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CONVERGENCE AND ERROR ESTIMATES IN FINITE VOLUME SCHEMES FOR GENERAL MULTIDIMENSIONAL SCALAR CONSERVATION LAWS I. EXPLICITE MONOTONE SCHEMES (*)

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Abstract. — *We study here the convergence of Finite Volume schemes of monotone type for general multidimensional conservation laws. By generalizing a previous result of Kuznetsov for Finite Difference schemes, we obtain under general assumptions error bounds in* $h^{1/4}$ *when the initial condition lies in* $BV(\mathbb{R}^d)$; *convergence follows for initial conditions in* $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Résumé. — *On étudie ici la convergence de schémas aux Volumes Finis de type monotone pour des lois de conservation multi-dimensionnelles générales. En généralisant un résultat antérieur de Kuznetsov pour des schémas aux Différences Finies, on obtient sous des hypothèses générales des majorations d'erreur en* $h^{1/4}$ *lorsque la condition initiale est dans* $BV(\mathbb{R}^d)$; *la convergence en découle pour des conditions initiales dans* $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

1. INTRODUCTION

We consider here a general nonlinear hyperbolic scalar equation, with initial condition :

$$\begin{cases} u_t(x, t) + \operatorname{div}(F(u(x, t))) = 0, \; x \in \mathbb{R}^d, \; t \in \mathbb{R}^+ , \\ u(x, 0) = u_0(x), \; x \in \mathbb{R}^d , \end{cases} \tag{1.1}$$

where F is a smooth $\mathbb{R}^d$ valued function and u is scalar.

We are interested in the numerical approximation of the entropy weak solution (in the Kruzkov [14] sense) of this problem. We consider Finite Volume explicit schemes of monotone type, defined as follows :

Let T_h be a triangulation of $\mathbb{R}^d \left(\mathbb{R}^d = \bigcup_{K \in T_h} K \right)$.

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We denote by $m(K)$ the measure in $\mathbb{R}^d$ of the cell K, by $e_i(K)$ ($i = 1, 2, \ldots, n_K$) the faces of K, by $n_{e_i, K}$ the outward unit normal to the face $e_i(K)$, by $m(e)$ the measure in $\mathbb{R}^{d-1}$ of the face e. Let S_h denote the set of the faces of the triangulation.

The family of triangulation T_h is regular in the following sense :

There exists two positive real a_- and a_+ such that :

$$\begin{aligned} &\text{(i)} \quad a_- h^d \leq m(K) \leq a_+ h^d \quad \forall K \in T_h \\ &\text{(ii)} \quad a_- h^{d-1} \leq m(e) \leq a_+ h^{d-1} \quad \forall e \in S_h . \end{aligned} \tag{1.2}$$

These conditions are classical in Finite Element analysis. They are not stringent, they just avoid local degeneracy of the mesh. We also suppose that n_K is uniformly bounded by n_0.

Let $\{t^n, n \in N\}$ be an increasing sequence of time values, we consider a Finite Volume method of monotone type. It yields a piecewise constant approximation u_h :

$$\begin{cases} u_h(x, t) = u_K^n \quad \forall (x, t) \in K \times [t^n, t^{n+1}[\\ u_K^{n+1} = u_K^n - \dfrac{k^n}{m(K)} \displaystyle\sum_{e \in \partial K} g(n_{e,K}, u_K^n, u_{K_e}^n)\, m(e) \\ u_h(x, 0) = \dfrac{1}{m(K)} \displaystyle\int_K u(x, 0)\, dx \equiv u_K^0 \quad \forall x \in K \end{cases} \tag{1.3}$$

where $k^n = t^{n+1} - t^n$ is the time step.

We study the convergence of u_h as $h \to 0$, we shall suppose that the sequence $\{k^n\}$ is such that there exists two positive functions of h, $k^+(h)$ and $k^-(h)$ satisfying :

$$0 < k^-(h) \leq k^n \leq k^+(h) \tag{1.4}$$

$g(n_{e,K}, u_K^n, u_{K_e}^n)$ (K_e is the neighbourgh of K along the face e) is the numerical flux of a 3 point monotone scheme in conservation form for the 1D scalar conservation law :

$$u_t(x, t) + \partial_x((F(u(x, t)) \cdot n_{e,K})) = 0 \tag{1.5}$$

such that :

$$\begin{aligned} &\text{(i)} \quad g(n, u, v) = -g(-n, v, u) \\ &\text{(ii)} \quad g(n, u, u) = F(u) \cdot n . \end{aligned} \tag{1.6}$$

We shall suppose that $g(n, u, v)$ is a Lipschitz continuous function of u and v :

For any $M > 0$, there exists a positive constant Q_0, which is an increasing function of M such that :

$$\forall u, v, u', v' \text{ such that } |u|, |u'|, |v|, |v'| \leqslant M, \quad \forall n \in \mathbb{R}^d$$

$$|g(n, u, v) - g(n, u', v')| \leqslant Q_0(M)(|u - u'| + |v - v'|). \quad (1.7)$$

When the triangulation is a cartesian grid, the Finite Volume method reduces to classical Finite-Difference monotone schemes. L^1 convergence has been analysed by Kuznetsov and Volosin [13] and later on by Crandall and Majda [7] for monotone schemes. L^1 error estimates in $h^{1/2}$ have also been established by Kuznetsov [12].

Recently, following Szepessy [17] in his analysis of the convergence of the stream-line diffusion Finite Element method, several authors ([5], [3], [6], [1]) get some convergence results for Finite Volume schemes by using the uniqueness result of Diperna [10] for measure valued solutions of (1.1).

Here we take a different approach. We extend results of Kuznetsov [12] to Finite Volume schemes and prove the following result :

THEOREM 1.1 : *Let $u_h(x, t)$ be an approximate solution of* (1.1) *with initial data in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, computed by the monotone Finite Volume scheme* (1.3).

We suppose that the following CFL *condition is satisfied (for some constant ε, $0 < \varepsilon < 1$):*

$$\sup_{\substack{|u|, |v| < \|u^0\|_\infty \\ \varepsilon \in S_h}} \frac{k^+(h)}{h_e^-} |Q_e(u, v)| \leqslant 1 - \varepsilon \quad (1.8)$$

If the initial data lies in $BV(\mathbb{R}^d)$ and if the triangulation is regular, then for any bounded Ω there exists two positive constants $K(u^0, T)$ and $\nu(\Omega, u^0, T)$ such that :

For any $t \leqslant T$

$$\|u(., t) - u_h(., t)\|_{L^1(\Omega)} \leqslant (1 + \sqrt{m(\Omega)})\, K(u^0, T) \times \\ \times \frac{\sqrt{h + k^+}}{(k^-)^{1/4}} + \nu(\Omega, u^0, T) \quad (1.9)$$

where $u(., t)$ is the entropy satisfying solution of (1.1) *and ν is such that :*

(i) $\lim_{\text{diam}(\Omega) \to \infty} \nu(\Omega, u^0, T)\, \text{diam}(\Omega)^{d-1} = 0$

(ii) $\nu(\Omega, u^0, T) = 0$ *if u^0 has a compact support and*

diam (Ω) *is sufficiently large .*

In particular for $(h/k^\pm) = \mathcal{O}(1)$ we get a L^1 rate of convergence in $h^{1/4}$.

If the initial data only lies in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and if the triangulation is regular, then* u_h *converges uniformly (for bounded t) in* $L^1(\mathbb{R}^d)$ *towards* u *:*

$$\lim_{h \to 0} \sup_{0 \leq t \leq T} \|u(.,t) - u_h(.,t)\|_{L^1(\mathbb{R}^d)} = 0 \tag{1.10}$$

as soon as :

$$\lim_{h \to 0} \frac{h + k^+}{\sqrt{k^-}} = 0 . \tag{1.11}$$

We refer to the Section 2.3 for the exact defintitions of Q_e the numerical viscosity by edge, and of h_e^-. The paper is organized as follows :

The second section introduces main features of monotone schemes, in particular L^1 contraction properties. We also recall an estimate involving the numerical viscosity that we have previously established in a joint paper [1] with Benharbit and Chalabi. This estimate is a crucial argument in the proof of the estimates of the last section (Proposition 4.1). The third section is devoted to an extension of Kuznetsov results on error bounds, in particular we relax his main assumption on the approximate solution, we don't need that it belongs to BV, we just require it is L^1 Lipchitz in time.

In the last section we establish a weak entropy production bound which allow us to prove Theorem 1.1 by applying the results of the previous section.

We now make some comments related to our results.

Remark 1.1 : Taking monotone scheme is not essential although it is actually used in our proof to establish L^1 lipchitz in time estimates. The hypothesis of Theorem 3.1 can be weakened in a way that allow us to extend our estimates to general E-scheme for which results of Section 2.3 and Proposition 4.1 are still valid. That will be analysed in a forecomming paper, together with implicit Finite Volume schemes and some higher order extensions.

Remark 1.2 : For general E-schemes we have established in [1] convergence (without error estimates) of Finite Volume methods under less restrictive assumptions :

— no regularity hypothesis as (1.2) are needed

Our present convergence rate estimate is significant even if $\lim_{h \to 0} h/k^- = 0$, which could be necessary in applications with stiff source terms. In particular we get convergence for any $\varepsilon > 0$ such that $k^+ = k^- = h^{2-\varepsilon}$.

