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ON THE COUPLING OF ELLIPTIC AND HYPERBOLIC NONLINEAR DIFFERENTIAL EQUATIONS (*)

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Abstract. — *In this paper we consider a boundary value problem for a second order nonlinear differential equation which degenerates into a nonlinear first order one in a given subdomain. To find out the coupling conditions at the interface, we transform the elliptic-hyperbolic problem into an elliptic-elliptic one by adding a small artificial viscosity. As this viscosity vanishes, the regularized problem degenerates into the original one yielding some conditions at the interface. When the flux function is not monotone, we obtain a unique solution for the problem satisfying a generalized entropy condition.*

Résumé. — *Dans cet article nous considérons un problème aux limites non linéaire du second ordre qui dégénère, dans une partie fixée du domaine, en un problème du premier ordre. Nous utilisons l'introduction d'une faible viscosité artificielle pour transformer ce problème mixte elliptique-hyperbolique en un problème totalement elliptique. Cela nous permet d'obtenir les conditions d'interfaces pour le problème originel par passage à la limite lorsque la viscosité additionnelle tend vers zéro. On montre alors que le problème admet une solution et une seule vérifiant une condition d'entropie généralisée.*

1. INTRODUCTION

This paper deals with the coupling of a nonlinear elliptic equation with a nonlinear hyperbolic one in a one-dimensional domain. This type of problem arises from several simplified physical models, like the infiltration process in a heterogeneous soil formed by two layers, such that in the second layer we

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can neglect the effects of diffusivity, or fluid-dynamical problems for viscous, compressible flows in the presence of a body so that near this body the viscosity effects have to be accounted for and at a distance from it these effects can be neglected. The problem in hand reads as follows.

Find a pair of functions u, v defined in $[\alpha, \beta]$ and $[\beta, \gamma]$ such that :

$$-(\mu(u)u')' + K(u)' + b(x, u) = g, \quad \text{in} \quad (\alpha, \beta), \tag{1.1a}$$

$$K(v)' + b(x, v) = g, \quad \text{in} \quad (\beta, \gamma), \tag{1.1b}$$

$$u(\alpha) = 0, \tag{1.1c}$$

$$sg(w(\gamma))[K(w(\gamma)) - K(c)] \geq 0, \tag{1.1d}$$

for all c between $w(\gamma)$ and 0,

where :

i) α, β, γ are real numbers : $\alpha < \beta < \gamma$.
ii) μ, K, b are functions such that :

$$\mu \in C^2(R), \quad \mu(t) \geq \mu_0 > 0,$$

$$K \in C^2(R),$$

$$b \in C^2([\alpha, \gamma] \times R), \quad b_w(x, w) \geq \nu > 0.$$

iii) $g \in C(\alpha, \gamma)$.

Problem (1.1) may be regarded as a time discretization of an evolution advection-diffusion problem, parabolic in (α, β) and hyperbolic in (β, γ), by an implicit method. This is the reason why we choose to impose a generalized Dirichlet condition at γ. If $K' > 0$, this condition is always satisfied and if $K'(w(\gamma)) < 0$, it gives $w(\gamma) = 0$.

Clearly, the formulation of the problem (1.1) is incomplete : it needs some coupling conditions at the interface β. In this work, our main concern is to find these conditions and to define what we mean by a solution of the elliptic-hyperbolic problem. This kind of problem has been considered until now in the linear case, first the case of the one-dimensional problem [3] and more recently the two-dimensional case [4]. These authors are interested in obtaining appropriate interface conditions which allow them to split the problem and use some domain decomposition methods. These methods are appropriate for fluid-dynamical problems, because they allow us to partition the domain into subdomains of simpler shape and, thanks to parallel computing, reduce the original problem to a sequence of subproblems which can be solved simultaneously.

As Gastaldi and Quarteroni have done in [3] for linear problems, we add a small artificial viscosity which transforms our elliptic-hyperbolic problem (1.1) into an elliptic-elliptic one. For the coupling between two elliptic problems, we require the continuity of the unknown and of the flux. As the small viscosity vanishes, the coupled elliptic-elliptic problem degenerates into the original one, yielding some conditions at the interface. These conditions we take as interface conditions for the elliptic-hyperbolic problem.

In section 2, we deal with the case of elliptic-hyperbolic nonlinear coupling with strictly monotone flux function $K(u)$. In this case, we obtain coupling conditions analogous to the linear case with a strictly positive or strictly negative advection function.

Section 3 is devoted to the study of a more difficult case, when the flux function is not a monotone function. In this situation, there are two difficulties in defining a solution of the elliptic-hyperbolic problem. First, as we said, we need some coupling conditions at the interface, and moreover there is a problem with the uniqueness of the solution in the hyperbolic domain. An extension of the entropy solution (see [2]), which solves the uniqueness problem for nonlinear conservation laws, allows us also in this case to characterize the coupling conditions.

From now on, we denote by C any positive constant independent of the parameter ε.

2. ELLIPTIC-HYPERBOLIC PROBLEMS WITH MONOTONE FLUX FUNCTION

In this section, we deal with a situation in which discontinuities could not appear in the hyperbolic domain ; this is the case in problems with monotone flux function. The characteristic lines of the evolution hyperbolic problem enter the domain across $\{\beta\} \times (0, T)$ if $K' \geq 0$ and across $\{\gamma\} \times (0, T)$ if $K' \leq 0$. Taking this into account, we can expect different coupling conditions at the interface depending on the monotonicity of K, and for this reason we distinguish two cases.

2.1. First case : K is strictly increasing

As we said in the introduction, in order to find the coupling conditions and to prove the existence of a solution to the elliptic-hyperbolic problem, we add a small artificial viscosity which transforms our problem into an elliptic-elliptic one. For the coupling of two elliptic problems, we impose as interface conditions the continuity of the unknown and of the flux. So the regularized problem is :

Given $\varepsilon > 0$, find u_ε and v_ε such that :

$$- (\mu(u_\varepsilon) u_\varepsilon')' + K(u_\varepsilon)' + b(x, u_\varepsilon) = g , \quad \text{in} \quad (\alpha, \beta), \tag{2.1a}$$

$$- \varepsilon(\mu(v_\varepsilon) v_\varepsilon')' + K(v_\varepsilon)' + b(x, v_\varepsilon) = g , \quad \text{in} \quad (\beta, \gamma), \tag{2.1b}$$

$$u_\varepsilon(\alpha) = 0 , \tag{2.1c}$$

$$v_\varepsilon'(\gamma) = 0\,, \tag{2.1d}$$

$$u_\varepsilon(\beta) = v_\varepsilon(\beta)\,,\quad u_\varepsilon'(\beta) = \varepsilon v_\varepsilon'(\beta)\,. \tag{2.1e}$$

The Neumann condition at γ may be replaced by one of Dirichlet type and analogous results could be proved. Problem (2.1) admits the following equivalent weak formulation.

