

U. HORNUNG

W. JÄGER

A. MIKELIĆ

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REACTIVE TRANSPORT THROUGH AN ARRAY OF CELLS WITH SEMI-PERMEABLE MEMBRANES (*)

by U. HORNING ⁽¹⁾, W. JÄGER ⁽²⁾, A. MIKELIĆ ⁽³⁾

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Abstract. — A problem of diffusion, convection, and nonlinear chemical reactions in a periodic array of cells is studied. It is assumed that in the cells there are porous bodies which are surrounded by semi-permeable membranes, i.e., fluxes and concentrations are coupled nonlinearly at the interfaces between the cells and the surrounding fluid. We consider the limit when the number of cells tends to infinity and at the same time their size tends to zero while the volume fraction of the cells remains fixed. In the limit we get a nonlinearly coupled problem with two scales, a global and a local one. We prove well-posedness of the micro-problems, uniqueness of the macro-problem, and convergence of the homogenization process. In order to determine the form of the limit equations we use the newly developed technique of two-scale convergence. Other methods being used are the theory of semilinear parabolic systems, maximum principles, compactness, monotonicity, and the energy method.

Résumé. — On considère la diffusion, la convection et les réactions chimiques non linéaires à travers un arrangement périodique de cellules. On suppose que chaque cellule contient des corps poreux entourés de membranes semi-perméables, c'est-à-dire qu'il y a liaison non linéaire entre les flux et les concentrations sur les interfaces cellules-fluide. Nous considérons la limite, quand le nombre de cellules tend vers l'infini et en même temps lorsque leur volume tend vers zéro, tout en conservant la même fraction de volume. Notre problème initial se comporte à grande échelle comme un problème non linéaire où interviennent deux échelles d'espace, — une échelle globale et une échelle locale. Nous démontrons l'existence d'une solution qui satisfait le principe du maximum à ε fixé. Lorsque le paramètre ε tend vers zéro, on obtient la convergence du processus d'homogénéisation vers une solution « homogénéisée » unique. Pour déterminer la forme des équations limite, nous utilisons la technique de convergence à deux échelles.

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⁽¹⁾ Fak. f. Informatik, UniBwM, D-85577 Neubiberg, Germany.

e-mail : ulrich@informatik.unibw-muenchen.de

⁽²⁾ Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany.

e-mail : cm3@dhdurz1.bitnet

⁽³⁾ Laboratoire d'Analyse Numérique, Bât. 101, Université Lyon 1, 43, bd. du onze novembre, F-69622 Villeurbanne Cedex, France.

e-mail : andro@lan1.univ-lyon1.fr

1. INTRODUCTION

Mathematical modeling of reactive flow through porous media is of great practical importance in many physical, chemical, and biological applications. There is a need for deriving macroscopic laws for the processes in these geometrically complex media including flow, diffusion, convection, and chemical reactions. Homogenization is providing techniques to pass from microscopic models to macroscopic ones letting the proper scale parameter ε in the system tend to zero. Formally, the macroscopic model equations can be obtained by multiple scale expansions and averaging. It is the « zero order » approximation of the original model system. The macroscopic description does not cause problems with solving, e.g., nonlinear partial differential equations in complex domains. However, one has to pay for this simplification by being forced to describe the local structure of the medium and to solve additional equations formulated with respect to microscopic variables in a standard cell. For simplicity, it is assumed that the medium is composed periodically of standard cells of size ε . Let us assume that such a cell is a cube which is split up into a solid part (for instance a ball) and a fluid part. We assume that we have Stokes flow of a fluid in the fluid part. Substances are diffusing and reacting in the fluid and in the solid part. They are transported by the flow in the fluid part.

The following examples for real applications should be kept in mind : 1) The solid part is a ball of a material which is porous itself. The substances are chemical species diffusing and reacting inside and outside of the balls. At the boundary of the balls there is a change of material properties, expressed by a jump of the coefficients in the equations and by — in general — nonlinear transmission conditions. If there is no reaction taking place, one has the situation of a chromatograph used for the separation of chemical substances. In case of a porous catalyst the reactions are important. They may also just take place on the surface of the balls, e.g., if the catalytic particles are concentrated on the surface of the solid part. 2) The solid parts are biological cells separated by a membrane from the fluid part. In this case the permeability of the membranes which are porous media themselves plays an important role. The fluid outside of biological cells is the extracellular fluid transporting nutrients, activators, and inhibitors for development of the cells. Again, in this case also processes on the cell surfaces and flow inside of the cells are of interest.

Homogenization leads to mathematical results, such as to a macroscopic limit of the microscopic system when the model parameter ε (e.g., radius of the ball) tends to zero. In case of a model for a chromatograph, this limit process was also rigorously justified (see [36]) and the validity of the model was experimentally tested (see [32]). The results show that the model equations obtained by homogenization in the case of a periodic structure are describing the experimental results very precisely, see also [33].

In this paper we consider a periodic structure, diffusion, transport and nonlinear reactions in the fluid part, diffusion and nonlinear reactions in the solid part. On the interface between the fluid and the solid part, we assume continuity of the flux and additional — in general nonlinear — transmission conditions.

The case of reactions and diffusion on the surface of the solid part was studied in [19] and [20]. The formulation of the model equations and especially of the transmission conditions is kept rather general in order to include important applications. The main mathematical work to be done consisted in solving the model equations of the micro-process and estimating the solutions in proper norms uniformly with respect to ε . Then the convergence result of the microscopic solution towards a macroscopic solution is obtained. Here, we have used the only recently developed notion of *two-scale convergence* coupled with monotonicity methods and compensated compactness. The energy method and the div-curl-lemma are being used as standard arguments in this framework. It seems not to be known how one could obtain the new results without the concept of two-scale convergence. In our knowledge, this is the first time that convergence of the homogenization procedure is proved for problems with nonlinear reactive terms and nonlinear transmission conditions.

For a survey on homogenization applied to flow, diffusion, convection, and reactions in porous media see the papers [18]. Problems of related type were investigated in various papers. Two-phase flow was studied in [5], [8] and [9]. Miscible displacement problems were studied in [27] [24] [1] [16] and [17]. General textbooks on the method of homogenization are [7], [6] and [30].

The important features of the micro-model (section 3) can be described as follows. In the fluid part Ω^ε there is a fluid flowing according to the Stokes equations (equations 1). The concentrations of the various chemical species in the fluid part Ω^ε and the solid part Π^ε are v_j^ε and w_j^ε , resp. The variables v_j^ε satisfy equations with diffusive, convective and reactive terms in the fluid part Ω^ε , whereas the variables w_j^ε satisfy equations with diffusive and reactive terms in the solid part Π^ε (see equations 1). The transmission conditions on Γ^ε — the interface between the fluid and the solid part — are both the continuity of the normal mass flux and a second condition of special type. Here we consider six different cases : conditions of 1) Dirichlet type, 2, 3), 4) Neumann type, and 5), 6) of Signorini type. In case 5), e.g., the transmission condition is

$$s \leq 0 \quad \text{and} \quad q \geq 0 \quad \text{and} \quad sq = 0$$

where $s = a_j v_j^\varepsilon - w_j^\varepsilon$ is a weighted difference of the concentrations in the fluid and solid part, resp., and $q = -\varepsilon c_j \vec{\nu}^\varepsilon \cdot \nabla w_j^\varepsilon$ is the mass flux in normal direction. It must be emphasized that the essential point in formulating the

micro-model is the proper scaling. The only criterion for the proper choice of the scale parameter and its powers are measurements, calibration and validation of experimental data. It turns out that the transmission conditions of the micro-model appear in almost the same form as boundary conditions of the local problems in the macro-model. Problems with some similarity to this one were described in [15] and [14].

2. NOTATIONS AND ASSUMPTIONS

2.1 The Geometry of the Problem

First we define the geometry of the problem.

x : macro space variable

y : micro space variable

t : time variable

T : end of time-interval

Ω : bounded domain in \mathbb{R}^n

Q : $[0, T] \times \Omega =$ time-space-domain

U : $Q \times Y$

A : $\Omega \times I$

$\partial\Omega$: $\bar{\Gamma}_D \cup \bar{\Gamma}_N =$ piecewise smooth boundary of Ω , $\Gamma_D \cap \Gamma_N = \emptyset$

Z : $(0, 1)^n =$ unit cell in \mathbb{R}^n

Y : open subset of Z (with $\text{clos}(Y) \subset \ker(Z)$) = representative cell

X : $Z \setminus \bar{Y} =$ representative pore

Γ : $\partial Y = \partial X =$ piecewise smooth boundary of Y and of X

$\bar{\nu}$: outer normal on $\partial\Omega$ with respect to Ω

: or inner normal on Γ with respect to Y

\vec{e}_i : i -th unit vector in \mathbb{R}^n .

For any subset Ξ of Z and integer vectors $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we denote the shifted subset by

$$\Xi^k = \Xi + \sum_{i=1}^n k_i \vec{e}_i.$$

Let

$$\Gamma^* = \bigcup \{ \Gamma^k : k \in \mathbb{Z}^n \}.$$

We assume that a scale factor $\varepsilon > 0$ is given. Then the geometry within Ω is defined as follows :

Π^ε : $\cup \{ \varepsilon Y^k : Y^k \subset \Omega, k \in \mathbb{Z}^n \} =$ ensemble of cells

R^ε : $[0, T] \times \Pi^\varepsilon$

Ω^ε : $\Omega \setminus \bar{\Pi}^\varepsilon =$ ensemble of pores

$$\begin{aligned}
 Q^\varepsilon &: [0, T] \times \Omega^\varepsilon \\
 \Gamma^\varepsilon &: \partial \Pi^\varepsilon = \{ \varepsilon \Gamma^k : \varepsilon \Gamma^k \subset \Omega, k \in \mathbb{Z}^n \} \\
 &: \text{ensemble of the membranes} \\
 \bar{\nu}^\varepsilon &: \text{inner normal on } \Gamma^\varepsilon \text{ with respect to } \Pi^\varepsilon \\
 &: \text{outer normal on } \Gamma^\varepsilon \text{ with respect to } \Omega^\varepsilon \\
 \chi^\varepsilon(x) &= \begin{cases} 1, & x \in \Omega^\varepsilon \\ 0, & x \in \Pi^\varepsilon \end{cases} : \text{characteristic function of } \Omega^\varepsilon.
 \end{aligned}$$

2.2 Nonlinearities

For $j = 1, \dots, m$ have

$$\begin{aligned}
 f_j : \mathbb{R}^m &\rightarrow \mathbb{R} = \text{reaction rate in the pores} \\
 g_j : \mathbb{R}^m &\rightarrow \mathbb{R} = \text{reaction rate in the cells} .
 \end{aligned}$$

We are going to choose conditions on the functions f_j and g_j such that we get L^∞ -estimates and non-negativity of the solutions v_j^ε and w_j^ε (see proposition 3, theorem 1 and proposition 5). Any other type of conditions giving ε -independent uniform bounds in the same class of functional spaces would also do.

We assume that for given positive constants C_j^v and $C_j^w = a_j C_j^v (j \in J)$ there are constants $A_j, B_j \geq 0$ such that

$$\begin{aligned}
 f_j(\vec{z}) &= -A_j z_j + \tilde{f}_j(\vec{z}) \quad \forall \vec{z} \in [0, C_l^v]^m \\
 &\text{and } g_j(\vec{z}) = -B_j z_j + \tilde{g}_j(\vec{z}) \quad \forall \vec{z} \in [0, C_l^w]^m
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{f}_j(\vec{z}) &\geq 0 \text{ and } \tilde{g}_j(\vec{z}) \geq 0 \quad \forall \vec{z} \in \mathbb{R}^m \text{ with } \vec{z} \geq 0, \\
 A_j &\geq \frac{1}{C_j^v} \tilde{f}_j(\vec{z}) \quad \forall \vec{z} \in \mathbb{R}^m \text{ with } 0 \leq z_l \leq C_l^v \quad \forall l \in J,
 \end{aligned}$$

and

$$B_j \geq \frac{1}{C_j^w} \tilde{g}_j(\vec{z}) \quad \forall \vec{z} \in \mathbb{R}^m \text{ with } 0 \leq z_l \leq C_l^w \quad \forall l \in J.$$

We extend the functions f_i and g_j to all \mathbb{R}^m in the following way. Let

$$f_j(\vec{z}) = f_j(\vec{z}^F), \quad g_j(\vec{z}) = g_j(\vec{z}^G)$$

with

$$z_k^F = \begin{cases} 0, & \text{if } z_k < 0 \\ z_k, & \text{if } 0 \leq z_k \leq C_k^v, \\ C_k^v, & \text{if } z_k > C_k^v \end{cases},$$

and the \bar{z}^G being defined analogously.

The sets $M_j \subset \mathbb{R}^2$ are defined by one of the following conditions,

$$(s, q) \in M_j \text{ iff } \begin{cases} s = 0 & , \text{ case 1} \\ q = 0 & , \text{ case 2} \\ q = b_j s_+ & , \text{ case 3} \\ q = -b_j s_- & , \text{ case 4} \\ s \leq 0 \text{ and } q \geq 0 \text{ and } sq = 0 & , \text{ case 5} \\ s \geq 0 \text{ and } q \leq 0 \text{ and } sq = 0 & , \text{ case 6} \end{cases}$$

where b_j are positive constants. The lower plus denotes the positive part

$$s_+ = \begin{cases} s, & s > 0 \\ 0, & \text{else} \end{cases}$$

and $s_- = (-s)_+$, hence $s = s_+ - s_-$. We also use the sets $N_j \subset \mathbb{R}$ defined by

$$s \in N_j \text{ iff } \begin{cases} s = 0, & \text{case 1} \\ s \in \mathbb{R}, & \text{cases 2, 3, 4} \\ s \leq 0, & \text{case 5} \\ s \geq 0, & \text{case 6} \end{cases}.$$

We assume that the index set $J = \{1, \dots, m\}$ is given as the union $J = J_1 \cup \dots \cup J_6$ of subsets J_1, \dots, J_6 . We shall also use positive constants a_j in the micro-model which later will be assumed to have the value 1.

