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SEMIDISCRETE AND SINGLE STEP FULLY DISCRETE FINITE ELEMENT APPROXIMATIONS FOR SECOND ORDER HYPERBOLIC EQUATIONS WITH NONSMOOTH SOLUTIONS (*)

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Abstract — *Finite element approximations are analyzed for initial boundary value problems for second order hyperbolic equations with nonsmooth solutions For both semidiscrete and fully discrete schemes, convergence estimates in negative norms are derived for problems with L^2 initial data, using L^2 projections of the initial data as starting values*

Résumé. — *On analyse des approximations par éléments finis des solutions irrégulières des problèmes aux limites et aux conditions initiales pour les équations hyperboliques de second ordre Pour des schémas de semidiscrétisation et de discrétisation totale, on obtient des majorations de l'erreur par rapport à des normes d'ordre négatif Ces schémas utilisent la projection L^2 de la valeur initiale qui est dans L^2*

1. INTRODUCTION

1.1. Notation

We consider approximating the solution of the following initial boundary value problem. Let Ω be a bounded domain in R^N, with smooth boundary $\partial\Omega$ and let $0 < t^* < \infty$ be fixed. A function $u : (0, t^*] \to R^1$ is sought which satisfies

$$\begin{cases} u_{tt} + Lu = 0 & \text{in} \quad \Omega \times (0, t^*] , \\ u = 0 & \text{on} \quad \partial\Omega \times (0, t^*] , \\ u(0) = u^0 & \text{in} \quad \Omega , \\ u_t(0) = u_t^0 & \text{in} \quad \Omega , \end{cases} \tag{1.1}$$

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where u^0 and u_t^0 are given functions, and L denotes the second order elliptic operator

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\,u\,,$$

with $a_{ij} = a_{ji} \in C^{\infty}(\bar{\Omega})$, $i, j = 1, 2, \dots, N$; $a_0 \in C^{\infty}(\bar{\Omega})$ and $a_0 \geqslant 0$ on Ω. L is assumed to satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\,\xi_j \geqslant \alpha \sum_{i=1}^{N} \xi_i^2\,, \tag{1.2}$$

for all $x \in \bar{\Omega}$ and for all $(\xi_1, \dots, \xi_N) \in R^N$, for some constant $\alpha > 0$.

For $s \geqslant 0$, $H^s(\Omega)$ will denote the Sobolev space of order s, of real valued functions on Ω. The norm on $H^s(\Omega)$ we denote by $\|\cdot\|_{H^s(\Omega)}$. The inner product on $L^2(\Omega) = H^0(\Omega)$ we denote by $(\cdot\,,\cdot)$, and the associated norm by $\|\cdot\|$.

We introduce certain subspaces of the Sobolev space $H^s(\Omega)$, denoted by $\dot{H}^s(\Omega)$. In order to define $\dot{H}^s(\Omega)$, we first note that there exists a sequence $\{\lambda_j\}_{j \geqslant 1}$ in nondecreasing order of real positive eigenvalues of the operator L, and a corresponding sequence of eigenfunctions $\{\phi_j\}_{j \geqslant 1} \subset C^{\infty}(\Omega)$, satisfying

$$\begin{cases} L\phi_j = \lambda_j\,\phi_j & \text{in} \quad \Omega\,, \\ \phi_j \;\;= 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.3}$$

The set $\{\phi_j\}_{j \geqslant 1}$ is complete in $L^2(\Omega)$, and may be chosen orthonormal. Define for $s \geqslant 0$, the space

$$\dot{H}^s(\Omega) = \left\{ v : \|v\|_s = \left(\sum_{j=1}^{\infty} |(v, \phi_j)|^2\,\lambda_j^s\right)^{1/2} < \infty \right\}.$$

Then $\dot{H}^0(\Omega) = L^2(\Omega)$, and it may be shown, [7], that

$$\dot{H}^s(\Omega) = \{ v \in H^s(\Omega) : L^j\,v = 0 \quad \text{on} \quad \partial\Omega\,, \quad j < s/2 \}\,,$$

and that on $\dot{H}^s(\Omega)$, the norms $\|\cdot\|_s$ and $\|\cdot\|_{H^s(\Omega)}$ are equivalent.

