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ANALYSIS OF MULTILEVEL DECOMPOSITION ITERATIVE METHODS FOR MIXED FINITE ELEMENT METHODS (*)

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Abstract — *We propose and analyze some iterative algorithms for mixed finite element methods for second-order elliptic equations The method is based on some multilevel decompositions for the finite element space and is related to the standard multigrid and hierarchical basis multigrid methods We show that the algorithms converge with rate bounded by* $1 - \omega(2 - \omega)/(CJ)$, *where* J *is the number of levels and* $0 < \omega < 2$ *is a relaxation parameter No regularity assumption beyond that necessary to define the weak form is assumed Hence the result holds for problems with jump coefficients or rough solutions We also establish a uniform convergence rate, independent of the number of levels, by additionally assuming full regularity for the second-order elliptic equations*

1. INTRODUCTION

We are concerned with solution methods for mixed finite element methods for second-order elliptic equations. It is known that the use of mixed finite element methods can provide very accurate approximations for the flux variable, which is a physically interesting quantity in many applications. The theory of the mixed method has been well established by many researchers (*cf.* [1, 7, 10, 14, 15, 20]). However, the technique of the mixed method leads to saddle point problems whose numerical solution have been quite difficult.

Recently, great attention has been focused on the domain decomposition method for the mixed method (*cf.* [16, 19, 11]). Both the theory and the numerical experiments show substantial achievement in solving the mixed finite element discretization problem.

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In this paper, we propose and analyse some iterative algorithms based on certain multilevel decompositions for the finite element space. The method can be regarded as a multilevel extension of the two-level Schwarz algorithm (*cf.* [19, 11]). Also, the method can be considered as a multigrid method as indicated in [23] for the standard Galerkin finite element method.

For simplicity, we take as our model the homogeneous Neumann boundary value problem

$$\begin{aligned} -\nabla \, . \, (a(x) \nabla p) &= f \, , \quad \text{in} \quad \Omega \, , \\ a(x) \nabla p \, . \, \nu &= 0 \, , \quad \text{on} \quad \partial \Omega \, , \end{aligned} \tag{1.1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and ν denotes the unit outward normal vector to the boundary $\partial\Omega$. Here ∇ and $\nabla .$ denote the gradient and the divergence operators, respectively.

Let $c(x) = a(x)^{-1}$ and $(. , .)$ denote the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$. Set

$$\begin{aligned} \mathcal{V} &= H_0(\text{div} \, ; \Omega) \\ &= \{ \mathbf{v} \in L^2(\Omega)^2 \, ; \nabla . \mathbf{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{v} . \nu = 0 \text{ on } \partial\Omega \} \, , \end{aligned}$$

which is a Hilbert space equipped with norm :

$$\| \mathbf{v} \|_H = (\| \mathbf{v} \|_0^2 + \| \nabla . \mathbf{v} \|_0^2)^{1/2} \, .$$

Let $W = L_0^2(\Omega)$ be the closed subspace of $L^2(\Omega)$ consisting of functions with vanishing mean values. By introducing the flux

$$\mathbf{u} = - a \, \nabla p \, , \tag{1.2}$$

the problem (1.1) is equivalent to the determination of $(\mathbf{u} \, ; p) \in \mathcal{V} \times W$ such that

$$\begin{aligned} (c\mathbf{u}, \mathbf{v}) - (\nabla . \mathbf{v}, p) &= 0 \, , & \mathbf{v} \in \mathcal{V} \, , \\ (\nabla . \mathbf{u}, w) &= (f, w) \, , & w \in W \, . \end{aligned} \tag{1.3}$$

The first equation in (1.3) stems from testing (1.2), divided by $a(x)$, against $\mathcal{V}$ and the second from testing (1.1), after substitution using (1.2), against W.

Let $\mathcal{T}_h$ be a triangulation of Ω and $\mathcal{V}^h$ and W^h be the finite element spaces for the flux and the pressure variables, respectively, satisfying the Babuška-Brezzi stability condition. Then, the mixed finite element method for (1.3) seeks $(\mathbf{u}_h \, ; p_h) \in \mathcal{V}^h \times W^h$ such that

$$\begin{aligned} (c\mathbf{u}_h, \mathbf{v}) - (\nabla . \mathbf{v}, p_h) &= 0 \, , & \mathbf{v} \in \mathcal{V}^h \, , \\ (\nabla . \mathbf{u}_h, w) &= (f, w) \, , & w \in W^h \, . \end{aligned} \tag{1.4}$$

Examples of $\mathcal{V}^h$ and W^h which are stable will be illustrated in Section 2.

Following the idea presented in [19, 11], we shall first find a discrete flux $\mathbf{u}^*$ such that $\nabla \cdot \mathbf{u}^* = \nabla \cdot \mathbf{u}_h$ via some multilevel decomposition structure. Then we reduce the saddle point problem (1.4) to a positive definite problem for the flux variable by the use of $\mathbf{u}^*$. The new problem is defined on the divergence-free subspace of $\mathcal{V}^h$ and thus has difficulty in the selection of the right basis functions. This difficulty, however, can be overcome by using either the domain decomposition method or the stream-function space (see Section 2).

Two multilevel iterative algorithms are proposed in Section 3 for the reduced saddle point problem. To compare with the standard multigrid method, we apply the Schwarz alternating method and the additive Schwarz method for each level to replace the standard Gauss-Seidel and Jacobi smoothings. We show that, without any regularity assumptions, the two algorithms converge with rate bounded by $1 - \omega(2-\omega)/(CJ)$ for some constant C, where J is the number of levels and $0 < \omega < 2$ is a relaxation parameter. Furthermore, if the full H^2 regularity is satisfied for the operator $\nabla . (c\nabla)$ with homogeneous Dirichlet boundary condition, then uniform convergence is possible.

The paper is organized as follows. In § 2, we review two known families of mixed finite element spaces. In § 3, we propose two multilevel decomposition algorithms (MDAs) for (1.4). The convergence estimates are established in § 4. Finally, we present some numerical results in § 5 to illustrate the efficiency of our method.

2. MIXED FINITE ELEMENT SPACES

We outline two families of mixed finite element spaces in this section, one on triangles and one on rectangles.