Weaker restriction on the time step k^- could be obtained for some particular conservation laws such as those studied in [6] (i.e. $u_t + \operatorname{div}(V . f(u)) = 0$ with f scalar and $V \in \mathbb{R}^2$). In particular k^- could be replaced by h in the estimate (4.1) of Proposition 4.1, which allows convergence without any restriction concerning k^-.

Remark 1.3 : The convergence rate estimate in $h^{1/4}$ is clearly not sharp for L^1 norm. For implicit schemes in the linear case $h^{1/2}$ convergence rate have been obtained by Johnson and Pitkaranka [11] in their analysis of $P0$ discontinuous Galerkin method which are identical, there, to Finite Volume methods. This rate is sharp, and we could get it if we were able to get uniform BV (in space) estimate of approximate solutions. In such a situation we could easily modify the proof of Proposition 4.1 to get $(h + k^+)$ rate instead of $\frac{h + k^+}{k^-}$, which leads to a sharp convergence rate in $h^{1/2}$ for L^1 norm.

2. MONOTONE FINITE VOLUME SCHEMES

2.1. From one-dimensional flux to Finite Volume schemes

For any edge of the cell K we define a value $u_K^{n+1,e}$ in the following way :

$$u_K^{n+1,e} = u_K^n - \lambda_K^n(g(n_{e,K}, u_K^n, u_{K_e}^n) - \\ - F(u_K^n).n_{e,K}) \equiv G(\lambda_K^n, K, e, u_K^n, u_{K_e}^n) \quad (2.1)$$

where

$$\lambda_K^n = \frac{k^n\, m(\partial K)}{m(K)}. \quad (2.2)$$

Noticing that for any A in $\mathbb{R}^d$ we have :

$$\sum_{e \in \partial K} A.n_{e,K}\, m(e) = 0 .$$

A straightforward calculation show that :

$$u_K^{n+1} = \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} u_K^{n+1,e} . \quad (2.3)$$

2.2. Basic properties of monotone schemes

We suppose that the scheme (1.3) is derived from 1D monotone schemes, that means that $G(\lambda, K, e, u, v)$ is a non decreasing function of u and v. We recall the following.

PROPOSITION 2.1 : *Let* $S(\lambda, u_-, u_0, u_+) \equiv u_0 - \lambda(g(u_0, u_+) - g(u_-, u_0))$ *define a 3 points monotone scheme in conservation form such that :*

$$g(u, u) = f(u) \quad (2.4)$$

where the conservation law is

$$u_t + f(u)_x = 0 \tag{2.5}$$

then we have necessary :

$g(u, v)$ is a non decreasing (resp. *non increasing) function of* u(resp. v) (2.6)

Conversely, let us suppose that condition (2.6) *is satisfed, then we have the following inequalities :*

$\forall c \in \mathbb{R}$

$$\begin{aligned} S(\lambda, u_-, u_0, u_+) \vee c &\leq S(\lambda, u_- \vee c, u_0 \vee c, u_+ \vee c) \\ - S(\lambda, u_-, u_0, u_+) \wedge c &\leq - S(\lambda, u_- \wedge c, u_0 \wedge c, u_+ \wedge c) \end{aligned} \tag{2.7}$$

where $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$.

At most, if g is a Lipchitz function of u and v, the scheme is monotone provided we have the following CFL *condition :*

$$\lambda \max(K_u, K_v) \leq 1 \tag{2.8}$$

where K_u(resp. K_v) *is the Lipchitz constant of g for u*(resp. *for* v).

As a consequency of (2.7) in Proposition 2.1, we have some entropy inequalities :

PROPOSITION 2.2 : *Let us define $W(\lambda, u, v)$ as :*

$$W(\lambda, u, v) = u - \lambda(g(u, v) - f(u)). \tag{2.9}$$

Then :

For any $c \in \mathbb{R}$

$$|W(\lambda, u, v) - c| - |u - c| + \lambda[h_c(u, v) - H_c(u)] \leq 0 \tag{2.10}$$

where

$$H_c(u) = (f(u) - f(c)) \operatorname{sgn}(u - c) \tag{2.11}$$

and

$$h_c(u, v) = g(u \vee c, v \vee c) - g(u \wedge c, v \wedge c). \tag{2.12}$$

Practically, to get discrete entropy inequalities, it is important to remark that conditions (2.7) are valid as soon as g satisfy (2.6). In particular the CFL condition (2.8) is not necessary.

We need some definitions :

$$\begin{aligned} &\text{(i)} \quad C(u, v) = \frac{f(u) - g(u, v)}{v - u} \\ &\text{(ii)} \quad Q(u, v) = \frac{f(v) - 2\, g(u, v) + f(u)}{v - u} . \end{aligned} \tag{2.13}$$

C is the incremental coefficient, Q is the numerical viscosity, they are positive since the scheme is monotone.

In the following we assume that the monotonicity of the scheme is achieved as soon as :

$$\lambda \sup_{z \in I(u, v)} Q(u, v) \leqslant 1 \tag{2.14}$$

where $I(u, v) = \{w ; w = \theta u + (1 - \theta) v, \theta \in [0, 1]\}$.

In particular this assumption is true for Godunov scheme and for Lax-Friedrichs scheme $\left(Q^{LF} = \frac{1}{\lambda}\right)$.

Remark 2.1 : In practice we also consider a slightly different CFL condition :

$$\lambda \max_{z \in I(u, v)} |f'(z)| \leqslant 1 \tag{2.15}$$

which generally implies (2.14), we can also consider the following :

$$\lambda \max_{z, z' \in I(u, v)} |Q(z, z')| \leqslant 1 \tag{2.16}$$

which always implies (2.14).

We now give a more precise result involving the local entropy production. We have proven it in a joint paper with Benharbit and Chalabi [1]. We have established a quadratic estimate, controlled by the incremental coefficient, of the local entropy production. This result is valid for the class of E-schemes (whose numerical viscosity is greater than the numerical viscosity of the Godunov scheme) introduced by Osher [16], and it is well known (see [16]) that monotone schemes are particular E-schemes.

PROPOSITION 2.3 : *Let $g(u, v)$ be the Lipchitz numerical flux of an E-scheme, approximating* (2.5). *Provided the following* CFL *restriction :*

$$\lambda Q(u, v) \leqslant 1 - \varepsilon \tag{2.17}$$

we have :

$$|W(\lambda, u, v)|^2 - |u|^2 + \lambda [h(u, v) - H(u)] \leqslant - \frac{\varepsilon}{2} |u - v|^2 (C(u, v))^2 \lambda^2 \tag{2.18}$$

where $H(u)$ *is the entropy flux associated with the entropy* $\eta(u) = |u|^2$ *(i.e.* $H'(u) = 2\,uf'(u)$*) and* $h(u, v)$*, the associated numerical entropy flux, is a locally Lipchitz function such that :*

$$h(u, u) = H(u)\,.$$

2.3. Stability estimates

We extend definitions of the previous section :

$$\begin{aligned} &\text{(i)} \quad C(n, u, v) = \frac{F(u).n - g(n, u, v)}{v - u} \\ &\text{(ii)} \quad Q(n, u, v) = \frac{F(v).n - 2\,g(n, u, v) + F(u).n}{v - u}\,. \end{aligned} \tag{2.19}$$

C is the incremental coefficient, Q is the numerical viscosity, they are positive since the one dimensional scheme (2.1) is a monotone scheme.

As a consequency of (1.6) (i) we have :

$$Q(n, u, v) = Q(-n, v, u) = C(n, u, v) + C(-n, v, u)\,.$$

We shall use some notations :

$$\begin{aligned} &\text{(i)} \quad C_{e,K}^n = C(n_{e,K}, u_K^n, u_{K_e}^n) \\ &\text{(ii)} \quad Q_e^n = Q(n_{e,K}, u_K^n, u_{K_e}^n) = Q_e(u_K^n, u_{K_e}^n) \\ &\text{(iii)} \quad \Delta_{e,K}^n(u) = u_{K_e}^n - u_K^n \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\text{(i)} \quad h_e^- = \min\left\{\frac{m(K)}{m(\partial K)}, \frac{m(K_e)}{m(\partial K_e)}\right\} \\ &\text{(ii)} \quad h_e^+ = \max\left\{\frac{m(K)}{m(\partial K)}, \frac{m(K_e)}{m(\partial K_e)}\right\}. \end{aligned} \tag{2.21}$$

In view of analysing the convergence of the scheme we first establish two results :

PROPOSITION 2.4 : *Let* $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and* $u_h(x, t)$ *be the function defined by the scheme* (1.3) *with monotone fluxes. We suppose that the following* CFL *condition is satisfed :*

$$\sup_{\substack{|u|,\,|v| < \|u^0\|_\infty \\ \varepsilon \in S_h}} \frac{k^+(h)}{h_e^-}\,|Q_e(u, v)| \leqslant 1 \tag{2.22}$$

then :

$$\min_{K' \in N(K)} \{u_{K'}^n\} \leq u_K^{n+1} \leq \max_{K' \in N(K)} \{u_{K'}^n\} \tag{2.23}$$

where

$$N(K) = \{K\} \cup \{K_e, e \in \partial K\} .$$

For all $t \geq 0$ $u_h(., t) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$$\|u_h(., t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} . \tag{2.24}$$

Let $v_h(x, t)$ *a function defined with the same scheme* (1.3), *then*

$$\{v_K^n \leq u_K^n, \forall K \in T_h\} \Rightarrow \{v_K^{n+1} \leq u_K^{n+1}, \forall K \in T_h\} . \tag{2.25}$$