2.3 Unknowns of the Micro-Model

- $\vec{u}^\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}^n$: velocity of the fluid
 $p^\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}$: pressure within the fluid
 $v_j^\varepsilon : [0, T] \times \Omega^\varepsilon \rightarrow \mathbb{R}$: concentration of the j -th solute in the fluid
 $w_j^\varepsilon : [0, T] \times \Pi^\varepsilon \rightarrow \mathbb{R}$: concentration of the j -th solute in the cells.

We use the vectors $\vec{v}^\varepsilon = (v_1^\varepsilon, \dots, v_m^\varepsilon)$ and $\vec{w}^\varepsilon = (w_1^\varepsilon, \dots, w_m^\varepsilon)$.

2.4 Data of the Micro-Model

- $\vec{u}_D : \Gamma_D \rightarrow \mathbb{R}^n$: prescribed boundary values of the fluid velocity
- $v_{j,D} : [0, T] \times \Gamma_D \rightarrow \mathbb{R}$: prescribed boundary values of the concentration of the j -th solute
- $v_{j,I} : \Omega \rightarrow \mathbb{R}$: initial concentration of the j -th solute in the pores
- $w_{j,I} : \Omega \rightarrow \mathbb{R}$: initial concentration of the j -th solute in the cells
- d_j : j -th diffusion constant in the fluid
- c_j : j -th diffusion constant in the cells.

Assumptions :

$$\vec{u}_D \in \left(H^{\frac{1}{2}}(\Gamma) \right)^n, \int_{\Gamma_D} \vec{v} \cdot \vec{u}_D d\Gamma = 0,$$

$$v_{jI} \in H^2(\Omega^\epsilon), w_{jI} \in H^2(\Pi^\epsilon), a_j v_{jI} - w_{jI} \in N_j, v_{jI}|_{\Gamma_D} = v_{jD}(0) \quad \forall j \in J,$$

$$0 \leq v_{j,I} \leq C_j^v, 0 \leq w_{j,I} \leq C_j^w, \text{ and } C_j^v = a_j C_j^w \quad \forall j \in J.$$

We use the functions

$$\hat{v}_{jI} = d_j \Delta v_{jI} + f_j(\vec{v}_I), \hat{w}_{jI}^\epsilon = \epsilon^2 c_j \Delta w_{jI} + g_j(\vec{v}_I)$$

and the vectors $\vec{v}_I = (v_{1,I}, \dots, v_{m,I})$ and $\vec{w}_I = (w_{1,I}, \dots, w_{m,I})$.

2.5 Unknowns of the Macro-Model

- $\vec{u} : \Omega \rightarrow \mathbb{R}^n$: velocity of the fluid
- $p : \Omega \rightarrow \mathbb{R}$: pressure in the fluid
- $v_j : Q \rightarrow \mathbb{R}$: concentration of the j -th solute in the fluid
- $w_j : Q \times \Gamma \rightarrow \mathbb{R}$: concentration of the j -th solute in the cells.

We use the vectors $\vec{v} = (v_1, \dots, v_m)$ and $\vec{w} = (w_1, \dots, w_m)$.

2.6 Auxiliary Functions of the Macro-Model

For $j = 1, \dots, n$ let $\vec{\kappa}_j = (\kappa_j^1, \dots, \kappa_j^n) : \bar{X} \rightarrow \mathbb{R}^n$ and $\pi_j : X \rightarrow \mathbb{R}$ be the Z -periodic solution of the cell problem

$$\begin{cases} \Delta_y \vec{\kappa}_j(y) = \nabla_y \pi_j - \vec{e}_j, & y \in X \\ \nabla_y \cdot \vec{\kappa}_j(y) = 0 & , y \in X \\ \vec{\kappa}_j(y) = 0 & , y \in \Gamma. \end{cases}$$

Then the elements K_{ij} of the tensor K are defined as

$$K_{ij} = \int_X \kappa_{ij}(y) dy .$$

It is well known that K is a symmetric, positive definite matrix (see, e.g., [31]).

Let for $j = 1, \dots, n$ the function $\lambda_j : \bar{X} \rightarrow \mathbb{R}$ be a Z -periodic solution of the cell problem

$$\begin{cases} \Delta_y \lambda_j(y) = 0 & , y \in X \\ \vec{\nu} \cdot \nabla_y \lambda_j(y) = -\vec{\nu} \cdot \vec{e}_j, & y \in \Gamma . \end{cases}$$

We extend λ_j into Y such that $\Delta_y \lambda_j(y) = 0, y \in Y$. The elements D_{ij} of the tensor D are defined as

$$D_{ij} = |X| \delta_{ij} + \int_X \partial_i \lambda_j(y) dy ,$$

where δ_{ij} is the Kronecker- δ . An equivalent definition is

$$Dz = \int_X \nabla_y \mu_z dy \quad \forall z \in \mathbb{R}^n ,$$

where μ_z solves

$$\begin{cases} \Delta_y \mu_z(y) = 0 & , y \in X \\ \vec{\nu} \cdot \nabla_y \mu_z(y) = 0, & y \in \Gamma \end{cases}$$

such that $\mu_z - z \cdot y$ is Z -periodic, hence $\lambda_j(y) = \mu_{\vec{e}_j}(y) - y_j$ (see [12]). Here and in the following we use the abbreviations

$$|X| = \int_X dy, \quad |Y| = \int_Y dy .$$

Furthermore, D is a symmetric, positive definite matrix (see, e.g., [12]).

2.7 Function Spaces

We introduce the Hilbert spaces

$$\begin{aligned} V^\varepsilon &= \{ \varphi \in H^1(\Omega^\varepsilon) : \varphi = 0 \text{ on } \Gamma_D \}^m \times (H^1(\Pi^\varepsilon))^m \\ V &= \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D \}^m \\ W &= (L^2(\Omega; H^1(Y)))^m \\ H^\varepsilon &= (L^2(\Omega^\varepsilon))^m \times (L^2(\Pi^\varepsilon))^m \\ \mathcal{H}^\varepsilon &= L^2(0, T; H^\varepsilon) \\ \mathcal{V}^\varepsilon &= L^2(0, T; V^\varepsilon) \\ \mathcal{V} &= L^2(0, T; V \times W). \end{aligned}$$

We use the convex sets

$$K^\varepsilon = \{ (\vec{\varphi}, \vec{\psi}) : \vec{\varphi} \in \{ \varphi \in H^1(\Omega^\varepsilon) : \varphi = 0 \text{ on } \Gamma_D \}^m, \\ \vec{\psi} \in (H^1(\Pi^\varepsilon))^m, \text{ and } a_j \varphi_j - \psi_j \in N_j \text{ on } \Gamma^\varepsilon \quad \forall j \in J \}$$

and

$$K = \{ (\vec{\varphi}, \vec{\sigma}) : \vec{\varphi} \in V, \vec{\sigma} \in W, \varphi_j = 0 \text{ on } \Gamma_D \\ \text{and } a_j \varphi_j - \sigma_j \in N_j \text{ on } \Lambda \quad \forall j \in J \} .$$

We also use the convex sets

$$\mathcal{K}^\varepsilon = \{ (\vec{\varphi}, \vec{\psi}) \in \mathcal{V}^\varepsilon : (\vec{\varphi}, \vec{\psi}) \in K^\varepsilon \text{ a.e. on } (0, T) \}$$

and

$$\mathcal{K} = \{ (\vec{\varphi}, \vec{\psi}) \in \mathcal{V} : (\vec{\varphi}, \vec{\psi}) \in K \text{ a.e. on } (0, T) \} .$$

3. THE MICRO-MODEL

3.1 The Strong Formulation

Written in strong form, the problem is to find functions \vec{u}^ε , v^ε , and w^ε that satisfy the following system of equations.

The flow :

$$\begin{cases} \varepsilon^2 \Delta \vec{u}^\varepsilon(x) = \nabla p^\varepsilon(x), & x \in \Omega^\varepsilon \\ \nabla \cdot \vec{u}^\varepsilon(x) = 0 & , \quad x \in \Omega^\varepsilon \\ \vec{u}^\varepsilon(x) = 0 & , \quad x \in \Gamma^\varepsilon . \\ \vec{u}^\varepsilon(x) = \vec{u}_D & , \quad x \in \Gamma_D \\ \vec{u}^\varepsilon(x) = 0 & , \quad x \in \Gamma_N \end{cases} \quad (1)$$

The reactive transport :

PROBLEM 1 :

$$\left\{ \begin{array}{ll} \partial_t v_j^\varepsilon(t, x) = d_j \Delta v_j^\varepsilon(t, x) - \vec{u}^\varepsilon(x) \cdot \nabla v_j^\varepsilon(t, x) + f_j(\vec{v}^\varepsilon(t, x)), & t > 0, x \in \Omega^\varepsilon \\ \partial_t w_j^\varepsilon(t, x) = \varepsilon^2 c_j \Delta w_j^\varepsilon(t, x) + g_j(\vec{w}^\varepsilon(t, x)) & , t > 0, x \in \Pi^\varepsilon \\ d_j \vec{v}^\varepsilon \cdot \nabla v_j^\varepsilon(t, x) = \varepsilon^2 c_j \vec{v}^\varepsilon \cdot \nabla w_j^\varepsilon(t, x) & , t > 0, x \in \Gamma^\varepsilon \\ (a_j v_j^\varepsilon(t, x) - w_j^\varepsilon(t, x), -\varepsilon c_j \vec{v}^\varepsilon \cdot \nabla w_j^\varepsilon(t, x)) \in M_j & , t > 0, x \in \Gamma^\varepsilon \\ v_j^\varepsilon(t, x) = v_{j,D}(t, x) & , t > 0, x \in \Gamma_D \\ \vec{v} \cdot \nabla v_j^\varepsilon(t, x) = 0 & , t > 0, x \in \Gamma_N \\ v_j^\varepsilon(t, x) = v_{j,I}(x) & , t = 0, x \in \Omega^\varepsilon \\ w_j^\varepsilon(t, x) = w_{j,I}(x) & , t = 0, x \in \Pi^\varepsilon. \end{array} \right.$$

For simplicity of the proofs, and in order to avoid too many technical difficulties, we assume homogeneous Dirichlet boundary conditions for the concentrations, i.e. $v_{j,D} = 0$.

3.2 The Weak Formulation

PROBLEM 2 : *The weak problem for problem 1 is to find functions $(\vec{v}^\varepsilon, \vec{w}^\varepsilon) \in \mathcal{X}^\varepsilon$ with $(v_j^\varepsilon, w_j^\varepsilon) \in L^\infty(Q^\varepsilon) \times L^\infty(R^\varepsilon)$ and $v_j^\varepsilon \geq 0$, $w_j^\varepsilon \geq 0$, $(\partial_t v_j^\varepsilon, \partial_t w_j^\varepsilon) \in L^2(Q^\varepsilon) \times L^2(R^\varepsilon)$ such that*

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, \varphi_j - v_j^\varepsilon)_{Q^\varepsilon} + (\partial_t w_j^\varepsilon, \psi_j - w_j^\varepsilon)_{R^\varepsilon} \\ & + d_j (\nabla v_j^\varepsilon, \nabla (\varphi_j - v_j^\varepsilon))_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla (\psi_j - w_j^\varepsilon))_{R^\varepsilon} \\ & + (\vec{u}^\varepsilon \cdot \nabla v_j^\varepsilon, \varphi_j - v_j^\varepsilon)_{Q^\varepsilon} - (f_j(\vec{v}^\varepsilon), \varphi_j - v_j^\varepsilon)_{Q^\varepsilon} - (g_j(\vec{w}^\varepsilon), \psi_j - w_j^\varepsilon)_{R^\varepsilon} \\ & + \sum_{j \in J_3} \int_0^T \varepsilon b_j ((a_j v_j^\varepsilon - w_j^\varepsilon)_+, \varphi_j - v_j^\varepsilon - \psi_j + w_j^\varepsilon)_{\Gamma^\varepsilon} dt - \\ & - \sum_{j \in J_4} \int_0^T \varepsilon b_j ((a_j v_j^\varepsilon - w_j^\varepsilon)_-, \varphi_j - v_j^\varepsilon - \psi_j + w_j^\varepsilon)_{\Gamma^\varepsilon} dt \geq 0 \end{aligned}$$

$$\forall (\vec{\varphi}, \vec{\psi}) \in \mathcal{X}^\varepsilon \quad (2)$$

with $(\vec{v}^\varepsilon, \vec{w}^\varepsilon)|_{t=0} = (\vec{v}_I, \vec{w}_I)$ holds.