RT Triangular Elements : Let $\mathbf{x} = (x, y)$ be the space variable. The RT (Raviart-Thomas) space [20] of index j on the triangle K for the flux is defined by

$$\mathcal{V}^h(K) = P_j(K) \oplus \mathbf{x}\hat{P}_j(K),$$

where $\hat{P}_j(K)$ is the space of homogeneous polynomials of degree j on K. The corresponding space for the pressure is given by

$$W^h(K) = P_j(K).$$

A locally-defined projection operator Π_h can be given by the following degrees of freedom :

$$\langle \mathbf{v} . \nu, p \rangle_e, \quad p \in P_j(e) \quad \text{for all three edges},$$

$$(\mathbf{v}, \phi)_K, \quad \phi \in P_{j-1}(K)^2.$$

BDFM Elements : The BDFM (Brezzi-Douglas-Fortin-Marini) spaces (*cf.* [8]) are modifications of the rectangular RT spaces. The space of index j for the flux variable is defined by

$$\mathscr{V}^h(K) = P_j(K)\backslash\{y^j\} \times P_j(K)\backslash\{x^j\} ;$$

and the pressure space is defined by

$$W^h(K) = P_{j-1}(K) ,$$

where $P_i(K)$ denotes the polynomials of total degree no larger than i on K. A similar projection operator Π_h can be defined locally using the following degrees of freedom :

$$\langle \mathbf{v} \cdot \nu, p \rangle_e , \quad p \in P_{j-1}(e) \quad \text{for all four edges} ,$$
$$(\mathbf{v}, \phi)_K , \quad \phi \in P_{j-2}(K)^2 .$$

It is not hard to check that the operators Π_h are well defined on $\mathscr{V} \cap H^1(\Omega)^2$ and satisfy the following commutative property

$$Q_h \nabla . = \nabla . \Pi_h \quad \text{on} \quad \mathscr{V} \cap H^1(\Omega)^2 , \tag{2.1}$$

where Q_h is the local L^2 projection operator from W onto W^h. The stability of the finite element spaces described above stems from (2.1) and the local natures of Π_h and Q_h. For a detailed discussion, see [1, 7, 8, 9, 10, 14, 15, 20].

In the rest of this section, we briefly describe the stream-function space. A more detailed discussion can be found in [11]. Let $\mathscr{H}^h$ be the divergence-free subspace of $\mathscr{V}^h$; i.e.,

$$\mathscr{H}^h = \{\mathbf{v} \in \mathscr{V}^h ; \nabla . \mathbf{v} = 0\} .$$

It follows that any flux $\mathbf{v} \in \mathscr{H}^h$ can be expressed as the **curl** of a stream function $\phi \in H^1(\Omega)$. Furthermore, the stream function ϕ is uniquely determined in $H_0^1(\Omega)$, since the flux has zero boundary value in the normal direction to $\partial\Omega$. The stream-function space $\mathscr{S}^h$ is the set of all stream functions with vanishing boundary values. The following is a characterization of stream function spaces for some known families in the mixed finite element method.

THEOREM 2.1 : *Let $\mathscr{S}^h$ denote the stream-function space. Then,*

(1) *for the triangular RT element of index $j \geq 0$ (cf. [20]), we have*

$$\mathscr{S}^h = \left\{\phi \in H_0^1(\Omega) ; \phi|_K \in P_{j+1}(K), K \in \mathcal{T}_h\right\} ;$$

(2) *for the BDFM element of index j (cf. [8]), we have*

$$\mathcal{S}^h = \left\{ \phi \in H_0^1(\Omega)\,;\ \phi|_K \in P_{j+1}(K) \backslash \{x^{j+1}, y^{j+1}\},\ K \in \mathcal{T}_h \right\}\ ;$$

(3) *for the rectangular RT element of index r (cf. [20]), we have*

$$\mathcal{S}^h = \left\{ \phi \in H_0^1(\Omega)\,;\ \phi|_K \in Q_{r+1,\,r+1}(K),\ K \in \mathcal{T}_h \right\}\ ;$$

(4) *for the BDM element of index $j \geq 1$ (cf. [9]), we have*

$$\mathcal{S}^h = \left\{ \phi \in H_0^1(\Omega)\,;\ \phi|_K \in P_{j+1}(K),\ K \in \mathcal{T}_h \right\}\ .$$

Remark 2.1 : We can define the stream-function space for problems with nonhomogeneous Neumann or Dirichlet boundary conditions. We emphasize that the forthcoming algorithms in Section 3 can be applied to such problems without any difficulty.

3. MULTILEVEL DECOMPOSITION ITERATIVE METHODS

Let $\mathcal{T}_0$ be an intentionally coarse initial triangulation of Ω. For $i = 1, \ldots, J$, let $\mathcal{T}_i$ be the triangulation obtained by breaking every triangle (rectangle) of $\mathcal{T}_{i-1}$ into four subtriangles (subrectangles) by connecting the mid-points of each edge (opposite edges, respectively). Denote also by $\mathcal{T}_h = \mathcal{T}_J$ the finest triangulation of Ω. The triangles of the triangulation $\mathcal{T}_i$ are called level i elements. The vertices of level i elements are called level i nodes.

Let $\mathcal{V}_i^h \times W_i^h$ be the finite element space associated with $\mathcal{T}_i$. For any $f^h \in W^h$, let

$$f_i^h = Q_i^h f^h - Q_{i-1}^h f^h\,, \quad i = 1, \ldots, J\,, \tag{3.1}$$

where Q_i^h is the standard L^2 projection operator onto the space W_i^h and $Q_{-1}^h = 0$.

LEMMA 3.1 : *Let f_i^h be as above. Then*

$$f^h = \sum_{i=0}^{J} f_i^h\,. \tag{3.2}$$

Furthermore, the functions f_i^h, $i = 1, \ldots, J$, have vanishing mean values on each element of level $i - 1$.

Proof : Note that Q_J^h is the identity operator on $W^h \equiv W_J^h$. It follows from (3.1) that

$$f^h \equiv Q_J^h f^h = \sum_{i=0}^{J} f_i^h .$$

Also, (3.1) implies that the mean value of f_i^h vanishes on each element of level $i-1$ since the spaces W_k^h are nested, and then $Q_{i-1}^h f^h$ is the L^2 projection of $Q_i^h f^h$ in W_{i-1}^h. □

As in the Schwarz alternating algorithm (*cf.* [11]), we first find a discrete flux whose divergence is f^h. Our method is based on the decomposition (3.2). Let $\mathcal{M}_0^h = \mathcal{V}_0^h$ and $\mathcal{M}_i^h$ be the subspace of $\mathcal{V}_i^h$ consisting of those fluxes whose boundary values in the normal component of the boundary of level $i-1$ elements are zero ; i.e.,

$$\mathcal{M}_i^h = \left\{ \mathbf{v} \in \mathcal{V}_i^h, \mathbf{v} \,.\, \nu_{\partial T} = 0 \,, \text{ on } \partial T \text{ and } T \in \mathcal{T}_{i-1} \right\} , \tag{3.3}$$

for $i = 1, \ldots, J$. The corresponding pressure spaces are defined by taking $\tilde{W}_0^h = W_0^h$ and

$$\tilde{W}_i^h = \left\{ w \in W_i^h, \int_T w \, dx = 0 \,, \quad \text{for all} \quad T \in \mathcal{T}_{i-1} \right\} , \tag{3.4}$$

for $i = 1, \ldots, J$. It follows from Lemma 3.1 that $f_i^h \in \tilde{W}_i^h$.