Remark 2.2 : By using the locally Lipchitz continuity of $g(n, u, v)$ (see (1.7)), we could replace the C.F.L. condition (2.22) by :

$$2\, Q_0(\|u^0\|_\infty) \sup_{e \in S_h} \frac{k^+(h)}{h_e^-} \leq 1 .$$

As a consequency of Proposition 2.4 and Crandall-Tartar lemma (see [7] for its proof) we have :

PROPOSITION 2.5 : *We suppose that the* CFL *condition* (2.22) *is satisfied, then the scheme* (1.3) *is* L_1 *contracting :*

For any u^0 *and* $v^0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, *for any* t *and* $\tau \geq 0$

$$\|u_h(., t+\tau) - v_h(., t+\tau)\|_{L^1(\mathbb{R}^d)} \leq \|u_h(., t) - v_h(., t)\|_{L^1(\mathbb{R}^d)} . \tag{2.26}$$

Moreover if $u^0 \in BV(\mathbb{R}^d)$, *and* $T > 0$, *then there exists a positive constant* K_1 *depending on* a_-, a_+, d *such that :*

For any t, $\tau \geq 0$ *with* $t + \tau \leq T$

$$\|u_h(., t+\tau) - u_h(., t)\|_{L^1(\mathbb{R}^d)} \leq$$
$$\leq \max(\tau, k^+(h))\, Q_0(\|u^0\|_\infty)\, K_1(a_-, a_+, d) \|u^0\|_{BV(\mathbb{R}^d)} \tag{2.27}$$

where the constant $Q_0(\|u^0\|_\infty)$ *is defined in* (1.7).

To establish Proposition 2.5 we need :

LEMMA 2.1 : (Crandall-Tartar) *Let* Ω *a measure space and* $C \subset L^1(\Omega)$ *have the property that* $f \vee g \in \mathscr{C}$ *whenever* $f, g \in \mathscr{C}$. *Let* $T : \to L^1(\Omega)$ *satisfy*

$$\int_\Omega T(f) = \int_\Omega f \quad for \quad f \in \mathscr{C} . \tag{2.28}$$

Then the following three properties of T are equivalent :

$$\begin{aligned}
&\text{(a)} \quad f, g \in \mathscr{C} \text{ and } f \leqslant g \text{ a.e. implies } T(f) \leqslant T(g) \text{ a.e.} \\
&\text{(b)} \quad \int_{\Omega} (T(f) - T(g))^{+} \leqslant \int_{\Omega} (f - g)^{+} \text{ for } f, g \in \mathscr{C} \qquad (2.29)\\
&\text{(c)} \quad \int_{\Omega} |T(f) - T(g)| \leqslant \int_{\Omega} |f - g| \text{ for } f, g \in \mathscr{C} .
\end{aligned}$$

Proof of Proposition 2.4 : Inequalities (2.23) are an easy consequence of monotonicity. By (2.3), we get :

$$u_K^{n+1} = \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} G(\lambda_K^n, K, e, u_K^n, u_{K_e}^n) \equiv \mathscr{G}_{K^n}(u_{K_1}^n, \ldots, u_{K_{\operatorname{card}(N(K))}}^n) . \qquad (2.30)$$

Since $G(\lambda, K, e, u, v)$ is a non decreasing function of u and v, therefore $\mathscr{G}$ is a non decreasing function of $u_{K_i}^n$ $(K_i \in N(K))$. Inequalities (2.23) and (2.25) follow easily for monotone schemes.

It remains to prove the L^1 stability estimate.

By repeated use of the entropy discrete inequality (2.10) for $c = 0$, we get :

$$|u_K^{n+1}| \, m(K) \leqslant |u_K^n| \, m(K) + k^n \sum_{e \in \partial K} m(e) \, h_0(n_{e,K}, u_K^n, u_{K_e}^n)$$

where the entropy numerical flux satisfies :

$$h_0(n, u, v) = - h_0(-n, v, u) .$$

Let $B(R)$ be the centered ball of radius R. By summing the previous inequalities over $K \subset B(R)$ we get :

$$\sum_{\substack{K \in T_h \\ K \subset B(R)}} |u_K^{n+1}| \, m(K) \leqslant \sum_{\substack{K \in T_h \\ K \subset B(R)}} |u_K^n| \, m(K) + k^n \sum_{e \in \Gamma_{h,R}} m(e) \, h_0(n_{e,K}, u_K^n, u_{K_e}^n)$$

where $\Gamma_{h,R} =$ is a closed hypersurface of $\mathbb{R}^d$ in the neighbourhood of $\partial B(R)$ (at a distance less than Ch).

As R tends to infinity the first terms converges towards $\|u_h(.,\, t^{n+1})\|_{L^2(\mathbb{R}^d)}$, the second one converges towards $\|u_h(.,\, t^n)\|_{L^1(\mathbb{R}^d)}$.

To estimate the third term, we remark that :

$$\sum_{e \in \Gamma_{h,R}} m(e) \, h_0(n_{e,K}, u_K^n, u_{K_e}^n) =$$

$$= \sum_{e \in \Gamma_{h,R}} m(e) \, (h_0(n_{e,K}, u_K^n, u_{K_e}^n) - H_0(0).n_{e,K}) + \sum_{e \in \Gamma_{h,R}} m(e) \, H_0(0).n_{e,K} .$$

The last term in the r.h.s. of this identity is equal to :

$$\int_{\Gamma_{h,R}} H_0(0).n = 0 .$$

Thanks to the Lipchitz continuity of h_0 and H_0, we have :

$$\left| h_0(n_{e,K}, u_K^n, u_{K_e}^n) - H_0(0).n_{e,K} \right| \leq C(\|u_0\|_\infty)\left(\left|u_K^n - u_{K_e}^n\right| + \left|u_K^n\right|\right).$$

Thus, the first term in the r.h.s. of the identity is bounded by

$$k^n C'(\|u_0\|_\infty) \sum_{e \in \Gamma_{h,R}} m(e)\left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right).$$

This terms tends towards 0 as R goes to infinity if $u_h(.,t^n) \in L^1(\mathbb{R}^d)$. Consequently, $u_h(.,t^{n+1}) \in L^1(\mathbb{R}^d)$ and

$$\|u_h(.,t^{n+1})\|_{L^1(\mathbb{R}^d)} \leq \|u_h(.,t^n)\|_{L^1(\mathbb{R}^d)}$$

(2.24) follows immediately. □

Proof of Proposition 2.5 : Let $\{k^n, n = 0, ..., N-1\}$ be a sequel such that the CFL condition (2.22) is satisfied for u_0 and v_0. We also define the set $\mathcal{M}$ as :

$$\mathcal{M} = \left\{ u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) ;\ \forall K \in T_h\ u|_K = \text{Const.} \right\} .$$

To establish (2.26), we apply Lemma 2.1 with $T = \mathcal{G}_{k^n}$ and $\mathcal{C} = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Thanks to Proposition 2.4, (2.29) (a) is true, thus taking $f = u_h(.,t^n)$ and $g = v_h(.,t^n)$, (2.29) (c) is also true. That means :

$$\|u_h(.,t^{n+1}) - v_h(.,t^{n+1})\|_{L^1(\mathbb{R}^d)} \leq \|u_h(.,t^n) - v_h(.,t^n)\|_{L^1(\mathbb{R}^d)} .$$

The inequality (2.26) directly follows.

To get (2.27), we apply Lemma 2.1 with $T = \mathcal{G}_{k^{n-1}}$ and $\mathcal{C} = \mathcal{M}$. As previously, (2.29) (c) is true, we take there $f = u_h(.,t^n)$ and $g = u_h(.,t^{n-1})$, thus we get :

$$\|\mathcal{G}_{k^{n-1}}(u_h(.,t^n)) - \mathcal{G}_{k^{n-1}}(u_h(.,t^{n-1}))\|_{L^1(\mathbb{R}^d)} \leq$$
$$\leq \|u_h(.,t^n) - u_h(.,t^{n-1})\|_{L^1(\mathbb{R}^d)} . \quad (2.31)$$

We have :

$$u_h(.,t^n) = \mathcal{G}_{k^{n-1}}(u_h(.,t^{n-1})) .$$

An easy calculation prove that :

$$\mathcal{G}_{k^{n-1}}(u_h(.,t^n)) - u_h(.,t^n) = \frac{k^{n-1}}{k^n}(u_h(.,t^{n+1}) - u_h(.,t^n)) . \quad (2.32)$$

Taking account of (2.32) into (2.31), it follows that :

$$\left\|u_h(.,t^{n+1})-u_h(.,t^n)\right\|_{L^1(\mathbb{R}^d)} \leqslant$$

$$\leqslant \frac{k^n}{k^{n-1}}\left\|u_h(.,t^n)-u_h(.,t^{n-1})\right\|_{L^1(\mathbb{R}^d)}. \quad (2.33)$$

Applying (2.33) for $p = 1, \ldots, n$, we get :

$$\left\|u_h(.,t^{n+1})-u_h(.,t^n)\right\|_{L^1(\mathbb{R}^d)} \leqslant \frac{k^n}{k^0}\left\|u_h(.,t^1)-u_h(.,t^0)\right\|_{L^1(\mathbb{R}^d)}.$$