For $s < 0$, $\dot{H}^s(\Omega)$ is defined as the dual of $\dot{H}^{-s}(\Omega)$ with respect to $L^2(\Omega)$. The norm on $\dot{H}^{-s}(\Omega)$ is given by

$$\|v\|_{-s} = \left(\sum_{j=1}^{\infty} |(v, \phi_j)|^2\,\lambda_j^{-s}\right)^{1/2}, \quad s \geqslant 0\,.$$

The solution of (1.1) is formally given by

$$u(t) = \sum_{j=1}^{\infty} [(u^0, \phi_j) \cos \lambda_j^{1/2} t + \lambda_j^{-1/2}(u_t^0, \phi_j) \sin \lambda_j^{1/2} t] \phi_j ,$$

for $t \geq 0$, from which it follows that for $0 \leq t \leq t^*$,

$$\|u(t)\|_s^2 + \|u_t(t)\|_{s-1}^2 = \|u^0\|_s^2 + \|u_t^0\|_{s-1}^2 \quad \text{for all} \quad s \geq 0 . \qquad (1.4)$$

The solution operator $T : L^2(\Omega) \to L^2(\Omega)$ of the associated elliptic boundary value problem is defined by

$$a(Tf, v) = (f, v), \quad \text{for all} \quad v \in \dot{H}^1(\Omega) \quad \text{for given} \quad f \in L^2(\Omega) \qquad (1.5)$$

where $a(.,.)$ denotes the bilinear form

$$a(w, v) = \int_\Omega \left\{ \sum_{i,j=1}^{N} a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0 w v \right\} dx , \quad \text{for } w, v \in H^1(\Omega) . \qquad (1.6)$$

T has a discrete spectrum of real positive eigenvalues $\{\mu_j\}_{j \geq 1}$, where $\mu_j = \lambda_j^{-1}$ with λ_j given by (1.3). Let $\mathscr{L}$ denote the operator on $L^2(\Omega) \times L^2(\Omega)$

$$\mathscr{L} = \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix} .$$

In terms of the operator $\mathscr{L}$, (1.1) is equivalent to

$$\begin{cases} \mathscr{U}_t = \mathscr{L}\mathscr{U}, & t > 0 \\ \mathscr{U}(0) = \mathscr{U}^0 \end{cases} \qquad (1.7)$$

where $\mathscr{U}(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}$ and $\mathscr{U}^0 = \begin{pmatrix} u^0 \\ u_t^0 \end{pmatrix}$

Let $0 < h \leq 1$ be a parameter, and $\{T_h\}_{0 < h \leq 1}$ a family of finite dimensional operators approximating the operator T. In particular, let $\{S_h^r(\Omega)\}_{0 < h \leq 1} \subset \dot{H}^1(\Omega)$ be a standard finite element space of piecewise polynomial functions of degree $r - 1$, with the approximation property

$$\inf_{\chi \in S_h^r(\Omega)} \{\|w - \chi\| + h\|w - \chi\|_1\} \leq C h^s \|w\|_{H^s(\Omega)} ,$$

for all $w \in \dot{H}^1(\Omega) \cap H^s(\Omega)$, for some constant C independent of h, $1 \leq s \leq r$. The operators $T_h : L^2(\Omega) \to S_h^r(\Omega)$ are defined by

$$a(T_h f, \chi) = (f, \chi), \quad \text{for all } \chi \in S_h^r(\Omega), \quad \text{for given } f \in L^2(\Omega) . \qquad (1.8)$$

The family $\{T_h\}_{0<h\leq 1}$ has the following properties :

$$T_h \text{ is symmetric, positive semidefinite on } L^2(\Omega) \text{ and positive definite on } S_h^r(\Omega) \tag{1.9}$$

$$\| (T - T_h) f \| \leq C h^s \|f\|_{H^{s-2}(\Omega)}, \quad \text{for all } f \in H^{s-2}(\Omega),\ 1 \leq s \leq r, \tag{1.10}$$

for some constant C independent of h.

