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**THE WARPING, THE TORSION
AND THE NEUMANN PROBLEMS IN A
QUASI-PERIODICALLY PERFORATED DOMAIN (*)**

by M. L. MASCARENHAS ^(1,2) and D. POLIŠEVSKI ⁽³⁾

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Abstract. — *Using the two-scale convergence method we study the homogenized behaviour of the warping and the torsion problems, in a two-dimensional domain with a quasi-periodic perforation, as well as the relationship between the two homogenized problems. The Neumann problem is also analyzed.*

Key words : Homogenization, quasi-periodic, warping function, torsion function, Neumann problem.

Résumé. — *Avec la technique de la convergence à double échelle, on étudie le comportement homogénéisé des problèmes de gauchissement et de torsion dans un domaine bi-dimensionnel perforé quasi périodiquement, ainsi que la relation entre les deux problèmes homogénéisés. On analyse également le problème de Neumann.*

1. INTRODUCTION

The influence of the microscopic structure, the so-called, material distribution, on the global behaviour of the elastic materials is of great importance in the structural optimization problems. Many and important contributions have been made in this direction, both from the analytical and numerical points of view. An extensive bibliography may be found in [6].

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Quasi-periodic structures are frequently used in structural optimization problems. In [7] we present a first approach to the torsion problem, in a quasi-periodically perforated domain, generalizing the classical work [4]. In the present paper, based on two-scale convergence method introduced by G. Nguetseng (see [10] and [11]) and recently developed by G. Allaire (see [1] and [2]), and using the uniform extension and compactness property of some class of domains, due to D. Chenais (see [3]), we adopt a different approach, which has less restrictive assumptions on the variation of the holes and which can also be extended to the three-dimensional case. The same method is used to homogenize the two-dimensional warping and Neumann problems. Moreover, we find the relationship between the homogenized coefficients, following some ideas already presented in [8], for the periodic case.

In section 2 we give a very brief summary of the results of two-scale convergence and of uniform extension and compactness for some class of domains, which are necessary to prove the homogenization process.

In section 3 we present the relationship between the torsion and warping problems, as well as some sufficient conditions for the convergence process. In section 4 we obtain their homogenized behaviour and the relationship between the homogenized coefficients. The Neumann problem is treated in section 5.

2. SOME RELEVANT RESULTS

The two-scale convergence method

The two-scale convergence method was introduced by G. Nguetseng, in 1989 (see [10] and [11]), and more recently developed by G. Allaire (see [1] and [2]), in view of the periodic and quasi-periodic homogenization problems.

In this paragraph we briefly present, with no demonstrations, the results we need to prove our homogenization process.

DEFINITION 2.1 : *Let Ω be an open set of \mathbb{R}^N and $Y = [0, 1]^N$. A sequence u_ε in $L^2(\Omega)$ two-scale converges to $u(x, y)$ in $L^2(\Omega \times Y)$ if, for all $\varphi \in \mathcal{D}(\Omega ; C^\infty_\#(Y))$:*

$$\int_{\Omega} u_\varepsilon(x) \varphi \left(x, \frac{x}{\varepsilon} \right) dx \rightarrow \int_{\Omega \times Y} u(x, y) \varphi(x, y) dx dy .$$

THEOREM 2.2 : *If u_ε is a bounded sequence in $L^2(\Omega)$ then there exists a subsequence ε' of ε , and some u in $L^2(\Omega \times Y)$ such that, as ε' goes to zero, u_ε two-scale converges to u .*

PROPOSITION 2.3 : *If u_ε two-scale converges to u , then u_ε converges weakly in $L^2(\Omega)$ to $u_0 = \int_Y u \, dy$. Moreover*

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega \times Y)} \geq \|u_0\|_{L^2(\Omega)}.$$

PROPOSITION 2.4 : *Let u_ε be a sequence in $H^1(\Omega)$, weakly converging to $u \in H^1(\Omega)$, then u_ε two-scale converges to u and ∇u_ε two-scale converges to $\nabla u + \nabla_y u_1$ for $u_1 \in L^2(\Omega ; H^1_\#(Y)/\mathbb{R})$.*

The uniform extension and compactness property

The following results are due to D. Chenais and their proofs may be found in [3].

DEFINITION 2.5 : *Let $h > 0$ and $\theta \in]0, \pi/2[$ be two given numbers and ξ a given element in \mathbb{R}^N such that $\|\xi\| = 1$. We call cone of angle θ , height h and axis ξ the set*

$$C(\xi, \theta, h) = \{x \in \mathbb{R}^N : (x, \xi) > \|x\| \cos \theta; \|x\| < h\}.$$

DEFINITION 2.6 : *Let $\theta \in]0, \pi/2[$, $h > 0$ and $0 < r \leq h/2$ be three given numbers. A subset Ω of \mathbb{R}^N is said to satisfy the cone property for θ , h and r , if*

$$\forall x \in \partial\Omega, \exists C_x = C(\xi_x, \theta, h), \text{ such that} \\ \forall y \in B(x, r) \cap \Omega, \quad y + C_x \subset \Omega.$$

Where $B(x, r)$ denotes the open ball of radius r and center x in \mathbb{R}^N .

DEFINITION 2.7 : *Let \mathcal{D} be a fixed open and bounded subset of \mathbb{R}^N . For each θ , h and r we define :*

$$\pi(\theta, h, r) = \{\Omega \subset \mathcal{D} : \Omega \text{ satisfies the cone property for } \theta, h \text{ and } r\}$$

and $U(\theta, h, r)$ the set of all the characteristic functions of elements of $\pi(\theta, h, r)$, with the $L^2(\mathcal{D})$ topology.

DEFINITION 2.8 : *For each $x \in \mathbb{R}^2$ and $\delta, k \in \mathbb{R}^+$, define*

$$P_{\delta, k}(x) = \{y \in \mathbb{R}^2 : |y_1 - x_1| < \delta \text{ and } |y_2 - x_2| < k \cdot \delta\}$$

and $Lip(k, \delta)$ the set of all open subset Ω of \mathcal{D} , such that : for all $x \in \partial\Omega$, there exists a local coordinate system and a function

$$\varphi_x :]x_1 - \delta, x_1 + \delta[\rightarrow \mathbb{R},$$

lipschitz of constant k , satisfying :

$$y \in P_{\delta, k}(x) \cap \Omega \quad \text{iff} \quad y \in P_{\delta, k}(x) \quad \text{and} \quad y_2 > \varphi_x(y_1).$$

PROPOSITION 2.9 : $\forall \theta, h, r \quad \exists k, \delta : \pi(\theta, h, r) \subset \text{Lip}(k, \delta)$ and, reciprocally,

$$\forall k, \delta \quad \exists \theta, h, r : \text{Lip}(k, \delta) \subset \pi(\theta, h, r).$$

THEOREM 2.10 : Let $\theta \in]0, \pi/2[$, $h > 0$ and $0 < r \leq h/2$.