In this paper we are going to study the situation «U» in which $J_3 = J_4 = \emptyset$, i.e. we are primarily interested in the problem of unilateral boundary conditions. The situation «N» in which $J_5 = J_6 = \emptyset$ is used as a

regularization or penalization of the situation « U ». Since the set \mathcal{X}^ε is a cone, in the situation « U » an equivalent formulation of the variational inequality 2 is the inequality

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, \varphi_j)_{Q^\varepsilon} + (\partial_t w_j^\varepsilon, \psi_j)_{R^\varepsilon} \\ & + d_j (\nabla v_j^\varepsilon, \nabla \varphi_j)_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla \psi_j)_{R^\varepsilon} \\ & + (\vec{u}^\varepsilon \cdot \nabla v_j^\varepsilon, \varphi_j)_{Q^\varepsilon} - (f_j(\vec{v}^\varepsilon), \varphi_j)_{Q^\varepsilon} - (g_j(\vec{w}^\varepsilon), \psi_j)_{R^\varepsilon}) \geq 0 \\ & \forall (\vec{\varphi}, \vec{\psi}) \in \mathcal{X}^\varepsilon \quad (3) \end{aligned}$$

together with the equality

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, v_j^\varepsilon)_{Q^\varepsilon} + (\partial_t w_j^\varepsilon, w_j^\varepsilon)_{R^\varepsilon} + \\ & + d_j (\nabla v_j^\varepsilon, \nabla v_j^\varepsilon)_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla w_j^\varepsilon)_{R^\varepsilon} - \\ & - (f_j(\vec{v}^\varepsilon), v_j^\varepsilon)_{Q^\varepsilon} - (g_j(\vec{w}^\varepsilon), w_j^\varepsilon)_{R^\varepsilon}) = 0, \quad (4) \end{aligned}$$

where the convection term with \vec{u}^ε has dropped, since \vec{u}^ε is divergence free. In the situation « N » the set \mathcal{X}^ε is a linear space ; therefore instead of the inequality 2 we get the variational equation

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, \varphi_j)_{Q^\varepsilon} + (\partial_t w_j^\varepsilon, \psi_j)_{R^\varepsilon} \\ & + d_j (\nabla v_j^\varepsilon, \nabla \varphi_j)_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla \psi_j)_{R^\varepsilon} \\ & + (\vec{u}^\varepsilon \cdot \nabla v_j^\varepsilon, \varphi_j)_{Q^\varepsilon} - (f_j(\vec{v}^\varepsilon), \varphi_j)_{Q^\varepsilon} - (g_j(\vec{w}^\varepsilon), \psi_j)_{R^\varepsilon}) \\ & + \sum_{j \in J_3} \int_0^T \varepsilon b_j ((a_j v_j^\varepsilon - w_j^\varepsilon)_+, \varphi_j - \psi_j)_{\Gamma^\varepsilon} dt - \\ & - \sum_{j \in J_4} \int_0^T \varepsilon b_j ((a_j v_j^\varepsilon - w_j^\varepsilon)_-, \varphi_j - \psi_j)_{\Gamma^\varepsilon} \times \\ & \times dt = 0 \quad \forall (\vec{\varphi}, \vec{\psi}) \in \mathcal{X}^\varepsilon. \quad (5) \end{aligned}$$

4. THE MACRO-MODEL

4.1 The Strong Formulation

The strong form of the problem is to find functions \vec{u} , v , and w that satisfy the following system of equations.

The flow :

$$\begin{cases} \vec{u}(x) = -K\nabla p(x), & x \in \Omega \\ \nabla \cdot \vec{u}(x) = 0 & , x \in \Omega \\ \vec{v} \cdot \vec{u}(x) = u_{vD}(x), & x \in \Gamma_D \\ \vec{v} \cdot \vec{u}(x) = 0 & , x \in \Gamma_N \end{cases} . \quad (6)$$

PROBLEM 3 : *The global problem :*

$$\begin{cases} |X| \partial_t v_j(t, x) + S_j(t, x) = d_j \nabla \cdot (D \nabla v_j(t, x)) - \\ \quad - \vec{u}(x) \cdot \nabla v_j(t, x) + |X| f_j(\vec{v}(t, x)), & t > 0, x \in \Omega \\ v_j(t, x) = v_{j,D}(t, x) & , t > 0, x \in \Gamma_D \\ \vec{v} \cdot \nabla v_j(t, x) = 0 & , t > 0, x \in \Gamma_N \\ v(t, x) = v_{j,I}(x) & , t = 0, x \in \Omega \end{cases}$$

where the sink terms

$$S_j(t, x) = - \int_{\Gamma} c_j \vec{v} \cdot \nabla_y w_j(t, x, y) d\Gamma(y), \quad t > 0, x \in \Omega$$

are defined in terms of the local problems :

$$\begin{cases} \partial_t w_j(t, x, y) = c_j \Delta_y w_j(t, x, y) + g_j(\vec{w}(t, x, y)), & t > 0, x \in \Omega, y \in Y \\ (a_j v_j(t, x) - w_j(t, x, y), -c_j \vec{v} \cdot \nabla_y w_j(t, x, y)) \in M_j, & t > 0, x \in \Omega, y \in \Gamma \\ w_j(t, x, y) = w_{j,I}(x) & , t = 0, x \in \Omega, y \in Y. \end{cases}$$

4.2 The Weak Formulation

PROBLEM 4 : *The weak problem for problem 3 is to find functions $(\vec{v}, \vec{w}) \in \mathcal{X}$ with $v \in L^\infty(Q)$, $w \in L^\infty(U)$, $v_j \geq 0$, $w_j \geq 0$, $\partial_t v_j \in L^2(Q)$ and $\partial_t w_j \in L^2(U)$ such that*

$$\begin{aligned} \sum_{j \in J} (|X| (\partial_t v_j + S_j, \varphi_j)_Q + d_j (D \nabla v_j, \nabla \varphi_j)_Q + \\ + (\vec{u} \cdot \nabla v_j, \varphi_j)_Q - |X| (f_j(\vec{v}), \varphi_j)_Q) = 0 \quad \forall \vec{\varphi} \in V, \quad (7) \end{aligned}$$

and

$$\begin{aligned} & \sum_{j \in J} ((\partial_t w_j, \sigma_j - w_j)_U + c_j (\nabla_y w_j, \nabla_y (\sigma_j - w_j))_U - \\ & \quad - (g_j(\vec{w}), \sigma_j - w_j)_U) + \sum_{j \in J_3} \int_0^T b_j ((a_j v_j - w_j)_+, \sigma_j - w_j)_{R^\varepsilon} dt + \\ & \quad + \sum_{j \in J_4} \int_0^T b_j ((a_j v_j - w_j)_-, \sigma_j - w_j)_{R^\varepsilon} dt \\ & \geq 0 \quad \forall \sigma \text{ with } (\vec{v}, \vec{\sigma}) \in \mathcal{X} \end{aligned} \tag{8}$$

where $(\vec{v}, \vec{w})|_{t=0} = (\vec{v}_I, \vec{w}_I)$ hold.

5. EXISTENCE FOR THE MICRO-MODEL

Here we treat only the situation « U », since the other cases are simpler. Without loss of generality, we can assume $a_j = 1$. This can be easily achieved by renaming $a_j v_j$ by v_j ; of course one has to redefine the nonlinear functions f_j appropriately. We start by introducing the following linearized problem.

PROBLEM 5 : Let

$$\vec{\Phi} = (\vec{F}, \vec{G}) = (F_1, \dots, F_m, G_1, \dots, G_m) \in (L^\infty(Q^\varepsilon))^m \times (L^\infty(R^\varepsilon))^m$$

be given. Then we look for functions

$$\gamma = (\vec{\alpha}, \vec{\beta}) = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m) \in \mathcal{X}^\varepsilon$$

with $\partial_t \alpha_j, \partial_t \beta_j \in L^2(Q)$ such that the variational inequality

$$\begin{aligned} & \sum_{j \in J} ((\partial_t \alpha_j, \varphi_j)_{Q^\varepsilon} + (\partial_t \beta_j, \psi_j)_{R^\varepsilon} + \\ & \quad + d_j (\nabla \alpha_j, \nabla \varphi_j)_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla \beta_j, \nabla \psi_j)_{R^\varepsilon} \\ & \quad + (\vec{u}^\varepsilon \cdot \nabla \alpha_j, \varphi_j)_{Q^\varepsilon} + (A_j \alpha_j - F_j, \varphi_j)_{Q^\varepsilon} + (B_j \beta_j - G_j, \psi_j)_{R^\varepsilon}) \geq 0 \\ & \quad \forall (\vec{\phi}, \vec{\psi}) \in \mathcal{X}^\varepsilon \end{aligned} \tag{9}$$

with $(\vec{\alpha}, \vec{\beta})|_{t=0} = (\vec{v}_I, \vec{w}_I)$ holds and an equality for $\varphi_j = \alpha_j$ and $\psi_j = \beta_j$.

Now we introduce the linear operator $E^\varepsilon : V^\varepsilon \rightarrow (V^\varepsilon)'$ by

$$\begin{aligned} & (E^\varepsilon(\vec{\alpha}, \vec{\beta}), (\vec{\varphi}, \vec{\psi}))_{(V^\varepsilon)', V^\varepsilon} = \\ & = \sum_{j \in J} (d_j(\nabla \alpha_j, \nabla \varphi_j)_{\Omega^\varepsilon} + \varepsilon^2 c_j(\nabla \beta_j, \nabla \psi_j)_{\Pi^\varepsilon} \\ & \quad + (\vec{u}^\varepsilon \cdot \nabla \alpha_j, \varphi_j)_{\Omega^\varepsilon} + (A_j \alpha_j, \varphi_j)_{\Omega^\varepsilon} + (B_j \beta_j, \psi_j)_{\Pi^\varepsilon}) \\ & \quad \forall (\vec{\alpha}, \vec{\beta}), (\vec{\varphi}, \vec{\psi}) \in V^\varepsilon. \quad (10) \end{aligned}$$

This operator E^ε is a continuous and coercive linear operator; the set K^ε is a closed convex subset of V^ε , and hence I_{K^ε} is a convex lower semicontinuous function on V^ε ; further, Φ is an element of the Hilbert space H^ε . Using these notations we can reformulate problem 5 in the following abstract way.

PROBLEM 6 : Find a function $\gamma \in \mathcal{X}^\varepsilon$ such that

$$\left(\frac{d}{dt} \gamma + E^\varepsilon \gamma, \delta - \gamma \right)_{(V^\varepsilon)', V^\varepsilon} \geq (\Phi, \delta - \gamma)_{H^\varepsilon} \text{ a.e. on } (0, T) \quad \forall \delta \in \mathcal{X}^\varepsilon$$

with $\gamma|_{t=0} = \gamma_I$.

Here we have used $\gamma_I = (\vec{v}_I, \vec{w}_I)$. We prove the existence of a solution for this problem in two steps; first we show that there is a solution of the following weak version of problem 6. For this we introduce the operator $\mathcal{E}^\varepsilon : \mathcal{V}^\varepsilon \rightarrow (\mathcal{V}^\varepsilon)'$ defined by

$$(\mathcal{E}^\varepsilon \gamma, \delta)_{(\mathcal{V}^\varepsilon)', \mathcal{V}^\varepsilon} = \int_0^T (E^\varepsilon \gamma, \delta)_{(V^\varepsilon)', V^\varepsilon} dt.$$

PROBLEM 7 : Find a function $\gamma \in \mathcal{X}^\varepsilon$ such that

$$\begin{aligned} \left(\frac{d}{dt} \delta + \mathcal{E}^\varepsilon \gamma, \delta - \gamma \right)_{(\mathcal{V}^\varepsilon)', \mathcal{V}^\varepsilon} & \geq (\Phi, \delta - \gamma)_{\mathcal{H}^\varepsilon} - \frac{1}{2} \|\delta(0) - \gamma_I\|_{H^\varepsilon}^2 \\ & \forall \delta \in \mathcal{X}^\varepsilon \text{ with } \frac{d}{dt} \delta \in (\mathcal{V}^\varepsilon)' \end{aligned}$$

and $\gamma|_{t=0} = (\vec{v}_I, \vec{w}_I)$.

PROPOSITION 1 : Let $\gamma_I \in \bar{K}^{\varepsilon H^\varepsilon}$ (=closure of K^ε in H^ε) and let $\Phi \in (\mathcal{V}^\varepsilon)'$. Then there exists a unique solution $\gamma \in C([0, T]; H^\varepsilon)$ of problem 7.

Proof : The existence is a consequence of Corollaire II.1 in [10] page 77 applied to the evolution triple $V^\varepsilon \subset H^\varepsilon \subset (V^\varepsilon)'$. The uniqueness follows from [22] pages 268-270. Q.E.D.

PROPOSITION 2 : Let $\gamma_j \in K^\epsilon$ and $\Phi \in \mathcal{H}^\epsilon$. Then there exists a unique solution $\gamma \in C([0, T]; V^\epsilon)$ of problem 7 such that $\frac{d}{dt} \gamma \in \mathcal{H}^\epsilon$.

Proof: This result follows from Corollaire II.2 in [10] page 92 in the following way. We take the weak solution $\gamma = (\vec{\alpha}, \vec{\beta})$ from proposition 1. Then we define the selfadjoint operator

$$(\mathcal{E}^\epsilon(\vec{\alpha}, \vec{\beta}), (\vec{\phi}, \vec{\psi}))_{(\mathcal{V}^\epsilon)', \mathcal{V}^\epsilon} = (\mathcal{E}^\epsilon(\vec{\alpha}, \vec{\beta}), (\vec{\phi}, \vec{\psi}))_{(\mathcal{V}^\epsilon)', \mathcal{V}^\epsilon} - \int_0^T (\vec{u}^\epsilon \cdot \nabla \alpha_j, \varphi_j)_{\Omega^\epsilon} dt$$

and the new force term

$$\tilde{\Phi} = (F_1 - \vec{u}^\epsilon \cdot \nabla \alpha_1, \dots, F_m - \vec{u}^\epsilon \cdot \nabla \alpha_m, G_1, \dots, G_m)$$

and apply Corollaire II.2. Q.E.D.