We also define the following forms on $L^2(\Omega) \times L^2(\Omega)$:

$$((\Phi, \Psi)) = (\Phi_1, \Psi_1) + (T\Phi_2, \Psi_2), \tag{1.11}$$

and

$$((\Phi, \Psi))_h = (\Phi_1, \Psi_1) + (T_h \Phi_2, \Psi_2). \tag{1.12}$$

In this work, we will use the following identity. Here, A and A_h are two operators with

$$((A\Phi, \Psi)) = ((\Phi, A^* \Psi)) \quad \text{and} \quad ((A_h \Phi, \Psi))_h = ((\Phi, A_h^* \Psi))_h.$$

$$\begin{aligned}
&(((A - A_h)\Phi, \Psi))_h = \\
&= ((A\Phi, \Psi))_h - ((A_h \Phi, \Psi))_h \\
&= ((A\Phi, \Psi))_h - ((A\Phi, \Psi)) + ((A\Phi, \Psi)) - ((A_h \Phi, \Psi))_h \\
&= ((A\Phi, \Psi))_h - ((A\Phi, \Psi)) + ((\Phi, A^* \Psi)) - ((\Phi, A_h^* \Psi))_h \\
&= ((T_h - T)(A\Phi)_2, \Psi_2) + ((\Phi, A^* \Psi)) - ((\Phi, A^* \Psi))_h \\
&\quad + ((\Phi, A^* \Psi))_h - ((\Phi, A_h^* \Psi))_h \\
&= ((T - T_h)(A\Phi)_2, \Psi_2) + ((T - T_h)\Phi_2, (A^* \Psi)_2) \\
&\quad + ((\Phi, (A^* - A_h^*)\Psi))_h .
\end{aligned} \tag{1.13}$$

1.2. Summary of Results

In Section 2 we derive the following estimate for semidiscrete approximations.

$$\sup_{0 \leq t \leq t^*} \|u(t) - u_h(t)\|_{-(r+1)} \leq C(t^*) h^r (\|u^0\| + \|u_t^0\|). \tag{1.14}$$

These estimates are obtained with L^2 projections of the initial data as starting values.

In Section 3, we consider single step fully discrete approximations which are based on rational approximations to e^{-z}. The rational functions satisfy the estimate

$$|r_\nu(iy) - e^{-iy}| \leq C_\nu |y|^{\nu+1}, \quad |y| \leq \sigma,$$

for constants $C_\nu < \infty$, $\sigma > 0$ and $\nu \geq 1$. We prove the estimate

$$\max \|W_1^n - u(nk)\|_{-\mu} \leq C(t^*)(h^r + k^\nu)(\|u^0\| + \|u_t^0\|)$$

where $\mu = \max\{r+1, \nu+1\}$, k denotes the discrete time step and W_1^n, the approximation at time level $t = nk$. The fully discrete estimates are also obtained using L^2 projections of the initial data as starting values.

In [4], Geveci proved energy and negative norm estimates for the solution of (1.1). However, the results in [4] do not include the case when the initial data u^0 and u_t^0 are in L^2 which is the case considered in this work.

In [2], discrete negative norm errors were computed for a two step fully discrete approximation to the solution of (1.1). The computed results are similar to the theoretical results for single step fully discrete approximations derived in this work.