Multilevel Decomposition Algorithm (MDA) (Part 1) : Let f^h be the L^2 projection of the right-hand side function f of (1.1) in W^h. Let $f_i^h \in \tilde{W}_i^h$ be as in (3.2).

(1) For $i = 0, \ldots, J$, solve $(\mathbf{u}_i^h \,;\, \theta_i^h) \in \mathcal{M}_i^h \times \tilde{W}_i^h$ by

$$\begin{aligned} (\tilde{c}\mathbf{u}_i^h, \mathbf{v}) - (\nabla \,.\, \mathbf{v}, \theta_i^h) &= 0 \,, \qquad && \mathbf{v} \in \mathcal{M}_i^h , \\ (\nabla \,.\, \mathbf{u}_i^h, w) &= (f_i^h, w) \,, && w \in \tilde{W}_i^h , \end{aligned} \tag{3.5}$$

where $\tilde{c}$ is an arbitrary positive function defined on Ω.

(2) Set

$$\mathbf{u}^* = \sum_{i=0}^{J} \mathbf{u}_i^h . \tag{3.6}$$

LEMMA 3.2 : *Let* $\mathbf{u}^*$ *be obtained by above algorithm. Then*

$$\nabla \,.\, \mathbf{u}^* = f^h . \tag{3.7}$$

Proof: It is obvious that

$$\nabla \cdot \mathbf{u}_i^h = f_i^h$$

for $i = 0, \ldots, J$. Thus, it follows from (3.2) that (3.7) is valid. □

Remark 3.1 : The solution of (3.5) can be obtained by solving some local problems. To see this, we note that the space $\mathcal{M}_i^h$ is the direct sum of subspaces defined on disjoint subdomains $\mathcal{T}_\kappa$, where $\mathcal{T}_\kappa$ are elements of level $i - 1$. Thus, the problem (3.5) is equivalent to subproblems restricted to the subdomain $\mathcal{T}_\kappa$ (with vanishing boundary value in the normal direction). The only exception is the solution of $(\mathbf{u}_0^h \,;\, \theta_0^h)$ which is defined on the coarse level and, therefore, can not be split into local problems. However, the coarse level problem should not cost much to compute.

Remark 3.2 : As already noted in [11], the coefficient $\tilde{c}$ in (3.5) may differ from c in the computation. This is true because our object here is only to find a flux $\mathbf{u}_i^h$ satisfying the second equation of (3.5). For convenience, one may take $\tilde{c} = 1$ or $\tilde{c} = c$ in the computation.

Remark 3.3 : The MDA (Part 1) provides a way to find the desired $\mathbf{u}^*$ based on the natural multilevel decomposition (3.1) for f^h. In practice, there are other methods available. The numerical experiments in § 5 use a different approach to construct $\mathbf{u}^*$ for rectangular domains.

By letting $\hat{\mathbf{u}}^h = \mathbf{u}_h - \mathbf{u}^*$, the mixed finite element discretization problem (1.4) is equivalent to seeking $(\hat{\mathbf{u}}^h \,;\, p_h) \in \mathcal{V}^h \times W^h$ satisfying

$$\begin{aligned} (c\hat{\mathbf{u}}^h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p^h) &= -\,(c\mathbf{u}^*, \mathbf{v})\,, \quad && \mathbf{v} \in \mathcal{V}^h \\ (\nabla \cdot \hat{\mathbf{u}}^h, w) &= 0\,, && w \in W^h\,, \end{aligned} \tag{3.8}$$

which is equivalent to the determination of an $\hat{\mathbf{u}}^h \in \mathcal{H}^h$ such that

$$(c\hat{\mathbf{u}}^h, \mathbf{v}) = -\,(c\mathbf{u}^*, \mathbf{v})\,, \quad \mathbf{v} \in \mathcal{H}^h\,. \tag{3.9}$$

It is clear that the problem (3.9) is positive definite for the new flux $\hat{\mathbf{u}}^h$. However, we have trouble in constructing a good basis for the divergence-free subspace $\mathcal{H}^h$. We have proposed (*cf.* [11]) to solve (3.9) by employing the stream-function space for $\mathcal{H}^h$, which effectively leads to a standard Galerkin method for a second-order elliptic equation. Thus, all the results in solution method developed for the Galerkin method can be applied directly to the reduced saddle-point problem (which is elliptic on the stream-function space).

However, here we would like to study the problem (3.9) in its present form. Following the idea presented in [11] for the Schwarz alternating method, we propose two iterative algorithms for solving (3.9) based on a certain multilevel structures of the space $\mathcal{V}^h$. The methods are similar to the multilevel decomposition iterative procedure proposed in [23, 24] for the Galerkin method. In general, the technique can be regarded as an extension of the standard multigrid method. The idea here is to replace the standard smoothing (e.g. the Gauss-Seidel and Jacobi) in the multigrid method by the Schwarz alternating method or additive Schwarz method for each level.

To illustrate the procedure, let $\mathcal{N}_i$ be the set of level i nodes for $i = 0, 1, \dots, J$. Associated with each node $x_{ik} \in \mathcal{N}_i$, let $\Omega_{i,k}$ be the subdomain of Ω consisting of level i elements having x_{ik} as a common vertex. It is clear that $\{\Omega_{i,k}\}_{k=1}^{m_i}$ forms an overlapping decomposition of Ω, where m_i is the number of level i nodes. Let $\mathcal{V}_{i,k} \times W_{i,k}$ be the corresponding finite element space of level i defined on $\Omega_{i,k}$ with the natural partition induced from $\mathcal{T}_i$. Accordingly, let $\mathcal{H}_{i,k}$ be the divergence-free subspace of $\mathcal{V}_{i,k}$. The second part of the MDA can then be stated as follows.

Multilevel Decomposition Algorithm-1 (MDA-1) (Part 2) : Given $\hat{\mathbf{u}}_n^h \in \mathcal{H}^h$, an approximation of the solution of (3.5), we seek the next approximate solution $\hat{\mathbf{u}}_{n+1}^h \in \mathcal{H}^h$ as follows :

(1) Define $Z_0 \in \mathcal{H}^h$ by

$$Z_0 = \hat{\mathbf{u}}_n^h + \omega P_0(\hat{\mathbf{u}}^h - \hat{\mathbf{u}}_n^h) .$$

(2) For $i = 1, \dots, J$, let $Y_0 = Z_{i-1}$ and

$$Y_k = Y_{k-1} + \omega P_{i,k}(\hat{\mathbf{u}}^h - Y_{k-1}), \quad k = 1, \dots, m_i ,$$

where $P_{i,k}$ is the projection operator onto $\mathcal{H}_{i,k}$ with respect to the $(\cdot, \cdot)_c \equiv (c \cdot, \cdot)$ inner product. Then, we let $Z_i = Y_{m_i}$.

