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ANALYSIS OF MIXED METHODS USING CONFORMING AND NONCONFORMING FINITE ELEMENT METHODS (*)

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Abstract. — *An abstract framework under which an equivalence between mixed finite element methods and certain modified versions of conforming and nonconforming finite element methods is established for second order elliptic problems with variable coefficients. It is shown, based on the equivalence, that mixed methods can be implemented through usual conforming or nonconforming methods modified in a cost-free manner and that new error estimates for these methods can be derived. The Raviart-Thomas, Brezzi-Douglas-Marini, and Marini-Pietra mixed methods for second order elliptic problems are analyzed by means of the present techniques.*

Résumé. — *On établit un cadre abstrait pour établir, dans le cas de problèmes elliptiques du 2^e ordre à coefficients variables, l'équivalence entre des méthodes d'éléments finis mixtes et certaines versions modifiées de méthodes d'éléments finis conformes et non conformes. En se basant sur cette équivalence, on montre que des méthodes mixtes peuvent être mises en œuvre à partir des méthodes conformes et non conformes habituelles, à moindre coût ; on montre également que des nouvelles estimations d'erreur peuvent être obtenues. On analyse par ces techniques, dans le cadre des problèmes elliptiques du 2^e ordre, les méthodes mixtes de Raviart-Thomas, Brezzi-Douglas-Marini et Marini-Pietra*

1. INTRODUCTION

It has been observed that in many cases mixed finite element methods give better approximations for the flux variable associated with the solution of a second order elliptic problem than classical Galerkin methods [2], [3], [13].

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However, the mixed formulation is more difficult to handle and, in general, is more expensive from a computational point of view [2], [11]. For second order elliptic problems with piecewise constant coefficients, it has been observed [2], [11] that this drawback can be circumvented by observing a certain equivalence between mixed methods and some modified versions of standard nonconforming methods. Arnold and Brezzi [2] showed, for example, that the Raviart-Thomas mixed method of lowest order is equivalent to the usual P_1-nonconforming method modified by augmenting the classical P_1-nonconforming space with P_3-bubbles and then proved that the equivalence is useful not only for implementing the mixed method but also for deriving error estimates.

Variable coefficients may significantly complicate the equivalence above and thus the performance of the mixed methods. Indeed, in the method of Arnold and Brezzi [2], the weighted averages of the inverse of the coefficients enter the numerical schemes through a projection on the Raviart-Thomas space.

The main purpose of this paper is to develop, in a rather general setting with minimal hypotheses, error estimates and implementations of mixed finite element methods for second order elliptic problems with variable coefficients. We shall develop an abstract framework under which an equivalence between mixed methods and certain modified versions of conforming and nonconforming finite element methods can be established. It will be shown that our abstract theory includes not only the existing analysis for the Raviart-Thomas method but also provides an approach to the analysis of other mixed methods such as the Brezzi-Douglas-Marini and Marini-Pietra methods [3], [12]. More specifically, it is proven, by means of the present techniques, that the lowest-order Brezzi-Douglas-Marini and Marini-Pietra methods are equivalent to modified conventional P_3-conforming and P_2-nonconforming finite element methods, respectively. It should be emphasized that the field of application of our abstract results is quite large even through only three families of mixed finite elements are analyzed here. Recently, Arbogast [1] has independently considered many of the same questions with different emphases.

We shall also show that the difficulties with the variable coefficients noted above can be avoided by projecting these coefficients into the finite element space of the scalar variable and that the introduction of the projection of the coefficients in the mixed methods above does not result in a reduction of the order of convergence of the method and can lead to great savings in computational effort. Moreover, the desirable features for piecewise-constant coefficients are shared by the approximation procedure considered for variable coefficients. In particular, based on the equivalence above, it is proven that the approximate solution for the flux variable produced by both methods can be computed from the solution of the usual conforming or nonconforming methods in an inexpensive manner, that a superconvergent approximation of the scalar variable by means of post-processing can be obtained, and that new duality error estimates for the methods can be obtained under a certain assumption on the triangulation of the domain.

The rest of this paper is organized as follows. In the next section an abstract theory of an equivalence between mixed methods and modified conventional finite element methods is established. Then, in § 3-6, an application of the results to the three families of mixed methods mentioned above is presented. Finally, in § 7, the mixed methods for second order elliptic problems with variable coefficients are discussed.

Throughout this paper we shall use the notation $\| \cdot \|_{s,K}$ and $\| \cdot \|_{s,\infty,K}$ for the norms on the Sobolev spaces $H^s(K)$ and $W^{s,\infty}(K)$, respectively, for $s \geq 0$ and some set $K \subset \mathbb{R}^2$. We shall also denote by $\| \cdot \|_K$ and $(\cdot , \cdot)_K$ the norm and the scalar product on $L^2(K)$. The subscript K will be omitted when it is Ω. Finally, the notation $\| \cdot \|_{-s}$ will indicate the norm on the dual space $H^{-s}(\Omega) = (H_0^s(\Omega))'$, $s \geq 0$.

2. AN ABSTRACT THEORY

In this section we shall first develop an abstract framework under which an equivalence between mixed finite element methods and certain modified versions of conventional finite element methods for (2.2) below can be established. Then, based on the equivalence, we shall obtain a duality error estimate for the methods introduced. In order to fix ideas, in the present and next four sections, the coefficient a will be assumed piecewise-constant. The extension of the results to the case of variable coefficients will be discussed in the last section.

Let Ω be a domain in $\mathbb{R}^2$, let $f \in L^2(\Omega)$, and let a be a smooth function on Ω such that

$$0 < a_* \leq a \leq a^* < \infty \quad \text{on } \Omega . \tag{2.1}$$

Consider the Dirichlet problem

$$- \operatorname{div} (a \nabla u) = f \quad \text{in} \quad \Omega , \tag{2.2a}$$

$$u = 0 \quad \text{on} \quad \partial\Omega . \tag{2.2b}$$

It is well known that problem (2.2) has a unique solution u.

Set

$$V = H(\operatorname{div} ; \Omega) = \{ \tau \in (L^2(\Omega))^2 : \operatorname{div} \tau \in L^2(\Omega) \} ,$$

with the usual norm

$$\| \tau \|_V = \left(\sum_{i=1}^{2} \| \tau_i \|^2 + \|\operatorname{div} \tau \|^2 \right)^{1/2} ,$$

where $\tau = (\tau_1, \tau_2)$, and let

$$W = L^2(\Omega) .$$

Introducing $\sigma = - a \nabla u$, a formulation of (2.2) appropriate for the mixed method is then :

$$\text{Find } (\sigma, u) \in V \times W \text{ such that}$$

$$(\alpha \sigma, \tau) - (u, \operatorname{div} \tau) = 0 , \qquad \forall \tau \in V , \tag{2.3a}$$

$$(\operatorname{div} \sigma, v) = (f, v) , \qquad \forall v \in W , \tag{2.3b}$$

where $\alpha = a^{-1}$.

For the discretization of (2.3), let $T_h = \{T\}$ be a regular partition of Ω into triangles or rectangles of diameter not greater than h, $0 < h < 1$, such that if T is a boundary element, the boundary edge can be curved, and let E_h denote the set of edges of triangles or rectangles of T_h with the decomposition

$$E_h^{\partial} = \{e \in E_h : e \in \partial\Omega\} , \quad E_h^0 = E_h \backslash E_h^{\partial} .$$

Associated with T_h, we introduce the finite element spaces

$$V_h = \{\tau \in (W)^2 : \tau|_T \in V(T) , \quad \forall T \in T_h\} ,$$

$$W_h = \{v \in W : v|_T \in W(T) , \quad \forall T \in T_h\} ,$$

where $V(T)$ and $W(T)$ are finite dimensional, polynomial spaces on T such that

(H1) $$\operatorname{div} V(T) \subseteq W(T) .$$

Note that we do not require that $V_h \subset V$. Namely, the normal components of elements in V_h are not assumed to be continuous across the interelement boundaries. For simplicity, we assume that there is an integer which bounds the degrees of the polynomials in the finite dimensional spaces introduced in this section.