Fully discrete approximations to (1.1) when the initial data are nonsmooth (e.g., when u^0 is discontinous) typically contain large oscillations. The presence of oscillations suggests that a post-processing procedure such as described in Bramble and Schatz [3] may be used to construct a better approximation. Pre-processing and post-processing were used with finite difference schemes for approximately solving hyperbolic equations in Lax and Mock [6]. Also, in Johnson and Nävert [5] post-processing based on a negative norm error estimate was applied to finite element approximations for advection-diffusion problems. In future work, we will consider the application of post-processing techniques to finite element approximations of the solution of (1.1).

Throughout the paper, C will denote a general constant, not necessarily the same in any two places.

2. SEMIDISCRETE APPROXIMATIONS

The semidiscrete approximation for the solution u of (1.1) is defined as the mapping $u^h : [0, t^*] \to S_h^r(\Omega)$ satisfying

$$\begin{aligned} T_h u_{tt}^h + u^h &= 0, \ 0 < t \leq t^*, \\ u^h(0) &= Pu^0, \\ u_t^h(0) &= Pu_t^0, \end{aligned} \tag{2.1}$$

where P denotes the $L^2(\Omega)$ projection operator onto $S_h^r(\Omega)$. We formulate (2.1) as a first order system. Set

$$U^h(t) = \begin{pmatrix} u^h(t) \\ u_t^h(t) \end{pmatrix}, \quad t \geq 0 .$$

(2.1) is equivalent to

$$\begin{aligned} U_t^h(t) &= \mathscr{L}_h U^h \\ U^h(0) &= P U^0 \end{aligned} \tag{2.2}$$

where $U^0 = \begin{pmatrix} u^0 \\ u_t^0 \end{pmatrix}$, $\mathscr{L}_h = \begin{pmatrix} 0 & I \\ -L_h & 0 \end{pmatrix}$ and L_h is the inverse of T_h on $S_h^r(\Omega)$.

THEOREM 2.1 : *Let u be the solution of* (1.1) *and let u^h be the semidiscrete approximation defined by* (2.1) *or* (2.2). *Suppose that $u^0 \in L^2(\Omega)$ and $u_t^0 \in L^2(\Omega)$. Then there exists a constant $C = C_r(t^*)$, such that*

$$\sup_{0 \leq t \leq t^*} \|u(t) - u^h(t)\|_{-(r+1)} \leq C h^r [\|u^0\| + \|u_t^0\|] . \tag{2.3}$$

Proof : Note that $((\mathscr{L}\Phi, \Psi)) = -((\Phi, \mathscr{L}\Psi))$ for Φ, $\Psi \in \dot{H}^2(\Omega) \times L^2(\Omega)$ and $((\mathscr{L}_h \Phi, \Psi))_h = -((\Phi, \mathscr{L}_h \Psi))_h$ for Φ, $\Psi \in S_h^r(\Omega) \times S_h^r(\Omega)$. It follows that $((e^{\mathscr{L}t} \Phi, \Psi)) = ((\Phi, e^{-\mathscr{L}t} \Psi))$ and $((e^{\mathscr{L}_h t} \Phi, \Psi))_h = ((\Phi, e^{-\mathscr{L}_h t} \Psi))_h$. Since $U(t) = e^{\mathscr{L}t} U^0$ and $U^h(t) = e^{\mathscr{L}_h t} P U^0$, it follows from (1.13) that

$$\begin{aligned} &((U(t) - U^h(t), \Psi))_h = \\ &= ((T_h - T) u_t(t), \Psi_2) + \left((T - T_h) u_t^0, \left(e^{-\mathscr{L}t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \right)_2 \right) \\ &\qquad + ((U^0, (e^{-\mathscr{L}t} - e^{-\mathscr{L}_h t} P) \Psi))_h \\ &= (u_t(t), (T_h - T) \Psi_2) + \left(u_t^0, (T - T_h) \left(e^{-\mathscr{L}t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \right)_2 \right) \\ &\qquad + ((U^0, (e^{-\mathscr{L}t} - e^{-\mathscr{L}_h t} P) \Psi))_h . \end{aligned} \tag{2.4}$$