(3) Set $\hat{\mathbf{u}}_{n+1}^h = Z_J$.

Remark 3.4 : The projection operator $P_{i,k}$ is defined locally on each (macro-element) $\Omega_{i,k}$. In practice, we have to solve a local problem on $\Omega_{i,k}$ to determine this operator ; the computation of such local problems is cheap. Mathew [19] proposed to find $P_{i,k}$ by solving a saddle point problem on $\Omega_{i,k}$. More precisely, for any $\chi \in \mathcal{V}^h$, $P_{i,k}\chi$ is the same as $\mathbf{u}_{i,k} \in \mathcal{H}_{i,k}$ defined by

$$\begin{aligned} (\mathbf{u}_{i,k}, \mathbf{v})_c - (\nabla \cdot \mathbf{v}, p_{i,k}) &= (\chi, \mathbf{v})_c , && \mathbf{v} \in \mathcal{V}_{i,k} , \\ (\nabla \cdot \mathbf{u}_{i,k}, w) &= 0 , && w \in W_{i,k} , \end{aligned} \tag{3.10}$$

where, of course, $p_{i,k} \in W_{i,k}$. The authors of [11] suggested an approach by using the stream-function space ; i.e., by letting $\mathcal{S}_{i,k}$ be the stream function space of $\mathscr{H}_{i,k}$, the determination of $P_{i,k}\chi$ is given by seeking **curl** ϕ, with $\phi \in \mathscr{S}_{i,k}$, satisfying

$$(\mathbf{curl}\ \phi, \mathbf{curl}\ \psi)_c = (\chi, \mathbf{curl}\ \psi)_c, \quad \text{for all} \quad \psi \in \mathscr{S}_{i,k}. \tag{3.11}$$

Since the Schwarz alternating method is an analogue of the SOR iterative method for matrix computation (*cf.* [5, 22, 23]), we see that the MDA-1 (Part 2) is actually an analogue of the multigrid algorithm based on the SOR smoothing for each level. Due to this connection, we propose to modify MDA-1 (Part 2) by using the additive Schwarz (*cf.* [12]) on each level i. To do so, let

$$R_i = \sum_{k=1}^{m_i} P_{k,i}. \tag{3.12}$$

It is clear that the operator R_i is selfadjoint and nonnegative with respect to the $(\cdot,\cdot)_c$ inner product. The properties of R_i can be summarized as follows.

LEMMA 3.3 : *Let R_i be as above and λ_i be the largest eigenvalue of R_i. If $T_i = \lambda_i^{-1} R_i$, then*

(1) *there exists a constant C independent of i such that*

$$\lambda_i \leq C,$$

(2) *the operator T_i is selfadjoint with respect to the $(\cdot,\cdot)_c$ inner product and such that*

$$\sigma(T_i) \subset [0, 1],$$

where $\sigma(T_i)$ denotes the spectrum of T_i.

Proof: It suffices to show that λ_i has a uniform upper bound C. Actually, we have

$$\begin{aligned}(R_i \mathbf{v}, \mathbf{v})_c &= \sum_{k=1}^{m_i} (P_{i,k}\mathbf{v}, \mathbf{v})_c = \sum_{k=1}^{m_i} (P_{i,k}\mathbf{v}, P_{i,k}\mathbf{v}) \\ &\leq \sum_{k=1}^{m_i} (c\mathbf{v}, \mathbf{v})_{\Omega_{i,k}} \leq C(\mathbf{v}, \mathbf{v})_c,\end{aligned}$$

which implies the uniform boundedness of λ_i. □

Multilevel Decomposition Algorithm-2 (MDA-2) (Part 2) : Given $\hat{\mathbf{u}}_n^h \in \mathscr{H}^h$, an approximation of the solution of (3.5), we seek the next approximate solution $\hat{\mathbf{u}}_{n+1}^h \in \mathscr{H}^h$ as follows :

(1) Define $Z_0 \in \mathscr{H}^h$ by

$$Z_0 = \hat{\mathbf{u}}_n^h + \omega P_0(\hat{\mathbf{u}}^h - \hat{\mathbf{u}}_n^h).$$

(2) For $i = 1, \ldots, J$, define Z_i by

$$Z_i = Z_{i-1} + \omega T_i (\hat{\mathbf{u}}^h - Z_{i-1}) .$$

(3) Set $\hat{\mathbf{u}}^h_{n+1} = Z_J$.

4. CONVERGENCE ANALYSIS

Since the multilevel iterative method is applied to (3.9), which is selfadjoint and positive definite on the Hilbert space $\mathcal{H}^h$, the general result developed in [5, 23] (see also [6, 22, 24]) can be employed to establish the convergence.

For completeness, we cite the result of [23] as follows : Let $a(\cdot, \cdot)$ be a symmetric and coercive bilinear form defined on a Hilbert space $\mathcal{V}$. Assume that $\mathcal{V}_i$, $i = 1, \ldots, J$, are closed subspaces of $\mathcal{V}$ satisfying

$$\mathcal{V} = \sum_{i=1}^{J} \mathcal{V}_i .$$

Let P_i be the projection operator with respect to the form $a(\cdot, \cdot)$. The main result in [23] is concerned with the norm estimate of the product operator E :

$$E = (I - \omega P_J)(I - \omega P_{J-1}) \ldots (I - \omega P_1) ,$$

where ω is any real number in (0, 2).

Assume that for any $v \in \mathcal{V}$ there exist $v_i \in \mathcal{V}_i$, for $i = 1, \ldots, J$, such that $v = \sum_{i=1}^{J} v_i$ satisfying

$$\sum_{i=1}^{J} \|v_i\|^2 \leq C_1 \|v\|^2 , \tag{4.1}$$

$$\sum_{i=2}^{J} \|P_i w_i\|^2 \leq C_0 \|v\|^2 , \tag{4.2}$$

and

$$\sum_{j=1}^{J-1} \|P_j w_{j+1}\|^2 \leq C_2 \|v\|^2 \tag{4.3}$$

for some constants C_1, C_0, and C_2, where $w_j = \sum_{k=j}^{J} v_k$ and $\| \cdot \| = a(\cdot, \cdot)^{1/2}$. Then we have

THEOREM 4.1 : *Assume that* (4.1), (4.2), *and* (4.3) *hold. Then*

$$\|Eu\|^2 \leq \gamma \|u\|^2 , \quad for\ all \quad u \in \mathcal{V} , \tag{4.4}$$

where

$$\gamma = 1 - \frac{\omega (2 - \omega)}{2(\omega^2 C_2 + C_1)} \tag{4.5}$$

or

$$\gamma = 1 - \frac{1}{C_0} \quad if \quad \omega = 1 . \tag{4.6}$$

We show in the following that the error reduction operators of MDA's (Part 2) have the product form E.