PROPOSITION 3 : Let

$$\|F_j\|_{L^\infty(Q^\epsilon)} \leq A_j C_j^v \quad \text{and} \quad \|G_j\|_{L^\infty(R^\epsilon)} \leq B_j C_j^w \quad \forall j \in J.$$

Then the solution $\gamma = (\alpha, \beta)$ in proposition 2 satisfies

$$\alpha_j \leq C_j^v \text{ a.e. on } Q^\epsilon \text{ and } \beta_j \leq C_j^w \text{ a.e. on } R^\epsilon.$$

Furthermore, if $F_j \geq 0$ and $G_j \geq 0$, then $\alpha_j \geq 0$ and $\beta_j \geq 0 \quad \forall j \in J$.

Proof: We restrict ourselves to the case $j \in J_5$, since the other cases are similar. The variational inequality 9 is also valid for $[0, t] \times \Omega^\epsilon$ and $[0, t] \times \Pi^\epsilon$ instead of Q^ϵ and R^ϵ , resp. Let $\bar{\alpha}_j = (\alpha_j - C_j^v)_+$ and $\bar{\beta}_j = (\beta_j - C_j^w)_+$. First we use the test functions $\varphi_j = \alpha_j - \bar{\alpha}_j$ and $\psi_j = \beta_j - \bar{\beta}_j$ for the inequality and $\varphi_j = \alpha_j$ and $\psi_j = \beta_j$ for the corresponding equality. By subtracting the two we get

$$\begin{aligned} \sum_{j \in J} \int_0^t ((\partial_t \alpha_j, \bar{\alpha}_j)_{\Omega^\epsilon} + (\partial_t \beta_j, \bar{\beta}_j)_{\Pi^\epsilon}) d\tau &\leq \\ &\leq - \sum_{j \in J} \int_0^t ((\vec{u}^\epsilon \cdot \nabla \alpha_j, \bar{\alpha}_j)_{\Omega^\epsilon} - \\ &\quad - (A_j \alpha_j - F_j, \bar{\alpha}_j)_{\Omega^\epsilon} - (B_j \beta_j - G_j, \bar{\beta}_j)_{\Pi^\epsilon}) d\tau. \end{aligned}$$

From this we have

$$\begin{aligned} \frac{1}{2} \sum_{j \in J} \int_0^t \left(\frac{d}{d\tau} \left(\|\bar{\alpha}_j\|_{\Omega^\varepsilon}^2 + \|\bar{\beta}_j\|_{H^\varepsilon}^2 \right) + (u^\varepsilon, \nabla |\bar{\alpha}_j|^2)_{\Omega^\varepsilon} \right) d\tau &\leq \\ &\leq \sum_{j \in J} \int_0^t \left((A_j \alpha_j - F_j, \bar{\alpha}_j)_{\Omega^\varepsilon} + (B_j \beta_j - G_j, \bar{\beta}_j)_{H^\varepsilon} \right) d\tau. \end{aligned}$$

By integrating over $[0, t]$ we get

$$\|\bar{\alpha}_j(t)\|_{\Omega^\varepsilon}^2 + \|\bar{\beta}_j(t)\|_{H^\varepsilon}^2 \leq \|\bar{\alpha}_j(0)\|_{\Omega^\varepsilon}^2 + \|\bar{\beta}_j(0)\|_{H^\varepsilon}^2 = 0$$

because of the assumptions on F_j , G_j , \vec{v}_l , and \vec{w}_l . Hence we get the upper bounds for α_j and β_j . In order to prove the non-negativity, we use $\bar{\alpha}_j = (\alpha_j)_-$ and $\bar{\beta}_j = (\beta_j)_-$ and plug the test functions $\varphi_j = -\bar{\alpha}_j$ and $\psi_j = -\bar{\beta}_j$ into the variational inequality 9 and get

$$\begin{aligned} \sum_{j \in J} \int_0^t \left((\partial_t \alpha_j, \bar{\alpha}_j)_{Q^\varepsilon} + (\partial_t \beta_j, \bar{\beta}_j)_{R^\varepsilon} \right) d\tau + \\ + \sum_{j \in J} \int_0^t \left((\vec{u}^\varepsilon \cdot \nabla \alpha_j, \bar{\alpha}_j)_{Q^\varepsilon} - (A_j \alpha_j - F_j, \bar{\alpha}_j)_{Q^\varepsilon} - \right. \\ \left. - (B_j \beta_j - G_j, \bar{\beta}_j)_{R^\varepsilon} \right) d\tau \leq 0. \end{aligned}$$

Using $\bar{\alpha}_j(0) = (v_{jl})_- = 0$ and $\bar{\beta}_j(0) = (w_{jl})_- = 0$ we conclude

$$\begin{aligned} \frac{1}{2} \sum_{j \in J} \left(\|\bar{\alpha}_j(t)\|_{\Omega^\varepsilon}^2 + \|\bar{\beta}_j(t)\|_{H^\varepsilon}^2 \right) &\leq \\ &\leq \sum_{j \in J} \int_0^t \left(A_j \|\bar{\alpha}_j\|_{\Omega^\varepsilon}^2 + B_j \|\bar{\beta}_j\|_{H^\varepsilon}^2 - (F_j, \bar{\alpha}_j)_{\Omega^\varepsilon} - (G_j, \bar{\beta}_j)_{H^\varepsilon} \right) d\tau. \end{aligned}$$

Since $F_j, G_j \geq 0$, Gronwall's inequality yields $\bar{\alpha}_j = \bar{\beta}_j = 0$. Q.E.D.

THEOREM 1 : *The micro-model problem 1 has at least one solution.*

Proof : We consider the mapping $\mathcal{G} : \mathcal{H}^\varepsilon \rightarrow C([0, T]; \mathcal{V}^\varepsilon)$ defined by $\Phi = (\vec{F}, \vec{G}) \rightarrow \mathcal{G}(\Phi) = \gamma = (\vec{\alpha}, \vec{\beta})$ according to theorem 2. From theorem 3 we get the following estimates

$$\begin{aligned} \|\alpha_j\|_{L^\infty(0, T; L^2(\Omega^\varepsilon))} &\leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}, \\ \|\beta_j\|_{L^\infty(0, T; L^2(H^\varepsilon))} &\leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}, \\ \|\alpha_j\|_{L^2(0, T; H^1(\Omega^\varepsilon))} &\leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}, \end{aligned}$$

and

$$\|\beta_j\|_{L^2(0, T; H^1(\Pi^\varepsilon))} \leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}.$$

We also have

$$\|\alpha_j\|_{L^\infty(Q^\varepsilon)} \leq C_3$$

and

$$\|\beta_j\|_{L^\infty(R^\varepsilon)} \leq C_3.$$

The smoothness of \vec{u}^ε enables us to also get the estimates

$$\left\| \frac{d\alpha_j}{dt} \right\|_{L^2(Q^\varepsilon)} \leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}$$

and

$$\left\| \frac{d\beta_j}{dt} \right\|_{L^2(R^\varepsilon)} \leq C_1 + C_2 \|\Phi\|_{\mathcal{H}^\varepsilon}.$$

All these estimates imply that the mapping \mathcal{G} maps a ball of radius r in \mathcal{H}^ε into a ball of radius $\max \{C_1 + C_2 r, C_3\}$ in the space

$$\mathcal{W}^\varepsilon = \left\{ \gamma = (\vec{\alpha}, \vec{\beta}) \in \mathcal{V}^\varepsilon : \alpha_j \in L^\infty(Q^\varepsilon), \beta_j \in L^\infty(R^\varepsilon), \frac{d\gamma}{dt} \in \mathcal{H}^\varepsilon \right\},$$

which is compactly imbedded in \mathcal{V}^ε .

For a given γ we define

$$\mathcal{F}(\gamma) = \Phi = (\vec{F}, \vec{G}) = (F_1, \dots, F_m, G_1, \dots, G_m)$$

by $F_j = \tilde{f}_j(\alpha)$ and $G_j = \tilde{f}_j(\beta) \forall j \in J$. We also define $\mathcal{F}^C(\gamma)$ by

$$F_j^C = \begin{cases} \tilde{f}_j & \text{if } |\alpha| \leq C \\ \tilde{f}_j^C & \text{if } |\alpha| > C \end{cases} \quad \text{and} \quad G_j^C = \begin{cases} \tilde{g}_j & \text{if } |\alpha| \leq C \\ \tilde{g}_j^C & \text{if } |\alpha| > C \end{cases},$$

where

$$\tilde{f}_j^C = \max \{ \tilde{f}_j(\alpha) : |\alpha| \leq C \} \quad \text{and} \quad \tilde{g}_j^C = \max \{ \tilde{g}_j(\alpha) : |\alpha| \leq C \}.$$

Then the mapping $\mathcal{G} \circ \mathcal{F}^C$ maps a bounded set in \mathcal{H}^ε into a compact set in \mathcal{H}^ε . It remains to check that $\mathcal{G} \circ \mathcal{F}^C$ is continuous. Let ζ^k be a sequence

converging in \mathcal{H}^ε to ζ and let $\gamma^k = \mathcal{G} \circ \mathcal{F}^C(\zeta^k)$. Then we can extract a subsequence γ^{k_i} converging weakly in $\tilde{\mathcal{W}}$ and strongly in \mathcal{H}^ε to some γ , where $\tilde{\mathcal{W}} = \left\{ \gamma \in \mathcal{V}^\varepsilon : \frac{d\gamma}{dt} \in \mathcal{H}^\varepsilon \right\}$. Obviously γ is a solution of problem 5 which is unique. Therefore, the whole sequence converges to γ , and one has $\gamma = \mathcal{G} \circ \mathcal{F}^C(\zeta)$. Thus $\mathcal{G} \circ \mathcal{F}^C$ is continuous. Now Schauder's fixed point theorem implies that $\mathcal{G} \circ \mathcal{F}^C$ has at least one fixed point.

The estimates from theorem 3 ensure that the solution γ does not depend on $C \geq C_0$ for some C_0 . Therefore, $\mathcal{G} \circ \mathcal{F}$ also has at least one fixed point. Q.E.D.

6. A PRIORI ESTIMATES

PROPOSITION 4 : *Let $(\vec{u}^\varepsilon, p^\varepsilon) \in (H_0^1(\Omega^\varepsilon))^n \times L_0^2(\Omega^\varepsilon)$ be a weak solution of system 1. Then we have the estimates*

$$\|\vec{u}^\varepsilon\|_{(L^2(\Omega^\varepsilon))^n} \leq C \quad \text{and} \quad \|p^\varepsilon\|_{L_0^2(\Omega^\varepsilon)} \leq C .$$

Proof : This is a direct consequence of results in [26] which generalize results in [35] from homogeneous to non-homogeneous boundary conditions. For the case of more complicated geometries of the cells see the paper [2]. Q.E.D.

PROPOSITION 5 : *For any solution of the micro-model problem 2 the estimates*

$$\|v_j^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \leq C_j^v, \quad \|w_j^\varepsilon\|_{L^\infty(\mathbb{R}^\varepsilon)} \leq C_j^w, \quad (11)$$

$$\|v_j^\varepsilon\|_{L^2(0, T; H^1(\Omega^\varepsilon))} \leq C, \quad \varepsilon \|\nabla w_j^\varepsilon\|_{L^2(0, T; L^2(\Pi^\varepsilon))} \leq C, \quad (12)$$

$$\sqrt{\varepsilon} b_j \| (v_j^\varepsilon - w_j^\varepsilon)_\pm \|_{L^2(0, T; L^2(\Gamma^\varepsilon))} \leq C. \quad (13)$$

and

$$\|\partial_t v_j^\varepsilon\|_{L^\infty(0, T; L^2(\Omega^\varepsilon))} \leq C, \quad \|\partial_t w_j^\varepsilon\|_{L^\infty(0, T; L^2(\Pi^\varepsilon))} \leq C,$$

$$\|\nabla \partial_t v_j^\varepsilon\|_{L^2(0, T; L^2(\Omega^\varepsilon))} \leq C, \quad \varepsilon \|\nabla \partial_t w_j^\varepsilon\|_{L^2(0, T; L^2(\Pi^\varepsilon))} \leq C. \quad (14)$$

hold independently of ε .

Proof: First we treat the situation « N ». We plug the test functions $\varphi_j = v_j^\varepsilon$ and $\psi_j = w_j^\varepsilon$ into equation 5 (with t instead of T) and get

$$\begin{aligned} & \frac{1}{2} \sum_{j \in J} (\|v_j^\varepsilon(t)\|_{\Omega^\varepsilon}^2 + \|w_j^\varepsilon(t)\|_{\Pi^\varepsilon}^2) + \\ & + \sum_{j \in J} \int_0^t (d_j \|\nabla v_j^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon^2 c_j \|\nabla w_j^\varepsilon\|_{\Pi^\varepsilon}^2 + A_j \|v_j^\varepsilon\|_{\Omega^\varepsilon}^2 + B_j \|w_j^\varepsilon\|_{\Pi^\varepsilon}^2) d\tau \\ & + \sum_{j \in J} \int_0^t \varepsilon b_j ((v_j^\varepsilon - w_j^\varepsilon)_+)^2 d\tau = \\ & = \frac{1}{2} \sum_{j \in J} (\|v_{jI}\|_{\Omega^\varepsilon}^2 + \|w_{jI}\|_{\Pi^\varepsilon}^2) + \\ & + \sum_{j \in J} \int_0^t ((\tilde{f}_j(\vec{v}^\varepsilon), v_j^\varepsilon)_{\Omega^\varepsilon} + (\tilde{g}_j(\vec{w}^\varepsilon), w_j^\varepsilon)_{\Pi^\varepsilon}) d\tau . \end{aligned}$$

From proposition 3 we have the estimates 11 ; therefore, we get also the estimates 12 and 13.