There exists a constant $k = k(\theta, h, r)$, depending only on θ , h and r , such that

$$\forall \Omega \in \pi(\theta, h, r) \quad \exists p_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^N)$$

linear and continuous extension operator, such that

$$\|p_\Omega\| \leq k, \quad \forall \Omega \in \pi(\theta, h, r).$$

THEOREM 2.11 : $U(\theta, h, r)$ is compact in $L^2(\mathcal{D})$.

3. THE TORSION AND WARPING FUNCTIONS. A PRIORI ESTIMATES

Let Ω be an open bounded connected lipschitz set of \mathbb{R}^2 and $Y = [0, 1]^2$.

Let $\gamma = \gamma(x, \lambda)$ be a C^1 function of x in $\bar{\Omega}$, with values in the space of simple, closed and piecewise $-C^1$ curves, *i.e.*,

$$\gamma \in C^1(\bar{\Omega}; C_s^1(0, 1; Y)),$$

satisfying $\gamma(\bar{\Omega} \times [0, 1]) \subset \text{int } Y$.

For each $x \in \bar{\Omega}$, let $1 - \chi(x, \cdot)$ represent the characteristic set function of $T(x)$ the closure of the interior part of $\gamma(x, \cdot)$. Extending $\chi(x, \cdot)$ by Y -periodicity to the entire \mathbb{R}^2 , let T be the closed subset of $\bar{\Omega} \times \mathbb{R}^2$ defined by $1 - \chi(\cdot, \cdot)$; then $\mathcal{O} = (\Omega \times \mathbb{R}^2) \setminus T$ is a connected open subset of $\Omega \times \mathbb{R}^2$, Y -periodic in the sense of its characteristic function $\chi(\cdot, \cdot)$, with respect to the second variable, and whose boundary, in $(\Omega \times Y)$, is given by

$$\partial\mathcal{O} = \{(x, y) : x \in \Omega, y = \gamma(x, \lambda), \lambda \in [0, 1]\}.$$

We denote

$$\theta(x) = \int_Y \chi(x, y) dy. \quad (3.1)$$

For each $\varepsilon > 0$, \mathbb{R}^2 is covered by squares $Y_{\varepsilon k} = \varepsilon k + \varepsilon Y$, where $k \in \mathbb{Z}^2$. Let \mathbb{Z}_ε represent the set of all $k \in \mathbb{Z}^2$ such that $Y_{\varepsilon k}$ is included in $\bar{\Omega}$. Define

$$\chi_\varepsilon(x) = \begin{cases} \chi\left(x, \frac{x}{\varepsilon}\right), & \text{if } x \in Y_{\varepsilon k} \text{ and } k \in \mathbb{Z}_\varepsilon, \\ 1, & \text{if } x \in \Omega \cap Y_{\varepsilon k} \text{ and } k \notin \mathbb{Z}_\varepsilon. \end{cases} \tag{3.2}$$

LEMMA 3.1 : χ_ε two-scale converges to $\chi(x, y)$.

Proof: Following Theorem 2.2 one has, up to a subsequence, $\chi_\varepsilon \rightarrow \xi(x, y)$, with $\xi \in L^2(\Omega \times Y)$. Let $\phi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, with support in T , and $\varphi \in \mathcal{D}(\Omega)$. For ε small enough one has

$$\int_\Omega \chi_\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx = \int_\Omega \chi\left(x, \frac{x}{\varepsilon}\right) \phi\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx = 0,$$

which implies $\int_{\Omega \times Y} \xi(x, y) \varphi(x, y) dx dy = 0$, for all $\phi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, $\text{supp } \phi \subset \subset T(x)$, $\varphi \in \mathcal{D}(\Omega)$, and, then $\xi(x, y) = 0$ in T . Analogously one proves that $\xi(x, y) = 1$ in \mathcal{O} . \square

Let Ω_ε be the subset of Ω defined by the characteristic function χ_ε . Let $T_{\varepsilon k}$, with $k \in \mathbb{Z}_\varepsilon$, represent each of the connected components of $\Omega \setminus \Omega_\varepsilon$. Remark that $\frac{1}{\varepsilon} T_{\varepsilon k}$ is a slight deformation of $T(\varepsilon k)$; $\frac{1}{\varepsilon} T_{\varepsilon k}$ is the interior part of the curve $y = \sigma(\varepsilon k, \lambda)$, implicitly defined by $y = \gamma(\varepsilon k + \varepsilon y, \lambda)$ when $y \in Y$ and $\lambda \in [0, 1]$, and, in fact, very near to $\gamma(\varepsilon k, \lambda)$, in the sense of the $C^1_s([0, 1]; Y)$ -norm. As in [7] (Lemma 2.2) one may obtain, for a constant independent of ε :

$$d_{\mathcal{H}}(\partial T_{\varepsilon k}, \partial(\varepsilon T(\varepsilon k))) < C \varepsilon^2, \tag{3.3}$$

where $d_{\mathcal{H}}$ means the Hausdorff distance between two sets.

Consider, now, the warping problem in Ω_ε :

$$\begin{cases} -\Delta w_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \partial_n w_\varepsilon = x_2 n_1 - x_1 n_2, & \text{on } \partial \Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} w_\varepsilon dx = 0. \end{cases} \tag{3.4}$$

The corresponding variational formulation is to find $w_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} w_\varepsilon dx = 0 \text{ and}$$

$$\int_{\Omega_\varepsilon} \nabla w_\varepsilon \nabla \varphi \, dx = \int_{\partial\Omega_\varepsilon} (x_2 n_1 - x_1 n_2) \varphi \, d\sigma, \quad \forall \varphi \in H^1(\Omega_\varepsilon). \quad (3.5)$$

The associate torsion problem in Ω_ε is :

$$\begin{cases} -\Delta \psi_\varepsilon = 2, & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon = 0, & \text{on } \partial\Omega, \\ \psi_\varepsilon = C_{\varepsilon k}, & \text{on } \partial T_{\varepsilon k}, \quad k \in \mathbb{Z}_\varepsilon, \\ \int_{\partial T_{\varepsilon k}} \partial_n \psi_\varepsilon = 2 |T_{\varepsilon k}|, & k \in \mathbb{Z}_\varepsilon, \end{cases} \quad (3.6)$$

where $|T_{\varepsilon k}|$ means the measure of $T_{\varepsilon k}$. Defining,

$$\Phi(\Omega_\varepsilon) = \left\{ v \in H^1(\Omega_\varepsilon) : v \text{ is zero on } \partial\Omega \text{ and } v \text{ is constant on each } \partial T_{\varepsilon k}, \quad k \in \mathbb{Z}_\varepsilon \right\},$$

the variational formulation of problem (3.6) is

$$\begin{cases} \psi_\varepsilon \in \Phi(\Omega_\varepsilon), \\ \int_{\Omega_\varepsilon} \nabla \psi_\varepsilon \nabla \varphi = - \int_{\Omega_\varepsilon} x \cdot \nabla \varphi, \quad \forall \varphi \in \Phi(\Omega_\varepsilon). \end{cases} \quad (3.7)$$