LEMMA 4.1 : *Let the MDA's (Part 2) be as in Section 3. Set*

$$\tilde{E}_i = \prod_{k=1}^{m_i} (I - \omega P_{i,k}) , \tag{4.7}$$

for $i = 1, \dots, J$ *and* $\tilde{E}_0 = (I - \omega P_0)$. *Then the MDA-1 (Part 2) has the following error reduction operator*

$$E = \prod_{i=0}^{J} \tilde{E}_i . \tag{4.8}$$

The MDA-2 (Part 2) has an error reduction operator

$$E = \prod_{i=0}^{J} (I - \omega T_i) . \tag{4.9}$$

The proof of (4.8) and (4.9) is straightforward, since the MDA's (Part 2) are special cases of the product algorithm (*cf.* [5, 23]).

To estimate the norm of the product operator E for MDA-1 (Part 2), it suffices to check the conditions of Theorem 4.1. First of all, we point out that the bilinear form in this application is given by $(. , .)_c \equiv (c . , .)$. Naturally, denote by $\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})_c^{1/2}$ the norm of the flux $\mathbf{v} \in \mathcal{H}^h$. Note that E is the product of operators $I - P_{i,k}$. Hence, we need to show that for any $\mathbf{v} \in \mathcal{H}^h$ there exist $\mathbf{v}_{i,k} \in \mathcal{H}_{i,k}$ and $\mathbf{v}_0 \in \mathcal{H}_0$ such that $\mathbf{v} = \mathbf{v}_0 + \sum_{i=1}^{J} \sum_{k=1}^{m_i} \mathbf{v}_{i,k}$, satisfying analogues of (4.1), (4.2), and (4.3) with « good » estimates of the constants.

LEMMA 4.2 : *For any* $\mathbf{v} \in \mathscr{H}^h$, *there exists a decomposition* $\{\mathbf{v}_{i,k}\}$, *described as above, which satisfies the assumptions* (4.1), (4.2), *and* (4.3) *with*

$$C_i = O(J), \quad i = 0, 1, 2. \tag{4.10}$$

Proof: Let $\sigma \in \mathscr{S}^h$ be the stream function of $\mathbf{v} \in \mathscr{H}^h$. Let $\sigma_0 = \tilde{Q}_0 \sigma \in \mathscr{S}_0$ and $\sigma_i = (\tilde{Q}_i - \tilde{Q}_{i-1}) \sigma \in \mathscr{S}_i$, for $i = 1, \dots, J$, where $\tilde{Q}_i$ is the $c(x)$-weighted L^2 projection operator onto the stream-function space $\mathscr{S}_i$ for level i. It is clear that

$$\sigma = \sum_{i=0}^{J} \sigma_i . \tag{4.11}$$

To define a decomposition for v, let $\{\phi_{i,k}\}$ be a partition of unity subordinated to the decomposition $\{\Omega_{i,k}\}_{k=1}^{m_i}$ for level i described in Section 3. Set

$$\mathbf{v}_{i,k} = \mathbf{curl}\, I_{h_i}(\phi_{i,k}\, \sigma_i), \tag{4.12}$$

where I_{h_i} is the nodal interpolation operator onto the stream-function space $\mathscr{S}_i$. It follows from (4.12) and (4.11) that $\{\mathbf{v}_{i,k}\}$ forms a decomposition of v. Now we show that the assumptions (4.1), (4.2), and (4.3) are satisfied with C_i estimated by (4.10).

For simplicity, we establish the estimate of C_1 only ; the analysis can be extended to C_0 and C_2 without any difficulty. Thus, we want to show that there exists a constant C such that

$$\|\mathbf{v}_0\|^2 + \sum_{i=1}^{J} \sum_{k=1}^{m_i} \|\mathbf{v}_{i,k}\|^2 \leq CJ \|\mathbf{v}\|^2 . \tag{4.13}$$

Actually, it is straightforward to see that there exists a constant C such that

$$|\nabla \phi_{i,k}| \leq C h_i^{-1} . \tag{4.14}$$

It follows that

$$\|\mathbf{v}_{i,k}\|^2 \leq C \left(\int_{\Omega_{i,k}} c(x) |\nabla \sigma_i|^2 \, dx + h_i^{-2} \int_{\Omega_{i,k}} c(x) |\sigma_i|^2 \, dx \right)$$

for some constant C. Thus,

$$\begin{aligned} \sum_{k=1}^{m_i} \|\mathbf{v}_{i,k}\|^2 &\leq \sum_{k=1}^{m_i} \left(\int_{\Omega_{i,k}} c(x) |\nabla \sigma_i|^2 \, dx + h_i^{-2} \int_{\Omega_{i,k}} c(x) |\sigma_i|^2 \, dx \right) \\ &\leq C \left(\|\sigma_i\|_{1,c}^2 + h_i^{-2} \|\sigma_i\|_{0,c}^2 \right), \end{aligned} \tag{4.15}$$

where $\| \cdot \|_{\kappa, c}$ denote the weighted norms in $H^{\kappa}(\Omega)$ with weight function $c(x)$ for $\kappa = 0,1$. Using the estimate

$$\begin{aligned} \|\sigma_l\|_{0,c} &= \left\|(\tilde{Q}_l - \tilde{Q}_{l-1})\,\sigma\right\|_{0,c} \\ &\leq C h_l \|\sigma\|_{1,c} \leq C h_l \|\mathbf{v}\| \end{aligned} \tag{4.16}$$

we obtain

$$\sum_{k=1}^{m_l} \|\mathbf{v}_{l,k}\|^2 \leq C \|\mathbf{v}\|^2 . \tag{4.17}$$

Also, it is easy to see that

$$\|\mathbf{v}_0\| = \left\|\mathbf{curl}\ (\tilde{Q}_0\,\sigma)\right\| \leq \left\|\tilde{Q}_0\,\sigma\right\|_{1,c} \leq C\,\|\sigma\|_{1,c} \leq C\,\|\mathbf{v}\| . \tag{4.18}$$

Thus, combining (4.16) and (4.17) completes the proof of (4.13). □

Remark 4.1 : The estimates (4.16) and (4.18) are obvious for any constant coefficient $c(x)$. In the presence of a large jump of $c(x)$, they are still valid under some situations. See [6] for a discussion.

As a consequence of Lemma 4.2 and Theorem 4.1 we have the following result for the convergence of the MDA-1 (Part 2).