It is well-known that, when dealing with discretizations of the mixed formulation (2.3), the linear algebraic systems produced by usual mixed finite element methods are generally indefinite. A way to overcome this difficulty is the introduction of Lagrange multipliers on the interelement boundaries in order to relax the continuity requirement on the normal components of the approximate solutions associated with the flux variables across these boundaries [10]. This leads to defining the multiplier space

$$\Lambda_h = \{\mu \in L^2(E_h) : \mu|_e \in \Lambda(e) , \quad \forall e \in E_h^0 ; \quad \mu|_e = 0, \quad \forall e \in E_h^{\partial}\} ,$$

where $\Lambda(e)$ is some polynomial space on the set e, and the norm of Λ_h is given by

$$|\mu|_h^2 = \sum_{e \in E_h^0} \|\mu\|_e^2 . \tag{2.4}$$

We are now ready to state the mixed-hybrid formulation for approximating the solution of (2.2) [2] :

Find $(\sigma_h, u_h, \lambda_h) \in V_h \times W_h \times \Lambda_h$ such that

$$(\alpha \sigma_h, \tau) - \sum_T \{(u_h, \operatorname{div} \tau)_T - (\lambda_h, \tau . n_T)_{\partial T}\} = 0, \qquad \forall \tau \in V_h , \tag{2.5a}$$

$$\sum_T (v, \operatorname{div} \sigma_h)_T = (f, v), \quad \forall v \in W_h , \tag{2.5b}$$

$$\sum_T (\mu, \sigma_h . n_T)_{\partial T} = 0, \qquad \forall \mu \in \Lambda_h , \tag{2.5c}$$

where n_T denotes the outward unit normal to T. We shall assume that the problem has a unique solution for each $f \in L^2(\Omega)$. This can easily be established [12] under the assumptions (H1) and that for each $v \in W(T)$ and $\mu \in \Lambda(\partial T)$ such that

$$\|\nabla v\|_T + \|\mu\|_{\partial T} \neq 0 ,$$

there exists $\tau \in V(T)$ satisfying

$$(\nabla v, \tau)_T + (\mu, \tau . n_T)_{\partial T} \neq 0 ,$$

and that

$$\gamma_0(W(T)) \subseteq \Lambda(\partial T) ,$$

where $\Lambda(\partial T) = \prod_{e \in \partial T} \Lambda(e)$ and γ_0 denotes the trace on ∂T.

We shall introduce another discrete formulation for approximating the solution of (2.2) which we shall prove to be equivalent to (2.5). To that end, we now define the « intermediate » multiplier space

$$\tilde{\Lambda}_h = \left\{\mu \in L^2(E_h) : \mu \big|_e \in \tilde{\Lambda}(e), \ \forall e \in E_h^0 ; \mu \big|_e = 0, \ \forall e \in E_h^\partial \right\} ,$$

where again $\tilde{\Lambda}(e)$ is a polynomial space on the set e such that

(H2) $\quad \Lambda(e) \subseteq \tilde{\Lambda}(e) , \ \tau . n_e \in \tilde{\Lambda}(e) , \ \tau \in V(T) , \ e \in E_h^0 , \ T \in T_h ,$

with n_e being a unit vector normal to e. Let now P_h and $R_h(\tilde{R}_h)$ denote the orthogonal projections onto W_h and $\Lambda_h(\tilde{\Lambda}_h)$ with respect to the norms

$\|\cdot\|$ and $|\cdot|_h$, respectively. Then, define $M_h \subset L^2(\Omega)$ to be a finite dimensional space such that the following two assumptions are satisfied :

(H3) $$\tilde{R}_h \psi \in \Lambda_h, \quad \forall \psi \in M_h .$$

(H4) For each $v \in W_h$ and $\mu \in \Lambda_h$, there is a unique φ in M_h satisfying

$$P_h \varphi = v, \quad \tilde{R}_h \varphi = \mu .$$

For a given $\tau \in (W)^2$, denote $P_V \tau$ the L^2-projection of τ in V_h. Another approximation procedure for (2.2) is then defined by seeking $\psi_h \in M_h$ such that [2]

$$\sum_T (aP_V(\nabla \psi_h), \nabla \varphi)_T = (P_h f, \varphi), \quad \forall \varphi \in M_h . \tag{2.6}$$

We are now in a position to prove the following equivalence theorem.

THEOREM 2.1 : *Assume that assumptions* (H1)-(H4) *are satisfied. Let* $(\sigma_h, u_h, \lambda_h)$ *be the unique solution of system* (2.5) *and let* $\psi_h \in M_h$ *be determined by*

$$P_h \psi_h = u_h, \quad \tilde{R}_h \psi_h = \lambda_h . \tag{2.7}$$

Then ψ_h *is the unique solution of* (2.6). *Moreover,* σ_h *is related to* ψ_h *by*

$$\sigma_h = - aP_V(\nabla \psi_h) . \tag{2.8}$$

Proof : Note that ψ_h is uniquely determined by (H4). From (H1), (H2), (2.7), and Green's formula, (2.5*a*) becomes

$$\sum_T (\nabla \psi_h, \tau)_T + (\alpha \sigma_h, \tau) = 0, \quad \forall \tau \in V_h .$$

This shows that $\alpha \sigma_h$ is the L^2-projection of $-\nabla \psi_h$ in V_h since $\alpha = a^{-1}$ is piecewise constant, and thus (2.8) holds.

Next, by (H1) and (2.5b) with $v = P_h \varphi$, $\varphi \in M_h$, we have

$$\sum_T (\operatorname{div} \sigma_h, \varphi)_T = (P_h f, \varphi), \quad \forall \varphi \in M_h .$$

Hence, by Green's formula,

$$\sum_T \left\{ (\varphi - \tilde{R}_h \varphi, \sigma_h . n_T)_{\partial T} + (\tilde{R}_h \varphi, \sigma_h . n_T)_{\partial T} - (\sigma_h, \nabla \varphi)_T \right\} =$$
$$= (P_h f, \varphi), \quad \forall \varphi \in M_h ,$$

which together with (H2), (H3), and (2.5c) implies that

$$-\sum_T (\sigma_h, \nabla\varphi)_T = (P_h f, \varphi), \quad \forall \varphi \in M_h .$$

This yields that ψ_h is a solution of (2.6) by (2.8).

In order to prove the uniqueness, let $\tilde{\psi}_h$ be another solution of (2.6) and define $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{\lambda}_h)$ by

$$\tilde{\sigma}_h = - aP_V(\nabla\tilde{\psi}_h), \tag{2.9a}$$

$$\tilde{u}_h = P_h \tilde{\psi}_h, \tag{2.9b}$$

$$\tilde{\lambda}_h = \tilde{R}_h \tilde{\psi}_h . \tag{2.9c}$$

By (2.9) and Green's formula, we see that

$$\begin{aligned}(\alpha\tilde{\sigma}_h, \tau) &- \sum_T \left\{(\tilde{u}_h, \operatorname{div}\tau)_T - (\tilde{\lambda}_h, \tau \cdot n_T)_{\partial T}\right\} \\ &= - \sum_T \left\{(\nabla\tilde{\psi}_h, \tau)_T + (\tilde{\psi}_h, \operatorname{div}\tau)_T - (\tilde{\psi}_h, \tau \cdot n_T)_{\partial T}\right\} \\ &= 0, \quad \forall \tau \in V_h .\end{aligned}$$

by (H1) and (H2). Next, for each $v \in W_h$, we define $\chi \in M_h$ such that

$$P_h \chi = v, \quad \tilde{R}_h \chi = 0 . \tag{2.10}$$

Then, using (2.10), (H1), Green's formula, (H2), (2.9a), and (2.6),

$$\begin{aligned}\sum_T (\operatorname{div}\tilde{\sigma}_h, v)_T &= \sum_T (\operatorname{div}\tilde{\sigma}_h, \chi)_T \\ &= \sum_T \left\{(\tilde{\sigma}_h \cdot n_T, \chi)_{\partial T} - (\tilde{\sigma}_h, \nabla\chi)_T\right\} \\ &= \sum_T (aP_V(\nabla\tilde{\psi}_h), \nabla\chi)_T \\ &= (P_h f, \chi) \\ &= (f, v), \quad \forall v \in W_h .\end{aligned}$$