Find $w_\varepsilon \in H(\alpha, \gamma)$ such that :

$$\int_\alpha^\gamma \lambda_\varepsilon(x)\,\mu\,(w_\varepsilon(x))\,w_\varepsilon'(x)\,\eta' + \int_\alpha^\gamma K(w_\varepsilon)'\,\eta + \int_\alpha^\gamma b(x, w_\varepsilon(x))\,\eta = \int_\alpha^\gamma g\eta\,,\quad \forall \eta \in H(\alpha, \gamma)\,, \tag{2.2}$$

where

$$H(\alpha, \gamma) = \{w \in H^1(\alpha, \gamma) : w(\alpha) = 0\}\,,$$

and

$$\lambda_\varepsilon(x) = \begin{cases} 1, & x \in (\alpha, \beta)\,, \\ \varepsilon, & x \in (\beta, \gamma)\,. \end{cases}$$

Note that w_ε solves problem (2.2) if and only if $u_\varepsilon = w_\varepsilon|_{[\alpha, \beta]}$ and $v_\varepsilon = w_\varepsilon|_{[\beta, \gamma]}$ solve problem (2.1).

Using an existence result for solution of generalized boundary value problems obtained in [7], p. 222 by weak comparison functions and using also an inverse monotonicity argument for the functional $B(w, \eta) = \int_\alpha^\gamma \lambda_\varepsilon(x)\,\mu\,(w)\,w'\,\eta' + \int_\alpha^\gamma K(w)'\,\eta + \int_\alpha^\gamma b(x, w)\,\eta - \int_\alpha^\gamma g\eta$, the following result is proved in [1],

PROPOSITION 2.1 : *Problem* (2.2) *has a unique solution, w_ε. Furthermore,*

$$|w_\varepsilon| \leq c_0,\quad \textit{where}\quad c_0 = \max\left\{\frac{1}{\nu}\,|b(x, 0) - g(x)|\right\}\,. \tag{2.3}$$

Some bounds are obtained now. These allow us to take the limit as ε goes to zero in (2.2) and to prove the existence of a solution to the elliptic-hyperbolic problem (1.1).

From now on, we make the following assumption

$$g \in C^1(R)\,. \tag{2.4}$$

LEMMA 2.2 : *There is a constant $C > 0$ such that*

$$\|w_\varepsilon'\|_{L^1(\alpha, \gamma)} \leq C\,. \tag{2.5}$$

Proof : We use the same technique as Lorenz in [5] to obtain a similar result for the solutions of nonlinear singularly perturbed problems. Differentiate the equations (2.1*a*) and (2.1*b*), multiply the results by $sg(\phi(w_\varepsilon)')$, where $\phi(s) = \int_0^s \mu(t)\,dt$, and integrate between α and β and between β and γ to obtain :

$$\nu \|w_\varepsilon'\|_{L^1(\alpha,\beta)} \leq c_1(\beta - \alpha) - \int_\alpha^\beta K(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') + \\ + \int_\alpha^\beta \phi(w_\varepsilon)''' \, sg(\phi(w_\varepsilon)')$$

and

$$\nu \|w_\varepsilon'\|_{L^1(\beta,\gamma)} \leq c_1(\gamma - \beta) - \int_\beta^\gamma K(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') + \\ + \varepsilon \int_\beta^\gamma \phi(w_\varepsilon)''' \, sg(\phi(w_\varepsilon)')$$

with

$$c_1 = \max \left\{ |b_x(x, u) - g'(x)| : (x, u) \in [\alpha, \gamma] \times [-c_0, c_0] \right\} .$$

Let $p_\delta(\,.\,)$ be a regularization of the sign function, that is :

$$p_\delta \in C^1(R), \quad p_\delta(z) = sg(z) \quad \text{for} \quad |z| \geq \delta \quad \text{and} \quad |z| = 0 , \\ 0 \leq p_\delta'(t) \leq \frac{C}{\delta} .$$

Using p_δ and applying Lebesgue's dominated convergence theorem for $\delta \to 0$, we have :

$$\int_\alpha^\beta \phi(w_\varepsilon)''' \, sg(\phi(w_\varepsilon)') \leq \phi(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') \big|_\alpha^\beta$$

and

$$\int_\beta^\gamma \phi(w_\varepsilon)''' \, sg(\phi(w_\varepsilon)') \leq \phi(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') \big|_\beta^\gamma$$

and, by Sacks' lemma :

$$\int_\alpha^\beta K(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') \leq K(w_\varepsilon)' \, sg(\phi(w_\varepsilon)') \big|_\alpha^\beta$$

and

$$\int_\beta^\gamma K(w_\varepsilon)'' \, sg(\phi(w_\varepsilon)') \leq K(w_\varepsilon)' \, sg(\phi(w_\varepsilon)')\big|_\beta^\gamma .$$

Finally, because of the coupling conditions at β, we obtain,

$$\nu \|w_\varepsilon'\|_{L^1(\alpha,\gamma)} \leq$$
$$\leq c_1(\gamma - \alpha) + [g(\alpha) - b(\alpha, w_\varepsilon(\alpha))] \, sg(\phi(w_\varepsilon)'(\alpha)) - [g(\gamma) - b(\gamma, w_\varepsilon(\gamma))] \, sg(\phi(w_\varepsilon)'(\gamma)) .$$

LEMMA 2.3 : *There is a constant $C > 0$ such that*

$$\|u_\varepsilon'\|_{L^2(\alpha,\beta)} \leq C , \tag{2.6}$$

$$\sqrt{\varepsilon}\|v_\varepsilon'\|_{L^2(\beta,\gamma)} \leq C , \tag{2.7}$$

$$\|K(v_\varepsilon)'\|_{L^2(\beta,\gamma)} \leq C . \tag{2.8}$$

Proof : Taking the function $\eta = \phi(w_\varepsilon)$ in (2.2), the bounds (2.6) and (2.7) follow from (2.3), (2.5) and $\phi' \geq \mu_0$.