THEOREM 4.2 : *There exists a constant C such that the convergence of the MDA-1 (Part 2) is bounded by*

$$\gamma_2 = 1 - \frac{\omega(2-\omega)}{CJ} . \tag{4.19}$$

We now turn to the convergence analysis of MDA-2 (Part 2). As indicated by Lemma 4.1, this algorithm has the error reduction operator E defined by (4.9), in which the operators T_l are no longer projectors. Thus, Theorem 4.1 can not be applied directly to provide a convergence estimate. However, it is possible to obtain an estimate by combining the idea presented in [5] and the proof of Theorem 4.1 in [23].

THEOREM 4.3 : *There exists a constant C such that the convergence of the MDA-2 (Part 2) is bounded by*

$$\gamma_3 = 1 - \frac{\omega(2-\omega)}{CJ} . \tag{4.20}$$

We need the following result to establish (4.20).

LEMMA 4.3 : *For any* $\mathbf{v}_\iota \in \mathscr{H}_\iota^h$, *let* σ_ι *be the stream function of* $\mathbf{v}_\iota$. *Then there exists a constant C such that*

$$|(\mathbf{u}, \mathbf{v}_\iota)_c| \leq C\,(T_\iota\,\mathbf{u}, \mathbf{u})_c^{1/2}\,(\|\mathbf{v}_\iota\|^2 + h_\iota^{-2}\|\sigma_\iota\|_{0,c}^2)^{1/2}, \quad \textit{for all} \quad u \in \mathscr{H}^h. \tag{4.21}$$

Proof : As in the proof of Lemma 4.2, let $\{\phi_{\iota,k}\}_{k=1}^{m_\iota}$ be the partition of unity subordinated to the decomposition $\{\Omega_{\iota,k}\}_{k=1}^{m_\iota}$ and

$$\mathbf{v}_{\iota,k} = \mathbf{curl}\, I_{h_\iota}(\phi_{\iota,k}\,\sigma_\iota)\,.$$

Similarly to (4.15), we have

$$\sum_{k=1}^{m_\iota} \|\mathbf{v}_{\iota,k}\|^2 \leq C\,(\|\sigma_\iota\|_{1,c}^2 + h_\iota^{-2}\|\sigma_\iota\|_{0,c}^2)\,, \tag{4.22}$$

for some constant C. It follows that

$$\begin{aligned}
|(\mathbf{u}, \mathbf{v}_\iota)_c| &\leq \sum_{k=1}^{m_\iota} |(\mathbf{u}, \mathbf{v}_{\iota,k})_c| \leq \sum_{k=1}^{m_\iota} |(P_{\iota,k}\,\mathbf{u}, \mathbf{v}_{\iota,k})_c| \\
&\leq \sum_{k=1}^{m_\iota} \|P_{\iota,k}\,\mathbf{u}\|\,\|\mathbf{v}_{\iota,k}\| \leq \left(\sum_{k=1}^{m_\iota} \|P_{\iota,k}\,\mathbf{u}\|^2\right)^{1/2} \left(\sum_{k=1}^{m_\iota} \|\mathbf{v}_{\iota,k}\|^2\right)^{1/2} \\
&\leq C\,(T_\iota\,\mathbf{u}, \mathbf{u})_c^{1/2}\,(\|\sigma_\iota\|_{1,c}^2 + h_\iota^{-2}\|\sigma_\iota\|_{0,c}^2)^{1/2} \\
&\leq C\,(T_\iota\,\mathbf{u}, \mathbf{u})_c^{1/2}\,(\|\mathbf{v}_\iota\|^2 + h_\iota^{-2}\|\sigma_\iota\|_{0,c}^2)^{1/2}\,,
\end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem 4.5 : First of all, it follows from Lemma 3.3 and [Lemma 2.1, 5] that

$$\sum_{\iota=0}^{J} (\omega T_\iota\,E_{\iota-1}\,\mathbf{v}, E_{\iota-1}\,\mathbf{v})_c \leq (\|\mathbf{v}\|^2 - \|E\mathbf{v}\|^2)/(2-\omega)\,, \quad \mathbf{v} \in \mathscr{H}^h, \tag{4.23}$$

where, as in [5, 23], $E_\iota = \prod_{k=0}^{\iota} (I - \omega T_k)$. Further, let σ be the stream function of $\mathbf{v}$ and $\tilde{Q}_\iota\,\sigma$ be the weighted L^2 projection of σ in the stream-function space $\mathscr{S}_\iota$. We clearly have

$$\mathbf{v} = \sum_{\iota=0}^{J} \mathbf{v}_\iota\,, \tag{4.24}$$

where $\mathbf{v}_l = \mathbf{curl}\,(\tilde{Q}_l\,\sigma - \tilde{Q}_{l-1}\,\sigma)$ and $\tilde{Q}_{-1} = 0$. By (4.24), we obtain

$$\begin{aligned}\|\mathbf{v}\|^2 = (\mathbf{v}, \mathbf{v})_c &= \sum_{l=0}^{J} (\mathbf{v}, \mathbf{v}_l)_c \\ &= (\mathbf{v}, \mathbf{v}_0)_c + \sum_{l=1}^{J} (E_{l-1}\,\mathbf{v}, \mathbf{v}_l)_c + \sum_{l=1}^{J} ((I - E_{l-1})\,\mathbf{v}, \mathbf{v}_l)_c \\ &= I_0 + I_1 + I_2\,. \end{aligned} \tag{4.25}$$

I_0 can be estimated as follows :

$$\begin{aligned} I_0 = (\mathbf{v}, \mathbf{v}_0)_c &= (P_0\,\mathbf{v}, \mathbf{v}_0)_c \\ &\leqslant (P_0\,\mathbf{v}, \mathbf{v})_c^{1/2}\,\|\mathbf{v}_0\| \leqslant C\,\sqrt{\omega^{-1}}(T_0\,\mathbf{v}, \mathbf{v})_c^{1/2}\,\|\mathbf{v}\|\,. \end{aligned} \tag{4.26}$$

To estimate I_2, we notice that

$$I - E_{l-1} = \sum_{k=0}^{l-1} \omega T_k\,E_{k-1}\,.$$

Thus, by changing the order of the double sum and then applying the Schwarz inequality we obtain

$$\begin{aligned} I_2 = \sum_{l=1}^{J} ((I - E_{l-1})\,\mathbf{v}, \mathbf{v}_l)_c &= \sum_{l=1}^{J}\sum_{k=0}^{l-1} (\omega T_k\,E_{k-1}\,\mathbf{v}, \mathbf{v}_l) \\ &= \sum_{k=1}^{J} (\omega T_k\,E_{k-1}\,\mathbf{v}, \mathbf{curl}\,(\sigma - \tilde{Q}_k\,\sigma))_c \\ &\leqslant C\,\sqrt{\omega J}\,\|\mathbf{v}\|\left(\sum_{l=1}^{J} (\omega T_l\,E_{l-1}\,\mathbf{v}, E_{l-1}\,\mathbf{v})_c\right)^{1/2}. \end{aligned} \tag{4.27}$$