Finally, for any $\mu \in \Lambda_h$, choose $\chi \in M_h$ satisfying

$$P_h \chi = 0, \quad \tilde{R}_h \chi = \mu . \tag{2.11}$$

Hence,

$$\begin{aligned}\sum_T (\tilde{\sigma}_h \cdot n_T, \mu)_{\partial T} &= \sum_T (\tilde{\sigma}_h \cdot n_T, \chi)_{\partial T} \\ &= \sum_T \left\{(\tilde{\sigma}_h, \nabla\chi)_T + (\operatorname{div}\tilde{\sigma}_h, \chi)_T\right\} .\end{aligned}$$

Applying (2.6), (2.9*a*), (H1), and (2.11), we find that

$$\sum_T (\tilde{\sigma}_h \cdot n_T, \mu)_{\partial T} = 0 , \quad \forall \mu \in \Lambda_h .$$

Combine the results above ; thus, $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{\lambda}_h)$ is a solution of system (2.5). But, by the uniqueness of the solution of (2.5), we see that $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{\lambda}_h) = (\sigma_h, u_h, \lambda_h)$, and so, by (H4), $\tilde{\psi}_h = \psi_h$. This completes the proof of the theorem.

We now turn to the derivation of an error estimate for the problem (2.6). To that end, we shall state further assumptions which will be required in the proof of our next theorem.

(H5) For any smooth function ϕ there exists a function ϕ_h in M_h such that

$$P_h(\phi - \phi_h) = 0 , \quad \tilde{R}_h(\phi - \phi_h) = 0 .$$

(H6) For $\varphi \in M_h$,

$$\sum_{e \in E_h^0} (\mu, [\varphi])_e = 0 , \quad \forall \mu \in \Lambda_h ,$$

where $[\varphi]$ stands for the value of the jump discontinuity of φ on the interelement boundaries.

(H7) $$P_0(e) \subseteq \Lambda(e) , \quad e \in E_h^0 ,$$

where $P_0(e)$ denotes the set of constants on e, so that

$$|v - R_h v|_h \leq C h^{1/2} \left(\sum_T \|\nabla v\|_T^2 \right)^{1/2} , \quad \forall v \in \prod_T H^1(T) , \ T \in T_h . \quad (2.12)$$

The solution ψ_h of (2.6) is assumed to satisfy the relation

(H8) $$P_V(\nabla \psi_h) = \nabla \psi_h ,$$

and the approximation property

(H9) $$\|\nabla(u - \psi_h)\|_h = \left(\sum_T \|\nabla(u - \psi_h)\|_T^2 \right)^{1/2} \leq C h^{r-1} \|u\|_r ,$$

for some $r \geq 1$.

The domain Ω will be said to be 2-regular if the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(a \nabla p) &= q \quad \text{in} \quad \Omega , \\ p &= 0 \quad \text{on} \quad \partial\Omega , \end{aligned}$$

is uniquely solvable for $q \in L^2(\Omega)$ and if

$$\|p\|_2 \leq C \|q\| .$$

THEOREM 2.2 : *Under the hypotheses of* (H1)-(H9), *if u and ψ_h are the solutions of* (2.2) *and* (2.6), *respectively, then,*

$$\|u - \psi_h\| \leq C\,(h^r \|u\|_r + \|f - P_h f\|_{-2}), \tag{2.13}$$

provided that Ω is 2-regular, where C is a generic constant independent of h.

Proof : Let $w = u - \psi_h$. Consider the auxiliary Dirichlet problem :

$$\text{Find } \phi \in H_0^1(\Omega) \text{ such that}$$
$$-\operatorname{div}(a \nabla \phi) = w \quad \text{in} \quad \Omega ,$$
$$\phi = 0 \quad \text{on} \quad \partial\Omega .$$

By the assumed elliptic regularity, we have

$$\|\phi\|_2 \leq C \|w\| . \tag{2.14}$$

Now,

$$\begin{aligned} \|w\|^2 &= \sum_T (a \nabla \phi, \nabla w)_T - \sum_T (a \nabla \phi \,.\, n_T, w)_{\partial T} \\ &\equiv R_1 - R_2 . \end{aligned} \tag{2.15}$$

Let ϕ_h be the function of ϕ in M_h according to (H5). Then, by (2.2), (2.6), and (H8),

$$\begin{aligned} R_1 &= \sum_T (a \nabla \phi, \nabla w)_T \\ &= (f - P_h f, \phi) + (P_h f, \phi - \phi_h) - \sum_T (a \nabla(\phi - \phi_h), \nu \psi_h)_T . \end{aligned} \tag{2.16}$$

Since, by (H1), (H2), (H5), and (2.8),

$$\sum_T (a \nabla(\phi - \phi_h), \nabla \psi_h)_T =$$
$$= \sum_T \left\{ (\phi - \phi_h, \operatorname{div} \sigma_h)_T - (\phi - \phi_h, \sigma_h \,.\, n_T)_{\partial T} \right\} = 0 ,$$

and

$$(P_h f, \phi - \phi_h) = 0 ,$$

we find that

$$|R_1| = |(f - P_h f, \phi)| \leq \|f - P_h f\|_{-2} \, \|\phi\|_2 . \tag{2.17}$$

Next, using (H6),

$$R_2 = \sum_T (a \nabla \phi \, . \, n_T, w)_{\partial T}$$
$$= \sum_T (a(\nabla \phi \, . \, n_T - R_h\{\nabla \phi \, . \, n_T\}), w - R_h w)_{\partial T} ,$$

so that, applying (2.12),

$$|R_2| \leq Ch \| \phi \|_2 \, \| \nabla w \|_h . \tag{2.18}$$

Now, combine (2.14)-(2.15), (2.17)-(2.18), and (H9) to obtain the desired result (2.13), and the proof of the theorem has been completed.

Remark : If $\tilde{\Lambda}_h = \Lambda_h$, the assumption (H5) follows immediately follows from (H4). As seen in the next section, this is the case in most applications. The hypothesis (H6) requires that the elements of M_h have a certain continuity across the interelement boundaries. The assumption (H7) is trivially satisfied for all the existing mixed spaces. The relation (H8) may be shown for some mixed spaces under a certain assumption on the triangulation of the domain. The estimate (H9) can be easily verified by the equivalence above between (2.5) and (2.6) and a known error estimate for the mixed method (2.5). Finally, the duality estimate (2.13) has been proved using the discretization formulation (2.6), which cannot be naturally derived from the original mixed formulation (2.5).

3. THE LOWEST-ORDER RAVIART-THOMAS METHOD I

In this section and the next three sections we shall apply the results of the previous section to several examples. We shall consider the lowest-order Raviart-Thomas and Brezzi-Douglas-Marini methods [13], [3] and the mixed method recently introduced by Marini and Pietra [12] since these methods are the most useful in practice. But, as mentioned in the introduction, the results in the previous section can be applied to other more general mixed spaces with higher indexes. For simplicity, we shall assume that Ω is a convex, polygonal domain. However, it will become clear that the same analysis can also be done for more general domains where $u \in H^2(\Omega)$.

Let $T_h = \{T\}$ be a triangular decomposition of Ω. The spaces $V(T)$, $W(T)$, $\Lambda(\partial T)$, and $\tilde{\Lambda}(\partial T)$ are defined by

$$V(T) = (P_0(T))^2 + (x, y) P_0(T) ,$$
$$W(T) = P_0(T) ,$$
$$\Lambda(\partial T) = \tilde{\Lambda}(\partial T) = P_0(\partial T) = \prod_{e \in \partial T} P_0(e) ,$$

where $P_k(T)$ denotes the set of polynomials of degree not greater than k, $k \geqslant 0$, on T. The assumption (H1) follows from the relation

$$\operatorname{div} V(T) = W(T),$$

and it is easy to see that (H2) holds.