Take the $L^2(\beta, \gamma)$ scalar product of (2.1*b*) and $\phi(v_\varepsilon)'$ and, because of (2.5), the inequality

$$\int_\beta^\gamma K(v_\varepsilon)' \, \phi(v_\varepsilon)' \leq C ,$$

is obtained. Thus,

$$\int_\beta^\gamma [K(v_\varepsilon)']^2 \leq C \int_\beta^\gamma K(v_\varepsilon)' \, v_\varepsilon' \leq \frac{C}{\mu_0} \int_\beta^\gamma K(v_\varepsilon)' \, \phi(v_\varepsilon)' \leq C .$$

Now we can prove the main result which gives existence and uniqueness of a solution to the original problem (1.1).

THEOREM 2.4 : *u_ε and v_ε converge as $\varepsilon \to 0$ to a pair of functions u and v which satisfy* (1.1*a*), (1.1*b*), (1.1*c*) *and the interface conditions,*

$$u(\beta) = v(\beta) , \tag{2.9}$$

$$u'(\beta) = 0 . \tag{2.10}$$

Proof : As a consequence of the previous bounds, we can show the existence of $u \in H^1(\alpha, \beta)$, $v \in L^2(\beta, \gamma)$ and $\chi \in H^1(\delta, \gamma)$ such that,

i) $u_\epsilon \to u$ in $L^2(\alpha, \beta)$ and everywhere,
ii) $u_\varepsilon' \to u'$ weakly in $L^2(\alpha, \beta)$,

iii) $v_\varepsilon \to v$ weakly in $L^2(\beta, \gamma)$,

iv) $K(v_\varepsilon) \to \chi$ weakly in $H^1(\beta, \gamma)$ and everywhere.

The continuity and the strict monotonicity of K imply that

v) $v = K^{-1}(\chi)$ in (β, γ),

vi) $v_\varepsilon \to v$ everywhere in (β, γ).

The boundary condition (1.1c) and the interface condition (2.9) follow from i) and vi).

Integrating by parts and taking the limit in the obtained expression, we have

$$\int_\alpha^\beta \mu(u)\, u'\, \eta' - \int_\alpha^\beta K(u)\, \eta' - \int_\beta^\gamma K(v)\, \eta' + K(v)(\gamma)\, \eta(\gamma) +$$
$$+ \int_\alpha^\beta b(x, u)\, \eta + \int_\beta^\gamma b(x, v)\, \eta = \int_\alpha^\gamma g\eta, \quad \forall \eta \in C_0^\infty(\alpha, \gamma). \quad (2.11)$$

We have used that $\lim\limits_{\varepsilon \to 0^+} \int_\beta^\gamma \varepsilon \mu(w_\varepsilon)\, w_\varepsilon'\, \eta' = 0$, from (2.7).

From (2.11), equations (1.1*a*) (1.1*b*) follow easily. In order to get the interface condition (2.10) at β, we take a function $\eta \in C_0^\infty(\alpha, \gamma)$ such that $\eta(\beta) = 1$ in (2.11) and integrate by parts ; then

$$- \int_\alpha^\beta (\mu(u)\, u')'\, \eta + \int_\alpha^\beta K(u)'\, \eta + \int_\beta^\gamma K(v)'\, \eta + \int_\alpha^\beta b(x, u)\, \eta +$$
$$+ \int_\beta^\gamma b(x, v)\, \eta + \mu(u(\beta))\, u'(\beta) = \int_\alpha^\gamma g\eta,$$

and, by (1.1*a*) and (1.1*b*), the interface condition (2.10) follows.

In order to prove the uniqueness, let us suppose that there are two solutions (u, v) and $(\bar{u}, \bar{v})$.

Multiplying the differential equation (1.1*a*) by $p_\delta^+(\phi(u) - \phi(\bar{u}))$, where p_δ^+ is a regularization of the Heaviside function, that is to say, a function which satisfies,

$$p_\delta^+ \in C^1(R), \quad p_\delta^+(t) = \mathrm{sg}^+(t)$$
$$\text{if} \quad t \geq \delta \quad \text{and} \quad t \leq 0, \quad 0 \leq p_\delta^{+\prime}(t) \leq \frac{C}{\delta}. \quad (2.12)$$

Integrating the difference by parts, we have :

$$[K(u(\beta)) - K(\bar{u}(\beta))]\, p_\delta^+(\phi(u(\beta)) - \phi(\bar{u}(\beta))) -$$
$$- \int_\alpha^\beta [K(u) - K(\bar{u})]\, p_\delta^+(\phi(u) - \phi(\bar{u}))' +$$
$$+ \int_\alpha^\beta [b(x, u) - b(x, \bar{u})]\, p_\delta^+(\phi(u) - \phi(\bar{u})) \leq 0,$$

and taking the limit as $\delta \to 0^+$ it results :

$$\int_\alpha^\beta [b(x, u) - b(x, \bar{u})] \operatorname{sg}^+ (\phi(u) - \phi(\bar{u})) \leq 0 ,$$

and by the monotonicity of b and ϕ,

$$\int_\alpha^\beta [u - \bar{u}]^+ \leq 0 .$$

Then $u \leq \bar{u}$. In the same way, we can prove the other inequality. Thus $u = \bar{u}$.

Now v and $\bar{v}$ satisfy the differential equation (1.1b) and the same initial condition. Multiplying the differential equations which satisfy v and $\bar{v}$ by $p_\delta^+ (K(v) - K(\bar{v}))$, integrating the difference by parts and taking the limit as $\delta \to 0$ we obtain,

$$\int_\beta^\gamma [b(x, v) - b(x, \bar{v})]^+ \leq 0 .$$

We would like to remark that the coupling conditions we have found allow us to split the problem and use different numerical methods to integrate each subproblem.