As for I_1, we use (4.21) and the estimate

$$\left\|\tilde{Q}_l\,\sigma - \tilde{Q}_{l-1}\,\sigma\right\|_{0,\,c} \leqslant C h_l\,\|\sigma\|_{1,\,c} \leqslant C h_l\,\|\mathbf{v}\|$$

to obtain

$$\begin{aligned} I_1 &= \sum_{l=1}^{J} (E_{l-1}\,\mathbf{v}, \mathbf{v}_l)_c \\ &\leqslant \sum_{l=1}^{J} (T_l\,E_{l-1}\,\mathbf{v}, E_{l-1}\,\mathbf{v})_c^{1/2}\left(\|\mathbf{v}_l\|^2 + h_l^{-2}\left\|\tilde{Q}_l\,\sigma - \tilde{Q}_{l-1}\,\sigma\right\|_{0,\,c}^2\right)^{1/2} \\ &\leqslant C\,\sum_{l=1}^{J} (T_l\,E_{l-1}\,\mathbf{v}, E_{l-1}\,\mathbf{v})_c^{1/2}\,\|\mathbf{v}\| \\ &\leqslant C\,\sqrt{J/\omega}\;\|\mathbf{v}\|\left(\sum_{l=1}^{J} (\omega T_l\,E_{l-1}\,\mathbf{v}, E_{l-1}\,\mathbf{v})_c\right)^{1/2}. \end{aligned} \tag{4.28}$$

Thus, combining (4.25) with (4.26), (4.23), and (4.28) yields

$$\|\mathbf{v}\|^2 \leq CJ\omega^{-1} \sum_{i=0}^{J} (\omega T_i E_{i-1} \mathbf{v}, E_{i-1} \mathbf{v})_c ,$$

which, together with (4.23), completes the proof of the theorem. □

The estimates (4.19) and (4.20) are established without any regularity assumption beyond subspaces of $H(\mathrm{div}, \Omega)$, necessary to define the weak form. However, they are level dependent, though the dependence is rather weak. In the rest of this section, we show that a uniform convergence rate is possible for the MDA's (Part 2) if more regularity is satisfied. To do so, let P_i, $i = 0, \ldots, J$, be projection operators from $\mathscr{H}^h$ onto $\mathscr{H}_i$ defined by

$$(P_i \mathbf{u}, \mathbf{v})_c = (\mathbf{u}, \mathbf{v})_c , \quad \text{for all} \quad \mathbf{u} \in \mathscr{H}^h, \quad \mathbf{v} \in \mathscr{H}_i . \tag{4.29}$$

Similarly, let $\tilde{P}_i$, $i = 0, \ldots, J$, be projection operators from $\mathscr{S}^h$ onto $\mathscr{S}_i$ defined by

$$(\mathbf{curl}\, \tilde{P}_i \phi, \mathbf{curl}\, \psi)_c = (\mathbf{curl}\, \phi, \mathbf{curl}\, \psi)_c , \quad \text{for all} \quad \psi \in \mathscr{S}_i , \tag{4.30}$$

where $\phi \in \mathscr{S}^h$.

Since $\mathscr{S}_i$ is the stream-function space of $\mathscr{H}_i$, we immediately obtain from (4.29) and (4.30) that

$$P_i \mathbf{u} = \mathbf{curl}\, \tilde{P}_i \sigma , \tag{4.31}$$

for $i = 0, \ldots, J$, where σ is the stream function of $\mathbf{u}$.

THEOREM 4.4 : *Assume that there exists a constant C such that*

$$\left\| \tilde{P}_i \sigma - \tilde{P}_{i-1} \sigma \right\|_{0,c} \leq C h_i \left\| \tilde{P}_i \sigma - \tilde{P}_{i-1} \sigma \right\|_{1,c} , \quad \sigma \in \mathscr{S}^h . \tag{4.32}$$

Then, there exists a constant C such that the convergence of the MDA-2 (Part 2) is bounded by

$$\gamma_4 = 1 - \frac{\omega(2-\omega)}{C} . \tag{4.33}$$

Proof : Since the proof is similar to that of Theorem 4.3, we merely outline the proof by pointing out the differences.

The inequality (4.23) is still valid. But we shall decompose $\mathbf{v}$ by (4.24) with

$$\mathbf{v}_i = P_i \mathbf{v} - P_{i-1} \mathbf{v} . \tag{4.34}$$

Thus, we have (4.25) with same I_0, I_1, and I_2. However, the term I_2 vanishes since by (4.29)

$$I_2 = \sum_{\iota=1}^{J} ((I - E_{\iota-1})\mathbf{v}, \mathbf{v}_\iota)_c$$
$$= \sum_{\iota=1}^{J} (\omega T_k E_{k-1}\mathbf{v}, \mathbf{v} - P_k \mathbf{v})_c = 0 .$$

The term I_0 can be estimated by (4.26). The estimate of I_1 is given as follows. Since, by (4.34) and (4.31), the stream function of $\mathbf{v}_\iota$ is $\sigma_\iota = \tilde{P}_\iota \sigma - \tilde{P}_{\iota-1} \sigma$, where $\sigma \in \mathcal{S}^h$ is the stream function of $\mathbf{v}$, we can use Lemma 4.3 to get

$$I_1 = \sum_{\iota=1}^{J} (E_{\iota-1}\mathbf{v}, \mathbf{v}_\iota)_c$$
$$\leq C \sum_{\iota=1}^{J} (T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v})_c^{1/2} \left(\|\mathbf{v}_\iota\|^2 + h_\iota^{-2} \|\tilde{P}_\iota \sigma - \tilde{P}_{\iota-1}\sigma\|_{0,c}^2 \right)^{1/2} .$$

Thus, it follows from (4.32) and the Schwarz inequality that

$$I_1 \leq C \sum_{\iota=1}^{J} (T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v})_c^{1/2} \|\mathbf{v}_\iota\|$$
$$\leq C \sqrt{\omega^{-1}} \left(\sum_{\iota=1}^{J} (\omega T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v})_c \right)^{1/2} \left(\sum_{\iota=1}^{J} \|\mathbf{v}_\iota\|^2 \right)^{1/2} .$$

Furthermore, we notice that $\|\mathbf{v}_\iota\|^2 = \|P_\iota \mathbf{v}\|^2 - \|P_{\iota-1}\mathbf{v}\|^2$. Thus,

$$I_1 \leq C \sqrt{\omega^{-1}} (\|\mathbf{v}\|^2 - \|P_0 \mathbf{v}\|^2)^{1/2} \left(\sum_{\iota=1}^{J} (\omega T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v})_c \right)^{1/2}$$
$$\leq C \sqrt{\omega^{-1}} \|\mathbf{v}\| \left(\sum_{\iota=1}^{J} (\omega T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v})_c \right)^{1/2} ,$$

which, along with (4.26) and (4.25), implies that

$$\|\mathbf{v}\|^2 \leq C \omega^{-1} \sum_{\iota=0}^{J} (\omega T_\iota E_{\iota-1}\mathbf{v}, E_{\iota-1}\mathbf{v}) . \tag{4.35}$$