We now turn to define the space M_h. For each T in T_h, let $(\lambda_1, \lambda_2, \lambda_3)$ represent the barycentric coordinates of a point of T, and let

$$N_{3h} = \{v : v|_T = \gamma_T \lambda_1 \lambda_2 \lambda_3,\ \gamma_T \in \mathbb{R},\quad \forall T \in T_h\}.$$

Then, define M_h by [2]

$$M_h = M_{NC} \oplus N_{3h},$$

where M_{NC} is the usual nonconforming space ; i.e.,

$$M_{NC} = \{v \in L^2(\Omega) : v|_T \in P_1(T),\quad \forall T \in T_h,$$

v is continuous at the midpoints of sides in E_h^0 and vanishes at the midpoints of sides in $E_h^\partial\}$.

Note that the space M_h is the classical P_1-nonconforming space augmented with P_3-bubbles. The hypothesis (H3) is trivial, since $\Lambda_h = \tilde{\Lambda}_h$ and thus $R_h = \tilde{R}_h$. The assumption (H4) was shown in [2].

Consequently, Theorem 2.1 is applicable and shows that the lowest-order Raviart-Thomas method is equivalent to a modified P_1-nonconforming method. It follows from (2.8) that

$$\sigma_h = -aP_V(\nabla \psi_h).$$

Furthermore, it can be shown [11] using the equivalence between systems (2.5) and (2.6) that the approximate solution σ_h can be computed by the simple formula

$$\sigma_h(x) = -a\,\nabla z_h + (P_h f)_T\,(x - x_T)/2\,,\quad x \in T\,, \tag{3.1}$$

where x_T is the barycenter of the triangle T, and $z_h \in M_{NC}$ is the solution of

$$\sum_T (a\,\nabla z_h, \nabla v)_T = (P_h f, v)\,,\quad \forall v \in M_{NC}\,. \tag{3.2}$$

Since it is easily seen from the definition of N_{3h} that the assumption (H8) is not valid, the duality estimate (2.13) cannot be derived naturally in the present case.

4. THE LOWEST-ORDER RAVIART-THOMAS METHOD II

In this section we shall reanalyze the lowest-order Raviart-Thomas method by means of modifying the space M_h. We shall show that, while the features of the previous section are preserved here, the new approach allows for Theorem 2.2 to be used to derive a new duality estimate for the scalar variable.

The spaces V_h, W_h, Λ_h, and $\tilde{\Lambda}_h$ are defined as in the previous section, but M_h is modified as follows. On the triangle T there exists a quadratic function (unique up to a multiplicative constant) $\phi_{0,T}(x)$ which vanishes at the two Gaussian quadrature points of each side of T. It can be written explicitly as [9]

$$\phi_{0,T}(x) = 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

which has been scaled so that its value is unity at the barycenter of T. Then, we introduce

$$N_{2h} = \{v : v|_T = \gamma_T \phi_{0,T}(x), \quad \gamma_T \in \mathbb{R}, \quad \forall T \in T_h\},$$

and

$$M_h = M_{NC} \oplus N_{2h},$$

where M_{NC} is defined as in the previous section. M_h is now the usual nonconforming space augmented with the P_2-bubble functions.

Again, (H3) is trivially valid and (H4) can be seen from the next lemma.

LEMMA 4.1 : *Let $T \in T_h$ be a triangle with edges e_i $(i = 1, 2, 3)$. Then for all $p_i \in L^2(e_i)$ $(i = 1, 2, 3)$ and $q \in L^2(T)$, there exists a unique $\chi \in M(T) = \{v|_T : v \in M_h\}$ satisfying*

$$(\chi - p_i, 1)_{e_i} = 0, \qquad i = 1, 2, 3, \tag{4.1}$$

$$(\chi - q, 1)_T = 0. \tag{4.2}$$

Proof : Clearly, the system given by (4.1) and (4.2) is a square linear system with four equations and unknowns. Hence, to prove existence and uniqueness of χ, it suffices to show that $\chi = 0$ if $q = 0$ and $p_i = 0$ $(i = 1, 2, 3)$.

Let $\chi = \chi_1 + \gamma_T \phi_{0,T}$ such that $\chi_1 \in M_{NC}$ and $\gamma_T \in \mathbb{R}$. Then, conditions (4.1) with $p_i = 0$ and the vanishing of the average value of $\phi_{0,T}$ on each edge imply that

$$(\chi_1, 1)_{e_i} = 0, \quad i = 1, 2, 3.$$

Consequently, it follows that $\chi_1 = 0$. As a result of this, $\chi = \gamma_T \phi_{0,T}$. Hence, by (4.2) with $q = 0$, $\gamma_T = 0$ and $\chi = 0$, and the proof has been completed.

Consequently, Theorem 2.1 shows again that the Raviart-Thomas method of lowest-order is equivalent to a modified P_1-nonconforming method amplified by P_2-bubbles this time. Moreover, based on the present equivalence, it can also be shown that the simple implementation (3.1) for σ_h is preserved here [4].

To apply Theorem 2.2, we must check that hypotheses (H5)-(H9) are valid. First, notice that (H5) is valid by Lemma 4.1. Next, it is immediate from the definition of M_h that the jumps of elements in M_h have zero mean values on interelement boundaries, so that (H6) is valid. Also, (H7) is obviously satisfied. It thus remains to check (H8) and (H9). For (H8), we need the next result.

LEMMA 4.2 : *If all triangles in the triangulation T_h of Ω are equilateral, then*

$$P_V(\nabla v) = \nabla v , \quad \forall v \in M_h . \tag{4.3}$$

Proof : It suffices to prove (4.3) for the P_2-bubbles by the definition of M_h. Let $T \in T_h$ be an equilateral triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Since $\lambda_3 = 1 - \lambda_1 - \lambda_2$, it follows from the definition of $\phi_{0,T}(x)$ that

$$\phi_{0,T}(x) = -1 + 6(\lambda_1 + \lambda_2) - 6(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) .$$

Hence, it suffices to consider the function

$$\phi(x) = \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 .$$

Let

$$a_1 = y_2 - y_3 , \qquad a_2 = y_3 - y_1 ,$$
$$b_1 = x_3 - x_2 , \qquad b_2 = x_1 - x_3 ,$$
$$D = |a_1 b_2 - a_2 b_1| .$$

Then, a calculation shows that

$$\nabla \phi = (\phi_{11} x + \phi_{12} y + \phi_1, \phi_{12} x + \phi_{22} y + \phi_2) ,$$

where

$$\phi_{11} = 2(a_1^2 + a_1 a_2 + a_2^2)/D^2 ,$$
$$\phi_{12} = (2 a_1 b_1 + a_1 b_2 + a_2 b_1 + 2 a_2 b_2)/D^2 ,$$
$$\phi_{22} = 2(b_1^2 + b_1 b_2 + b_2^2)/D^2 ,$$

and ϕ_1 and ϕ_2 are some constants. Since T is equilateral, it can be easily calculated that $\phi_{12} = 0$ and $\phi_{11} = \phi_{22} = 3 h_T^2/2 D^2$. That is, $\nabla \phi \in V(T)$, and the proof is completed.

As a result of (4.3), (H8) is valid and the system (2.6) may be rewritten as :

$$\text{Find } \psi_h \in M_h \text{ such that}$$
$$\sum_T (a \nabla \psi_h, \nabla v)_T = (P_h f, v), \quad \forall v \in M_h . \tag{4.4}$$

Hence, we see that the method is just a slightly modified version of the usual nonconforming method.

THEOREM 4.3 : *If u and ψ_h are the solutions of (2.2) and (4.4), respectively, and all triangles in the triangulation T_h of Ω are equilateral, then,*

$$\|\nabla u - \nabla \psi_h\|_h = \left(\sum_T \|\nabla u - \nabla \psi_h\|_T^2\right)^{1/2} \leq C h \|u\|_2 , \tag{4.5}$$

$$\|u - \psi_h\| \leq C h^2 \|f\|_1 , \tag{4.6}$$

with C independent of u and h.