It is well known that both problems (3.5) and (3.7) have unique solutions which are related by :

PROPOSITION 3.2 : *For each $\varepsilon > 0$, the warping and torsion functions satisfy, for almost all $x \in \Omega_\varepsilon$, the relation :*

$$\nabla w_\varepsilon - Rx = R \nabla \psi_\varepsilon, \quad \text{with } R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.8)$$

Proof : From problem (3.4) $\nabla w_\varepsilon - Rx$ is a divergence-free function in Ω_ε which has zero normal trace on $\partial\Omega_\varepsilon$; so, following [5] (corollary 3.1, p. 26), there exists a unique stream function, solution of problem (3.6) and, consequently, equal to ψ_ε , that satisfies, in Ω_ε ,

$$\begin{cases} \partial_1 w_\varepsilon - x_2 = \partial_2 \psi_\varepsilon, \\ \partial_2 w_\varepsilon + x_1 = -\partial_1 \psi_\varepsilon, \end{cases} \quad (3.9)$$

i.e., (3.8). \square

Consider the natural extension of ψ_ε to all Ω that takes, on each hole $T_{\varepsilon k}$, $k \in \mathbb{Z}_\varepsilon$, the constant value $C_{\varepsilon k}$. One has $\psi_\varepsilon \in H^1(\Omega)$. The same extension can be considered for any function of the space $\Phi(\Omega_\varepsilon)$ and the

variational formulation (3.8) may be written as follows :

$$\begin{cases} \psi_\varepsilon \in \Phi(\Omega_\varepsilon), \\ \int_{\Omega_\varepsilon} \nabla \psi \nabla \varphi \, dx = 2 \int_{\Omega_\varepsilon} \varphi, \quad \forall \varphi \in \Phi(\Omega_\varepsilon). \end{cases} \quad (3.10)$$

LEMMA 3.3 : *The norms $\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ and $\|\psi_\varepsilon\|_{H^1(\Omega)}$ are bounded independently of ε .*

Proof : Integrating by parts the right-hand sides of (3.5) and (3.10) and making respectively, $\varphi = w_\varepsilon$ and $\varphi = \psi_\varepsilon$, one obtains, for a constant C independent of ε ,

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \quad \text{and} \quad \|\nabla \psi_\varepsilon\|_{L^2(\Omega)} \leq C$$

which proves the result since, in the last inequality, one may use Poincaré's inequality in Ω . \square

Consider, now, the following two problems, where C represents always a constant, independent of ε :

$$\begin{cases} \text{Find } \xi_\varepsilon \in L^2(\Omega); \\ \|\xi_\varepsilon\|_{L^2(\Omega)} \leq C, \\ \operatorname{div} [\nabla \psi_\varepsilon + x\chi_\varepsilon + \xi_\varepsilon(1 - \chi_\varepsilon)] = 0 \text{ in } \Omega. \end{cases} \quad (3.11)$$

and

$$\begin{cases} \text{Find } v_\varepsilon \in H^1(\Omega); \\ \|v_\varepsilon\|_{H^1(\Omega)} \leq C, \\ v_\varepsilon = w_\varepsilon \text{ in } \Omega_\varepsilon. \end{cases} \quad (3.12)$$

Both problems are related as follows :

PROPOSITION 3.4 : *Problem (3.11) has a solution if and only if problem (3.12) is solvable.*

Proof : Let ξ_ε be a solution of problem (3.11) and consider $\eta_\varepsilon = (\nabla w_\varepsilon)^\sim + R\xi_\varepsilon(1 - \chi_\varepsilon)$, where $(\nabla w_\varepsilon)^\sim$ is the zero extension of ∇w_ε to the entire Ω and R is the orthonormal matrix in (3.8), whose transpose coincides with $(-R)$. Let $\varphi \in H_0^1(\Omega)$, then

$$\begin{aligned} \int_{\Omega} \eta_\varepsilon R \nabla \varphi \, dx &= \int_{\Omega} -R \eta_\varepsilon \nabla \varphi \, dx = \\ &= \int_{\Omega} [-R(\nabla w_\varepsilon)^\sim + \xi_\varepsilon(1 - \chi_\varepsilon)] \nabla \varphi \, dx. \end{aligned}$$

Using equality (3.8) in the last integral, one obtains

$$\int_{\Omega} \eta_{\varepsilon} R \nabla \varphi \, dx = \int_{\Omega} [\nabla \psi_{\varepsilon} + x \chi_{\varepsilon} + \xi_{\varepsilon} (1 - \chi_{\varepsilon})] \nabla \varphi \, dx,$$

which, by (3.11) is equal to zero. Then η_{ε} is orthogonal to any divergence-free function having its normal trace equal to zero (see [5], Thm. 3.4, p. 31) and one may conclude that η_{ε} is a gradient, i.e., there exists $u_{\varepsilon} \in H^1(\Omega)$, such that

$$(\nabla w_{\varepsilon})^{\sim} + R \xi_{\varepsilon} (1 - \chi_{\varepsilon}) = \nabla u_{\varepsilon}.$$

Defining $v_{\varepsilon} = u_{\varepsilon} - \frac{1}{|\Omega_{\varepsilon}|} \int_{\Omega_{\varepsilon}} u_{\varepsilon}$, one may conclude that $v_{\varepsilon} \chi_{\varepsilon}$ satisfies problem (3.5) and, then, $v_{\varepsilon} \chi_{\varepsilon} = w_{\varepsilon}$. Since ξ_{ε} and $(\nabla w_{\varepsilon})^{\sim}$ are bounded in $L^2(\Omega)$ the same happens with $\nabla u_{\varepsilon} = \nabla v_{\varepsilon}$ and, using Poincaré's inequality in Ω , we find that $\|v_{\varepsilon}\|_{H^1(\Omega)}$ is bounded independently of ε .

Conversely, the same kind of argument holds: if v_{ε} is a solution of problem (3.12), $\xi_{\varepsilon} = -R \nabla v_{\varepsilon}$ will be a solution of problem (3.11). \square

In order to solve problem (3.11) we shall introduce the following supplementary hypothesis (H):

« There exist, as in Definition 2.7 for $\mathcal{D} = Y$, three numbers θ , h and r such that, for ε small enough, the family $\left\{ \frac{1}{\varepsilon} T_{\varepsilon k} \right\}_{k \in \mathbb{Z}_{\varepsilon}}$ is included in $\pi(\theta, h, r)$. »

LEMMA 3.5 : If $\{T_{\alpha}\}_{\alpha \in A}$ is a family of connected elements of $\pi(\theta, h, r)$ then there exists a real constant C , independent of α , such that

$$\sup_{\substack{u \in H^1(T_{\alpha}) \\ \int_{T_{\alpha}} u = 0 \\ u \neq 0}} \frac{\|u\|_{H^1(T_{\alpha})}}{\|\nabla u\|_{L^2(T_{\alpha})}} \leq C. \quad (3.13)$$

Proof : Using Theorem 2.10, let $\{p_{\alpha}\}_{\alpha \in A}$ be a family of linear continuous extension operators such that $\|p_{\alpha}\| \leq k$, where k is a constant independent of $\alpha \in A$.