The next step is to prove estimates for the time derivatives of v_j^ε and w_j^ε . For that purpose we rewrite 5 in equivalent form :

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, \varphi_j)_{\Omega^\varepsilon} + (\partial_t w_j^\varepsilon, \psi_j)_{\Pi^\varepsilon} + \\ & + d_j (\nabla v_j^\varepsilon, \nabla \varphi_j)_{\Omega^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla \psi_j)_{\Pi^\varepsilon} \\ & + (\vec{u}^\varepsilon \cdot \nabla v_j^\varepsilon, \varphi_j)_{\Omega^\varepsilon} - (f_j(\vec{v}^\varepsilon), \varphi_j)_{\Omega^\varepsilon} - (g_j(\vec{w}^\varepsilon), \psi_j)_{\Pi^\varepsilon}) \\ & + \sum_{j \in J_3} \varepsilon b_j ((v_j^\varepsilon - w_j^\varepsilon)_+, \varphi_j - \psi_j)_{\Gamma^\varepsilon} = 0 \text{ a.e. on } (0, T) \quad \forall (\vec{\varphi}, \vec{\psi}) \in V^\varepsilon \end{aligned}$$

with $(\vec{v}^\varepsilon, \vec{w}^\varepsilon)|_{t=0} = (\vec{v}_I, \vec{w}_I)$. We differentiate this equation (more precisely, we take difference quotients in time and pass to the limit) and get

$$\begin{aligned} & \sum_{j \in J} ((\partial_{tt} v_j^\varepsilon, \varphi_j)_{\Omega^\varepsilon} + (\partial_{tt} w_j^\varepsilon, \psi_j)_{\Pi^\varepsilon} + \\ & + d_j (\nabla \partial_t v_j^\varepsilon, \nabla \varphi_j)_{\Omega^\varepsilon} + \varepsilon^2 c_j (\nabla \partial_t w_j^\varepsilon, \nabla \psi_j)_{\Pi^\varepsilon} \\ & + (\vec{u}^\varepsilon \cdot \nabla \partial_t v_j^\varepsilon, \varphi_j)_{\Omega^\varepsilon} - (\nabla f_j(\vec{v}^\varepsilon) \partial_t \vec{v}^\varepsilon, \varphi_j)_{\Omega^\varepsilon} - (\nabla g_j(\vec{w}^\varepsilon) \partial_t \vec{w}^\varepsilon, \psi_j)_{\Pi^\varepsilon}) \\ & + \sum_{j \in J_3} \varepsilon b_j \left(\frac{(v_j^\varepsilon - w_j^\varepsilon)_+}{|v_j^\varepsilon - w_j^\varepsilon|} \partial_t (v_j^\varepsilon - w_j^\varepsilon)_+, \varphi_j - \psi_j \right)_{\Gamma^\varepsilon} = \\ & = 0 \text{ a.e. on } (0, T) \quad \forall (\vec{\varphi}, \vec{\psi}) \in V^\varepsilon . \end{aligned}$$

Now we use the test functions $\varphi_j = \partial_t v_j^\varepsilon$ and $\psi_j = \partial_t w_j^\varepsilon$ and get by integration over time

$$\begin{aligned}
& \frac{1}{2} \sum_{j \in J} (\|\partial_t v_j^\varepsilon(t)\|_{\Omega^\varepsilon}^2 + \|\partial_t w_j^\varepsilon(t)\|_{\Gamma^\varepsilon}^2) + \\
& + \sum_{j \in J} \int_0^t (d_j \|\nabla \partial_t v_j^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon^2 c_j \|\nabla \partial_t w_j^\varepsilon\|_{\Gamma^\varepsilon}^2) d\tau \\
& + \sum_{j \in J_3} \int_0^t \varepsilon b_j \left\| \frac{(v_j^\varepsilon - w_j^\varepsilon)_+}{|v_j^\varepsilon - w_j^\varepsilon|} |\partial_t (v_j^\varepsilon - w_j^\varepsilon)_+| \right\|_{\Gamma^\varepsilon}^2 d\tau \\
& = \frac{1}{2} \sum_{j \in J} (\|\hat{v}_{jI}\|_{\Omega^\varepsilon}^2 + \|\hat{w}_{jI}\|_{\Gamma^\varepsilon}^2) \\
& + \sum_{j \in J} \int_0^t ((\nabla f_j(\vec{v}^\varepsilon) \partial_t \vec{v}^\varepsilon, \partial_t \vec{v}^\varepsilon)_{\Omega^\varepsilon} + (\nabla g_j(\vec{w}^\varepsilon) \partial_t \vec{w}^\varepsilon, \partial_t \vec{w}^\varepsilon)_{\Gamma^\varepsilon}) d\tau .
\end{aligned}$$

We get immediately the estimates 14. Now we come to the situation « U ». From the equation in 4 (with t instead of T) we get

$$\begin{aligned}
& \frac{1}{2} \sum_{j \in J} (\|v_j^\varepsilon(t)\|_{\Omega^\varepsilon}^2 + \|w_j^\varepsilon(t)\|_{\Gamma^\varepsilon}^2) + \\
& + \sum_{j \in J} \int_0^t (d_j \|\nabla v_j^\varepsilon\|_{\Omega^\varepsilon}^2 + \varepsilon^2 c_j \|\nabla w_j^\varepsilon\|_{\Gamma^\varepsilon}^2 + A_j \|v_j^\varepsilon\|_{\Omega^\varepsilon}^2 + B_j \|w_j^\varepsilon\|_{\Gamma^\varepsilon}^2) d\tau \\
& = \frac{1}{2} \sum_{j \in J} (\|v_{jI}^2\|_{\Omega^\varepsilon}^2 + \|w_{jI}^\varepsilon\|_{\Gamma^\varepsilon}^2) + \\
& + \sum_{j \in J} \int_0^t ((\tilde{f}_j(\vec{v}^\varepsilon), v_j^\varepsilon)_{\Omega^\varepsilon} + (\tilde{g}_j(\vec{w}^\varepsilon), w_j^\varepsilon)_{\Gamma^\varepsilon}) d\tau .
\end{aligned}$$

From theorem 3 we know the estimates 11 from which we get the bounds 12. In order to prove estimates for the time derivatives of v_j^ε and w_j^ε , we consider the transmission conditions of cases 5 and 6 as limits $b_j \rightarrow \infty$ of the cases 3 and 4. Since the estimates for these cases do not depend on b_j , we get for case « U » also the estimates 14. Q.E.D.

Remark : We have not investigated the question of uniqueness of the solutions \vec{v}^ε and \vec{w}^ε of the micro-model for case « U ». The estimates of proposition 5 apply only to those functions that can be obtained as limits $b_j \rightarrow \infty$ of the corresponding problems in the situation « N ». Therefore, the convergence result of section 8 and 9 is applicable only to these solutions of the micro-model.

7. UNIQUENESS FOR THE MACRO-MODEL

THEOREM 2 : *The macro-model problem 4 has at most one solution.*

Proof : Since the variational inequality 8 is of the type of a « moving obstacle » problem, we transform it into one with « zero obstacle ». Let $\vec{\zeta} = \vec{w} - \vec{v}$ (remember $a_j = 1 \forall j \in J$) ; then we get instead of the inequality 8

$$\sum_{j \in J} ((\partial_t \zeta_j, \sigma_j - \zeta_j)_U + c_j (\nabla_y w_j, \nabla_y (\sigma_j - \zeta_j))_U) \geq \sum_{j \in J} (-\partial_t v_j + g_j(\vec{\zeta} + \vec{v}), \sigma_j - \zeta_j)_U \quad \forall \vec{\sigma} \in W \text{ with } \vec{\sigma} \geq 0 \text{ a.e. on } \Gamma, \quad (15)$$

where now $\vec{\zeta}(0) = \vec{w}_I - \vec{v}_I$ in Y and $\vec{\zeta} \geq 0$ a.e. on Γ . We suppose that there are two solutions $(\vec{v}^1, \vec{\zeta}^1)$ and $(\vec{v}^2, \vec{\zeta}^2)$ of the variational equality 7 and the variational inequalities 15, resp. Then we define $\vec{v} = \vec{v}^1 - \vec{v}^2$ and $\vec{\zeta} = \vec{\zeta}^1 - \vec{\zeta}^2$. We plug $\vec{\sigma} = \vec{\zeta}^2$ into the inequality 15 with $\vec{\zeta}^1$ and $\vec{\sigma} = \vec{\zeta}^2$ into inequality 15 with $\vec{\zeta}^2$. Thus we get

$$\sum_{j \in J} \int_0^t ((\partial_t v_j, \zeta_j)_U + (\partial_t \zeta_j, \zeta_j)_U + c_j (\nabla_y \zeta_j, \nabla_y \zeta_j)_U) d\tau \leq \sum_{j \in J} \int_0^t (g_j(\vec{\zeta} + \vec{v}^1) - g_j(\vec{\zeta} + \vec{v}^2), \zeta_j)_U d\tau. \quad (16)$$

We rewrite the source terms S_j^i (with $j \in J, i = 1, 2$) as

$$S_j^i = \int_Y (\partial_t w_j^i - g_j(\vec{w}^i)) dy = |Y| \partial_t v_j^i + \int_Y \partial_t \zeta_j^i dy - \int_Y g_j(\vec{w}^i) dy$$

and get from equation 7 the equations

$$\begin{aligned} \sum_{j \in J} \int_0^t ((\partial_t v_j^i, v_j)_\Omega + (\partial_t \zeta_j^i, v_j)_U) d\tau = \\ = \sum_{j \in J} \int_0^t (-d_j (D \nabla v_j^i, \nabla v_j)_\Omega + |X| (f_j(\vec{v}^i), v_j)_\Omega + \\ + (g_j(\vec{w}^i), v_j)_U - (\vec{u} \cdot \nabla v_j^i, v_j)_\Omega) d\tau \end{aligned}$$

for $i = 1, 2$. Subtracting these two equations and using $(\vec{u} \cdot \nabla v_j, v_j)_\Omega = 0$ we get

$$\begin{aligned} \sum_{j \in J} \int_0^t ((\partial_t v_j, v_j)_\Omega + (\partial_t \zeta_j^i, v_j)_U) d\tau = \sum_{j \in J} \int_0^t (-d_j (D \nabla v_j, \nabla v_j)_\Omega + \\ + |X| (f_j(\vec{v}^1) - f_j(\vec{v}^2), v_j)_\Omega + (g_j(\vec{w}^1) - g_j(\vec{w}^2), v_j)_U) d\tau. \quad (17) \end{aligned}$$

Subtracting the inequality (16) and equation (17) we obtain

$$\begin{aligned}
& \sum_{j \in J} ((v_j(t), \zeta_j(t))_U + \\
& + \frac{1}{2} (v_j(t), v_j(t))_\Omega + \frac{1}{2} (\zeta_j(t), \zeta_j(t))_U \\
& + \int_0^t (d_j(D \nabla v_j, \nabla v_j)_\Omega + c_j(\nabla_y \zeta_j, \nabla_y \zeta_j)_U) d\tau) \\
& \leq \sum_{j \in J} \int_0^t (|X| (f_j(\vec{v}^1) - f_j(\vec{v}^2), v_j)_\Omega + (g_j(\vec{w}^1) - g_j(\vec{w}^2), v_j + \zeta_j)_U) d\tau \\
& \leq C \sum_{j \in J} \int_0^t (\|v\|_{L^2(\Omega)}^2 + \|\zeta\|_{L^2(U)}^2) d\tau.
\end{aligned}$$

To this inequality we apply

$$|(v_j, \zeta_j)_U| \leq \frac{|Y|}{1-\delta} \|v_j\|_\Omega^2 + \frac{1-\delta}{2} \|\zeta_j\|_U^2$$

for arbitrary $0 < \delta < 1$ and get

$$\begin{aligned}
\frac{\delta}{2} \|\zeta_j(t)\|_U^2 + \frac{1}{2} \left(|X| - |Y| \frac{\delta}{1-\delta} \right) \|v_j(t)\|_\Omega^2 & \leq \\
& \leq C \sum_{j \in J} \int_0^t (\|v_j\|_{L^2(\Omega)}^2 + \|\zeta_j\|_{L^2(U)}^2) d\tau.
\end{aligned}$$

Choosing δ such that $|X| - |Y| \frac{\delta}{1-\delta} > 0$ and applying Gronwall's inequality we obtain $\vec{v} = 0$ and $\vec{\zeta} = 0$. Q.E.D.

8. CONVERGENCE

PROPOSITION 6 : *Let \vec{u}^ε be extended by zero to $\Omega \setminus \Omega^\varepsilon$. Then there exists an extension \tilde{p}^ε of the pressure p^ε such that*

$$\begin{aligned}
\vec{u}^\varepsilon & \rightharpoonup \vec{u} \quad \text{weakly in } (L^2(\Omega))^n \\
\tilde{p}^\varepsilon & \rightarrow p \quad \text{strongly in } L_0^2(\Omega)
\end{aligned}$$

with $(\vec{u}, p) \in (L^2(\Omega))^n \times L_0^2(\Omega)$ being a solution of system 6.

Proof: This result is contained in [26] and [35]. An explicit formula for the extension of the pressure can be found in [23]. Q.E.D.

Let us note that the result of the previous proposition remains also valid for the case of Navier-Stokes equations with the convergence of the pressure taking place in some appropriate functional spaces ; for more details see [25]. For simplicity of the notations we are going to identify p^ε and \tilde{p}^ε .