Let us suppose now that $u_h^0 \in BV(\mathbb{R}^d)$, considering that :

$$\left\|u_h(.,t^1)-u_h(.,t^0)\right\|_{L^1(\mathbb{R}^d)} = k^0 \sum_K \left|\sum_{e\in\partial K} m(e)\, C^0_{e,K}\, \Delta^0_e u\right| \leqslant$$

$$\leqslant k^0 \sum_{e\in S_h} m(e)\, Q^0_e \left|\Delta^0_e u\right|$$

$$\leqslant k^0 \sup_{\{e\in S_h\}}\{Q^0_e\} \sum_{e\in S_h} m(e)\left|\Delta^0_e u\right| \leqslant k^0 \sup_{\{e\in S_h\}}\{Q^0_e\}\left\|u_h^0\right\|_{BV(\mathbb{R}^d)}$$

we obtain :

$$\left\|u_h(.,t^{n+1})-u_h(.,t^n)\right\|_{L^1(\mathbb{R}^d)} \leqslant k^n \sup_{\{e\in S_h\}}\{Q^0_e\}\left\|u_h^0\right\|_{BV(\mathbb{R}^d)}.$$

The CFL condition (2.22) insures that :

$$\left\|u_h(\,.\,,t)\right\|_{L^\infty(\mathbb{R}^d)} \leqslant \left\|u_h^0\right\|_\infty$$

thus, taking account of (1.7), we get :

$$\sup_{\{e\in S_h\}}\{Q^0_e\} \leqslant 2\,Q_0(\left\|u_h^0\right\|_\infty). \quad (2.34)$$

It follows that :

$$\left\|u_h(.,t^{n+1})-u_h(\cdot\,,t^n)\right\|_{L^1(\mathbb{R}^d)} \leqslant 2\,k^n\,Q_0(\left\|u_h^0\right\|_\infty)\left\|u_h^0\right\|_{BV(\mathbb{R}^d)}. \quad (2.35)$$

Since $u_h^0 \in L^1(\mathbb{R}^d)$, $\forall \nu > 0$ there exists $R(\nu) > 0$, such that :

$$\left\|\tilde{u}_h^0 - u_h^0\right\|_{L^1(\mathbb{R}^d)} \leqslant \nu$$

where

$$\tilde{u}^0 = \begin{cases} u^0 \text{ if } x \in B(R(\nu)) \\ 0 \text{ if } x \in B(R(\nu))\,. \end{cases}$$

Thanks to the results of Cockburn [8] on the continuity of the

L^2 projection on a bounded domain, we have :

$$\|\tilde{u}_h^0\|_{BV(B(R(\nu)))} \leq K_2(a_-, a_+, d)\ \|\tilde{u}^0\|_{BV(B(R(\nu)))} = \\ = K_2 \|u^0\|_{BV(B(R(\nu)))} \leq K_2 \|u^0\|_{BV(\mathbb{R}^d)}$$

where K_2 does not depends on $R(\nu)$.

Taking account of (2.35) and (2.34) we finally obtain (with obvious notations) :

$$\begin{aligned}\|u_h^{n+1} - u_h^n\|_{L^1(\mathbb{R}^d)} &\leq \|u_h^{n+1} - \tilde{u}_h^{n+1}\|_{L^1(\mathbb{R}^d)} + \\ &\quad + \|\tilde{u}_h^{n+1} - \tilde{u}_h^n\|_{L^1(\mathbb{R}^d)} + \|u_h^n - \tilde{u}_h^n\|_{L^1(\mathbb{R}^d)} \\ &\leq 2\|u_h^0 - \tilde{u}_h^0\|_{L^1(\mathbb{R}^d)} + \|\tilde{u}_h^{n+1} - \tilde{u}_h^n\|_{L^1(\mathbb{R}^d)} \\ &\leq 2\|u_h^0 - \tilde{u}_h^0\|_{L^1(\mathbb{R}^d)} + \|\tilde{u}_h^{n+1} - \tilde{u}_h^n\|_{L^1(\mathbb{R}^d)} \\ &\leq 2\ \nu + 2\ k^n\ Q_0(\|\tilde{u}_h^0\|_\infty)\|\tilde{u}_h^0\|_{BV(\mathbb{R}^d)} \leq 2\ k^n\ Q_0(\|u_h^0\|_\infty)\|u_h^0\|_{BV(\mathbb{R}^d)}\ .\end{aligned}$$

Letting ν tends to zero, we get easily that :

For $p > 0$, any integer

$$\|u_h^{n+p} - u_h^n\|_{L^1(\mathbb{R}^d)} \leq (t^{n+p} - t^n)\, K_3(a_-, a_+, d)\, \frac{h}{k^0}$$

which leads to the expected result. □

In [2] we have used Proposition 2.3 to establish a weak BV estimate and prove convergence of general E-scheme, we recall this result and the sketch of its proof, since it will be used as a key ingredient in the proof of Proposition 4.1.

PROPOSITION 2.6 : *Let us suppose that $u^0 \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and that we have the CFL condition :*

$$\sup_{\substack{|u|,\ |v| \leq \|u^0\|_\infty \\ e \in S_h}} \frac{k^+(h)}{h_e^-}\,|Q_e(u, v)| \leq 1 - \varepsilon \tag{2.36}$$

then, for any $T > 0$:

$$\sum_{e \in S_h,\ \{n\,;\,t^n \leq T\}} m(e)\, k^n\, \lambda_e^n(\tilde{Q}_e^n)^2\, |\Delta_{e,K}^n(u)|^2 \leq \frac{4}{\varepsilon}\, \|u_h^0\|^2_{L^2(\mathbb{R}^d)} \tag{2.37}$$

where :

$$\lambda_e^n = \frac{k^n}{h_e^+} \tag{2.38}$$

Proof of Proposition 2.6 : Thanks to the local entropy production estimate (2.18) and to the following convexity inequality :

$$|u_K^{n+1}|^2 \leqslant \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} |u_K^{n+1,e}|^2$$

we get :

$$|u_K^{n+1}|^2 - |u_K^n|^2 + \frac{k^n}{m(K)} \sum_{e \in \partial K} m(e)\, h(n_{e,K}, u_K^n, u_{K_e}^n) \leqslant$$

$$\leqslant -\frac{\varepsilon}{2} \sum_{e \in \partial K} |\Delta_{e,K}^n(u)|^2 (C_{e,K}^n)^2 \lambda_{K^2}^n \frac{m(e)}{m(\partial K)},$$

multiplying the previous inequality by $m(K)$ and summing over $K \in T_h \cap \Omega$ (where Ω is a bounded open set of $\mathbb{R}^d$), we get :

$$\sum_{K \in T_h \cap \Omega} m(K)|u_K^{n+1}|^2 + \frac{\varepsilon}{2} \sum_{K \in T_h \cap \Omega} k^n \sum_{e \in \partial K} |\Delta_{e,K}^n(u)|^2 (C_{e,K}^n)^2 \lambda_K^n m(e) +$$

$$+ k^n \sum_{e \in \partial \Omega} m(e)\, h(n_{e,K}, u_K^n, u_{K_e}^n) \leqslant \sum_{K \in T_h \cap \Omega} m(K)\, |u_K^n|^2$$

where $K(e)$ is the polyhedron containing e, when e is on the boundary $\partial \Omega$.

Noticing that $\lambda_K^n (C_{e,K}^n)^2 + \lambda_{K_e}^n (C_{e,K_e}^n)^2 \geqslant \frac{1}{2} \lambda_e^n (Q_e^n)^2$, we obtain :

$$\sum_{K \in T_h \cap \Omega} m(K)|u_K^{n+1}|^2 + \frac{\varepsilon}{4} \sum_{e \in S_h \cap \Omega} k^n m(e)\, |\Delta_{e,K}^n(u)|^2 (Q_e^n)^2 \lambda_e^n +$$

$$+ k^n \sum_{e \in \partial \Omega} m(e)\, h(n_{e,K}, u_K^n, u_{K_e}^n) \leqslant \sum_{K \in T_h \cap \Omega} m(K)\, |u_K^n|^2 .$$

Summing up these inequalities over n such that $t^n \leqslant T$, we get :

$$\sum_{K \in T_h \cap \Omega} m(K)|u_K^N|^2 + \frac{\varepsilon}{4} \sum_{e \in S_h \cap \Omega,\ \{n ; t^n \leqslant T\}} k^n m(e)\, |\Delta_{e,K}^n(u)|^2 (\tilde{Q}_e^n)^2 \lambda_e^n +$$

$$+ \sum_{e \in \partial \Omega,\ \{n ; t^n \leqslant T\}} k^n m(e)\, h(n_{e,K}, u_K^n, u_{K_e}^n) \leqslant \sum_{K \in T_h \cap \Omega} m(K)\, |u_K^0|^2 .$$

Making Ω tends to $\mathbb{R}^d$, the estimate (2.37) follows easily by considering that $h(n, u, v)$ is a Lipschitz function of u and v, and that u_h^n is uniformly bounded in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

3. ERROR BOUNDS FOR A FAMILY OF APPROXIMATE SOLUTIONS

Following Kutznetzov in his study of finite difference schemes we shall establish :

THEOREM 3.1 : *Let u_h a family of approximate solution of (1.1) over $\mathbb{R}^d \times [0, T]$ such that there exists some positive constants C_1, C_2, C_3, and two functions $\varphi_1(\tau, h)$ and $\varphi_2(h)$ which satisfy :*

$$\|u_h\|_{L^1(\mathbb{R}^d)} \leq C_1 \tag{3.1}$$

$$\|u_h(., t+\tau) - u_h(., t)\|_{L^1(\mathbb{R}^d)} \leq \varphi_1(\tau, h)\, C_2 \tag{3.2}$$