Proof : First, (4.5) follows directly from the relation

$$\sigma_h = - a \nabla \psi_h ,$$

and the known error estimate [2], [7]

$$\|\sigma - \sigma_h\| \leq C h \|u\|_2 ;$$

consequently, (H9) is satisfied with $r = 2$. Hence, apply Theorem 2.2 and an approximation property of P_h to obtain

$$\begin{aligned} \|u - \psi_h\| &\leq C (h^2 \|u\|_2 + \|f - P_h f\|_{-2}) \\ &\leq C (h^2 \|u\|_2 + \|f - P_h f\|_{-1}) \\ &\leq C h^2 \|f\|_1 . \end{aligned}$$

and the proof is complete.

5. THE LOWEST-ORDER BREZZI-DOUGLAS-MARINI METHOD

Let $T_h = \{T\}$ be again a decomposition of Ω into triangles, and set

$$\begin{aligned} V(T) &= (P_1(T))^2 , \\ W(T) &= P_0(T) , \\ \Lambda(\partial T) &= \tilde{\Lambda}(\partial T) = \prod_{e \in \partial T} P_1(e) . \end{aligned}$$

The assumptions (H1) and (H2) hold trivially.

In order to introduce M_h, let Γ_h denote the collection of the vertices of triangles of T_h, and let $\tilde{\Delta}$ be a function from Γ_h into $\mathbb{R}$. Then, define

$$M_h^{\tilde{\Delta}} = \left\{ v \in C^0(\bar{\Omega}) : v\big|_T \in P_3(T),\ \forall T \in T_h\,;\ v(i) = \tilde{\Delta}(i),\ \forall i \in \Gamma_h \right\},$$

and

$$M_h = H_0^1(\Omega) \cap M_h^{\tilde{\Delta}}.$$

Observe that M_h depends on the function $\tilde{\Delta}$. For example, let $\tilde{\Delta}$ be the zero function on Γ_h ; then,

$$\begin{aligned} M_h^0 &= \left\{ v \in C^0(\bar{\Omega}) : v\big|_T \in P_3(T),\ \forall T \in T_h\,;\ v(i) = 0,\ \forall i \in \Gamma_h \right\}, \\ &= \left\{ v \in C^0(\bar{\Omega}) : v\big|_T \in \text{span } \{9\,\lambda_i\,\lambda_j(3\,\lambda_i - 1)/2,\ 27\,\lambda_1\,\lambda_2\,\lambda_3, \right. \\ &\qquad\qquad\qquad\qquad \left. i, j = 1, 2, 3, \quad i \neq j\},\ \forall T \in T_h \right\}. \end{aligned}$$

The next lemma is just one form of the standard uniqueness theorem for determining a cubic polynomial on a triangle.

LEMMA 5.1. *Let $\tilde{\Delta}$ be any given function on Γ_h. Then for $v \in W_h$ and $\mu \in \Lambda_h$, there is a unique $\chi \in M_h$ such that*

$$P_h \chi = v\,, \quad R_h \chi = \mu\,.$$

As a consequence of the lemma, the assumption (H4) is valid. Therefore, as $\Lambda_h = \tilde{\Lambda}_h$ and thus (H3) is true, we apply Theorem 2.1 to conclude that the lowest-order Brezzi-Douglas-Marini method is equivalent to a modified P_3-conforming finite element method.

Note that there is no simple formula for the computation of the approximate solution σ_h produced by the Brezzi-Douglas-Marini method like (3.1) in the case of the Raviart-Thomas method. This may account for a difference in the computation between these two mixed methods.

6. THE MARINI-PIETRA METHOD

With T_h defined as before, for each triangle $T \in T_h$ we let

$$\begin{aligned} V(T) &= \text{span } \{\tau^1, \tau^2, \tau^3\}\,, \\ W(T) &= P_0(T)\,, \\ \Lambda(\partial T) &= P_0(\partial T)\,, \\ \tilde{\Lambda}(\partial T) &= P_1(\partial T)\,, \end{aligned}$$

where

$$\tau^1 = (1, 0)\,, \quad \tau^2 = (0, 1)\,, \quad \tau^3 = (\tau_1^3, \tau_2^3)\,,$$

such that $\tau^3 \in (P_1(T))^2$ and, for a chosen edge e of T,

$$(\tau^3 \cdot n_e, 1)_e = 1 , \tag{6.1a}$$

$$(\tau^3 \cdot n_e, 1)_{\tilde{e}} = 0, \quad \forall \tilde{e} \neq e , \tag{6.1b}$$

$$(\tau_1^3, 1)_T = (\tau_2^3, 1)_T = 0 . \tag{6.1c}$$

The condition (6.1) determines a one-dimensional manifold ; τ^3 can be chosen arbitrarily as an element of this manifold. In particular, τ^3 can be chosen as the element of minimum norm [12], for example.

From the choice above, it is obvious that (H1) and (H2) are valid.

To construct M_h, let $R_h = R_h^0$ and $\tilde{R}_h = R_h^1$ indicate the usual orthogonal projections onto Λ_h and $\tilde{\Lambda}_h$, respectively, with respect to the norm $| \cdot |_h$. For each $T \in T_h$, let a_{i1}^T and a_{i2}^T be the two Gaussian quadrature points of each side e_i of T, $i = 1, 2, 3$, and let

$$M(T) = \left\{ v : v \in P_2(T), v(a_{i1}^T) = v(a_{i2}^T), \quad i = 1, 2, 3 \right\} .$$

Note that, since the six nodal values satisfy [9]

$$\sum_{i=1}^{3} \left\{ v(a_{i2}^T) - v(a_{i1}^T) \right\} = 0 , \quad \forall v \in P_2(T) ,$$

$M(T)$ is four-dimensional. Then, we introduce

$$M_h = \left\{ v \in L^2(\Omega) : v|_T \in M(T) , \quad \forall T \in T_h , \right.$$

v is continuous at the two Gaussian quadrature points of sides in E_h^0 and vanishes at the two Gaussian quadrature points of sides in $\left. E_h^\partial \right\}$.

Hence, M_h is a modified P_2-nonconforming space.

LEMMA 6.1 : *Let $T \in T_h$ be a triangle with sides e_i ($i = 1, 2, 3$). Then for any $p_i \in P_0(e_i)$ ($i = 1, 2, 3$) and $q \in L^2(T)$, there exists a unique $\chi \in M(T)$ such that*

$$(\chi - p_i, z)_{e_i} = 0 , \quad \forall z \in P_1(e_i) , \quad i = 1, 2, 3 \tag{6.2a}$$

$$(\chi - q, 1)_T = 0 . \tag{6.2b}$$

Remark : By the definition of $M(T)$, equation (6.2*a*) in fact has only three linearly independent equations. Also, as the P_2-bubble $\phi_{0,T}(x)$ vanishes at the six Gaussian quadrature points, (6.2*b*) is needed to uniquely determine $\chi \in M(T)$.

Proof of Lemma 6.1 : Clearly, by the definition of $M(T)$, the system given by (6.2) is a square linear system with four equations and unknowns. Hence, to prove existence and uniqueness of χ, it suffices to show that $\chi = 0$ if $q = 0$ and $p_i = 0$ $(i = 1, 2, 3)$.

First, condition (6.2*a*) with $p_i = 0$ implies that

$$(\chi, z)_{e_i} = 0, \quad \forall z \in P_1(e_i), \quad i = 1, 2, 3 ;$$

consequently, there is $\gamma_T \in \mathbb{R}$ such that $\chi = \gamma_T \, \phi_{0,T}(x)$. Hence, by (6.2*b*) with $q = 0$, $\gamma_T = 0$. Namely, $\chi = 0$, and the proof has been completed.

It now becomes apparent that the hypothesis (H4) is just a simple consequence of the above lemma.

LEMMA 6.2 : *For all* $v \in M_h$,

$$R_h^1 v = R_h^0 v .$$

This result is immediate from the definition of M_h, and so (H3) is satisfied.

Using Theorem 2.1, we see that the Marini-Pietra method is equivalent to a modified P_2-nonconforming method.

Set

$$X_h = \left\{ v \in C^0(\bar{\Omega}) : v|_T \in M(T), \ \forall T \in T_h, \ v|_{\partial\Omega} = 0 \right\} .$$

Then, it is interesting to note that [9]

$$M_h = X_h \oplus N_{2h} ,$$

where N_{2h} is defined as in the second example.