If the right hand side of (3.13) is not bounded independently of α , there exist two sequences $\alpha_n \in A$ and $u_n \in H^1(T_{\alpha_n})$, $\int_{T_{\alpha_n}} u_n = 0$, $u_n \neq 0$, such that $\|u_n\|_{H^1(T_{\alpha_n})} = 1$ and $\|\nabla u_n\|_{L^2(T_{\alpha_n})} < \frac{1}{n}$.

Using Theorem 2.11 we may suppose that, up to a subsequence, there exists a connected $T_0 \in \pi(\theta, h, r)$ such that, in terms of the characteristic functions, one has

$$\chi_{T_{\alpha_n}} \rightarrow \chi_{T_0}, \quad \text{in } L^2(Y). \tag{3.14}$$

In fact, as we prove in the Appendix, the subset of connected elements of $\pi(\theta, h, r)$ is closed, in the sense of its characteristic functions, for the $L^2(Y)$ topology.

Since $1 = \|u_n\|_{H^1(T_{\alpha_n})} \leq \|p_{\alpha_n} u_n\|_{H^1(Y)} \leq k$, we may also suppose that, up to a subsequence, there exists a $u_0 \in H^1(Y)$ such that

$$p_{\alpha_n} u_n \rightharpoonup u_0, \quad \text{weakly in } H^1(Y),$$

which implies

$$\nabla(p_{\alpha_n} u_n) \rightharpoonup \nabla u_0, \quad \text{weakly in } L^2(Y) \tag{3.15}$$

and, by compact injection,

$$p_{\alpha_n} u_n \rightarrow u_0, \quad \text{in } L^2(Y). \tag{3.16}$$

From (3.14) and (3.15) one obtains

$$\nabla(p_{\alpha_n} u_n) \cdot \chi_{T_{\alpha_n}} \rightarrow \nabla u_0 \cdot \chi_{T_0}, \quad \text{weakly in } L^2(Y).$$

Since

$$\left\| \nabla(p_{\alpha_n} u_n) \cdot \chi_{T_{\alpha_n}} \right\|_{L^2(Y)} = \|\nabla u_n\|_{L^2(T_{\alpha_n})} \rightarrow 0,$$

one has $\nabla u_0 \cdot \chi_{T_0} = 0$, which means that u_0 is constant on T_0 .

Since

$$0 = \int_{T_{\alpha_n}} u_n = \int_Y \chi_{T_{\alpha_n}} \cdot p_{\alpha_n} u_n \rightarrow \int_Y \chi_{T_0} \cdot u_0 = u_0 |T_0|,$$

$|T_0| \neq 0$ and T_0 is connected, one concludes that $u_n \equiv 0$ on T_0 .

We have, then, from (3.14) and (3.16),

$$\|u_n\|_{L^2(T_{\alpha_n})} = \int_Y |\chi_{T_{\alpha_n}} \cdot p_{\alpha_n} u_n|^2 \rightarrow \int_Y \chi_{T_0} |u_0|^2 = 0,$$

and, consequently,

$$1 = \|u_n\|_{H^1(T_{\alpha_n})} = \|u_n\|_{L^2(T_{\alpha_n})} + \|\nabla u_n\|_{L^2(T_{\alpha_n})} \rightarrow 0,$$

which is absurd. \square

PROPOSITION 3.6 : *Admit that hypothesis (H) is satisfied ; then problem (3.11) has a solution.*

Proof : Consider, for each $k \in \mathbb{Z}_\varepsilon$, the restriction of $\psi_\varepsilon + \frac{\|x\|^2}{2}$ to $Y_{\varepsilon k}$ and let $\zeta_{\varepsilon k}$ be defined, in $Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}$, by

$$\zeta_{\varepsilon k}(y) = \frac{1}{\varepsilon} \psi_\varepsilon(k\varepsilon + \varepsilon y) + \frac{1}{2\varepsilon} \|\varepsilon k + \varepsilon y\|^2 ;$$

then

$$\begin{cases} \nabla_y \zeta_{\varepsilon k}(y) = \nabla \psi_\varepsilon(k\varepsilon + \varepsilon y) + (\varepsilon k + \varepsilon y) \\ \Delta_y \zeta_{\varepsilon k}(y) = \varepsilon (\Delta \psi_\varepsilon(k\varepsilon + \varepsilon k) + 2) = 0 \end{cases} . \quad (3.17)$$

Let $\theta_{\varepsilon k} \in H^1\left(\frac{1}{\varepsilon} T_{\varepsilon k}\right)$ be the unique solution of the following compatible Neumann cell-problem :

$$\begin{cases} -\Delta_y \theta_{\varepsilon k} = 0, & \text{in } \frac{1}{\varepsilon} T_{\varepsilon k}, \\ \partial_{n_y} \theta_{\varepsilon k} = \partial_{n_y} \zeta_{\varepsilon k}, & \text{on } \partial \frac{1}{\varepsilon} T_{\varepsilon k}, \\ \int_{\frac{1}{\varepsilon} T_{\varepsilon k}} \theta_{\varepsilon k} dy = 0. \end{cases} \quad (3.18)$$

Since hypothesis (H) holds, let $p_{\frac{1}{\varepsilon} T_{\varepsilon k}}$ and k be the linear continuous extensions and constant mentioned in Theorem 2.10.

Multiplying (3.18) by $\theta_{\varepsilon k}$ and using (3.17), one obtains

$$\begin{aligned} \|\nabla_y \theta_{\varepsilon k}\|_{L^2\left(\frac{1}{\varepsilon} T_{\varepsilon k}\right)}^2 &= \int_{\partial \frac{1}{\varepsilon} T_{\varepsilon k}} \partial_{n_y} \zeta_{\varepsilon k} \cdot \theta_{\varepsilon k} = \int_{Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}} \nabla_y \zeta_{\varepsilon k} \cdot p_{\frac{1}{\varepsilon} T_{\varepsilon k}} \theta_{\varepsilon k} \leq \\ &\leq k \|\nabla_y \zeta_{\varepsilon k}\|_{L^2\left(Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}\right)} \cdot \|\theta_{\varepsilon k}\|_{H^1\left(\frac{1}{\varepsilon} T_{\varepsilon k}\right)}. \end{aligned}$$

Using hypothesis (H), Lemma 3.5 holds for the family $\left\{\frac{1}{\varepsilon} T_{\varepsilon k}\right\}_{k \in \mathbb{Z}_\varepsilon}$, and we obtain

$$\|\nabla_y \theta_{\varepsilon k}\|_{L^2\left(\frac{1}{\varepsilon} T_{\varepsilon k}\right)} \leq C \|\nabla_y \zeta_{\varepsilon k}\|_{L^2\left(Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}\right)}, \quad (3.19)$$

where C is a constant independent of ε and k .