Finally, combining (4.23) and (4.35) yields the estimate (4.33). □

5. NUMERICAL EXPERIMENTS

In this section, we present some numerical results to illustrate our theory established in this paper. For simplicity, we consider the problem (1.2) with $c = 1$. The domain $\Omega = (0,1) \times (0, 1)$ is the unit square. The homogeneous Neumann boundary condition is imposed on $\partial\Omega$. The right-hand side function is

$$f(x, y) = 2\,\pi^2 \cos \pi x \cos \pi y\,, \quad (x, y) \in \Omega\,,$$

and the exact solution is

$$p(x, y) = \cos \pi x \cos \pi y\,.$$

It is clear from the definition of the flux that $\mathbf{u} = (u_1, u_2)$ where

$$u_1 = \pi \sin \pi x \cos \pi y\,, \quad u_2 = \pi \cos \pi x \sin \pi y\,.$$

The RT space of the lowest order is used in the computation. We begin with a uniform 2×2 grid. The code is used to refine each coarse element (square) into four congruent small squares, obtaining a fine mesh with $(2^J + 1)^2$ nodal points. The number J is said to be the number of levels of the refinement. In the MDA (Part 1), the code does not quite follow the idea presented in the paper because of the use of a different decomposition (the procedure suggested in the paper is more applicable to adaptively refined meshes). More precisely, for the computation presented, the discrete flux $\mathbf{u}^*$ satisfying (3.1) is obtained according to the following decomposition for Ω. Let $\mathcal{T}_h$ be the fine mesh of Ω. Define $\mathcal{T}_0$ to be the collection of 2^J « thin » strips (see *fig.* 1). Each element (strip) has a partition inherited from $\mathcal{T}_h$. Thus, we can use the MDA (Part 1) to construct the desired discrete flux $\mathbf{u}^*$.

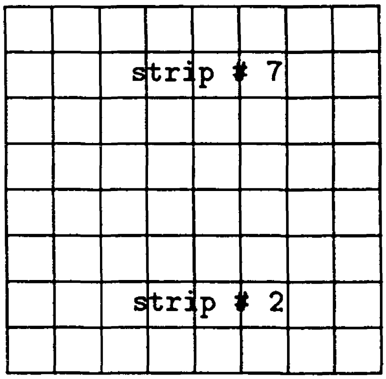

Figure 1. — Illustration of an 8 × 8 rectangular partition.

After we have $\mathbf{u}^*$, we apply the MDA (Part 2) to approximate the new flux $\hat{\mathbf{u}}^h$. We summarize the numerical result obtained by using MDA-1 (Part 2) in Tables 5.1-5.2 from which we see that the relaxation parameter ω can speed up the convergence of the algorithm. But we see no difference in the convergence estimate.

The number of accurate digits is defined by

$$\text{Digits} = -\log\left(\frac{\|\mathbf{u}_n^h - \mathbf{u}\|_0}{\|\mathbf{u}\|_0}\right),$$

where $\|\cdot\|_0$ is the standard L^2 norm and $\mathbf{u}_n^h = \hat{\mathbf{u}}_n^h + \mathbf{u}^*$ is the approximate solution of the finite element approximation $\mathbf{u}^h$.

The results in Tables 5.3-5.4 are obtained by using MDA-2 with different choices of λ_i. We found that the best choice for this number is 1.

The average rate of convergence of the MDA's is presented in Table 5.5. We emphasize that, according to our theory, the rate of convergence of the MDA's is independent of the number of levels in our computational example. This is verified by the numbers in the Table as well.

Table 5.1. — Convergence of the MDA-1 with $\omega = 1$.

Iteration	1	2	3	4	5	6	7	8	9
Digits ($J = 5$)	0.71	1.44	2.20	2.96	3.36	3.39	3.40	—	—
Digits ($J = 6$)	0.71	1.44	2.19	2.98	3.70	3.97	4.00	—	—
Digits ($J = 7$)	0.71	1.44	2.19	2.98	3.75	4.33	4.56	—	—
Digits ($J = 8$)	0.71	1.44	2.19	2.97	3.76	4.40	4.89	5.15	5.20

Table 5.2. — Convergence of the MDA-1 with $\omega = 1.2$.

Iteration	1	2	3	4	5	6	7	8	9
Digits ($J = 5$)	0.95	1.97	2.71	3.22	3.37	3.39	3.40	—	—
Digits ($J = 6$)	0.95	1.96	2.76	3.41	3.84	3.98	4.00	—	—
Digits ($J = 7$)	0.95	1.95	2.78	3.45	4.01	4.47	4.59	4.60	—
Digits ($J = 8$)	0.95	1.95	2.78	3.46	4.03	4.64	5.12	5.20	—

Table 5.3. — Convergence of the MDA-2 with $\lambda_i^{-1} = 1$.

Iteration	2	3	4	5	6	7	8	9	10
Digits ($J = 5$)	1.23	1.84	2.47	3.05	3.35	3.39	—	—	—
Digits ($J = 6$)	1.22	1.84	2.47	3.09	3.67	3.96	3.99	4.00	—
Digits ($J = 7$)	1.22	1.84	2.46	3.09	3.72	4.30	4.57	4.60	—
Digits ($J = 8$)	1.22	1.84	2.46	3.09	3.73	4.36	4.94	5.18	5.20

Table 5.4. — Convergence of the MDA-2 with $\lambda_i^{-1} = 0.9$.

Iteration	7	8	9	10	11	12	13	14	15
Digits ($J = 5$)	3.22	3.36	3.39	3.40	—	—	—	—	—
Digits ($J = 6$)	3.33	3.70	3.92	3.98	4.00	—	—	—	—
Digits ($J = 7$)	3.34	3.76	4.14	4.43	4.56	4.59	4.60	—	—
Digits ($J = 8$)	3.34	3.77	4.17	4.55	4.87	5.08	5.17	5.19	5.20

Table 5.5. — The average rate of convergence of MDA.

	$\delta(J = 5)$	$\delta(J = 6)$	$\delta(J = 7)$	$\delta(J = 8)$
MDA-1 ($\omega = 1$)	0.33	0.27	0.27	0.26
MDA-1 ($\omega = 1.2$)	0.33	0.27	0.27	0.22
MDA-2 ($\lambda_i^{-1} = 0.9$)	0.46	0.43	0.44	0.45
MDA-2 ($\lambda_i^{-1} = 1$)	0.33	0.32	0.31	0.30

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