7. VARIABLE COEFFICIENTS

In this section we shall briefly extend the results of the previous sections on piecewise constants to the case of variable coefficients. As an example, we shall consider the lowest-order Raviart-Thomas method in detail. Other methods can be analyzed analogously. For more information on variable coefficient mixed finite element methods, refer to [4].

7.1. Basic error estimates

Let $T_h = \{T\}$ be a quasiregular partition of Ω into triangles. Set

$$\bar{V}_h = \left\{ \tau \in V : \tau|_T \in V(T), \quad \forall T \in T_h \right\} ,$$

with $V(T)$ defined as in § 3, and set $\alpha_h = P_h \alpha$. Note that the normal components of each element in $\bar{V}_h$ are now required to be continuous across the interelement boundaries.

We now introduce a modified mixed formulation for approximating the solution of (2.2) :

Find $(\bar{\sigma}_h, \bar{u}_h) \in \bar{V}_h \times W_h$ such that

$$(\alpha_h \bar{\sigma}_h, \tau) - (\operatorname{div} \tau, \bar{u}_h) = 0, \qquad \forall \tau \in \bar{V}_h, \tag{7.1a}$$

$$(\operatorname{div} \bar{\sigma}_h, v) = (f, v), \qquad \forall v \in W_h, \tag{7.1b}$$

where we have projected the coefficient into the space W_h. Observe that, when a is piecewise constant, (7.1) is the standard mixed finite element method [2], [7].

We now state some error estimates.

THEOREM 7.1 : *Problem* (7.1) *has a unique solution* $(\bar{\sigma}_h, \bar{u}_h)$. *Moreover, there exists* $C > 0$, *independent of h, such that*

$$\|\sigma - \bar{\sigma}_h\| \leq C(\|\alpha - \alpha_h\| + \|\sigma - \Pi_h \sigma\|), \tag{7.2}$$

$$\|\operatorname{div}(\sigma - \bar{\sigma}_h)\| \leq C \|(I - P_h) \operatorname{div} \sigma\| \leq Ch\|f\|_1, \tag{7.3}$$

$$\|u - \bar{u}_h\| \leq Ch(\|u\|_2 + |\alpha|_1), \tag{7.4}$$

$$\|P_h u - \bar{u}_h\| \leq C_1 h^2(\|u\|_2 + \|f\|_1 + |\alpha|_1), \tag{7.5}$$

where (σ, u) *is the solution of* (2.3), I *is the identity operator,* $C_1 = C_1(\|a\|_{1,\infty})$, *and* Π_h *will be defined below.*

Note that it follows from (2.1) that

$$\alpha_h \geq (a^*)^{-1} > 0, \tag{7.6}$$

so that the existence and uniqueness of a solution to (7.1) can be demonstrated in a standard way (see, e.g., [7]). Estimates (7.2)-(7.5) can be obtained by making use of the duality ideas of Douglas and Roberts [7]. We shall here use a more direct approach to obtain these estimates. This approach is easy to understand and is simpler than that given in [7], [8], [13].

Let $\Pi_h : H^1(\Omega) \to \bar{V}_h$ be the Raviart-Thomas projection, [7], [13], which satisfies

$$\|\tau - \Pi_h \tau\| \leq C\|\tau\|_1 h, \quad \tau \in (H^1(\Omega))^2, \tag{7.7}$$

$$\|\operatorname{div}(\tau - \Pi_h \tau)\| \leq C\|\operatorname{div} \tau\|_1 h, \quad \tau \in (H^1(\Omega))^2, \operatorname{div} \tau \in H^1(\Omega), \tag{7.8}$$

$$\operatorname{div} \Pi_h = P_h \operatorname{div}, \ (H^1(\Omega))^2 \to W_h. \tag{7.9}$$

We shall also require the approximation property

$$\|v - P_h v\|_{-s} \leq C\|v\|_1 h^{s+1}, \quad s = 0, 1, \tag{7.10}$$

if $v \in H^1(\Omega)$.

Proof of Theorem 7.1 : Let $x = \sigma - \bar{\sigma}_h = (\sigma - \Pi_h \sigma) + (\Pi_h \sigma - \bar{\sigma}_h) = y + z$ and $\xi = u - \bar{u}_h = (u - P_h u) + (P_h u - \bar{u}_h) = \eta + \zeta$. These errors satisfy the equations given by subtracting (7.1) from (2.3) and using (7.9) :

$$(\alpha_h x, \tau) - (\zeta, \operatorname{div} \tau) = (\{\alpha_h - \alpha\} \sigma, \tau), \quad \forall \tau \in \bar{V}_h, \quad (7.11a)$$

$$(v, \operatorname{div} z) = 0, \quad \forall v \in W_h. \quad (7.11b)$$

Take the test functions $\tau = z$ in (7.11*a*) and $v = \zeta$ in (7.11*b*) and add to have

$$\begin{aligned}(\alpha_h x, z) &= (\{\alpha_h - \alpha\} \sigma, z) \\ &\leq C \|\alpha_h - \alpha\|^2 + \varepsilon \|z\|^2,\end{aligned}$$

where ε is a positive constant which may be taken as small as we please. Consequently, since $(\alpha_h z, z) = (\alpha_h x, z) - (\alpha_h y, z)$, it follows that

$$\|z\|^2 \leq C \left\{\|\alpha_h - \alpha\|^2 + \|y\|^2\right\} + \varepsilon \|z\|^2,$$

and that

$$\|x\| \leq \|y\| + \|z\| \leq C (\|\alpha_h - \alpha\| + \|y\|);$$

i.e., (7.2) holds.

Next, (7.11*b*) shows that $\operatorname{div} \bar{\sigma}_h = P_h \operatorname{div} \sigma$; consequently,

$$\|\operatorname{div} x\| = \|\operatorname{div} \sigma - P_h \operatorname{div} \sigma\|,$$

which gives (7.3) by (7.10) with $s = 0$.

Now, let $\tilde{\tau} \in \bar{V}_h$ in (7.11*a*) be a function associated with ζ such that [13]

$$\operatorname{div} \tilde{\tau} = \zeta \quad \text{and} \quad \|\tilde{\tau}\|_{H(\operatorname{div}; \Omega)} \leq C \|\zeta\|;$$

then,

$$\begin{aligned}\|\zeta\|^2 = (\zeta, \operatorname{div} \tilde{\tau}) &= (\alpha_h x, \tilde{\tau}) - (\{\alpha_h - \alpha\} \sigma, \tilde{\tau}) \\ &\leq C \left\{\|x\|^2 + \|\alpha_h - \alpha\|^2\right\} + \varepsilon E \|\tilde{\tau}\|^2 \\ &\leq C \left\{\|x\|^2 + \|\alpha_h - \alpha\|^2\right\} + \varepsilon E \|\zeta\|^2;\end{aligned}$$

i.e., by (7.2), (7.7), and (7.10), (7.4) holds.

In order to show (7.5), we rewrite (7.11) as

$$(\alpha x, \tau) - (\zeta, \operatorname{div} \tau) = (\{\alpha_h - \alpha\} \bar{\sigma}_h, \tau), \quad \forall \tau \in \bar{V}_h, \quad (7.12a)$$

$$(v, \operatorname{div} x) = 0, \quad \forall v \in W_h. \quad (7.12b)$$

Note that, by (7.2), (7.7), (7 10), quasiregularity of T_h, and the boundedness of Π_h,

$$\begin{aligned}\|\bar{\sigma}_h\|_\infty &\leq \|z\|_\infty + \|\Pi_h \sigma\|_\infty \\ &\leq Ch^{-1}(\|x\| + \|y\|) + \|\Pi_h \sigma\|_\infty \\ &\leq C\end{aligned} \tag{7.13}$$

We are now in a position to prove the estimate (7.5) by means of a duality argument different from that given in [7].