We present some numerical results obtained for the problem

$$\begin{gathered} - w'' + \left(\frac{w^3}{3} + w \right)' + (1 + x^2) w = 1 , \quad x \in (-1, 0) , \\ \left(\frac{w^3}{3} + w \right)' + (1 + x^2) w = 1 , \quad x \in (0, 1) , \\ w(-1) = -1 , \quad w(1) = 1 . \end{gathered} \tag{2.13}$$

These results support the theoretical results obtained. Problem (2.13) is solved in two ways. In one approach, we deal with the elliptic regularization of problem (2.13) with $\varepsilon = 10^{-6}$. This regularized problem is solved by the Engquist-Osher difference scheme on the following mesh, which always contains the point $x = \beta$;

$$I_h = \{x_j : x_0 = \alpha , x_N = \beta, x_M = \gamma, x_{j+1} = x_j + h_j, 0 \leq j \leq M - 1 \} ,$$

where

$$h = (h_0, h_1, \dots, h_N, \dots, h_{M-1})^t \in R^M ,$$

with $\sum_{i=0}^{N-1} h_i = \beta - \alpha$ and $\sum_{i=N}^{M-1} h_i = \gamma - \beta$. Let $\bar{h}_j = \dfrac{h_{j-1} + h_j}{2}$.

The scheme used is the following,

$$w_0 = -1$$

$$-\frac{1}{\bar{h}_j}\left[\frac{w_{j-1}-w_j}{h_{j-1}}+\frac{w_{j+1}-w_j}{h_j}\right]+\frac{1}{\bar{h}_j}(K(w_j)-K(w_{j-1}))+b(x_j, w_j)=0\,,$$

$$1 \leqslant j \leqslant N-1\,,$$

$$-\left[\frac{w_{N-1}-w_N}{h_{N-1}}+\frac{\varepsilon w_{N+1}-w_N}{h_N}\right]+K(w_N)-K(w_{N-1})=0\,,$$

$$-\frac{\varepsilon}{\bar{h}_j}\left[\frac{w_{j-1}-w_j}{h_{j-1}}+\frac{w_{j+1}-w_j}{h_j}\right]+\frac{1}{\bar{h}_j}(K(w_j)-K(w_{j-1}))+b(x_j, w_j)=0\,,$$

$$N+1 \leqslant j \leqslant M-1\,,$$

$$w_M = 1,$$

with $K(w) = \frac{w^3}{3} + w$. The equation at $x_N = \beta$ is a discretization of the continuity of the flux across this point.

The second approach consists in integrating the elliptic problem in (α, β) with the condition $u'(\beta) = 0$ in a first step and afterwards solving the hyperbolic problem with $v(\beta) = u(\beta)$ using a one-step backward method. Results shown in figure 1 are obtained on a uniform mesh with step-size $h = 1/40$, using both approaches. The differences between them are about 10^{-5}.

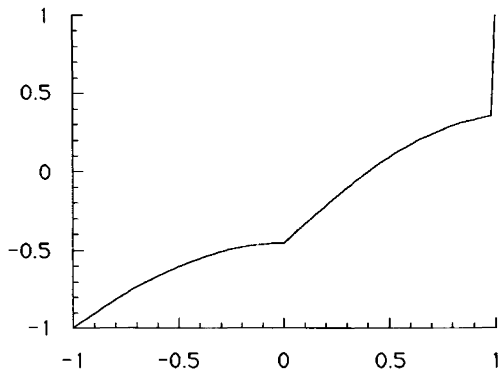

Figure 1.

2.2. Second case : *K* is strictly decreasing

In this case the regularized problem is :

Given $\varepsilon > 0$, find u_ε, v_ε such that

$$-(\mu(u_\varepsilon)u_\varepsilon')' + K(u_\varepsilon)' + b(x, u_\varepsilon) = g, \quad \text{in} \quad (\alpha, \beta), \tag{2.14a}$$

$$-\varepsilon(\mu(v_\varepsilon)v_\varepsilon')' + K(v_\varepsilon)' + b(x, v_\varepsilon) = g, \quad \text{in} \quad (\beta, \gamma), \tag{2.14b}$$

$$u_\varepsilon(\alpha) = 0, \tag{2.14c}$$

$$v_\varepsilon(\gamma) = 0, \tag{2.14d}$$

$$u_\varepsilon(\beta) = v_\varepsilon(\beta), \quad u_\varepsilon'(\beta) = \varepsilon v_\varepsilon'(\beta). \tag{2.14e}$$

Problem (2.14) is equivalent to the following weak problem :

Find $w_\varepsilon \in H_0^1(\alpha, \gamma)$ such that

$$\int_\alpha^\gamma \lambda_\varepsilon(x)\,\mu(w_\varepsilon(x))\,w_\varepsilon'(x)\,\eta' + \int_\alpha^\gamma K(w_\varepsilon)'\,\eta + \int_\alpha^\gamma b(x, w_\varepsilon(x))\,\eta =$$

$$= \int_\alpha^\gamma g\eta, \quad \forall \eta \in H_0^1(\alpha, \gamma). \tag{2.15}$$

In the same way as for the problem (2.2), one shows that (2.15) has a unique solution w_ε and that the bound (2.3) holds. The asymptotic analysis is analogous. Moreover, we still assume (2.4).

LEMMA 2.5 : *There is a constant $C > 0$ such that*

$$\|w_\varepsilon'\|_{L^1(\alpha, \gamma)} \leq C, \tag{2.16}$$

$$\|u_\varepsilon'\|_{L^2(\alpha, \beta)} \leq C, \tag{2.17}$$

$$\sqrt{\varepsilon}\,\|v_\varepsilon'\|_{L^2(\beta, \gamma)} \leq C. \tag{2.18}$$

In the following lemma, we have a bound on the L^2-norm of $K(v_\varepsilon)'$ in a left neighbourhood of the right boundary γ. It allows us to keep the boundary condition at γ when we take the limit.