Define, for $k \in \mathbb{Z}_\varepsilon$,

$$\zeta_{\varepsilon k} = \begin{cases} \nabla_y \theta_{\varepsilon k}, & \text{in } \frac{1}{\varepsilon} T_{\varepsilon k}, \\ \nabla_y \zeta_{\varepsilon k}, & \text{in } Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}. \end{cases}$$

In view of (3.17) and (3.18)

$$\operatorname{div}_y \xi_{\varepsilon k} = 0, \text{ in } Y, \quad \|\xi_{\varepsilon k}\|_{L^2(Y)} \leq \bar{C} \|\nabla_y \zeta_{\varepsilon k}\|_{L^2(Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k})}, \quad (3.20)$$

where \bar{C} is a constant independent of ε and k .

Defining, $\xi_\varepsilon(x) = \nabla \psi_\varepsilon(x) + x$ for $x \in \Omega \cap Y_{\varepsilon k}$, $k \notin \mathbb{Z}_\varepsilon$, and $\xi_\varepsilon(x) = \xi_{\varepsilon k} \left(\frac{x}{\varepsilon} - k \right)$ for $x \in Y_{\varepsilon k}$, $k \in \mathbb{Z}_\varepsilon$, one has

$$\operatorname{div}_x (\nabla \psi_\varepsilon + x \chi_\varepsilon + \xi_\varepsilon (1 - \chi_\varepsilon)) = \operatorname{div}_x \xi_\varepsilon = 0, \text{ in } \Omega,$$

and finally,

$$\begin{aligned} \|\xi_\varepsilon\|_{L^2(\Omega)}^2 &= \sum_{\substack{\varepsilon k \\ k \notin \mathbb{Z}_\varepsilon}} \int_{Y_{\varepsilon k} \cap \Omega} |\nabla \psi_\varepsilon(x) + x|^2 dx + \sum_{\substack{\varepsilon k \\ k \in \mathbb{Z}_\varepsilon}} \int_{Y_{\varepsilon k}} \left| \xi_{\varepsilon k} \left(\frac{x}{\varepsilon} - k \right) \right|^2 dx \\ &= \sum_{\substack{\varepsilon k \\ k \notin \mathbb{Z}_\varepsilon}} \int_{Y_{\varepsilon k} \cap \Omega} |\nabla \psi_\varepsilon(x) + x|^2 dx + \sum_{\substack{\varepsilon k \\ k \in \mathbb{Z}_\varepsilon}} \frac{1}{\varepsilon} \int_{Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}} |\xi_{\varepsilon k}(y)|^2 dy, \end{aligned}$$

by (3.20)

$$\leq \sum_{\substack{\varepsilon k \\ k \notin \mathbb{Z}_\varepsilon}} \int_{Y_{\varepsilon k} \cap \Omega} |\nabla \psi_\varepsilon(x) + x|^2 dx + \bar{C} \sum_{\substack{\varepsilon k \\ k \in \mathbb{Z}_\varepsilon}} \frac{1}{\varepsilon} \int_{Y \setminus \frac{1}{\varepsilon} T_{\varepsilon k}} |\nabla_y \zeta_{\varepsilon k}(y)|^2 dy,$$

by (3.17)

$$\begin{aligned} &\leq \sum_{\substack{\varepsilon k \\ k \notin \mathbb{Z}_\varepsilon}} \int_{Y_{\varepsilon k} \cap \Omega} |\nabla \psi_\varepsilon(x) + x|^2 dx + \bar{C} \sum_{\substack{\varepsilon k \\ k \in \mathbb{Z}_\varepsilon}} \int_{Y \setminus T_{\varepsilon k}} |\nabla \psi_\varepsilon + x|^2 dx, \\ &\leq \max \{1; \bar{C}\} \int_{\Omega_\varepsilon} |\nabla \psi_\varepsilon + x|^2 dx, \\ &\leq \max \{1; \bar{C}\} \|\nabla \psi_\varepsilon + x\|_{L^2(\Omega)}^2, \end{aligned}$$

which, in view of Lemma 3.3, proves that the $L^2(\Omega)$ -norm of ξ_ε is bounded, independently of ε , and the proof is completed. \square

REMARK 3.7 : *Hypothesis (H) is essential since some geometries may generate unbounded sequences of the form (3.13) (see [12], ch. 6, p. 95).* \square

4. THE HOMOGENIZATION PROCESS

Following the two-scale convergence method we determine, in this section, the homogenized behaviour of the torsion and warping problems. Moreover we study the relationship between them.

The torsion problem

Let ξ_ε be the solution of problem (3.11). Using Theorem 2.2, there exists a function $\psi_2 \in L^2(\Omega \times Y)$, such that, up to a subsequence of $\varepsilon \rightarrow 0$, the following two-scale convergence holds

$$(\xi_\varepsilon - x)(1 - \chi_\varepsilon) \rightharpoonup \psi_2. \quad (4.1)$$

Since ψ_ε , already extended to the entire Ω , is bounded in $H_0^1(\Omega)$, independently of ε , we may also assume, using Proposition 2.4, the existence of $\psi \in H_0^1(\Omega)$ and $\psi_1 \in L^2(\Omega ; H_{\#}^1(y)/\mathbb{R})$ such that, up to a subsequence of $\varepsilon \rightarrow 0$, the following two-scale convergences take place

$$\psi_\varepsilon \rightharpoonup \psi, \quad \nabla \psi_\varepsilon \rightharpoonup \nabla \psi + \nabla_y \psi_1. \quad (4.2)$$

Using as test functions, in the definition of the two-scale convergence, some $\varphi \in \mathcal{D}(\Omega ; C_{\#}^\infty(Y))$, with support in \mathcal{O} or in T , one concludes that

$$\psi_2(x, y) = 0, \quad \text{a.e. in } \mathcal{O} \quad (4.3)$$

and

$$\nabla \psi(x) + \nabla_y \psi_1(x, y) = 0, \quad \text{a.e. in } T. \quad (4.4)$$

From (3.11) one has

$$\int_{\Omega} [\nabla \psi_\varepsilon + x + (\xi_\varepsilon - x)(1 - \chi_\varepsilon)] \nabla \varphi \, dx = 0, \quad \forall \varphi \in H_0^1(\Omega). \quad (4.5)$$

