sigma - \boldsymbol\sigma_h)$$
$$\leq Ch(\|\boldsymbol\eta\|_{2,\Omega} + \|\xi\|_{2,\Omega})\, h^r(\|\boldsymbol\sigma\|_{r+1,\Omega} + \|u\|_{r,\Omega})$$
$$+ Ch^2(\|\boldsymbol\eta\|_{2,\Omega} + \|\xi\|_{3,\Omega})\, h^{r-1}(\|\boldsymbol\sigma\|_{r,\Omega} + \|u\|_{r,\Omega})$$
$$\leq Ch^{r+1}\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{0,\Omega}(\|\boldsymbol\sigma\|_{r+1,\Omega} + \|u\|_{r,\Omega}) \quad (4.36)$$

and the desired result for $\|\boldsymbol\sigma - \boldsymbol\sigma_h\|_{0,\Omega}$ follows from (4.29), (4.36), (4.34) and (4.4). Setting G to be an arbitrary function in V and $\mathbf{F} = \mathbf{0}$ we get the estimate for $\|u - u_h\|_{-1,\Omega}$:

$$\|u - u_h\|_{-1, \Omega} = \sup_{G \neq 0} \frac{|(u - u_h, G)_{0, \Omega}|}{\|G\|_{1, \Omega}}$$
$$\leqslant \frac{C h^{r+1} \|G\|_{1, \Omega} (\|\boldsymbol{\sigma}\|_{r+1, \Omega} + \|u\|_{r, \Omega})}{\|G\|_{1, \Omega}}$$
$$\leqslant C h^{r+1} (\|\boldsymbol{\sigma}\|_{r+1, \Omega} + \|u\|_{r, \Omega}),$$

where the *a priori* estimate (4.15) has been used. □

5. CONCLUSIONS

We have presented an analysis of a least-squares mixed finite element method. The difference between the present paper and [18] is that a new boundary condition for $\boldsymbol{\sigma}$ is imposed and a new term is added to the bilinear form. Following this approach we were able to prove optimal L^2- and H^1-error estimates for the cases $k = r$ and $k + 1 = r$. The numerical experiments which we recently conducted confirm the theoretical rates of convergence and will be reported in a separate paper. Also, some important issues related to *a posteriori* error estimates are currently under consideration.

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