LEMMA 2.6 : *The set $\{K(v_\varepsilon)' \mid \varepsilon > 0\}$ is bounded in $L^2(\delta, \gamma)$, for any δ with $\beta < \delta < \gamma$.*

Proof : Let us introduce a function $\eta \in C_+^\infty([\beta, \gamma])$ such that $\eta(\beta) = 0$, $\eta(\gamma) = 1$ and $\eta' \geq 0$. Taking the $L^2(\beta, \gamma)$ scalar product of (2.14*b*) by $\eta\phi(v_\varepsilon)'$, and integrating by parts in the first term, we obtain

$$-\frac{\varepsilon}{2}[\phi(v_\varepsilon)'(\gamma)]^2 + \varepsilon\int_\beta^\gamma [\phi(v_\varepsilon)']^2\,\eta' +$$

$$+ \int_\beta^\gamma K(v_\varepsilon)'\,\phi(v_\varepsilon)'\,\eta + \int_\beta^\gamma b(x, v_\varepsilon)\,\phi(v_\varepsilon)'\,\eta = \int_\beta^\gamma g(x)\,\phi(v_\varepsilon)'\,\eta. \tag{2.19}$$

From the bounds (2.16), (2.18) and the expression (2.19), we get,

$$\int_\beta^\gamma |K(v_\varepsilon)' \phi(v_\varepsilon)' \eta| \leq C .$$

Then,

$$\int_\beta^\gamma [K(v_\varepsilon)']^2 \eta \leq C \int_\beta^\gamma |K'(v_\varepsilon)| [v_\varepsilon']^2 \eta \leq$$

$$\leq \frac{C}{\mu_0} \int_\beta^\gamma |K(v_\varepsilon)' \phi(v_\varepsilon)' \eta| \leq C .$$

THEOREM 2.7 : *There are unique* $u \in H^1(\alpha, \beta)$, $v \in C([\beta, \gamma])$ *which satisfy* (1.1*a*) (1.1*b*), (1.1*c*), (1.1*d*) *and the interface condition,*

$$-\mu(u(\beta))u'(\beta) + K(u(\beta)) = K(v(\beta)) . \tag{2.20}$$

Proof: As a consequence of the previous lemma, we can find $u \in H^1(\alpha, \beta)$, $v \in L^2(\beta, \gamma)$ and $\chi \in H^1(\delta, \gamma)$ such that

i) $u_\varepsilon \to u$ in $L^2(\alpha, \beta)$ and everywhere,
ii) $u_\varepsilon' \to u'$ weakly in $L^2(\alpha, \beta)$,
iii) $v_\varepsilon \to v$ weakly in $L^2(\beta, \gamma)$,
iv) $K(v_\varepsilon) \to \chi$ in $L^2(\delta, \gamma)$ and everywhere.

Because of the continuity and the strict monotonicity of K,
v) $v = K^{-1}(\chi) \in C([\beta, \gamma])$,
vi) $v_\varepsilon \to v$ everywhere in (δ, γ).

The boundary conditions $u(\alpha) = v(\gamma) = 0$ follow from i) and vi). The estimate (2.16) implies that $v \in BV(\beta, \gamma)$, which allows to define continuously $v(\beta)$ by its right limit.

Integrating by parts in (2.15) and taking the limit in the expression obtained, we have

$$\int_\alpha^\beta \mu(u) u' \eta' - \int_\alpha^\beta K(u) \eta' - \int_\beta^\gamma K(v) \eta' +$$

$$+ \int_\alpha^\beta b(x, u) \eta + \int_\beta^\gamma b(x, v) \eta = \int_\alpha^\gamma g\eta ,$$

$$\forall \eta \in C_0^\infty(\alpha, \gamma) , \tag{2.21}$$

due to the fact $\lim_{\varepsilon \to 0^+} \int_\beta^\gamma \varepsilon \mu(w_\varepsilon) w_\varepsilon' \eta' = 0$.

Equations (1.1*a*) and (1.1*b*) hold because of (2.21).

Taking a function $\eta \in C_0^\infty(\alpha, \gamma)$ in (2.21), such that $\eta(\beta) = 1$ and integrating by parts, we obtain

$$- \int_\alpha^\beta (\mu(u) u')' \eta + \int_\alpha^\beta K(u)' \eta +$$

$$+ \int_\beta^\gamma K(v)' \eta + \int_\alpha^\beta b(x, u) \eta + \int_\beta^\gamma b(x, v) \eta +$$

$$+ \mu(u(\beta)) u'(\beta) - K(u(\beta)) + K(v(\beta)) = \int_\alpha^\gamma g\eta ,$$

and, by (1.1*a*) and (1.1*b*), the interface condition follows.

In order to prove the uniqueness, let us suppose that there are two solutions (u, v) and $(\bar{u}, \bar{v})$.

Multiplying the differential equation (1.1*b*) satisfied by v and $\bar{v}$ by $p_\delta^+ (K(v) - K(\bar{v}))$, where p_δ^+ is the regularization of the Heaviside function defined in (2.12), and integrating the difference, we get :

$$\int_\beta^\gamma [K(v)' - K(\bar{v})'] p_\delta^+ (K(v) - K(\bar{v})) +$$

$$+ \int_\beta^\gamma [b(x, v) - b(x, \bar{v})] p_\delta^+ (K(v) - K(\bar{v})) = 0 . \quad (2.22)$$

Integrating by parts in (2.22) and taking the limit when $\delta \to 0^+$, it follows that

$$- [K(v(\beta)) - K(\bar{v}(\beta))] \operatorname{sg}^+ (K(\bar{v}(\beta)) - K(v(\beta))) +$$

$$+ \int_\beta^\gamma [b(x, v) - b(x, \bar{v})] \operatorname{sg}^+ (K(\bar{v}) - K(v)) = 0 .$$

Because of the strictly decreasing monotonicity of K and the increasing monotonicity of b we have that $v \leq \bar{v}$. In the same way we can prove the other inequality, and therefore $v = \bar{v}$.

Multiplying now the differential equations, which are satisfied by u and $\bar{u}$, by $p_\delta^+ (\phi(u) - \phi(\bar{u}))$ and integrating by parts the difference, and then taking the limit as $\delta \to 0^+$, we obtain

$$[K(v(\beta)) - K(\bar{v}(\beta))] sg^+ (\phi(u(\beta)) - \phi(\bar{u}(\beta))) +$$

$$+ \int_\alpha^\beta [b(x, u) - b(x, \bar{u})] sg^+ (\phi(u) - \phi(\bar{u})) \leq 0 , \quad (2.23)$$

where we have taken into account the interface condition. The inequality

$u \leq \bar{u}$ follows easily from (2.22) because we have proved that $v = \bar{v}$, and ϕ, b are increasing monotone functions.