For any bounded set Ω, for any function $g(y, s)$ in $L^\infty(t_0, t_1; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$, for any $\omega(x, t)$ symmetric (i.e. $\omega(x, t) = \omega(-x, t) = \omega(x, -t)$) function in $\mathscr{C}_0^\infty(\mathbb{R}^d \times \mathbb{R})$, there exists two positive constants $\mathscr{C}_3(t_0, t_1)$ and $\nu(\Omega, u^0)$ such that :

$$\begin{aligned} -\int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} &\{|u_h(x,t) - g(y,s)|\ \omega_t(\xi, \chi) + \\ &+ (H_{g(y,s)}(u_h(x,t)) - H_{g(y,s)}(0)) \,.\, grad_x(\omega(\xi, \chi))\}\, dx\, dy\, dt\, ds \\ &- \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, t_0) - g(y,s)|\ \omega(\xi, t_0 - s)\, dx\, dy\, ds \\ &+ \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, t_1) - g(y,s)|\ \omega(\xi, t_1 - s)\, dx\, dy\, ds \\ &\leq C_3(t_0, t_1) \sqrt{m(\Omega)}\, \varphi_2(h)\, \|\omega\|_{1,1} + \nu(\Omega, u^0)\, \|\omega\|_1 \end{aligned} \tag{3.3}$$

where $\xi = x - y$ and $\chi = t - s$.

Then if $u^0 \in BV(\mathbb{R}^d)$ we have the following error bound :

For any $t_0 \leq t_1 \leq T$, there exists a positive constant $K(u^0, T)$ such that :

For any $\varepsilon > 0$

$$\begin{aligned} \|u(., t_1) - u_h(., t_1)\|_{L^1(\Omega)} &\leq \|u(., t_0) - u_h(., t_0)\|_{L^1(\Omega)} + \\ &+ K(u^0, T)(1 + \sqrt{m(\Omega)})\left(\varepsilon + \varphi_1(\varepsilon, h) + \frac{\varphi_2(h)}{\varepsilon}\right) + \nu(\Omega, u^0)\,. \end{aligned} \tag{3.4}$$

where u is the entropy solution of (1.1).

Proof of Theorem 3.1 : Let u the entropy solution of (1.1). For any function $\phi(y, s) \in \mathscr{C}_0^\infty(\mathbb{R}^d \times \mathbb{R})$, for any $l \in \mathbb{R}$ we have Kruzkov entropy inequalities :

$$- \int_{\mathbb{R}^d \times [t_0, t_1]} \{ |u(y, s) - l| \, \phi_s(y, s) + H_l(u(y, s)) \cdot \operatorname{grad}_y(\phi(y, s)) \} \, dy \, ds$$

$$\leqslant + \int_{\mathbb{R}^d} \{ |u(y, t_0) - l| \, \phi(y, t_0) - |u(y, t_1) - l| \, \phi(y, t_1) \} \, dy . \quad (3.5)$$

Taking l and ϕ depending on (x, t), such that :

$$l(x, t) = u_h(x, t) \quad \text{and} \quad \phi(y, s, x, t) = \omega(\xi, \chi)$$

with ω symmetric $\in \mathscr{C}_0^\infty(\mathbb{R}^d \times \mathbb{R})$, we integrate (3.5) over Ω.

Taking account of $\phi_x + \phi_y = 0 = \phi_t + \phi_s$ we get :

$$- \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} \{ |u_h(x, t) - u(y, s)| \, \omega_t(\xi, \chi) +$$

$$+ H_{u_h(x, t)}(u(y, s)) \cdot \operatorname{grad}_x(\omega(\xi, \chi)) \} \, dx \, dy \, dt \, ds$$

$$\leqslant \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} \{ |u_h(x, t) - u(y, t_0)| \, \omega(\xi, t_0 - t)$$

$$- |u_h(x, t) - u(y, t_1)| \, \omega(\xi, t_1 - t) \} \, dx \, dy \, dt . \quad (3.6)$$

Let us define $J(u, u_h, \tau, \omega)$ as :

$$J(u, u_h, \tau, \omega) = \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, t) - u(y, \tau)| \, \omega(\xi, \tau - t) \, dx \, dy \, dt +$$

$$+ \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, \tau) - u(y, s)| \, \omega(\xi, \tau - s) \, dx \, dy \, ds \quad (3.7)$$

and $L(u, \omega)$ as :

$$L(u, \omega) = - \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} H_{u(y, s)}(0) \cdot \operatorname{grad}_x(\omega(\xi, \chi)) \, dx \, dy \, dt \, ds . \quad (3.8)$$

Making appear the l.h.s. of (3.3) (we take $g \equiv u$) in (3.6), we obtain :

$$J(u, u_h, t_1, \omega) \leqslant J(u, u_h, t_0, \omega) + L(u, \omega) +$$

$$+ C_3(t_0, t_1) \sqrt{m(\Omega)} \, \varphi_2(h) \, \|\omega\|_{1, 1} + \nu(\Omega, u^0) \, \|\omega\|_1 . \quad (3.9)$$

Now to get the expected result it is convenient to choose ω as follows. Let ζ (resp. θ) be a cut-off function in $\mathbb{R}^d$ (resp. $\mathbb{R}$). We assume that $\zeta \in \mathscr{C}_0^\infty(\mathbb{R}^d)$

(resp. $\theta \in \mathscr{C}_0^\infty(\mathbb{R})$),

$$\zeta(x) = \zeta(-x), \quad \text{support } \zeta \subset [-1, 1]^d, \quad \int_{\mathbb{R}^d} \zeta(x)\, dx = 1$$

$$\theta(t) = \theta(-t), \quad \text{support } \theta \subset [-1, 1], \quad \int_{\mathbb{R}} \theta(t)\, dt = 1 .$$

Set

$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d} \zeta\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \theta_\varepsilon(t) = \frac{1}{\varepsilon} \theta\left(\frac{t}{\varepsilon}\right)$$

where $\varepsilon > 0$ will be fixed later on and

$$\omega(x, t) = \zeta_\varepsilon(x)\, \theta_\varepsilon(t), \tag{3.10}$$

thus

$$\|\omega\|_{1,1} \leqslant \frac{C}{\varepsilon} \|\omega\|_1 = 1 . \tag{3.11}$$

Now we are equipped to estimate $J(u, u_h, t_i, \omega)$ $(i = 0, 1)$. With the particular choice (3.10) of ω get :

$$J(u, u_h, t_i, \omega) = \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times$$

$$\times \left\{ \iint_{\Omega \times \mathbb{R}^d} |u_h(x, t) - u(y, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt$$

$$+ \int_{t_0}^{t_1} \theta_\varepsilon(t_i - s) \left\{ \iint_{\Omega \times \mathbb{R}^d} |u_h(x, t_i) - u(y, s)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} ds$$

$$= \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \left\{ \iint_{\Omega \times \mathbb{R}^d} \{ |u_h(x, t) - u(y, t_i)| \right.$$

$$\left. + |u_h(x, t_i) - u(y, t)| \}\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt .$$

Now, by repeated use of the triangle inequality we get :

$$|u_h(x, t) - u(y, t_0)| + |u_h(x, t_0) - u(y, s)| \leqslant$$

$$\leqslant 2 |u(x, t_0) - u_h(x, t_0)| + 2 |u(x, t_0) - u(y, t_0)|$$

$$+ |u_h(x, t) - u_h(x, t_0)| + |u(y, t_0) - u(y, t)|$$

and

$$|u_h(x, t) - u(y, t_1)| + |u_h(x, t_1) - u(y, s)| \geqslant$$

$$\geqslant 2 |u(x, t_1) - u_h(x, t_1)| - 2 |u(x, t_1) - u(y, t_1)|$$

$$- |u_h(x, t) - u_h(x, t_1)| - |u(y, t_1) - u(y, t)| .$$

Thus, we obtain the following estimates for $J(u, u_h, t_i, \omega)$ $(i = 0, 1)$:

$$\text{(i)}\quad J(u, u_h, t_0, \omega) \leq 2\, J_1(u, u_h, t_0, \omega) + 2\, J_2(u, t_0, \omega) + \\ + J_3(u_h, t_0, \omega) - J_4(u, t_0, \omega)$$

$$\text{(ii)}\quad J(u, u_h, t_1, \omega) \geq 2\, J_1(u, u_h, t_1, \omega) - 2\, J_2(u, y_1, \omega) - \\ - J_3(u_h, t_1, \omega) - J_4(u, t_1, \omega) \tag{3.12}$$

with :

$$J_1(u, u_h, t_i, \omega) = \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times \\ \times \left\{ \iint_{\Omega \times \mathbb{R}^d} |u(x, t_i) - u_h(x, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt \tag{3.13}$$

$$J_2(u, t_i, \omega) = \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times \\ \times \left\{ \iint_{\Omega \times \mathbb{R}^d} |u(x, t_i) - u(y, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt \tag{3.14}$$

$$J_3(u_h, t_i, \omega) = \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times \\ \times \left\{ \iint_{\Omega \times \mathbb{R}^d} |u_h(x, t) - u_h(x, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt \tag{3.15}$$

$$J_4(u, t_i, \omega) = \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times \\ \times \left\{ \iint_{\Omega \times \mathbb{R}^d} |u(y, t_i) - u(y, t)|\ \zeta_\varepsilon(\xi)\, dx\, dy \right\} dt\,. \tag{3.16}$$

It remains to evaluate the corresponding integrals.