Let $\psi \in L^2(\Omega)$, and let $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that $-\operatorname{div}(a \nabla\varphi) = \psi$. then, it follows from (7 9) and (7 12) that with $\beta = \alpha_h - \alpha$:

$$\begin{aligned}(\zeta, \psi) &= (\zeta, -\operatorname{div}(a \nabla\varphi)) \\ &= (\zeta, -\operatorname{div}(\Pi_h\{a \nabla\varphi\})) \\ &= (\beta\bar{\sigma}_h, \Pi_h\{a \nabla\varphi\}) - (\alpha x, \Pi_h\{a \nabla\varphi\}) \\ &= (\beta\bar{\sigma}_h, \Pi_h\{a \nabla\varphi\} - a \nabla\varphi) - (\beta x, a \nabla\varphi) + (\beta\sigma, a \nabla\varphi) \\ &\quad + (\alpha x, a \nabla\varphi - \Pi_h\{a \nabla\varphi\}) + (\operatorname{div} x, \varphi - P_h \varphi),\end{aligned}$$

so that, by (7 7), (7 10), (7 13), and the approximation property

$$\|\beta\|_\infty \leq Ch\|\alpha\|_{1\ \infty},$$

we have

$$\begin{aligned}|(\zeta, \psi)| &\leq \|\beta\| \|\bar{\sigma}_h\|_\infty \|\Pi_h\{a \nabla\varphi\} - a \nabla\varphi\| \\ &\quad + \|\beta\|_\infty \|x\| \|a \nabla\varphi\| + \|\beta\|_{-1} \|a\sigma \nabla\varphi\|_1 \\ &\quad + \|\alpha\|_\infty \|x\| \|a \nabla\varphi - \Pi_h\{a \nabla\varphi\}\| + \|\operatorname{div} x\| \|\varphi - P_h \varphi\| \\ &\leq C_1(h\|\beta\| + \|\beta\|_{-1} + h\|x\| + h\|\operatorname{div} x\|)\|\varphi\|_2,\end{aligned}$$

with $C_1 = C_1(\|a\|_{1\ \infty})$. Finally, combine (7 2), (7.3), (7 7), (7 10), and the assumed elliptic regularity to obtain (7.5), and the proof has been completed.

7.2. Post-Processing

Associated with (2.5), we have the modified mixed-hybrid method .

Find $(\bar{\sigma}_h, \bar{u}_h, \bar{\lambda}_h) \in V_h \times W_h \times \Lambda_h$ such that

$$(\alpha_h \bar{\sigma}_h, \tau) - \sum_T \left\{(\bar{u}_h, \operatorname{div} \tau)_T - (\bar{\lambda}_h, \tau \cdot n_T)_{\partial T}\right\} = 0, \qquad \forall \tau \in V_h, \tag{7.14a}$$

$$\sum_T (v, \operatorname{div} \bar{\sigma}_h)_T = (f, v), \qquad \forall v \in W_h, \tag{7.14b}$$

$$\sum_T (\mu, \bar{\sigma}_h \cdot n_T)_{\partial T} = 0, \qquad \forall \mu \in \Lambda_h. \tag{7 14c}$$

Again, by (7.6), the existence and uniqueness of solution to (7.14) can be easily shown in a standard way. Furthermore, since equation (7.14*c*) imposes the required continuity on $\bar{\sigma}_h$, the pair $(\bar{\sigma}_h, \bar{u}_h)$ of (7.14) is the same as that of (7.1).

Set

$$|\mu|^2_{-1/2,h} = \sum_{e \in E_h^0} \|\mu\|_e^2 h_e .$$

The following lemma can be proved by the argument given in [2].

LEMMA 7.2 : *There is a constant C, independent of u and h, such that, for every $T \in T_h$ and every edge e of T,*

$$\left\|\bar{\lambda}_h - R_h u\right\|_e \leq C\,(h_T^{1/2}\|x\|_T + h_T^{-1/2}\|\zeta\|_T + h_T^{1/2}\|\alpha - \alpha_h\|_T)\,, \quad (7.15)$$

$$\left|\bar{\lambda}_h - R_h u\right|_{-1/2,h} \leq C\,(h\|x\| + \|\zeta\| + h\|\alpha - \alpha_h\|)\,. \quad (7.16)$$

As mentioned before, the advantage of the system associated with (7.14) is that the stiffness matrix is positive definite. Moreover, the multiplier above can be used to obtain by post-processing an approximate solution to u which is asymptotically more accurate than the approximation $\bar{u}_h$.

THEOREM 7.3 : *Let*

$$W_h^1 = \left\{v \in L^2(\Omega) : v|_T \in P_1(T), \quad \forall T \in T_h\right\}\,,$$

and let $u_h^* \in W_h^1$ be defined by

$$R_h u_h^* = \bar{\lambda}_h\,. \quad (7.17)$$

Then,

$$\|u - u_h^*\| \leq C_1 h^2(\|u\|_2 + \|f\|_1 + |\alpha|_1)\,, \quad (7.18)$$

if u is the solution of (2.2), *where* $C_1 = C_1(\|a\|_{1,\infty})$.

Proof : The existence and uniqueness of u_h^* are obvious. We also define $\tilde{u}_h \in W_h^1$ by

$$R_h \tilde{u}_h = R_h u\,; \quad (7.19)$$

by standard arguments (see, e.g., [6]),

$$\|u - \tilde{u}_h\| \leq C h^2 \|u\|_2\,. \quad (7.20)$$

Observe that, by (7.17) and (7.19),

$$R_h(u_h^* - \tilde{u}_h) = \bar{\lambda}_h - R_h u .$$

Then, by a simple scaling argument (see, e.g., [5]),

$$\|u_h^* - \tilde{u}_h\|_{0,T} \leq C h_T^{1/2} \sum_{i=1}^{3} \|\bar{\lambda}_h - R_h u\|_{e_i}, \tag{7.21}$$

for all $T \in T_h$ with edges e_i, $i = 1, 2, 3$. Combine (7.15), (7.20) and (7.21) with (7.2), (7.5), (7.7), and (7.10) to obtain the desired result (7.18), and the proof is complete.

Note that u_h^* approximates u with a higher order of accuracy than $\bar{u}_h$, as required, and is continuous at the midpoints of sides in E_h^0 and zero at the midpoints of sides in E_h^{∂}.

7.3. Implementation

Let M_h be defined as in § 3 or in § 4. Corresponding to (2.6), we define the analogue :

$$\text{Find } \bar{\psi}_h \in M_h \text{ such that}$$

$$\sum_T (\alpha_h^{-1} P_V(\nabla \bar{\psi}_h), \nabla v)_T = (P_h f, v), \quad \forall v \in M_h . \tag{7.22}$$

Then, in the same argument as in § 2, we have

THEOREM 7.4 : *Let* $(\bar{\sigma}_h, \bar{u}_h, \bar{\lambda}_h)$ *be the solution of* (7.14) *and let* $\bar{\psi}_h \in M_h$ *be given by*

$$P_h \bar{\psi}_h = \bar{u}_h, \quad \tilde{R}_h \bar{\psi}_h = \bar{\lambda}_h . \tag{7.23}$$

Then $\bar{\psi}_h$ *is the unique solution of* (7.22) *and*

$$\bar{\sigma}_h = - \alpha_h^{-1} P_V(\nabla \bar{\psi}_h) . \tag{7.24}$$

Let us now discuss the structure of (7.22). Let $N_h = N_{2h}$ or N_{3h}. For $v \in M_{NC}$, we have $P_V(\nabla v) = \nabla v$, a piecewise constant. Moreover, the gradient of a bubble function in N_h has zero mean value on each T. Indeed, for $v \in N_h$ and $q = (1, 0)$ or $(0, 1)$, we have

$$(\nabla v, q)_T = (v, q \cdot n_T)_{\partial T} - (v, \operatorname{div} q)_T = 0 ,$$

since v vanishes at the two Gaussian quadrature points of or on each side of T. Therefore, the solution of (7.22) may be determined as $z_h + \xi_h$ where

$(z_h, \xi_h) \in M_{NC} \times N_h$ is the unique solution of

$$\sum_T (\alpha_h^{-1} \nabla z_h, \nabla v)_T = (P_h f, v), \quad \forall v \in M_{NC}, \tag{7.25a}$$

$$\sum_T (\alpha_h^{-1} P_V(\nabla \xi_h), \nabla \varphi)_T = (P_h f, \varphi), \quad \forall \varphi \in N_h. \tag{7.25b}$$

In summary, we have

$$\bar{\sigma}_h = -\alpha_h^{-1}(\nabla z_h + P_V(\nabla \xi_h)), \tag{7.26}$$

$$\bar{u}_h = P_h(z_h + \xi_h), \tag{7.27}$$

$$\bar{\lambda}_h = R_h z_h, \tag{7.28}$$

where (z_h, ξ_h) satisfies (7.25). From the solution of (7.25) one can deduce the solution of (7.14). Moreover, based on (7.25)-(7.28), we have the simple solution given in the following proposition by the argument in [11].