In this case, it is also possible to split the problem. First, one can integrate the hyperbolic problem :

$$\begin{gathered}\text{Find a function } v \text{ defined in } [\beta, \gamma] \text{ such that}\\ K(v)' + b(x, v) = g\,, \quad \text{in} \quad (\beta, \gamma)\,,\\ v(\gamma) = 0\,,\end{gathered}$$

and then, knowing $v(\beta)$, one can integrate the elliptic problem :

$$\begin{gathered}\text{Find a function } v \text{ defined in } [\alpha, \beta] \text{ such that}\\ -(\mu(u)\, u')' + K(u)' + b(x, u) = g\,, \quad \text{in} \quad (\alpha, \beta)\,,\\ u(\alpha) = 0\,,\\ -\mu(u(\beta))\, u'(\beta) + K(u(\beta)) = K(v(\beta))\,.\end{gathered}$$

In figure 2, we present some numerical results obtained for the problem

$$\begin{aligned} - w'' - \left(\frac{w^3}{3}\right)' + w = 0\,, \quad & x \in (-1, 0)\,,\\ - \left(\frac{w^3}{3}\right)' + w = 0\,, \quad & x \in (0, 1)\,,\\ w(-1) = -1 \quad w(1) = 1\,. \end{aligned}$$

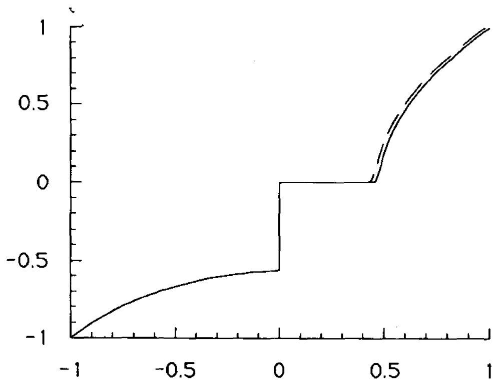

Figure 2.

The solid line shows the results obtained for the decoupled problem and the dashed line shows the results obtained for the regularized problem with $\varepsilon = 10^{-6}$ using the Engquist-Osher scheme.

3. ELLIPTIC-HYPERBOLIC PROBLEMS WITH ARBITRARY FLUX FUNCTION

In this section, we deal with more general problems which correspond to general flux functions. Again the same regularization technique gives us a solution for the elliptic-hyperbolic problem. Then we consider the problem :

Given $\varepsilon > 0$, find u_ε and v_ε such that :

$$-(\mu(u_\varepsilon)u_\varepsilon')' + K(u_\varepsilon)' + b(x, u_\varepsilon) = g, \quad \text{in} \quad (\alpha, \beta), \tag{3.1a}$$

$$-\varepsilon(\mu(v_\varepsilon)v_\varepsilon')' + K(v_\varepsilon)' + b(x, v_\varepsilon) = g, \quad \text{in} \quad (\beta, \gamma), \tag{3.1b}$$

$$u_\varepsilon(\alpha) = 0, \tag{3.1c}$$

$$v_\varepsilon(\gamma) = 0, \tag{3.1d}$$

$$u_\varepsilon(\beta) = v_\varepsilon(\beta), \quad u_\varepsilon'(\beta) = \varepsilon v_\varepsilon'(\beta) \tag{3.1e}$$

The equivalent weak problem is :

Find $w_\varepsilon \in H_0^1(\alpha, \gamma)$ such that :

$$\int_\alpha^\gamma \lambda_\varepsilon(x)\,\mu(w_\varepsilon(x))\,w_\varepsilon'(x)\,\eta' + \int_\alpha^\gamma K(w_\varepsilon)'\,\eta + \\ + \int_\alpha^\gamma b(x, w_\varepsilon(x))\,\eta = \int_\alpha^\gamma g\eta, \quad \forall \eta \in H_0^1(\alpha, \gamma). \tag{3.2}$$

As in the previous cases, we obtain again existence and uniqueness of a solution for the regularized problem given in proposition (2.1).

Now let us discuss the asymptotic behaviour of w_ε as ε goes to zero. We also make the assumption

$$g \in C^1(R),$$

and in the same way as in the previous cases we obtain the following results.

LEMMA 3.2 : *There is a constant C such that*

$$\text{i)} \quad \|w_\varepsilon\|_{W^{1,1}(\alpha, \gamma)} \leq C, \tag{3.3}$$

$$\text{ii)} \quad \|u_\varepsilon'\|_{L^2(\alpha, \beta)} \leq C, \tag{3.4}$$

$$\text{iii)} \quad \sqrt{\varepsilon}\,\|v_\varepsilon'\|_{L^2(\beta, \gamma)} \leq C. \tag{3.5}$$

We introduce the space

$$NBV(\alpha, \gamma) = \{u \in BV(\alpha, \gamma) : u(x) = \\ = u(x^+)\ \forall x \in (\alpha, \gamma), u(\gamma) = u(\gamma^-)\},$$

where $BV(\alpha, \gamma)$ denotes the functions of bounded variation on $[\alpha, \gamma]$. Now we are in a position to characterize the limit of w_ε as ε goes to zero.

THEOREM 3.3 : *There is a sequence* $\{w_\varepsilon\}_{\varepsilon>0}$ *which converges a.e. to a function* $w \in NBV(\alpha, \gamma)$ *such that* $w|_{(\alpha, \beta)} \in H^1(\alpha, \beta)$, $w(\alpha) = 0$ *and for all* $c \in R$ *and* $\varphi \in C_+^\infty(R)$,

$$
\begin{aligned}
&- \int_\alpha^\beta \mu(w)\, w'\, sg(w - c)\, \varphi' + \\
&+ \int_\alpha^\gamma sg(w - c)\{[K(w) - K(c)]\, \varphi' - [b(x, w) - g(x)]\, \varphi\} \\
&\geqslant sg(-c)\, \varphi(\alpha)[\mu(w(\alpha))\, w'(\alpha) - K(w(\alpha)) + K(c)] \\
&- sg(-c)\, \varphi(\gamma)[-K(w(\gamma)) + K(c)]\,.
\end{aligned} \tag{3.6}
$$