Let $\varphi = \phi(x) + \varepsilon \phi_1\left(x, \frac{x}{\varepsilon}\right)$, where $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega ; C_{\#}^\infty(Y))$. One obtains, from (4.5)

$$\int_{\Omega} [\nabla \psi_{\varepsilon} + x + (\xi_{\varepsilon} - x)(1 - \chi_{\varepsilon})] \times \\ \times \left[\nabla \phi + \varepsilon \nabla_x \phi_1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \phi_1 \left(x, \frac{x}{\varepsilon} \right) \right] dx = 0,$$

and, consequently, passing this to the limit,

$$\int_{\Omega \times Y} [\nabla \psi + \nabla_y \phi_1 + x + \psi_2] \cdot [\nabla \phi + \nabla_y \phi_1] dx dy = 0,$$

for all ϕ and ϕ_1 . It yields the following two-scale homogenized torsion system :

$$\begin{cases} -\operatorname{div}_x \left[\nabla \psi + \int_Y \psi_2 dy \right] = 2, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial \Omega, \\ -\operatorname{div}_y [\nabla_y \psi_1 + \psi_2] = 0, & \text{in } \Omega \times Y, \end{cases} \quad (4.6)$$

$\psi_1, \nabla_y \psi_1, \psi_2$ are Y -periodic, satisfying, respectively, (4.4) and (4.3).

Developing the cell-problem in (4.6) and conditions (4.3) and (4.4), and since $\nabla \psi(x)$ is the independent of y , $\psi_1 \in L^2(\Omega; H^1_{\#}(Y)/\mathbb{R})$ and $\psi_2 \in L^2(\Omega \times Y)$ must satisfy the following problems, a.e. in $x \in \Omega$,

$$\begin{cases} -\Delta_y \psi_1 = 0, & \text{in } Y \setminus T(x), \\ \psi_1 = -\nabla \psi(x) y, & \text{on } \partial T(x), \end{cases} \quad (4.7)$$

and

$$\begin{cases} -\operatorname{div}_y \psi_2 = 0, & \text{in } T(x), \\ \psi_2 n = (\nabla \psi + (\nabla_y \psi_1)|_{Y \setminus T(x)}) n, & \text{on } \partial T(x). \end{cases} \quad (4.8)$$

Defining, for $i = 1, 2$ and $x \in \Omega$, ϕ_i^x and ϕ_i^{x*} as the solutions of

$$\begin{cases} -\Delta_y \phi_i^x = 0, & \text{in } Y \setminus T(x), \\ \phi_i^x = -y_i, & \text{on } \partial T(x), \\ \phi_i^x, \nabla_y \phi_i^x & \text{are } Y\text{-periodic}, \end{cases} \quad (4.9)$$

and

$$\begin{cases} -\Delta_y \phi_i^{x*} = 0, & \text{in } T(x), \\ \partial_n \phi_i^{x*} = \partial_n \phi_i^x, & \text{on } \partial T(x), \\ \int_{T(x)} \phi_i^{x*} = 0, & \phi_i^{x*} \text{ } Y\text{-periodic}, \end{cases} \quad (4.10)$$

respectively, one has, a.e. in $x \in \Omega$,

$$\Psi_1(x, y) = \phi_1^x(y) \cdot \partial_1 \psi(x) + \phi_2^x(y) \cdot \partial_2 \psi(x), \text{ a.e. in } y \in Y \setminus T(x)$$

and

$$\psi_2(x, y) = \left(\begin{array}{l} (\partial_1^y \phi_1^{x*} + 1) \partial_1 \psi + \partial_1^y \phi_2^{x*} \partial_2 \psi \\ \partial_2^y \phi_1^{x*} \partial_1 \psi + (\partial_2^y \phi_2^{x*} + 1) \partial_2 \psi \end{array} \right) + \psi_2^{x*}(x, y),$$

a.e. in $y \in T(x)$, where ψ_2^* is the curl of some stream function, with respect to y , and, consequently,

$$\int_{T(x)} \psi_2^* dy = 0.$$

Since from (4.9) and (4.10),

$$\int_{T(x)} \partial_i^y \phi_j^{x*} dy = \int_{Y \setminus T(x)} \nabla_y \phi_j^x \cdot \nabla_y \phi_i^x dy,$$

the first equation of (4.6) takes, then, the form

$$\begin{cases} -\operatorname{div}_x(A \nabla \psi) = 2, \text{ in } \Omega, \\ \psi = 0, \text{ on } \partial \Omega, \end{cases} \quad (4.11)$$

where

$$\begin{aligned} A = A(x) &= [a_{ij}(x)]_{i,j=1,2}, \quad \text{and} \quad (\text{c.f. [7], § 4}) \\ a_{ij}(x) &= (2 - \theta(x)) \delta_{ij} + \int_{Y \setminus T(x)} \nabla_y \phi_i^x \nabla_y \phi_j^x dx. \end{aligned} \quad (4.12)$$

Equations (4.11) and (4.9) (or (4.11) represent the macroscopic and microscopic problems, associated to the homogenization process, respectively.

The warping problem

Let v_ε be the solution of problem (3.12) and let \tilde{w}_ε and $(\nabla w_\varepsilon)^\sim$ be the zero extensions to Ω of w_ε and ∇w_ε , respectively. Using Theorem 2.2 and Proposition 2.4 there exists w and σ in $L^2(\Omega \times Y)$, $v \in H^1(\Omega)$ and $v_1 \in L^2(\Omega; H_\#^1(Y)/\mathbb{R})$, such that, up to a subsequence of ε , the following two-scale convergences hold

$$\begin{aligned} \tilde{w}_\varepsilon &\rightharpoonup w, & (\nabla w_\varepsilon)^\sim &\rightharpoonup \sigma \\ v_\varepsilon &\rightharpoonup v, & \nabla v_\varepsilon &\rightharpoonup \nabla v + \nabla_y v_1. \end{aligned} \quad (4.13)$$

Using adequeted test functions one obtains :

$$\begin{aligned} w(x, y) &= v(x) \cdot \chi(x, y), \\ \sigma(x, y) &= [\nabla v(x) + \nabla_y v_1(x, y)] \chi(x, y), \quad \text{a.e. in } (\Omega \times Y). \end{aligned} \tag{4.14}$$

From (3.5) one has

$$\int_{\Omega} ((\nabla w_{\varepsilon})^{\sim} - Rx \chi_{\varepsilon}) \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1(\Omega). \tag{4.15}$$