The first one :

$$J_1(u, u_h, t_i, \omega) = \left(\int_{t_0}^{t_1} \theta_\varepsilon(t - t_i)\, dt \right) \times \\ \times \left[\int_\Omega |u(x, t_i) - u_h(x, t_i)| \left(\int_{\mathbb{R}^d} \zeta_\varepsilon(\xi)\, dy \right) dx \right] \\ = \int_\Omega |u(x, t_i) - u_h(x, t_i)|\, dx = \|u(., t_i) - u_h(., t_i)\|_{L^1(\Omega)}\,.$$

Thus, we have established :

$$J_1(u, u_h, t_i, \omega) = \|u(., t_i) - u_h(., t_i)\|_{L^1(\Omega)}. \tag{3.17}$$

The second one :

$$J_2(u, t_i, \omega) = \left(\int_{t_0}^{t_1} \theta_\varepsilon(t - t_i)\, dt\right) \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \times$$
$$\times \left\{\iint_{\Omega \times \mathbb{R}^d} |u(x, t_i) - u(y, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy\right\}$$
$$= \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \left\{\iint_{\Omega \times \mathbb{R}^d} |u(x, t_i) - u(y, t_i)|\ \zeta_\varepsilon(\xi)\, dx\, dy\right\}.$$

We recall that support $\zeta_\varepsilon \subset [-\varepsilon, \varepsilon]^d$ and that

$$\int_{\mathbb{R}^d} \zeta_\varepsilon(x)\, dx = 1 .$$

Hence

$$J_2(u, t_i, \omega) \leqslant \sup_{\|z\| \leqslant \varepsilon} \left\{\int_{\mathbb{R}^d} |u(y + z, t_i) - u(y, t_i)|\ dy\right\} \int_\Omega \zeta_\varepsilon(z)\, dz$$
$$\leqslant \sup_{\|z\| \leqslant \varepsilon} \left\{\int_{\mathbb{R}^d} |u(y + z, t_i) - u(y, t_i)|\ dy\right\}$$
$$\leqslant \varepsilon \|u(., t_i)\|_{BV(\mathbb{R}^d)} .$$

Thus,

$$J_2(u, t_i, \omega) \leqslant \varepsilon \|u(., t_i)\|_{BV(\mathbb{R}^d)} . \tag{3.18}$$

The third one :

$$J_3(u_h, t_i, \omega) =$$
$$= \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \left\{\int_\Omega |u_h(x, t) - u_h(x, t_i)| \left(\int_{\mathbb{R}^d} \zeta_\varepsilon(\xi)\, dy\right) dx\right\} dt$$
$$= \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i) \left\{\int_\Omega |u_h(x, t) - u_h(x, t_i)|\ dx\right\} dt .$$

We recall that support $\theta_\varepsilon \subset [-\varepsilon, \varepsilon]$ and that

$$\int_{\mathbb{R}} \theta_\varepsilon(t)\, dt = 1 .$$

Thus,

$$J_3(u_h, t_i, \omega) \leq \int_{t_0}^{t_1} \theta_\varepsilon(t - t_i)\, dt \times$$

$$\times \sup_{0 \leq \tau \leq \varepsilon} \left\{ \int_\Omega |u_h(x, t_i - \tau) - u_h(x, t_i)|\, dx \right\}$$

$$\leq \frac{1}{2} \sup_{0 \leq \tau \leq \varepsilon} \left\{ \int_\Omega |u_h(x, t_i - \tau) - u_h(x, t_i)|\, dx \right\}.$$

Taking account of (3.2) we finally get :

$$J_3(u_h, t_i, \omega) \leq C_2\, \varphi_1(\varepsilon, h). \tag{3.19}$$

The fourth one : The exact solution also satisfy estimates of Proposition 2.5 (Lipchitz L_1 continuity in time), thus we get similarly :

$$J_4(u, t_i, \omega) \leq \varepsilon K_1(t_0, t_1, u^0). \tag{3.20}$$

We also need to evaluate $L(u, \omega)$. Since we have :

$$H_u(0) = \operatorname{sgn}(u)\, (F(u) - F(0)) \equiv \Psi(u) \tag{3.21}$$

$H_u(0)$ is a Lipchitz function of u.

Thanks to the symmetry of ω we get easily that

$$L(u, \omega) = - \int_{\partial\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} \frac{1}{2} \{\Psi(u(x(\sigma) - z, s)) -$$

$$- \Psi(u(x(\sigma) + z, s))\}\, \omega(z, t - s)\, dx(\sigma)\, dz\, dt\, ds.$$

Hence, we have

$$|L(u, \omega)| \leq K(\|u\|_\infty) \int_{\partial\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} \times$$

$$\times |u(x(\sigma) - z, s) - u(x(\sigma) + z, s)|\, \omega(z, t - s)\, dx(\sigma)\, dz\, dt\, ds.$$

By using similar arguments as in the estimate of J_2 we establish that :

$$|L(u, \omega)| \leq \varepsilon K(\|u\|_\infty) \|u(., 0)\|_{BV(\mathbb{R}^d)}. \tag{3.22}$$

Now, combining together (3.9) and (3.12) with estimates (3.17), ..., (3.20) and (3.22) we get :

$$\|u(., t_1) - u_h(., t_1)\|_{L^1(\Omega)} \leq \|u(., t_0) - u_h(., t_0)\|_{L^1(\Omega)} +$$

$$+ C(u^0, T) \left(\varepsilon + \varphi_1(\varepsilon, h) + \frac{\varphi_2(h)}{\varepsilon} \sqrt{m(\Omega)} \right) + \nu(\Omega, u^0).$$

This completes the proof. □

4. CONVERGENCE AND ERROR BOUNDS FOR MONOTONE SCHEMES

4.1. A weak bound for entropy production

PROPOSITION 4.1 : *Let $g(y, s)$ any function in $L^{\infty}(t_0, t_1; L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d))$, $\omega(x, t)$ any symmetric (i.e. $\omega(x, t) = \omega(-x, t) = \omega(x, -t)$) function in $\mathscr{C}_0^{\infty}(\mathbb{R}^d \times \mathbb{R})$, Ω any bounded open subset of $\mathbb{R}^d$, $u_h(x, t)$ an approximate solution of (1.1) computed with the scheme (1.3) (supposed to be an E-scheme), $u^0 \in L^{\infty}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then there exists a positive constant K_4 such that :*

$$
\begin{aligned}
&- \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]^2} \{ |u_h(x, t) - g(y, s)| \; \omega_t(\xi, \chi) + \\
&\qquad + (H_{g(y, s)}(u_h(x, t)) - H_{g(y, s)}(0)) \, . \, \mathrm{grad}_x \, (\omega(\xi, \chi)) \} \, dx \, dy \, dt \, ds \\
&- \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, t_0) - g(y, s)| \; \omega(\xi, t_0 - s) \, dx \, dy \, ds \qquad (4.1) \\
&- \int_{\Omega \times \mathbb{R}^d \times [t_0, t_1]} |u_h(x, t_1) - g(y, s)| \; \omega(\xi, t_1 - s) \, dx \, dy \, ds \leq \\
&\qquad \leq K_4(u^0) \sqrt{m(\Omega)} \frac{h + k^+}{\sqrt{k^-}} (t_1 - t_0)^{\frac{1}{2}} \; \|\omega\|_{1,1} + \nu(\Omega, u^0, t_1 - t_0) \; \|\omega\|_1
\end{aligned}
$$

where $\xi = x - y$, $\chi = t - s$, and ν is such that :

$$
\begin{aligned}
&\text{(i)} \quad \lim_{diam\,(\Omega) \to \infty} \nu(\Omega, u^0, t_1 - t_0) \, \mathrm{diam}\, (\Omega)^{d-1} = 0 \qquad (4.2)\\
&\text{(ii)} \quad \nu(\Omega, u^0, t_1 - t_0) = 0 \textit{ if } u^0 \textit{ has a compact support and } \\
&\qquad\qquad \mathrm{diam}\, (\Omega) \textit{ is sufficiently large}.
\end{aligned}
$$

Proof of Proposition 4.1 : For simplicity we suppose that the time discretisation is such that :

$$t^0 = t_0, \; t^N = t_1, \quad k^n = t^{n+1} - t^n, \; n \in \{0, 1, \ldots, N\}$$

and that $\partial\Omega \subset S_h$.