Proof : The set $\{w_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{1,1}(\alpha, \gamma)$. Therefore, there exists a function $w \in NBV(\alpha, \gamma)$ such that (upon extracting a subfamily),

$$w_\varepsilon \to w \quad \text{in} \quad L^1(\beta, \gamma)\,.$$

On the other hand, from (3.4) and (2.3), we have

$$
\begin{aligned}
&w_\varepsilon \to w \quad \text{in} \quad L^2(\alpha, \beta)\,, \\
&w_\varepsilon' \to w' \quad \text{weakly in} \quad L^2(\alpha, \beta)\,.
\end{aligned}
$$

Multiplying equations (3.1*a*) and (3.1*b*) by $p_\delta(w_\varepsilon - c)\, \varphi$, where p_δ was defined in (2.12), then integrating by parts in (α, γ) and finally going to the limit as $\delta \to 0$, we obtain

$$
\begin{aligned}
&\phi(w_\varepsilon)'(\alpha)\, sg(-c)\, \varphi(\alpha) - \varepsilon\phi(w_\varepsilon)'(\gamma)\, sg(-c)\, \varphi(\gamma) + \\
&+ \int_\alpha^\gamma \lambda_\varepsilon\, \phi(w_\varepsilon)'\, sg(w_\varepsilon - c)\, \varphi' + [K(w_\varepsilon(\gamma)) - K(c)]\, sg(-c)\, \varphi(\gamma) \\
&- [K(w_\varepsilon(\alpha)) - K(c)]\, sg(-c)\, \varphi(\alpha) - \int_\alpha^\gamma [K(w_\varepsilon) - K(c)]\, sg(w_\varepsilon - c)\, \varphi' \\
&+ \int_\alpha^\gamma b(x, w_\varepsilon)\, sg(w_\varepsilon - c)\, \varphi - \int_\alpha^\gamma g\, sg(w_\varepsilon - c)\, \varphi \leqslant 0\,.
\end{aligned} \tag{3.7}
$$

The result follows by letting ε go to zero in this inequality.

THEOREM 3.4 : *There is only one function* $w \in NBV(\alpha, \gamma)$ *satisfying* $w(\alpha) = 0$, $w|_{(\alpha, \beta)} \in H^1(\alpha, \beta)$, *and condition* (3.6).

Proof : i) The inequality (3.6) is also valid for functions $\varphi_{\gamma,\delta}$ and $\varphi_{\beta,\delta}$, where

$$\varphi_{y,\delta} = \begin{cases} 0, & \text{if } x \leqslant y - \delta, \\ 1 + \dfrac{x-y}{\delta}, & \text{if } y - \delta \leqslant x \leqslant y, \\ 1 - \dfrac{x-y}{\delta}, & \text{if } y \leqslant x \leqslant y + \delta, \\ 0, & \text{if } x \geqslant y + \delta. \end{cases}$$

Substituting these functions in (3.7) and taking the limit as $\delta \to 0$, we get

$$sg(w(\gamma))[K(w(\gamma)) - K(c)] \geqslant 0, \tag{3.8}$$

for all c between $w(\gamma)$ and 0,

$$-\mu(w(\beta^-))\, w'(\beta^-) + K(w(\beta^-)) = K(w(\beta^+)) \tag{3.9a}$$

$$sg(w(\beta^-) - w(\beta^+))[K(w(\beta^+)) - K(c)] \geqslant 0,$$

$$\forall c \text{ between } w(\beta^-) \text{ and } w(\beta^+). \tag{3.9b}$$

ii) Let us assume that there exist two functions w and $\bar{w}$ in $NBV(\alpha, \gamma)$ such that $w(\alpha) = \bar{w}(\alpha) = 0$, $w|_{[\alpha,\beta]} \in H^1(\alpha, \beta)$, $\bar{w}|_{[\alpha,\beta]} \in H^1(\alpha, \beta)$ and both satisfy (3.6). Then,

$$\begin{aligned} & sg(w(\gamma) - \bar{w}(\gamma))[K(w(\gamma)) - K(\bar{w}(\gamma))] = \\ & = sg(w(\gamma) - c)[K(w(\gamma)) - K(c)] \\ & + sg(\bar{w}(\gamma) - c)[(\bar{w}(\gamma)) - K(c)] \geqslant 0. \end{aligned} \tag{3.10}$$

iii) w and $\bar{w}$ satisfy,

$$sg(w(\beta^-) - \bar{w}(\beta^-))\{-[\phi(w) - \phi(\bar{w})]'(\beta^-) + [K(w) - K(\bar{w})](\beta^-)\} +$$

$$+ \int_\alpha^\beta [b(x, w) - b(x, \bar{w})]\, sg(w - \bar{w}) \leqslant 0. \tag{3.11}$$

iv) w and $\bar{w}$ satisfy,

$$\begin{aligned} & sg(w(\beta^+) - \bar{w}(\beta^+))[K(w(\beta^+)) - K(\bar{w}(\beta^+))] - \\ & - sg(w(\gamma) - \bar{w}(\gamma))[K(w(\gamma)) - K(\bar{w}(\gamma))] \\ & \geqslant \int_\beta^\gamma sg(w - \bar{w})[b(x, w) - b(x, \bar{w})]. \end{aligned} \tag{3.12}$$

v) From the inequalities i), ii), iii) and iv), we obtain

$$\nu \|w - \bar{w}\|_{L^1(\alpha, \gamma)} \leq$$
$$\leq [K(w(\beta^+)) - K(\bar{w}(\beta^+))]$$
$$\times [sg(w(\beta^+) - \bar{w}(\beta^+)) - sg(w(\beta^-) - \bar{w}(\beta^-))] . \quad (3.13)$$

The uniqueness follows from (3.13) if we prove that the second term is negative. Assume $sg(w(\beta^+) - \bar{w}(\beta^+)) \neq sg(w(\beta^-) - \bar{w}(\beta^-))$ and take, for example, $sg(w(\beta^+) - \bar{w}(\beta^+)) = 1$. We have six possible cases :

a) $w(\beta^-) \leq \bar{w}(\beta^-) \leq \bar{w}(\beta^+) < w(\beta^+)$.
If we consider (3.9) for w and $c = \bar{w}(\beta^+)$, we obtain

$$K(w(\beta^+)) - K(\bar{w}(\beta^+)) \leq 0 ,$$

and therefore $w = \bar{w}$.

b) $\bar{w}(\beta^+) \leq w(\beta^-) \leq \bar{w}(\beta^-) \leq w(\beta^+)$.
If $\bar{w}(\beta^+) < w(\beta^-)$ or $w(\beta^-) < \bar{w}(\beta^-)$, we deduce from (3.9) for $\bar{w}$ and $c = w(\beta^-)$ and (3.9) for w and $c = w(\beta^-)$ that

$$K(w(\beta^+)) - K(\bar{w}(\beta^+)) =$$
$$= - [K(\bar{w}(\beta^+)) - K(w(\beta^-))] + [K(w(\beta^+)) - K(w(\beta^-))] \leq 0 .$$

Therefore, $w = \bar{w}$.