LEMMA 1 : *There exists an extension operator $P \in \mathcal{L}(H^1(X), H^1(Z))$ such that*

$$\|\nabla P \varphi\|_{(L^2(Z))^n} \leq \|\nabla \varphi\|_{(L^2(X))^n} \quad \forall \varphi \in H^1(X).$$

Furthermore, there is an extension operator $P^\varepsilon \in \mathcal{L}(H^1(\Omega^\varepsilon), H^1(\Omega))$ such that

$$\|\nabla P^\varepsilon \varphi\|_{(L^2(\Omega))^n} \leq \|\nabla \varphi\|_{(L^2(\Omega^\varepsilon))^n} \quad \forall \varphi \in H^1(\Omega^\varepsilon).$$

Proof : For the construction of P see [12] pp. 603-604. The operator P^ε is defined in the following way ; for $\varphi \in H^1(\Omega^\varepsilon)$ let $y = \frac{x}{\varepsilon}$ and $\tilde{\varphi}(y) = \frac{1}{\varepsilon} \varphi(\varepsilon y)$; then $P^\varepsilon \varphi = \varepsilon P \tilde{\varphi}\left(\frac{x}{\varepsilon}\right)$, see also [20] Lemma 5. Q.E.D.

Notations : For the rest of this section we use the following functions : let $\mu_z^\varepsilon(x) = \varepsilon P \mu_z\left(\frac{x}{\varepsilon}\right)$ (see section 2.6 and Lemma 1), $\vec{\eta}_z = \nabla_y \mu_z$ (extended by zero in the interior of Y), and $\vec{\eta}_z^\varepsilon(x) = \vec{\eta}_z\left(\frac{x}{\varepsilon}\right)$.

PROPOSITION 7 : *There is a subsequence such that*

$$\begin{aligned} \mu_z^\varepsilon &\rightharpoonup \mu_z^* \quad \text{weakly in } H^1(\Omega), \\ \nabla \mu_z^\varepsilon &\rightharpoonup z \quad \text{weakly in } (L^2(\Omega))^n, \\ \vec{\eta}_z^\varepsilon &\rightharpoonup Dz \quad \text{weakly in } (L^2(\Omega))^n \end{aligned}$$

for any $z \in \mathbb{R}^n$ with some μ_z^* which satisfies $\nabla \mu_z^* = z$.

Proof : This follows by standard arguments, see, e.g. [12] page 597 or [20] proposition 1. Q.E.D.

From now on we identify v_j^ε with $P^\varepsilon v_j^\varepsilon$. Let the vector field $\vec{\xi}_j^\varepsilon$ be defined as ∇v_j^ε in Ω^ε and by $\varepsilon^2 \nabla w_j^\varepsilon$ in the interior of Π^ε . Then the extension lemma from [12] pp. 593-597 implies that $\vec{\xi}_j^\varepsilon$ and $\nabla \cdot \vec{\xi}_j^\varepsilon$ are uniformly bounded in $(L^2(Q))^n$ and $L^2(Q)$ resp.

LEMMA 2 : *There is an extension operator $\tilde{P} \in \mathcal{L}(H^1(Y), H^1(Z)) \cap \mathcal{L}(L^\infty(Y), L^\infty(Z))$ such that*

$$\|\nabla_x \tilde{P} \varphi\|_{(L^2(Z))^n} \leq \|\nabla_x \varphi\|_{(L^2(Y))^n}.$$

Furthermore, there is an extension operator $\tilde{P}^\varepsilon \in \mathcal{L}(H^1(\Pi^\varepsilon), H^1(\Omega)) \cap \mathcal{L}(L^\infty(\Pi^\varepsilon), L^\infty(\Omega))$ such that

$$\|\nabla_x \tilde{P}^\varepsilon \varphi\|_{(L^2(\Omega))^n} \leq \|\nabla_x \varphi\|_{(L^2(\Pi^\varepsilon))^n}.$$

Proof: There is an open set Y_0 with smooth boundary such that $\text{clos}(Y) \subset Y_0$ and $\text{clos}(Y_0) \subset \text{int}(Z)$. Then one can find an extension operator $P_0 \in \mathcal{L}(H^1(Y), H_0^1(Y_0)) \cap \mathcal{L}(L^\infty(Y), L^\infty(Y_0))$ with the desired properties. If one extends this trivially into all Z , one gets \tilde{P} . The construction of \tilde{P}^ε is obvious by summation over the individual cells. Q.E.D.

We now identify w_j^ε with its extension according to lemma 2; then w_j^ε is uniformly bounded in $L^\infty(Q)$, and $\varepsilon \nabla w_j^\varepsilon$ is uniformly bounded in $L^2(Q)$. Now we get

PROPOSITION 8 : *There is a subsequence such that*

$$\begin{aligned} v_j^\varepsilon &\rightharpoonup v_j^* && \text{weakly in } L^2(0, T; H^1(\Omega)), \\ v_j^\varepsilon &\rightharpoonup v_j^* && \text{weak}^* \text{ in } L^\infty(Q), \\ \partial_t v_j^\varepsilon &\rightharpoonup \partial_t v_j^* && \text{weakly in } L^2(Q), \\ v_j^\varepsilon &\rightarrow v_j^* && \text{strongly in } C([0, T]; L^2(\Omega)), \\ \vec{\xi}_j^\varepsilon &\rightharpoonup \vec{\xi}_j^* && \text{weakly in } (L^2(Q))^n \end{aligned}$$

with some v_j^* and $\vec{\xi}_j^*$, such that $\nabla \cdot \vec{\xi}_j^*$ is bounded in $L^2(Q)$.

Proof: This follows from proposition 5. Note that strong convergence in $C([0, T]; L^2(\Omega))$ is a consequence of proposition 5 and results from [34]. Q.E.D.

In order to prove the main convergence result of this paper, namely theorem 3, we use the notion of *two-scale convergence* which was introduced in [29] and developed further in [3]. The idea behind this concept was used in [4].

DEFINITION 1 : *The sequence $\{w^\varepsilon\} \subset L^2(Q)$ is said to two-scale converge to a limit $w \in L^2(Q \times Z)$ iff for any $\sigma \in C^\infty(Q; C_{\text{per}}^\infty(Z))$ (« per » denotes Z -periodicity) one has*

$$\lim_{\varepsilon \rightarrow 0} \int_Q w^\varepsilon(t, x) \sigma\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \int_Q \int_Z w(t, x, y) \sigma(t, x, y) dy dx dt.$$

LEMMA 3 : *From each bounded sequence in $L^2(Q)$ one can extract a subsequence which two-scale converges to a limit $w \in L^2(Q \times Z)$.*

Proof : See [29]. Q.E.D.

LEMMA 4 : Let w^ϵ and $\epsilon \nabla w^\epsilon$ be bounded sequences in $L^2(Q)$. Then there exists a function $w \in L^2(Q ; H^1_{\text{per}}(z))$ and a subsequence such that both w^ϵ and $\epsilon \nabla w^\epsilon$ two-scale converge to w and $\nabla_y w$, resp.

Proof : See [3] and [29]. Q.E.D.

Remark : Let $\sigma \in L^2_{\text{per}}(Z)$, define $\sigma^\epsilon(x) = \sigma\left(\frac{x}{\epsilon}\right)$, and let the sequence $\{w^\epsilon\} \subset L^2(Q)$ two-scale converge to a limit $w \in L^2(Q \times Z)$. Then $\{\sigma^\epsilon w^\epsilon\}$ two-scale converges to a limit σw .

PROPOSITION 9 : There is a subsequence such that

$$\begin{aligned} w_j^\epsilon &\rightarrow w_j^* \\ \partial_t w_j^\epsilon &\rightarrow \partial_t w_j^* \\ \epsilon \nabla w_j^\epsilon &\rightarrow \nabla_y w_j^* \\ w_j^\epsilon(T) - w_{jI} &\rightarrow w_j^*(T) - w_j^*(0) \end{aligned}$$

in the two-scale sense with some $w_j^* \in L^2(Q ; H^1_{\text{per}}(Z)) \cap H^1(0, T ; L^2(\Omega \times Z))$.

Proof : The first three statements follow from the estimates of proposition 5 and lemmas 3 and 4. The last one is obtained by choosing a $\sigma \in C^\infty(\Omega ; C^\infty_{\text{per}}(Z))$ and observing

$$\lim_{\epsilon \rightarrow 0} \int_Q \partial_t w^\epsilon(t, x) \sigma\left(x, \frac{x}{\epsilon}\right) dx dt = \int_Q \int_Z \partial_t w(t, x, y) \sigma(x, y) dy dx dt .$$

Integration by parts with respect to time yields the result. Q.E.D.

PROPOSITION 10 : The function w_j^* defined in proposition 9 satisfies

$$w_j^*|_{t=0} = w_{jI} \quad \text{a.e. on } \Omega \times Y .$$

Proof : Let $\vec{\varphi} \in (C^\infty_0(\Omega))^m$ and $\omega \in C^\infty_0(0, T)$ with $\omega(T) = 0$. Furthermore, let $\sigma \in C^\infty_0(Y)$. We extend σ by zero to $Z \setminus Y$ and Z -periodically to \mathbb{R}^n . We define σ^ϵ by $\sigma^\epsilon(x) = \sigma\left(\frac{x}{\epsilon}\right)$. Then we have

$$(\partial_t w_j^\epsilon, \varphi_j \sigma^\epsilon \omega)_{R^\epsilon} = - (w_{jI}, \varphi_j \sigma^\epsilon \omega(0))_{\Pi^\epsilon} - (w_j^\epsilon, \varphi_j \sigma^\epsilon \partial_t \omega)_{R^\epsilon} .$$

Passing to the limit yields

$$\begin{aligned} \int_U \partial_t w_j^* \sigma \, dy \, \varphi_j \, dx \, \omega \, dt &= \\ &= - \int_{\Omega} \int_Y w_{jI} \sigma \, dy \, \varphi_j \, dx - \int_U w_j^* \sigma \, dy \, \varphi_j \, dx \, \partial_t \omega \, dt \end{aligned}$$

and thus the equality to be proved. Q.E.D.

Propositions 9 and 10 now imply $w_j^\varepsilon(T) \rightarrow w_j^*(T)$ in the two-scale sense.

LEMMA 5: Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a continuous function satisfying $0 \leq \Phi(\lambda) \leq C(1 + |\lambda|^2)$ for all $\lambda \in \mathbb{R}^n$, $\sigma \in (C_0^\infty(Q; C_{\text{per}}^\infty(Z)))^n$, and $\sigma^\varepsilon(t, x) = \sigma\left(t, x, \frac{x}{\varepsilon}\right)$. Then

$$\lim_{\varepsilon \rightarrow 0} \int_Q \Phi(\sigma^\varepsilon) \, dx \, dt = \int_Q \int_Z \Phi(\sigma) \, dy \, dx \, dt. \tag{18}$$

Furthermore, let Φ in addition be convex. Then, if v^ε is a bounded sequence from $(L^2(\Omega))^n$ which two-scale converges towards v , we have

$$\liminf_{\varepsilon \rightarrow 0} \int_Q \Phi(v^\varepsilon) \, dx \, dt \geq \int_Q \int_Z \Phi(v) \, dy \, dx \, dt. \tag{19}$$

Proof: Firstly, let us note that σ can be approximated by linear combinations of the form

$$\sigma(t, x, y) \approx \sum_k \varphi^k(t, x) \vartheta^k(y)$$

(see [29]); therefore, it is sufficient to prove equation 18 for sums of this type. Now we approximate the integral

$$\int_Q \Phi\left(\sum_k \varphi^k(t, x) \vartheta^k\left(\frac{x}{\varepsilon}\right)\right) \, dx \, dt$$

by

$$\begin{aligned} \sum_{i=1}^{N(\varepsilon)} \int_0^T \int_{\varepsilon(Z+k_i)} \Phi\left(\sum_k \varphi^k(t, x) \vartheta^k\left(\frac{x}{\varepsilon}\right)\right) \, dx \, dt &= \\ &= \varepsilon^n \sum_{i=1}^{N(\varepsilon)} \int_0^T \int_Z \Phi\left(\sum_k \varphi^k(t, x) \vartheta^k(y)\right) \, dx \, dt \end{aligned}$$

and get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Q \Phi \left(\sum_k \varphi^k(t, x) \vartheta^k \left(\frac{x}{\varepsilon} \right) \right) dx dt &= \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} \int_0^T \int_{\varepsilon(Z+k_i)} \Phi \left(\sum_k \varphi^k(t, x) \vartheta^k \left(\frac{x}{\varepsilon} \right) \right) dx dt \\ &= \int_Q \int_Z \Phi \left(\sum_k \varphi^k(t, x) \vartheta^k(y) \right) dy dx dt, \end{aligned}$$

and equation 18 is proved.

Now we are going to prove the inequality 19. Standard convex analysis implies that Φ is a pointwise supremum of a family of continuous affine functions (see, e.g., [13]) and hence

$$\int_Q \Phi(v^\varepsilon) dx dt \geq \int_Q h(v^\varepsilon) dx dt,$$

where h is a continuous affine form on $L^2(\Omega)$. Therefore, we get easily

$$\liminf_{\varepsilon \rightarrow 0} \int_Q \Phi(v^\varepsilon) dx dt \geq \int_Q \int_Z h(v) dy dx dt,$$

and by taking the supremum over affine forms on the right hand side we have inequality 19. Q.E.D.

PROPOSITION 11: *For the subsequences in propositions 8 and 9 the relations*

$$v_j - w_j \in N_j \quad \forall j \in J$$

hold on Λ .