Let $\varphi = \phi(x) + \varepsilon \phi_1\left(x, \frac{x}{\varepsilon}\right)$, where $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega; C_{\#}^{\infty}(Y))$. Passing to the limit in (4.15) and using (4.13) and (4.14), one obtains

$$\int_{\Omega \times Y} [\nabla v + \nabla_y v_1 - Rx] \cdot \chi(x, y) [\nabla \phi + \nabla_y \phi_1] \, dx \, dy = 0,$$

for all ϕ and ϕ_1 . It yields the following two scale homogenized torsion system :

$$\left\{ \begin{aligned} &\int_{\Omega} \left[(\nabla v - Rx) \theta + \int_Y \nabla_y v_1 \cdot \chi \, dy \right] \cdot \nabla \phi \, dx = 0, \quad \forall \phi, \\ &\int_{\Omega} v \theta \, dx = 0, \\ &\int_Y [\nabla_y v_1 \chi + (\nabla v - Rx) \chi] \nabla_y \phi_1 \, dy = 0, \quad \forall \phi_1, \text{ in } \Omega, \\ &v_1 \text{ Y-periodic.} \end{aligned} \right. \tag{4.16}$$

Developing the third equation integral in (4.16) one sees that v_1 is given by

$$v_1(x, y) = (\partial_1 v(x) - x_2) \theta_1^x(y) + (\partial_2 v(x) + x_1) \theta_2^x, \text{ a.e. in } (x, y) \in \Omega \times Y,$$

where $\theta_i^x, i = 1, 2$ and $x \in \Omega$, are the unique solutions of

$$\left\{ \begin{aligned} &-\Delta_y \theta_i^x = 0, \text{ in } Y - T(x), \\ &\partial_n \theta_i^x = -n_i, \text{ on } \partial T(x), \\ &\int_{Y \setminus T(x)} \theta_i^x \, dy = 0, \\ &\theta_i^x, \nabla_y \theta_i^x \text{ Y-periodic.} \end{aligned} \right. \tag{4.17}$$

The first equation of (4.16) takes, then, the form

$$\begin{cases} -\operatorname{div}_x(A^* \nabla v) = -\operatorname{div}_x(A^* Rx), \text{ in } \Omega, \\ (A^* \nabla v) n = (A^* Rx) n, \text{ on } \partial\Omega, \\ \int_{\Omega} v \theta \, dx = 0, \end{cases} \quad (4.18)$$

where

$$A^* = A^*(x) = [a_{ij}^*(x)]_{i,j=1,2},$$

and

$$a_{ij}^*(x) = \theta(x) \delta_{ij} - \int_{Y \setminus T(x)} \nabla_y \theta_i^x \cdot \nabla_y \theta_j^x \, dy. \quad (4.19)$$

Equations (4.18) and (4.17) represent the macroscopic and the microscopic problems associated to the homogenization process, respectively.

The relation between the homogenized torsion and warping

The following proposition, whose proof was suggest by F. Murat, relates the matrices (4.12) and (4.19).

PROPOSITION 4.1: *Both matrices A and A^* , introduced in (4.12) and (4.19), respectively, are elliptic, symmetric and satisfy*

$$A = \frac{A^*}{\det A^*} \quad \text{or, equivalently,} \quad A^* = \frac{A}{\det A}. \quad (4.20)$$

Proof: For each fixed $x \in \Omega$ consider the following functions of y , periodically extended to the entire \mathbb{R}^2 :

$$\xi_1(y) = (\nabla_y \theta_i^x + e_i) \tilde{},$$

where « $\tilde{}$ » represents the zero extension of $(\nabla_y \theta_i^x + e_i)$ to $T(x)$,

$$\xi_2(y) = \nabla_y (\hat{\theta}_i^x + y_i),$$

where $\hat{\theta}_i^x$ is the extension of θ_i^x to a function of $H_{\#}^1(Y)$,

$$\eta_1(y) = (\nabla_y \phi_i^x) \hat{} + e_i,$$

where $(\nabla_y \phi_i^x) \hat{}$ is equal to $\nabla_y \phi_i^x$, in $Y \setminus T(x)$, and to $\nabla_y \phi_i^{x*}$, in $T(x)$,

$$\eta_2(y) = \nabla_y [(\phi_i^x + y_i) \hat{}],$$

where « \cdot » represents the extension of $(\phi_i^1 + y_i)$ to $T(x)$, by its constant value on the boundary.

The following equality holds :

$$\xi_y(y) R \eta_1(y) = \xi_2(y) R \eta_2(y), \text{ a.e. in } y \in Y, \tag{4.21}$$

where R is defined as in (3.8).

Since, from the variational formulations (4.9), (4.10) and (4.17), one has $\text{div}_y \xi_1 = \text{div}_y \eta_1 = 0$, in Y , and since ξ_2 and η_2 are gradients, the right hand side of equality (4.21) is the product of the divergence free vector ξ_1 by the curl free vector $R \eta_1$, and the left hand side is the product of the curl free vector ξ_2 by the divergence free vector $R \eta_2$. Using the compensated compactness div-curl-Theorem (see [9], Theorem 1, p. 490) we are allowed to pass to the limit, as ε goes to zero, in equality

$$\xi_1 \left(\frac{z}{\varepsilon} \right) R \eta_1 \left(\frac{z}{\varepsilon} \right) = \xi_2 \left(\frac{z}{\varepsilon} \right) R \eta_2 \left(\frac{z}{\varepsilon} \right), \quad z \in \mathbb{R}^2,$$

and obtain

$$\int_Y \xi_1 \, dy \, R \int_Y \eta_1 \, dy = \int_Y \xi_2 \, dy \, R \int_Y \eta_2 \, dy. \tag{4.22}$$

Using the definitions (4.12) and (4.19), and the variational formulations (4.9) and (4.17), expression (4.22) implies that $A^* R A = R$, and, consequently, any of the two equivalent equalities

$$A = \frac{A^*}{\det A^*} \quad \text{or} \quad A^* = \frac{A}{\det A}.$$

The ellipticity of A and A^* follows from the fact that, using the Einstein's convention on indexes,

$$\xi_i \xi_j \int_{Y \setminus T(x)} \nabla_y \phi_i^x \cdot \nabla_y \phi_j^x \, dy = \int_{Y \setminus T(x)} |\xi_k \nabla_y \phi_k^x|^2 \, dy \geq 0,$$

for all $\xi = (\xi_i)_{i=1,2} \in \mathbb{R}^2$.

The symmetry of A and A^* is obvious, from the definition. \square

Now the homogenization result may be stated as follows :

THEOREM 4.2 : Let \tilde{w}_ε be the zero extension of the solution w_ε of problem (3.4) to the entire domain Ω and let $\tilde{\psi}_\varepsilon$ be the extension, by its

constant value on each connected component of the boundary, of the solution ψ_ε of problem (3.6) to the entire domain Ω ; then the following convergences hold

$$\begin{aligned} \tilde{w}_\varepsilon &\rightharpoonup v\theta, & \text{weakly in } L^2(\Omega) \\ \tilde{w}_\varepsilon &\rightharpoonup \psi, & \text{weakly in } H_0^1(\Omega), \end{aligned} \quad (4.23)$$

where $\theta(x)$ is defined as in (3.1) and v and ψ are the unique solutions of the homogenized problems (4.18) and (4.11), respectively, whose matrices are related as in (4.20).