If $\bar{w}(\beta^-) < w(\beta^+)$, then from (3.9) for w and $c = \bar{w}(\beta^-)$ we also get $w = \bar{w}$.

In the same way, we prove that $w = \bar{w}$ for the cases :

c) $\bar{w}(\beta^+) \leq w(\beta^-) \leq w(\beta^+) \leq \bar{w}(\beta^-)$,
d) $\bar{w}(\beta^+) < w(\beta^+) \leq w(\beta^-) \leq \bar{w}(\beta^-)$,
e) $w(\beta^-) \leq \bar{w}(\beta^+) \leq \bar{w}(\beta^-) \leq w(\beta^+)$,
f) $w(\beta^-) \leq \bar{w}(\beta^+) \leq w(\beta^+) \leq \bar{w}(\beta^-)$.

The existence and uniqueness results obtained in theorems (3.3) and (3.4) characterize a solution of the elliptic-hyperbolic problem through a generalized entropy condition. In the next theorem, we give another characterization which explains why this function w is a solution of the elliptic-hyperbolic problem and gives some coupling conditions at the interface.

THEOREM 3.5 : *The only function $w \in NBV(\alpha, \gamma)$ satisfying $w(\alpha) = 0$, $w|_{[\alpha, \beta]} \in H^1(\alpha, \beta)$ and* (3.6), *is characterized by the following conditions :*

i) $-(\mu(w) w')' + K(w)' + b(x, w) = g(x), \quad x \in (\alpha, \beta)$,

$$\int_\beta^\gamma \{K(w) \varphi' - [b(x, w) - g(x)] \varphi\} = 0 , \quad \forall \varphi \in C_0^\infty(\beta, \gamma) .$$

ii) *For all discontinuities* $y \in (\beta, \gamma)\, K(w(y^-)) = K(w(y^+))$,

$$sg(w(y^+) - w(y^-))[K(w(y)) - K(c)] \leqslant 0$$

$$\forall c \quad \textit{between} \quad w(y^+) \quad \textit{and} \quad w(y^-)\,.$$

iii) *At* β, $-\mu(w(\beta^-))\,w'(\beta^-) + K(w(\beta^-)) = K(w(\beta^+))$, *and* $sg(w(\beta^-) - w(\beta^+))[K(w(\beta^+)) - K(c)] \geqslant 0$ *holds for all* c *between* $w(\beta^-)$ *and* $w(\beta^+)$.

iv) *At* γ $sg(w(\gamma))[K(w(\gamma)) - K(c)] \geqslant 0$ *holds for all* c *between* $w(\gamma)$ *and* 0.

Proof : The proof follows with the same ideas of [6] for stationary shock problems. The conditions at the interface have been proved in (3.9).

We present some numerical results obtained using an Engquist-Osher scheme for the regularized problem. On a mesh defined in (α, γ) this scheme reads,

$$-\frac{1}{\bar{h}_j}\left[\frac{\phi(w_{j-1}) - \phi(w_j)}{h_{j-1}} + \frac{\phi(w_{j+1}) - \phi(w_j)}{h_j}\right] + \\ + \frac{1}{\bar{h}_j}\left[\int_{w_{j-1}}^{w_j} K'_+(s)\,ds + \int_{w_j}^{w_{j+1}} K'_-(s)\,ds\right] + b(x_j, w_j) = 0\,,$$

$$1 \leqslant j \leqslant N-1\,,$$

$$-\left[\frac{\phi(w_{N-1}) - \phi(w_N)}{h_{N-1}} + \varepsilon\,\frac{\phi(w_{N+1}) - \phi(w_N)}{h_N}\right] + \\ + \left[\int_{w_{N-1}}^{w_N} K'_+(s)\,ds + \int_{w_N}^{w_{N+1}} K'_-(s)\,ds\right] = 0\,,$$

$$-\frac{\varepsilon}{\bar{h}_j}\left[\frac{\phi(w_{j-1}) - \phi(w_j)}{h_{j-1}} + \frac{\phi(w_{j+1}) - \phi(w_j)}{h_j}\right] + \\ + \frac{1}{\bar{h}_j}\left[\int_{w_{j-1}}^{w_j} K'_+(s)\,ds + \int_{w_j}^{w_{j+1}} K'_-(s)\,ds\right] + b(x_j, w_j) = 0\,,$$

$$N+1 \leqslant j \leqslant M-1\,.$$

We present the numerical results obtained for the problem

$$-w'' + \left(\frac{w^4}{4} - \frac{w^2}{2}\right)' + w = 0\,, \quad x \in (-1, 0)\,,$$

$$\left(\frac{w^4}{4} - \frac{w^2}{2} \right)' + w = 0 , \quad x \in (0, 1) ,$$

$$w(-1) = -1.6 , \quad w(1) = 1.4$$

considering the regularized problem for $\varepsilon = 10^{-6}$, and on a uniform mesh with step-size $h = 1/40$.

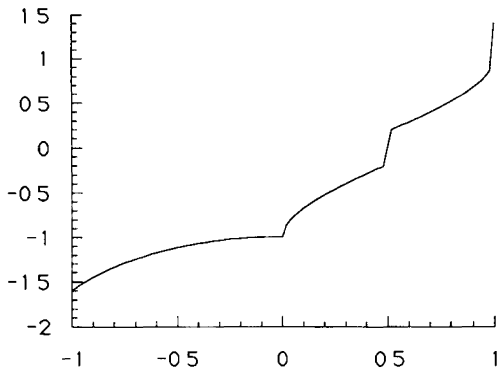

Figure 3.

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