For $(y, s) \in \mathbb{R}^d \times \mathbb{R}^+$, let us define $\mathscr{H}$, $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_g$ as :

$$
\begin{aligned}
\mathscr{H}(g, u_h, \omega)(y, s) = - \int_{\Omega \times [t_0, t_1]} &\{ |u_h(x, t) - g(y, s)| \; \omega_t(\xi, \chi) + \\
&+ H_{g(y, s)}(u_h(x, t)) \, . \, \mathrm{grad}_x \, (\omega(\xi, \chi)) \} \, dx \, dt
\end{aligned}
$$

$$\mathscr{H}_0(g, u_h, \omega)(y, s) = \int_\Omega |u_h(x, t_0) - g(y, s)| \; \omega(\xi, t_0 - s)\, dx$$

$$\mathscr{H}_1(g, u_h, \omega)(y, s) = \int_\Omega |u_h(x, t_1) - g(y, s)| \; \omega(\xi, t_1 - s)\, dx$$

$$\mathscr{H}_g(\omega)(y, s) = \int_{\Omega \times [t_0, t_1]} H_{g(y, s)}(0) \,.\, \mathrm{grad}_x\, (\omega(\xi, \chi))\, dx\, dt\,.$$

Considering that $u_h(x, t)$ is constant over $K \times [t^n, t^{n+1}[$, we get that

$$\mathscr{H}(g, u_h, \omega)(y, s) = - \sum_{\substack{n = 0, \dots, N-1 \\ K \in T_h\,;\, K \subset \Omega}} \int_{K \times [t^n, t^{n+1}]} \left\{ |u_K^n - g(y, s)| \; \omega_t(\xi, \chi) + \right.$$

$$\left. + H_{g(y, s)}(u_K^n) \,.\, \mathrm{grad}_x\, (\omega(\xi, \chi)) \right\} dx\, dt$$

$$= - \sum_{\substack{n = 0, \dots, N-1 \\ K \in T_h\,;\, K \subset \Omega}} (A_K^{n, 1}(y, s) + A_K^{n, 2}(y, s))$$

with $A_K^{n, 1}$ and $A_K^{n, 2}$ defined as follows :

$$A_K^{n, 1}(y, s) = \int_{K \times [t^n, t^{n+1}]} |u_K^n - g(y, s)| \; \omega_t(\xi, \chi)\, dx\, dt$$

$$A_K^{n, 2}(y, s) = \int_{K \times [t^n, t^{n+1}]} H_{g(y, s)}(u_K^n) \,.\, \mathrm{grad}_x\, (\omega(\xi, \chi))\, dx\, dt\,.$$

Considering now some averaged values of ω :

$$\omega_{K, n}(y, s) = \frac{1}{m(K)} \int_K \omega(\xi, t^n - s)\, dx$$

and

$$\omega_{e, n}(y, s) = \frac{1}{k^n\, m(e)} \int_{e \times [t^n, t^{n+1}]} \omega((x(\sigma) - y, \chi))\, d\sigma\, dt$$

we get :

$$A_K^{n, 1}(y, s) = |u_K^n - g(y, s)| \; m(K)\, (\omega_{K, n}(y, s) - \omega_{K, n+1}(y, s))$$

$$A_K^{n, 2}(y, s) = H_{g(y, s)}(u_K^n) \int_{\partial K \times [t^n, t^{n+1}]} n(x(\sigma))\, \omega(x(\sigma) - y, \chi)\, d\sigma\, dt$$

$$= H_{g(y, s)}(u_K^n) \left(\sum_{e \in \partial K} n_e\, \omega_{e, n}(y, s)\, m(e)\, k^n \right)$$

$$\sum_{\substack{n=0,\dots,N-1\\ K\in T_h;\, K\subset\Omega}} |u_K^n - g(y,s)|\, m(K)\,(\omega_{K,n}(y,s) - \omega_{K,n+1}(y,s)) =$$

$$= \sum_{K\in T_h;\, K\subset\Omega} m(K)\,|u_K^0 - g(y,s)|\,\omega_{K,0}(y,s)$$

$$+ \sum_{\substack{n=1,\dots,N\\ K\in T_h;\, K\subset\Omega}} m(K)\,\omega_{K,n}(y,s)\,(|u_K^n - g(y,s)| - |u_K^{n-1} - g(y,s)|)$$

$$- \sum_{K\in T_h;\, K\subset\Omega} m(K)\,|u_K^N - g(y,s)|\,\omega_{K,N}(y,s)\,.$$

Noticing that :

$$\sum_{K\in T_h;\, K\subset\Omega} m(K)\,|u_K^0 - g(y,s)|\,\omega_{K,0}(y,s) =$$

$$= \int_\Omega |u_h(x,t_0) - g(y,s)|\,\omega(\xi, t_0 - s)\,dx = \mathscr{H}_0(g, u_h, \omega)\,(y,s)$$

$$\sum_{K\in T_h;\, K\subset\Omega} m(K)\,|u_K^N - g(y,s)|\,\omega_{K,N}(y,s) =$$

$$= \int_\Omega |u_h(x,t_1) - g(y,s)|\,\omega(\xi, t_1 - s)\,dx = \mathscr{H}_1(g, u_h, \omega)\,(y,s)$$

$$\sum_{\substack{n=0,\dots,N-1\\ e\in\partial\Omega\cap S_h}} m(e)\,k^n\,\omega_{e,n}\,H_{g(y,s)}(0)\,.\,n_{e,K} =$$

$$= \sum_{\substack{n=0,\dots,N-1\\ e\in\partial\Omega\cap S_h}} \int_{e\times[t^n,t^{n+1}]} H_{g(y,s)}(0)\,.\,n_{e,K}\,\omega(\xi,\chi)\,dx(\sigma)\,dt$$

$$= \int_{\partial\Omega\times[t_0,t_1]} H_{g(y,s)}(0)\,.\,n(x(\sigma))\,\omega(\xi,\chi)\,dx(\sigma)\,dt$$

$$= \int_{\Omega\times[t_0,t_1]} H_{g(y,s)}(0)\,.\,\mathrm{grad}_x\,(\omega(\xi,\chi))\,dx\,dt = \mathscr{H}_g(\omega)\,(y,s)$$

and taking account of the conservativity property (1.6) (i) we get :

$$\mathscr{H}(g, u_h, \omega) = \mathscr{H}_0(g, u_h, \omega) - \mathscr{H}_1(g, u_h, \omega) - \mathscr{H}_g(\omega)\,(y,s) +$$

$$+ \sum_{\substack{n=0,\dots,N-1\\ K\in T_h;\, K\subset\Omega}} \Big\{ m(K)\,\omega_{K,n+1}(|u_K^{n+1} - g| - |u_K^n - g|)$$

$$+ \sum_{e\in\partial K} m(e)\,k^n\,\omega_{e,n}[h_g(n_{e,K}, u_K^n, u_{K_e}^n) - H_g(u_K^n)\,.\,n_{e,K}]\Big\}$$

$$- \sum_{\substack{n=0,\dots,N-1\\ e\in S_h\cap\partial\Omega}} m(e)\,k^n\,\omega_{e,n}(h_g(n_{e,K}, u_K^n, u_{K_e}^n) - H_g(0)\,.\,n_{e,K})\,. \quad (4.3)$$

Let us denote $\delta(u^0, g, y, s)$ by

$$\sum_{\substack{n=0,\dots,N-1 \\ e \in S_h \cap \partial\Omega}} m(e)\, k^n\, \omega_{e,n}(h_g(n_{e,K}, u_K^n, u_{K_e}^n) - H_g(0)\,.\,n_{e,K}) \equiv$$

$$\equiv \delta(u^0, g, y, s)\,. \quad (4.4)$$

As a consequency of (2.3) we have :

$$(|u_K^{n+1} - g(y, s)| - |u_K^n - g(y, s)|) \leqslant$$
$$+ \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} (|u_K^{n+1,e} - g(y, s)| - |u_K^n - g(y, s)|)\,. \quad (4.5)$$

Taking account of (4.5) in (4.3), we obtain :

$$\mathscr{H}(g, u_h, \omega)(y, s) - \mathscr{H}_0(g, u_h, \omega)(y, s) +$$
$$+ \mathscr{H}_1(g, u_h, \omega)(y, s) + \mathscr{H}_g(\omega)(y, s) \leqslant$$
$$\leqslant \sum_{\substack{n=0,\dots,N-1 \\ K \in T_h;\, K \subset \Omega}} m(K) \left\{ \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} (\omega_{K,n+1} - \omega_{e,n})(y, s) \times \right.$$
$$\times (|u_K^{n+1,e} - g(y, s)| - |u_K^n - g(y, s)|)$$
$$+ \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} \omega_{e,n}(y, s)\, [\,|u_K^{n+1,e} - g(y, s)| - |u_K^n - g(y, s)|$$
$$\left. + \lambda_K^n (h_g(n_{e,K}, u_K^n, u_{K_e}^n) - H_g(u_K^n)\,.\,n_{e,K})] \right\} - \delta(u^0, g, y, s)\,.$$

The term into [ ] in the r.h.s. of the previous inequality is nothing else than the entropy production across the edge e and it is negative (see (2.10)). Considering now the following triangle inequality :

$$|\,|u_K^{n+1,e} - g(y, s)| - |u_K^n - g(y, s)|\,| \leqslant$$
$$\leqslant |u_K^{n+1,e} - u_K^n| = \lambda_K^n\, C(n_{e,K}, u_K^n, u_{K_e}^n)\, |\Delta_e^n u|$$

we obtain

$$(\mathscr{H}(g, u_h, \omega) - \mathscr{H}_0(g, u_h, \omega) +$$
$$+ \mathscr{H}_1(g, u_h, \omega) + \mathscr{H}_g(\omega))(y, s) \leqslant -\delta(u^0, g, y, s) +$$
$$+ \sum_{\substack{n=0,\dots,N-1 \\ K \in T_h;\, K \subset \Omega}} m(K) \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} \lambda_K^n |\omega_{K,n+1}(y, s) -$$
$$- \omega_{e,n}(y, s)|\, C(n_{e,K}, u_K^n, u_{K_e}^n)\, |\Delta_e^n u|\,.$$