Moreover the following relation between v and ψ holds

$$A^*(\nabla v - Rx) = R \nabla \psi, \quad (4.24)$$

where R is as in (3.8).

Proof: The only things left to prove are, first, that convergences (4.23) hold for all the sequence ε and not only for a subsequence and, second, expression (4.24). In view of the ellipticity of A and A^* , the solutions of problems (4.11) and (4.18) are unique which implies that the hole sequence ε satisfies the required convergences.

Expression (4.24) is a straightforward consequence of expression (3.8), convergences (4.13), equalities (4.14), the definition of v_1 and of the matrix A^* . \square

5. THE NEUMANN PROBLEM

The Neumann problem may also be treated in the same context as the warping and torsion functions.

For $f \in L^2(\Omega)$ let u_ε be the unique solution of

$$\begin{cases} -\Delta u_\varepsilon = f\chi_{\varepsilon}, & \text{in } \Omega_\varepsilon, \\ \partial_n u_\varepsilon = 0, & \text{on } \partial T_{\varepsilon k}, k \in \mathbb{Z}_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

Defining $V(\Omega_\varepsilon) = \{v \in H^1(\Omega_\varepsilon) : v \text{ is equal to zero on } \partial\Omega\}$, the variational formulation of (5.1) is :

$$\begin{cases} u_\varepsilon \in V(\Omega_\varepsilon) \\ \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi \, dx = \int_{\Omega_\varepsilon} f \, dx, \quad \forall \varphi \in V(\Omega_\varepsilon). \end{cases} \quad (5.2)$$

The following theorem holds :

THEOREM 5.1 : *Let \tilde{u}_ε be the zero extension of the solution u_ε of problem (5.1) to the entire domain Ω . Then \tilde{u}_ε converges weakly in $L^2(\Omega)$ to $v\theta$, where θ is defined as in (3.1) and v is the unique solution of the following homogenized problem*

$$\begin{cases} -\operatorname{div} (A^* \nabla v) = \theta f, \text{ in } \Omega, \\ v = 0, \text{ on } \partial\Omega, \end{cases} \tag{5.3}$$

where the matrix A^* is defined as in (4.19).

Proof : Since the proof is analogous to the one made for the warping function, we only point out the main differences.

From the variational formulation (5.2) one obtains that the zero extension $(\nabla u_\varepsilon)^\sim$ of ∇u_ε to the entire Ω , is bounded in $L^2(\Omega)$, by some constant independent of ε .

We construct an $H_0^1(\Omega)$ bounded extension v_ε of u_ε by considering $\xi_\varepsilon \in L^2(\Omega)$ such that

$$\operatorname{div} [-R(\nabla w_\varepsilon)^\sim + \xi_\varepsilon(1 - \chi^\varepsilon)] = 0, \text{ in } \Omega, \tag{5.4}$$

and concluding that $(\nabla w_\varepsilon)^\sim + R\xi_\varepsilon(1 - \chi^\varepsilon)$ is the gradient of some function $v_\varepsilon \in H_0^1(\Omega)$.

The proof of the existence of ξ_ε is the same presented in Proposition 3.6 : we replace, in (3.18), $\nabla_y \xi_{\varepsilon k}$ by $(-R \nabla u_\varepsilon)$ ($k\varepsilon + \varepsilon y$), with R defined as in (3.8), and the same arguments hold, since $-R(\nabla u_\varepsilon)^\sim$ is bounded in $L^2(\Omega)$ and $-R \nabla u_\varepsilon$ is divergence-free in Ω_ε .

Using the two-scale convergence method, and the boundness conditions, there exist functions

$$u, \sigma \in L^2(\Omega \times Y), v \in H_0^1(\Omega) \text{ and } v_1 \in L^2(\Omega ; H_{\#}^1(Y)/\mathbb{R})$$

such that the following two-scale convergences hold :

$$\tilde{u}_\varepsilon \rightharpoonup u, \quad (\nabla u_\varepsilon)^\sim \rightharpoonup \sigma, \quad v_\varepsilon \rightharpoonup v, \quad \nabla v_\varepsilon \rightharpoonup \nabla v + \nabla_y v_1,$$

with

$$\begin{aligned} u(x, y) &= v(x) \chi(x, y), \\ \sigma(x, y) &= [\nabla v(x) + \nabla_y v_1(x, y)] \chi(x, y), \quad \text{a.e. in } \Omega \times Y. \end{aligned}$$

Passing to the limit in (5.2), we obtain

$$\int_{\Omega \times Y} [\nabla v + \nabla_y v_1] \chi(x, y) [\nabla \phi + \nabla_y \phi_1] dx dy = \theta f,$$

for all $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega; C_{\neq}^{\infty}(Y))$, which leads to

$$v_1(x, y) = \partial_1 v(x) \theta_1^x(y) + \partial_2 v(x) \theta_2^x(y),$$

a.e. in $(x, y) \in \Omega \times Y$, where θ_i^x , $i = 1, 2$, are defined in (4.17) and to the homogenized equation (5.3). \square

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APPENDIX

LEMMA A.1 : *Let $T \in \pi(\theta, h, r)$ then there exists a positive real number d_0 , depending only on θ, h and r , such that the distance between two distinct connected components of T is greater than or equal to d_0 .*

Proof : Let $k, \delta \in \mathbb{R}^+$: $\pi(\theta, h, r) \subset \text{Lip}(k, \delta)$.

Let A and B be two distinct connected components of T .

Let $a \in A$ and $b \in B$, $d(a, b) < \min \left\{ \frac{\delta}{2}, \frac{k \cdot \delta}{2} \right\} = d_0$.

There exists c , belonging to the segment $[a, b]$, such that $c \in \partial T$. Then $a, b \in P_{\delta k}(c) \cap T$, which is absurd, since, by Definition 2.8, $P_{\delta k}(c) \cap T$, is an open connected set. \square

PROPOSITION A.2 : *The set $\pi_1(\theta, h, r) = \{T \in \pi(\theta, h, r) : T \text{ is connected}\}$ is closed, in the sense of the characteristic functions, with respect to the induced $L^2(Y)$ topology.*

Proof : Let $T_1 \in \pi_1$ and $T \in \pi \setminus \pi_1$. Suppose, without loss of generality, that T has only two distinct connected components A and B . We know that $d(A, B) \geq d_0 > 0$.

Using a covering of T_1 made of balls $B\left(c, \frac{d_0}{4}\right)$, we conclude that either $\exists c \in T_1 : B\left(c, \frac{d_0}{4}\right) \subset \mathbb{R}^N \setminus T$, either $T_1 \subset Y \setminus A$, or $T_2 \subset Y \setminus B$.

Since $\left| B\left(c, \frac{d_0}{4}\right) \cap T_1 \right| \geq \frac{d_0}{2} \cdot \theta$, it follows that, in each case, T is exterior to Π_1 . \square

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