Proof: First we consider the case $j \in J_1$. Let $\vec{\rho} \in (C^\infty(Y))^n$, extend it Z -periodically and define $\vec{\rho}^\varepsilon(x) = \vec{\rho} \left(\frac{x}{\varepsilon} \right)$. Then with $\psi \in C_0^\infty(\Omega)$ and $\omega \in C_0^\infty(0, T)$ we get

$$\begin{aligned} 0 &= \int_0^T \varepsilon(v_j^\varepsilon - w_j^\varepsilon, \vec{v} \cdot \vec{\rho}^\varepsilon \psi \omega)_{\Gamma^\varepsilon} dt = \int_0^T \varepsilon(\nabla \cdot (\vec{\rho}^\varepsilon(v_j^\varepsilon - w_j^\varepsilon) \psi), \omega)_{\Pi^\varepsilon} dt \\ &= \int_0^T \varepsilon(\nabla \cdot \vec{\rho}^\varepsilon(v_j^\varepsilon - w_j^\varepsilon), \psi)_{\Pi^\varepsilon} \omega dt \\ &+ \int_0^T \varepsilon(\vec{\rho}^\varepsilon \cdot \nabla(v_j^\varepsilon - w_j^\varepsilon), \psi)_{\Pi^\varepsilon} \omega dt + O(\varepsilon). \end{aligned}$$

Now we determine the limits of these two terms separately. We have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \varepsilon (\nabla \cdot \vec{\rho}^\varepsilon (v_j^\varepsilon - w_j^\varepsilon), \psi \chi_{\Pi^\varepsilon})_\Omega \omega \, dt &= \\ &= \int_U \nabla_y \cdot \vec{\rho} (v_j^* - w_j^*) \, dy \, \psi \, dx \, \omega \, dt \end{aligned}$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \varepsilon (\vec{\rho}^\varepsilon \cdot \nabla (v_j^\varepsilon - w_j^\varepsilon), \psi \chi_{\Pi^\varepsilon})_\Omega \omega \, dt &= \\ &= \int_U \vec{\rho} \cdot \nabla_y (v_j^* - w_j^*) \, dy \, \psi \, dx \, \omega \, dt . \end{aligned}$$

Therefore, we conclude

$$\int_U \nabla_y \cdot (\vec{\rho} (v_j^* - w_j^*)) \, dy \, \psi \, dx \, \omega \, dt = 0$$

and hence

$$\int_\Gamma \vec{\nu} \cdot \vec{\rho} (v_j^* - w_j^*) \, d\Gamma(y) = 0 .$$

Since $\vec{\rho}$ was arbitrary, we have $v_j^* = w_j^*$ on Γ for $j \in J_1$.

The next case is $j \in J_5$. Here we choose test functions as before and assume $-\vec{\nu} \cdot \vec{\rho} \geq 0$ on Γ , $\psi \geq 0$ on Ω , and $\omega \geq 0$ on $(0, T)$. Then we have

$$0 \leq \int_0^T \varepsilon (-v_j^\varepsilon + w_j^\varepsilon, -\vec{\nu} \cdot \vec{\rho}^\varepsilon \psi \omega)_{\Gamma^\varepsilon} \, dt .$$

In exactly the same way as before we get

$$0 \leq \int_\Gamma -\vec{\nu} \cdot \vec{\rho} (w_j^\varepsilon - v_j^\varepsilon) \, d\Gamma(y) .$$

Since $\vec{\rho}$ was arbitrary, we get $v_j^* \leq w_j^*$ on Γ for $j \in J_5$. The case $j \in J_6$ is similar. Q.E.D.

PROPOSITION 12: *The functions w_j^* from proposition 9 satisfy the variational inequality 8, i.e.,*

$$\begin{aligned} \sum_{j \in J} ((\partial_t w_j^*, \sigma_j - w_j^*)_U + c_j(\nabla_y w_j^*, \nabla_y(\sigma_j - w_j^*))_U - \\ - (g_j(\vec{w}^*), \sigma_j - w_j^*)_U) \\ \geq 0 \quad \forall \sigma \quad \text{with} \quad (\vec{v}, \vec{\sigma}) \in \mathcal{X}. \end{aligned}$$

Proof: We choose $\sigma_j \in C_0^\infty(Q; C_{\text{per}}^\infty(Z))$ such that $\sigma_j = 0$ on Γ for $j \in J_1$, $\sigma_j \leq 0$ for $j \in J_5$, and $\sigma_j \geq 0$ for $j \in J_6$. We define $\sigma_j^\varepsilon(t, x) = \sigma_j\left(t, x, \frac{x}{\varepsilon}\right)$. Then after choosing $\varphi_j = v_j^\varepsilon$ and $\psi_j = \sigma_j^\varepsilon + v_j^\varepsilon$ as test functions we get

$$\begin{aligned} \sum_{j \in J} ((\partial_t w_j^\varepsilon, \sigma_j^\varepsilon + v_j^\varepsilon - w_j^\varepsilon)_{R^\varepsilon} + \\ + \varepsilon^2 c_j(\nabla w_j^\varepsilon, \nabla(\sigma_j^\varepsilon + v_j^\varepsilon - w_j^\varepsilon))_{R^\varepsilon} - (g_j(\vec{w}^\varepsilon), \sigma_j^\varepsilon + v_j^\varepsilon - w_j^\varepsilon)_{R^\varepsilon}) \geq 0. \quad (20) \end{aligned}$$

At this point we suppose that $-g$ is a strictly monotone function. The general case will be considered later. The monotonicity of $-g$ implies that the inequality 20 is equivalent to

$$\begin{aligned} \sum_{j \in J} ((\partial_t w_j^\varepsilon, \sigma_j^\varepsilon + v_j^\varepsilon)_{R^\varepsilon} + \\ + \varepsilon^2 c_j(\nabla w_j^\varepsilon, \nabla \sigma_j^\varepsilon + \nabla v_j^\varepsilon)_{R^\varepsilon} - (g_j(\vec{\sigma}^\varepsilon + \vec{v}^\varepsilon), \sigma_j^\varepsilon + v_j^\varepsilon - w_j^\varepsilon)_{R^\varepsilon}) \\ \geq \sum_{j \in J} (\varepsilon^2 c_j(\nabla w_j^\varepsilon, \nabla w_j^\varepsilon)_{R^\varepsilon} + (\partial_t w_j^\varepsilon, w_j^\varepsilon)_{R^\varepsilon}). \quad (21) \end{aligned}$$

Now we take the limits $\varepsilon \rightarrow 0$ of the terms on the left hand side separately. We have

$$\begin{aligned} (\partial_t w_j^\varepsilon, \sigma_j^\varepsilon + v_j^\varepsilon)_{R^\varepsilon} &\rightarrow (\partial_t w_j^*, \sigma_j + v_j^*)_U \\ (\varepsilon \nabla w_j^\varepsilon, \varepsilon \nabla(\sigma_j^\varepsilon + v_j^\varepsilon))_{R^\varepsilon} &\rightarrow (\nabla_y w_j^*, \nabla_y(\sigma_j + v_j^*))_U. \end{aligned}$$

Let us determine the limit of the term involving g_j . Firstly, the uniform L^∞ -bounds of $(v_j^\varepsilon, w_j^\varepsilon)$, proposition 8, and Lebesgue's dominated convergence theorem imply that the limit is the same, if we replace v_j^ε by v_j^* . Now, using the remark after lemma 3, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} (g_j(\vec{\sigma}^\varepsilon + \vec{v}^\varepsilon), \sigma_j^\varepsilon + v_j^\varepsilon)_{R^\varepsilon} = \\ = \lim_{\varepsilon \rightarrow 0} (g_j(\vec{\sigma}^\varepsilon + \vec{v}^*), \sigma_j^\varepsilon + v_j^*)_{R^\varepsilon} = (g_j(\vec{\sigma} + \vec{v}^*), \sigma_j + v_j^*)_U \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} (g_j(\vec{\sigma}^\varepsilon + \vec{v}^\varepsilon), w_j^\varepsilon)_{R^\varepsilon} = \lim_{\varepsilon \rightarrow 0} (g_j(\vec{\sigma}^\varepsilon + \vec{v}^*), w_j^\varepsilon)_{R^\varepsilon} = (g_j(\vec{\sigma} + \vec{v}^*), w_j^*)_U.$$

For limiting the quadratic terms we use the remark after lemma 3 and proposition 9 and get

$$\chi_{H^\varepsilon} \varepsilon \nabla w_j^\varepsilon \rightarrow \chi_Y \nabla_y w_j^* \quad \text{in the two-scale sense}$$

and

$$\chi_{H^\varepsilon} w_j^\varepsilon(T) \rightarrow \chi_Y w_j^*(T) \quad \text{in the two-scale sense .}$$

Now we directly apply formula 19 from lemma 5 and get

$$\liminf_{\varepsilon \rightarrow 0} \int_{R^\varepsilon} \varepsilon^2 |\nabla w_j^\varepsilon|^2 dx dt \geq \int_U |\nabla_y w_j^*|^2 dy dx dt$$

and

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} (\partial_t w_j^\varepsilon, w_j^\varepsilon)_{R^\varepsilon} &= \liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{2} \|w_j^\varepsilon(T)\|_{H^\varepsilon}^2 - \frac{1}{2} \|w_{jI}\|_{H^\varepsilon}^2 \right) \\ &\geq \frac{1}{2} \|w_j^*(T)\|_U^2 - \frac{|Y|}{2} \|w_{jI}\|_\Omega^2 = (\partial_t w_j^*, w_j^*)_U. \end{aligned}$$

By inserting all these relations into formula 21 we get

$$\begin{aligned} \sum_{j \in J} ((\partial_t w_j^*, \sigma_j + v_j^* - w_j^*)_U + \\ + c_j (\nabla_y w_j^*, \nabla_y (\sigma_j + v_j^* - w_j^*))_U - (g_j(\vec{\sigma} + \vec{v}^*), \sigma_j + v_j^* - w_j^*)_U) \geq 0. \end{aligned}$$

Standard convex analysis now implies

$$\begin{aligned} \sum_{j \in J} ((\partial_t w_j^*, \sigma_j + v_j^* - w_j^*)_U + c_j (\nabla_y w_j^*, \nabla_y (\sigma_j + v_j^* - w_j^*))_U - \\ - (g_j(\vec{w}^*), \sigma_j + v_j^* - w_j^*)_U) \geq 0. \end{aligned}$$

This inequality and proposition 11 imply the result.

If $-g$ is non-monotone, we use \tilde{g} as it was defined in section 2.2. Then we can choose a constant $C \geq 0$ such that $-\tilde{g}_j(\vec{z}) + Cz_j = -\bar{g}_j(\vec{z})$ is monotone for $j \in J$. Thus we get

$$\partial_t w_j^\varepsilon - C w_j^\varepsilon - c_j \Delta w_j^\varepsilon - \bar{g}_j(\vec{w}^\varepsilon) = 0.$$

After introducing the function $\bar{w}_j^\varepsilon = e^{-Ct} w_j^\varepsilon$ we get

$$\partial_t \bar{w}_j^\varepsilon - c_j \Delta \bar{w}_j^\varepsilon - e^{-Ct} \bar{g}_j(e^{Ct} \bar{w}^\varepsilon) = 0$$

and

$$e^{-Ct} \bar{g}_j(e^{Ct} \bar{z}) = (\nabla e^{-2Ct} G(e^{Ct} \bar{z}))_j = (\nabla \bar{G}(\bar{z}))_j.$$

In this way we have reduced the problem to the case of a monotone function – *g*. Q.E.D.

PROPOSITION 13 : *The function $\vec{\xi}_j^*$ from proposition 8 satisfies*

$$\vec{\xi}_j^* = D \nabla v_j^*. \tag{22}$$

Proof : From the definition of $\vec{\eta}_z^\varepsilon$ and the extension lemma from [12] we have $\nabla \cdot \vec{\eta}_z^\varepsilon = 0$ in Ω . Let $\vec{\varphi} \in (C_0^\infty(\Omega))^m$ and $\omega \in C_0^\infty(0, T)$. Then we get

$$(\vec{\eta}_z^\varepsilon, \nabla v_j^\varepsilon \varphi_j \omega)_Q + (\vec{\eta}_z^\varepsilon, v_j^\varepsilon \nabla \varphi_j \omega)_Q = 0.$$

Passing to the limit $\varepsilon \rightarrow 0$ and by using the « div-curl »-lemma from [28] we get

$$(\vec{\eta}_z^\varepsilon, \nabla v_j^\varepsilon \varphi_j \omega)_Q = (\nabla \mu_j^\varepsilon, \vec{\xi}_j^\varepsilon \varphi_j \omega)_Q + O(\varepsilon) \rightarrow (z, \vec{\xi}_j^* \varphi_j \omega)_Q$$

and

$$(\vec{\eta}_z^\varepsilon, v_j^\varepsilon \nabla \varphi_j \omega)_Q \rightarrow (Dz, v_j^* \nabla \varphi_j \omega)_Q.$$

Therefore, we have

$$(\xi_j^*, z \varphi_j \omega)_Q = (D \nabla v_j^*, z \varphi_j \omega)_Q.$$

Since $z, \varphi_j,$ and ω were arbitrary, we get the result. Q.E.D.

Remark : This is the only place in this paper where we use the energy method ; also we point out that in general $\nabla \cdot \vec{\xi}_j^\varepsilon \neq 0 \neq \nabla \cdot \vec{\xi}_j^*$.

PROPOSITION 14 : *For the limit of the subsequence in proposition 8 the relation 7, i.e.,*

$$\sum_{j \in J} (|X| (\partial_t v_j^* + S_j, \varphi_j)_Q + d_j (D \nabla v_j^*, \nabla \varphi_j)_Q + (\vec{u} \cdot \nabla v_j^*, \varphi_j)_Q - |X| (f_j(\vec{v}^*), \varphi_j)_Q) = 0 \quad \forall \vec{\varphi} \in \mathcal{V} \tag{23}$$

holds.