Taking account of

$$\left|h_g(n_{e,K}, u_K^n, u_{K_e}^n) - H_g(0) \cdot n_{e,K}\right| \leq K(\|u^0\|_\infty, \|g\|_\infty)\left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right)$$

we get :

$$-\delta(u^0, g, y, s) \leq K(\|u^0\|_\infty, \|g\|_\infty) \sum_{\substack{n=0,\dots,N-1 \\ e \in S_h \cap \partial\Omega}} m(e)\, k^n\, \omega_{e,n}\left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right)$$

and consequently :

$$\begin{aligned}\mathscr{H}(g, u_h, \omega)(y, s) - \mathscr{H}_0(g, u_h, \omega)(y, s) &+ \\ &+ \mathscr{H}_1(g, u_h, \omega)(y, s) + \mathscr{H}_g(\omega)(y, s)\end{aligned}$$

$$\begin{aligned}\leq \sum_{\substack{n=0,\dots,N-1 \\ K \in T_h;\, K \subset \Omega}} m(K) \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} \lambda_K^n \left|\omega_{K,n+1}(y, s) - \omega_{e,n}(y, s)\right| \times \\ \times C(n_{e,K}, u_K^n, u_{K_e}^n)\left|\Delta_e^n u\right| \\ + K(\|u^0\|_\infty, \|g\|_\infty) \sum_{\substack{n=0,\dots,N-1 \\ e \in S_h \cap \partial\Omega}} m(e)\, k^n\, \omega_{e,n}(y, s)\left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right).\end{aligned}$$

Integrating this last inequality over $\mathbb{R}^d \times [t_0, t_1]$ we finally obtain :

$$\begin{aligned}\mathscr{L}(g, u_h, \omega) &\leq \\ \leq \sum_{\substack{n=0,\dots,N-1 \\ K \in T_h;\, K \subset \Omega}} \sum_{e \in \partial K} m(e)\, k^n\, C_{e,K}^n \left|\Delta_e^n u\right| &\times \\ \times \left(\int_{\mathbb{R}^d \times [t_0, t_1]} \left|\omega_{K,n+1}(y, s) - \omega_{e,n}(y, s)\right| dy\, ds\right) \\ + \sum_{\substack{n=0,\dots,N-1 \\ e \in S_h \cap \partial\Omega}} m(e)\, k^n\left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right)\|\omega\|_1\end{aligned}$$

where

$$\begin{aligned}\mathscr{L}(g, u_h, \omega) \equiv \int_{\mathbb{R}^d \times [t_0, t_1]} (\mathscr{H}(g, u_h, \omega) - \mathscr{H}_0(g, u_h, \omega) &+ \\ + \mathscr{H}_1(g, u_h, \omega) + \mathscr{H}_g(\omega))(y, s)\, dy\, ds\,.\end{aligned}$$

An easy computation prove that :

$$\begin{aligned}\int_{\mathbb{R}^d \times [t_0, t_1]} \left|\omega_{K,n+1}(y, s) - \omega_{e,n}(y, s)\right| dy\, ds &\leq \\ \leq (K(a_-, a_+, d)\, h + k)\, \|\omega\|_{1,1}\,.\end{aligned}$$

With similar arguments as in the proof of L^1 bounds of Proposition 2.4 we prove easily that :

$$\nu(u^0, \Omega, t_1 - t_0) \equiv \sum_{\substack{n=0,\dots,N-1 \\ e \in S_h \cap \partial\Omega}} m(e)\, k^n \left(\left|u_K^n\right| + \left|u_{K_e}^n\right|\right)$$

satisfies (4.2) :

(i) $$\lim_{\operatorname{diam}(\Omega) \to \infty} \nu(\Omega, u^0, t_1 - t_0) \operatorname{diam}(\Omega)^{d-1} = 0$$

(ii) $\nu(\Omega, u^0, t_1 - t_0) = 0$ if u^0 has a compact support and diam (Ω) is sufficiently large .

Thus

$$\begin{aligned}
&\mathscr{L}(g, u_h, \omega) - \nu(u^0, \Omega, t_1 - t_0)\, \|\omega\|_1 \leqslant \\
&\leqslant (h + k^+)\, \|\omega\|_{1,1} \sum_{\substack{n=0,\dots,N-1 \\ K \in T_h;\, K \subset \Omega}} \sum_{e \in \partial K} m(e)\, k^n\, C_{e,K}^n \left|\Delta_e^n u\right| \\
&\leqslant (h + k^+)\, \|\omega\|_{1,1} \sum_{\substack{n=0,\dots,N-1 \\ e \in S_h;\, e \subset \Omega}} k^n\, m(e)\, Q_e^n \left|\Delta_e^n u\right| \\
&\leqslant (h + k^+)\, \|\omega\|_{1,1} \left(\sum_{\substack{n=0,\dots,N-1 \\ e \in S_h;\, e \subset \Omega}} k^n\, m(e)\, \lambda_e^n (Q_e^n)^2 \left|\Delta_e^n u\right|^2 \right)^{1/2} \times \\
&\qquad \times \left(\sum_{\substack{n=0,\dots,N-1 \\ e \in S_h;\, e \subset \Omega}} \frac{k^n\, m(e)}{\lambda_e^n} \right)^{1/2}.
\end{aligned}$$

Then, noticing that :

$$\begin{aligned}
\sum_{n,\, e \in S_h} \frac{k^n\, m(e)}{\lambda_e} &\leqslant \frac{1}{k^-} \sum_{n,\, e \in S_h} k^n \left\{ \frac{m(e)}{m(\partial K)} m(K) + \frac{m(e)}{m(\partial K_e)} m(K_e) \right\} = \\
&= T \frac{1}{k^-} \sum_{K \in T_h} m(K) \sum_{e \in \partial K} \frac{m(e)}{m(\partial K)} = T \frac{1}{k^-} m(\Omega)\,.
\end{aligned}$$

thanks to the entropy global production estimate (2.37) we finally get :

$$\begin{aligned}
&\mathscr{L}(g, u_h, \omega) - \nu(u^0, \Omega, t_1 - t_0)\, \|\omega\|_1 \leqslant \\
&\qquad \leqslant C'(u_0) \frac{h + k^+}{\sqrt{k^-}} \|\omega\|_{1,1} (t_1 - t_0)^{1/2} \left[\frac{n_0\, m(\Omega)\, a_+}{a_-} \right]^{1/2}.
\end{aligned}$$

This completes the proof. □

4.2. Error bounds for monotone schemes

We now proves the main result of this paper :

Proof of Theorem 1.1 : Thanks to Propositions 2.5 and 4.1 we can apply Theorem 3.1 with :

$$\varphi_1(\tau, h) = \max\ (\tau, k^+ (h)), \quad \varphi_2(h) = \frac{h + k^+}{\sqrt{k^-}} .$$

Thus, we can take $\varepsilon > 0$ which minimize the r.h.s. of the inequality (3.4) of Theorem 3.1. This r.h.s. is bounded by :

$$2\ \varepsilon + k^+ (h) + \frac{h + k^+}{\varepsilon \sqrt{k^-}} .$$

We get the minimum for :

$$\varepsilon^2 = \frac{h + k^+}{\sqrt{k^-}} .$$

(1.9) follows immediatly.

As a consequency of the error bound (1.9), we also get a convergence proof for approximate solutions of (1.1) with initial data in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The proof follows the one by Crandall and Majda in [7] :

For $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ we choose $u_{0,m} \in BV(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with compact support such that

$$\begin{cases} |u_{0,m}| \leq \|u_0\|_\infty & \text{and} \\ \|u_0 - u_{0,m}\|_{L^1(\mathbb{R}^d)} \to 0 & \text{as} \quad m \to 0 . \end{cases}$$

This can be done by standart methods. Denote the Finite Volume solution corresponding to the initial value $u_{0,m}$ by $u_{h,m}$. Considering all functions as functions of t with values in $L^1(\mathbb{R}^d)$, we knows from (1.8) that :

$$\lim_{h \to 0} u_{h,m}(t) = u_m(t)$$

exists uniformly for bounded t. From L^1 contractions property (2.26) of Proposition 2.5, it follows that

$$\sup_{0 \leq t} \|u_{h,l}(t) - u_{h,m}(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_{0,l} - u_{0,m}\|_{L^1(\mathbb{R}^d)} .$$

Thus, $\{u_m\}$ is Cauchy in $\mathscr{C}([0, \infty); L^1(\mathbb{R}^d))$ and consequently converges uniformly to a limit $u \in \mathscr{C}([0, \infty); L^1(\mathbb{R}^d))$.

By using the triangle inequality, we get :

$$\|u_h(t) - u(t)\|_{L^1(\mathbb{R}^d)} \leqslant$$

$$\leqslant \|u_h(t) - u_{h,m}(t)\|_{L^1(\mathbb{R}^d)} + \|u_{h,m}(t) - u_m(t)\|_{L^1(\mathbb{R}^d)} + \|u_m(t) - u(t)\|_{L^1(\mathbb{R}^d)}$$

$$\leqslant \|u_0 - u_{0,m}\|_{L^1(\mathbb{R}^d)} + \|u_{h,m}(t) - u_m(t)\|_{L^1(\mathbb{R}^d)} + \|u_m(t) - u(t)\|_{L^1(\mathbb{R}^d)} .$$

The first and third terms above can be made small uniformly in t taking m large. The second term can be made small, uniformly for t bounded, by taking h small $\left(\text{as soon as } \lim_{h \to 0} \frac{h + k^+}{\sqrt{k^-}} = 0\right)$.

It follows that u_h converges locally uniformly in $L^1(\mathbb{R}^d)$ towards u.

This completes the proof. □

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