PROPOSITION 7.5 : *In each T, σ_h at a point x is evaluated by the simple formula*

$$\bar{\sigma}_h = -\alpha_h^{-1} \nabla z_h + (P_h f)_T (x - x_T)/2, \quad x \in T,$$

where z_h satisfies (7.25*a*).

We shall now derive error estimates for (7.22).

THEOREM 7.6 : *If u and $\bar{\psi}_h$ are the solutions of* (2.2) *and* (7.22), *respectively, and if*

$$P_V(\nabla \bar{\psi}_h) = \nabla \bar{\psi}_h, \tag{7.29}$$

then,

$$\left\|\nabla(u - \bar{\psi}_h)\right\|_h = \left(\sum_T \left\|\nabla u - \nabla \bar{\psi}_h\right\|_T^2\right)^{1/2} \leq C_2 h(\|u\|_2 + \|\alpha\|_1), \tag{7.30}$$

$$\left\|v - \bar{\psi}_\eta\right\| \leq C_3 h^2(\|a\|_1 + \|f\|_1), \tag{7.31}$$

where $C_2 = C_2(\|u\|_{1,\infty})$ and $C_3 = C_3(\|u\|_{2,\infty}, \|a\|_{1,\infty})$.

Proof : First, by (2.1) and (7.10), note that

$$\|a - \alpha_h^{-1}\|_{-s} \leq C \|\alpha\|_1 h^{s+1}, \quad s = 0, 1, \tag{7.32}$$

$$\|a - \alpha_h^{-1}\|_\infty \leq C \|\alpha\|_{1,\infty} h. \tag{7.33}$$

Then, using (2.1), (7.32) with $s = 0$, (7.2), (7.7), and the triangle inequality,

$$\left\|\nabla (u - \bar{\psi}_h)\right\|_h \leqslant C \left\|\alpha_h^{-1} \nabla (u - \bar{\psi}_h)\right\|_h$$
$$\leqslant C \left(\left\| (a - \alpha_h^{-1}) \nabla u\right\| + \left\|\sigma - \bar{\sigma}_h\right\|\right)$$
$$\leqslant C_2 h (\|u\|_2 + \|\alpha\|_1) ;$$

i.e., (7.30) holds with $C_2 = C_2(\|u\|_{1, \infty})$.

In order to prove (7.31), we shall adapt the duality argument given in § 2. Let $w = u - \bar{\psi}_h$, and let $\phi \in H_0^1(\Omega)$ be such that

$$- \operatorname{div} (a \nabla \phi) = w \quad \text{in } \Omega ,$$

and

$$\|\phi\|_2 \leqslant C \|w\| . \tag{7.34}$$

As in (2.15), we write

$$\begin{aligned} \|w\|^2 &= \sum_T (a \nabla \phi, \nabla w)_T - \sum_T (a \nabla \phi \cdot n_T, w)_{\partial T} \\ &= \sum_T (\{a - \alpha_h^{-1}\} \nabla \phi, \nabla w)_T + \sum_T (\alpha_h^{-1} \nabla \phi, \nabla w)_T \\ &\quad - \sum_T (a \nabla \phi \cdot n_T w)_{\partial T} \\ &\equiv R_1 + R_2 + R_3 . \end{aligned} \tag{7.35}$$

Using (7.33), we see that

$$\begin{aligned} |R_1| &\leqslant \left\|a - \alpha_h^{-1}\right\|_\infty \|\nabla \phi\| \|\nabla w\|_h \\ &\leqslant C_1 h^2 \|\nabla w\|_h^2 + \varepsilon \|\phi\|_2^2 , \end{aligned} \tag{7.36}$$

where $C_1 = C_1(\|a\|_{1, \infty})$. The term R_3 can be treated in the same manner as in the second section to obtain

$$|R_3| \leqslant C h \|\phi\|_2 \|\nabla w\|_h . \tag{7.37}$$

For R_2, observe that

$$\begin{aligned} R_2 &= \sum_T (\alpha_h^{-1} \nabla \phi, \nabla (u - \bar{\psi}_h))_T \\ &= \sum_T (\{\alpha_h^{-1} - a\} \nabla \phi, \nabla u)_T + \sum_T \left\{(a \nabla \phi, \nabla u)_T - (\alpha_h^{-1} \nabla \phi, \nabla \bar{\psi}_h)_T\right\} \\ &\equiv R_2^1 + R_2^2 . \end{aligned} \tag{7.38}$$

Applying the same ideas as in (2.16)-(2.17), we get

$$|R_2^2| \leqslant C h^2 \|f\|_1 . \tag{7.39}$$

Finally, R_2^1 can be bounded as follows :

$$\begin{aligned} |R_2^1| &= \left|\sum_T (\{\alpha_h^{-1} - a\} \nabla\phi, \nabla u)_T\right| \\ &\leqslant \|\alpha_h^{-1} - a\|_{-1} \|\nabla u . \nabla\phi\|_1 \\ &\leqslant C_3 \|\alpha_h^{-1} - a\|_{-1}^2 + \varepsilon \|\phi\|_2^2 , \end{aligned} \tag{7.40}$$

where $C_3 = C_3(\|u\|_{2,\infty})$. Now, combine (7.30), (7.32), and (7.34)-(7.40) to yield the desired result (7.31) if h is sufficiently small, and the proof has been finished.

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REFERENCES

[1] T. Arbogast, *A new formulation of mixed finite element methods for second order elliptic problems* (to appear).

[2] D. N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods : implementation postprocessing and error estimates, *RAIRO Modél. Math. Anal. Numér., 19* (1985), pp. 7-32.

[3] F. Brezzi, J. Douglas Jr and L. Donatella Marini, Two families of mixed finite elements for second order elliptic problems, *Numer. Math., 47* (1985), pp. 217-235.

[4] Z. Chen, *On the relationship between mixed and Galerkin finite element methods,* Ph. D. thesis, Purdue University, West Lafayette, Indiana, August (1991).

[5] F. Brezzi and M. Fortin, *Hybrid and Mixed Finite Element Methods,* to appear.

[6] P. Ciarlet, *The Finite Element Method for Elliptic Problems,* North-Holland, Amsterdam, 1978.

[7] J. DOUGLAS Jr and J. E. ROBERTS, Global estimates for mixed methods for second order elliptic problems, *Math. Comp., 45* (1985), pp. 39-52.

[8] R. FALK and J. OSBORN, Error estimates for mixed methods, *RAIRO, Modél. Math. Anal. Numér., 14* (1980), pp. 249-277.

[9] M. FORTIN and M. SOULIE, A non-conforming piecewise quadratic finite element on triangles, *Internat. J. Numer. Methods Engrg., 19* (1983), pp. 505-520.

[10] B. X. FRAEIJS DE VEUBEKE, *Displacement and equilibrium models in the finite element method,* in Stress Analysis, O. C. Zienkiewicz and G. Holister (eds.), John Wiley, New York, 1965.

[11] L. DONATELLA MARINI, An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method, *SIAM J. Numer. Anal., 22* (1985), pp. 493-496.

[12] L. DONATELLA MARINI and P. PIETRA, An abstract theory for mixed approximations of second order elliptic problems, *Mat. Apl. Comput., 8* (1989), pp. 219-239.

[13] P. A. RAVIART and J. M. THOMAS, A mixed finite element method for second order elliptic problems, in Mathematical Aspects of the Finite Element Method, *Lecture Notes in Math. 606,* Springer-Verlag, Berlin and New York (1977), pp. 292-315.