Proof: Once again, we suppose that $-g$ is a monotone function. The general case will be reduced to this special case in the same way as in the proof of proposition 12.

Let $\vec{\varphi} \in (C_0^\infty(Q))^m$. Then we use $\vec{\varphi}$ for both test functions $\vec{\varphi}$ and $\vec{\psi}$ in the inequality 3 and get

$$\begin{aligned} & \sum_{j \in J} ((\partial_t v_j^\varepsilon, \varphi_j - v_j^\varepsilon)_{Q^\varepsilon} + (\partial_t w_j^\varepsilon, \varphi_j - w_j^\varepsilon)_{R^\varepsilon} \\ & + d_j (\nabla v_j^\varepsilon, \nabla (\varphi_j - v_j^\varepsilon))_{Q^\varepsilon} + \varepsilon^2 c_j (\nabla w_j^\varepsilon, \nabla (\varphi_j - w_j^\varepsilon))_{R^\varepsilon} \\ & + (\vec{u}^\varepsilon, \nabla v_j^\varepsilon (\varphi_j - v_j^\varepsilon))_{Q^\varepsilon} - (f_j(\vec{v}^\varepsilon), \varphi_j - v_j^\varepsilon)_{Q^\varepsilon} - (g_j(\vec{\varphi}), \varphi_j - w_j^\varepsilon)_{R^\varepsilon}) \\ & \geq 0. \end{aligned} \quad (24)$$

We have

$$(\vec{u}^\varepsilon, \nabla v_j^\varepsilon \varphi_j)_Q = - (\vec{u}^\varepsilon, \nabla \varphi_j v_j^\varepsilon)_Q \rightarrow (\vec{u}, \nabla \varphi_j v_j^*)_Q = (\vec{u}, \nabla v_j^* \varphi_j)_Q$$

as $\varepsilon \rightarrow 0$ and

$$\liminf_{\varepsilon \rightarrow 0} \int_Q c_j \varepsilon \nabla w_j^\varepsilon \cdot \varepsilon \nabla w_j^\varepsilon \chi_{\Pi^\varepsilon} dx dt \geq \int_U c_j |\nabla_y w_j^*|^2 dy dx dt.$$

Furthermore, by using the « div-curl » lemma from [28] we get

$$d_j (\nabla v_j^\varepsilon, \nabla v_j^\varepsilon)_{Q^\varepsilon} = d_j (\xi_j^\varepsilon, \nabla v_j^\varepsilon)_Q + O(\varepsilon) \rightarrow d_j (D \nabla v_j^*, \nabla v_j^*)_Q.$$

Now propositions 8, 9, and 12 together with the last three formulas imply by taking the limits in inequality 24

$$\begin{aligned} & \sum_{j \in J} (|X| (\partial_t v_j^*, \varphi_j - v_j^*)_Q \\ & + (\partial_t w_j^*, \varphi_j - w_j^*)_U + d_j (D \nabla v_j^*, \nabla (\varphi_j + v_j^*))_Q \\ & + (\vec{u} \cdot \nabla v_j^*, \varphi_j - v_j^*)_Q - |X| (f_j(\vec{v}^*), \varphi_j - v_j^*)_Q \\ & - c_j \|\nabla_y w_j^*\|_U^2 - (g_j(\vec{\varphi}), \varphi_j - w_j^*)_U) \\ & \geq 0 \end{aligned} \quad (25)$$

which can be written as

$$\begin{aligned} & \sum_{j \in J} (|X| (\partial_t v_j^* - f_j(\vec{v}^*), \varphi_j - v_j^*)_Q + d_j (D \nabla v_j^*, \nabla (\varphi_j - v_j^*))_Q \\ & + (\vec{u} \cdot \nabla v_j^*, \varphi_j - v_j^*)_Q + (\partial_t w_j^* - g_j(\vec{w}^*), \varphi_j - w_j^*)_U) \\ & \geq \sum_{j \in J} c_j (\nabla_y w_j^*, \nabla_y w_j^*)_U. \end{aligned} \quad (26)$$

We note that formula 8 implies

$$c_j(\nabla_y w_j^*, \nabla_y(v_j^* - w_j^*))_U + (\partial_t w_j^* - g_j(\vec{w}^*), v_j^* - w_j^*)_U \cong 0.$$

The last two relations give finally

$$\sum_{j \in J} (|X|(\partial_t v_j^* - f_j(\vec{v}^*), \varphi_j - v_j^*)_Q + d_j(D \nabla v_j^*, \nabla(\varphi_j - v_j^*))_Q + (\vec{u} \cdot \nabla v_j^*, \varphi_j - v_j^*)_Q + (\partial_t w_j^* - g_j(\vec{w}^*), \varphi_j - w_j^*)_U) \cong 0.$$

Q.E.D.

THEOREM 3 : *The limit functions \vec{v}^* and \vec{w}^* solve the macro-model problem 4.*

Proof : We have only to collect the results from propositions 10, 12, and 14. Q.E.D.

9. THE RESULT

THEOREM 4 : *Let the data satisfy the assumptions from section 2.4 and the functions \vec{f} and \vec{g} satisfy the conditions of section 2.2. Let $(\vec{u}^\varepsilon, p^\varepsilon)$ be a weak solution of system 1 and $(\vec{v}^\varepsilon, \vec{w}^\varepsilon)$ a solution of problem 2. Then there exist extensions of $\vec{u}^\varepsilon, p^\varepsilon, \vec{v}^\varepsilon,$ and \vec{w}^ε (denoted by the same symbols) such that*

$\vec{u}^\varepsilon \rightharpoonup \vec{u}$	<i>weakly in $(L^2(\Omega))^n$</i>
$p^\varepsilon \rightarrow p$	<i>strongly in $L^2_0(\Omega)$.</i>
$v_j^\varepsilon \rightharpoonup v_j$	<i>weakly in $L^2(0, T; H^1(\Omega))$</i>
$v_j^\varepsilon \rightharpoonup v_j$	<i>weak* in $L^\infty(Q)$</i>
$\partial_t v_j^\varepsilon \rightharpoonup \partial_t v_j$	<i>weakly in $L^2(Q)$</i>
$v_j^\varepsilon \rightarrow v_j$	<i>strongly in $C([0, T]; L^2(\Omega))$</i>
$w_j^\varepsilon \rightarrow w_j$	<i>in the two-scale sense</i>
$\partial_t w_j^\varepsilon \rightarrow \partial_t w_j$	<i>in the two-scale sense</i>
$\varepsilon \nabla_x w_j^\varepsilon \rightarrow \nabla_y w_j$	<i>in the two-scale sense</i>

for $j \in J$, where (\vec{u}, p) is a weak solution of system 6 and (\vec{v}, \vec{w}) is a solution of problem 4.

Proof : The convergence of the subsequence towards (\vec{v}, \vec{w}) follows from theorem 3. Since according to theorem 2 the macro-problem 4 has a unique solution, we get convergence of the whole sequence. Q.E.D.

REFERENCES

- [1] I. AGANOVIĆ, A. MIKELIĆ, 1992, Homogenization of nonstationary flow of a two-constituent mixture through a porous medium, *Asymptotic Analysis*, **6**, 173-189.
- [2] G. ALLAIRE, 1989, Homogenization of the Stokes flow in a connected porous medium, *Asympt. Anal.*, **2**, 203-222.
- [3] G. ALLAIRE, 1991, Homogénéisation et convergence à deux échelles. Application à un problème de convection diffusion, *C. R. Acad. Sci. Paris*, **312**, Ser. I, 581-586.
- [4] T. ARBOGAST, J. DOUGLAS, U. HORNUNG, 1990, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.*, **21**, 823-836.
- [5] T. ARBOGAST, J. DOUGLAS, U. HORNUNG, 1991, Modeling of naturally fractured reservoirs by formal homogenization techniques, Dautray R. (Ed.) *Frontiers in Pure and Applied Mathematics*, Elsevier, Amsterdam, 1-19.
- [6] N. BAKHVALOV, G. PANASENKO, 1989, *Homogenization : Averaging Processes in Periodic Media*, Kluwer, Dordrecht.
- [7] A. BENSOUSSAN, J. L. LIONS, G. PAPANICOLAOU, 1978, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam.
- [8] A. P. BOURGEAT, 1985, Nonlinear homogenization of two-phase flow equations J. H. Lightbourne, S. M. Rankin (Eds), *Physical Mathematics and Nonlinear Partial Differential Equations*, 207-212.
- [9] A. P. BOURGEAT, 1986, Homogenization of two-phase flow equations, *Proceedings Symposia Pure Mathem.*, **45**, 157-163.
- [10] H. BRÉZIS, 1972, Problèmes unilatéraux, *J. Math. pures et appl.*, **51**, 1-168.
- [11] E. CANON, W. JÄGER, *Homogenization for nonlinear adsorption-diffusion processes in porous media*, to appear.
- [12] D. CIORANESCU, J. SAINT-JEAN-PAULIN, 1979, Homogenization in open sets with holes, *J. Math. Anal. Appl.*, **71**, 590-607.
- [13] I. EKELAND, R. TEMAM, 1976, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam.
- [14] A. FRIEDMAN, P. KNABNER, A Transport Model with Micro- and Macro-Structure, *J. Differ. Equ.*, to appear.
- [15] K. GRÖGER, 1971, Zum Rand-Anfangswertproblem der Adsorption und Diffusion bei Festbettprozessen, *Mathem. Nachr.*, **49**, 251-259.
- [16] U. HORNUNG, 1991, Homogenization of Miscible Displacement in Unsaturated Aggregated Soils, G. Dal Maso, G. F. Dell'Antonio (Eds.) *Composite Media and Homogenization Theory, Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, 143-153.

- [17] U. HORNUNG, 1991, Miscible displacement in porous media influenced by mobile and immobile water, *Rocky Mountain J. Math.*, **21**, 645-669 corr. 1153-1158.
- [18] U. HORNUNG, 1992, Applications of the homogenization method to flow and transport in porous media Xiao Shutie (Ed.) *Summer School on Flow and Transport in Porous Media*, World Scientific Publisher, Singapore, 167-222.
- [19] U. HORNUNG, W. JÄGER, 1987, A model for chemical reactions in porous media J. Warnatz, W. Jäger (Eds.) *Complex Chemical Reaction Systems. Mathematical Modeling and Simulation*, *Chemical Physics*, **47**, 318-334.
- [20] U. HORNUNG, W. JÄGER, 1991, Diffusion, convection, adsorption, and reaction of chemicals in porous media, *J. Differ. Equat.*, **92**, 199-225.
- [21] U. HORNUNG, R. SHOWALTER, 1990, Diffusion models for fractured media, *J. Math. Anal. Applies*, **147**, 69-80.
- [22] J. L. LIONS, 1969, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod/Gauthier-Villars, Paris.
- [23] R. LIPTON, A. AVELLANEDA, 1990, A Darcy law for slow viscous flow past a stationary array of bubbles, *Proc. Royal Soc. Edinburgh*, **114A**, 71-79.
- [24] A. MIKELIĆ, 1989, A convergence theorem for homogenization of two-phase miscible flow through fractured reservoirs with uniform fracture distributions, *Applicable Analysis*, **33**, 203-214.
- [25] A. MIKELIĆ, 1991, Homogenization of nonstationary Navier-Stokes equations in a domain with grained boundary, *Ann. Mat. Pura e Appl.*, **158**, 167-179.
- [26] A. MIKELIĆ, I. AGANOVIĆ, 1987, Homogenization in a porous medium under a nonhomogeneous boundary condition, *Boll. Un. Mat. Ital. (A)* **1**, 171-180.
- [27] A. MIKELIĆ, I. AGANOVIĆ, 1988, Homogenization of stationary flow of miscible fluids in a domain with a grained boundary, *SIAM J. Math. Anal.*, **19**, 287-294.
- [28] F. MURAT, 1978, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa, Ser. 4*, **5**, 489-507.
- [29] G. NGUETSENG, 1989, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, **20**, 608-623.
- [30] O. A. OLEINIK, S. M. KOZLOV, V. V. ZHIKOV, 1991, *Homogenization of Differential Operators*, North-Holland, Amsterdam.
- [31] E. SANCHEZ-PALENCIA, 1980, Non-Homogeneous Media and Vibration Theory, *Springer Lecture Notes in Physics*, **129**.
- [32] K. SATTEL-SCHWIND, 1988, *Untersuchung über Diffusionsvorgänge bei der Gelpermeations-Chromatographie von Poly-p-Methylstyrol*, Dissertation, Fachbereich Chemie, Universität Heidelberg.
- [33] K. SIEBEL, 1988, *Diffusion in dispersen Medien. Homogenisierung*, Diplomarbeit, Fachbereich Mathematik, Universität Heidelberg.
- [34] J. SIMON, 1987, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura e Appl.*, **145**, 65-96.

- [35] L. TARTAR, 1980, Incompressible fluid flow in a porous medium — convergence of the homogenization process, E. Sanchez-Palencia (Ed.) « Non-homogeneous media and vibration theory » *Lecture Notes in Physics*, **127**, Springer, Berlin, 368-377.
- [36] C. VOGT, 1982, *A homogenization theorem leading to a Volterra integro — differential equation for permeation chromatography*, Preprint **155**, SFB 123, Universität Heidelberg.

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* G. ALLAIRE, Homogenization and two-scale convergence, *SIAM J. Math. Anal.*, **23** (1992) 1482-1518.