From Theorem 2.1 of Baker and Bramble [1] and the fact that $(e^{-\mathscr{L}t} - e^{-\mathscr{L}_h t} P) \Psi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (e^{\mathscr{L}t} - e^{\mathscr{L}_h t} P) \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix}$, we have

$$\sup_{0 \leq t \leq t^*} \left\| (e^{-\mathscr{L}t} - e^{-\mathscr{L}_h t}) \Psi \right\|_h \leq C h^r [\|\Psi_1\|_{r+1} + \|\Psi_2\|_r] \tag{2.5}$$

where $\| \cdot \|_h = ((\cdot, \cdot))_h^{1/2}$. From (1.10) and (1.4) it follows that

$$\left\| (T - T_h)\left(e^{-\mathscr{L}t}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\right)_2 \right\| \leqslant Ch^r \left\| \left(e^{-\mathscr{L}t}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\right)_2 \right\|_{r-2} \leqslant$$
$$\leqslant Ch^r(\|\Psi_1\|_{r-2} + \|\Psi_2\|_{r-1}) . \quad (2.6)$$

Since $\|U^0\|_h \leqslant C(\|u^0\| + \|u_t^0\|)$, it follows from (2.4), (2.5), (2.6) and the Cauchy-Schwarz inequality that with $\Psi = \begin{pmatrix}\Psi_1\\ 0\end{pmatrix}$

$$|(u(t) - u^h(t), \Psi_1)| \leqslant Ch^r(\|u^0\| + \|u_t^0\|)\|\Psi_1\|_{r+1} .$$

(2.3) follows from this estimate because $\|u(t) - u^h(t)\|_{-(r+1)} = \sup_{\Psi_1 \in H^{r+1}(\Omega)} \dfrac{|(u(t) - u^h(t))|}{\|\Psi_1\|_{r+1}}$ (see Thomée [7], page 80). ■

3. SINGLE STEP FULLY DISCRETE APPROXIMATIONS

Let r be a complex valued rational function defined for the complex variable z, satisfying

$$|r(iy) - e^{-iy}| \leqslant C|y|^{\nu+1} , \quad |y| \leqslant \sigma , \quad (3.1)$$

for constants $0 < C < \infty$, $\sigma > 0$ and $\nu > 0$, and

$$|r(iy)| \leqslant 1 . \quad \text{for all real } y \text{ with } |y| \leqslant \alpha , \quad (3.2)$$

where $0 < \alpha \leqslant \infty$.

For $k > 0$, the semidiscrete approximation $U^h(t)$ satisfies $U^h(t + k) = e^{k\mathscr{L}_h} U^h(t)$. Thus, the fully discrete approximation to (1.1), denoted by $\{W^n\}_{n \geqslant 0} \subset S_h^r(\Omega) \times S_h^r(\Omega)$, is defined by

$$\begin{aligned} W^{n+1} &= D^{-1}(k\mathscr{L})N(k\mathscr{L})W^n , \quad n = 0, 1, \ldots \\ W^0 &= PU^0 \end{aligned} \quad (3.3)$$

where $r(z) = D^{-1}(z)N(z)$ and D and N are minimal degree polynomials.

THEOREM 3.1 : *Let u be the solution of* (1.1) *and let $r(z)$ be a rational function satisfying* (3.1) *and* (3.2). *Let $\{W^n\}_{n \geqslant 0}$ be the sequence of* *approximations defined by* (3.3). *Then, there exists a constant* $C = C(r, \nu, t^*)$ *such that*

$$\sup_{0 \leq n \leq [t^*/k]} \|W_1^n - u(nk)\|_{-\mu} \leq C(h^r + k^\nu)(\|u^0\| + \|u_t^0\|) \tag{3.4}$$

where $\mu = \max\{r+1, \nu+1\}$.

Proof: Note that $U^h(nk) = e^{\mathcal{L}_h nk} P U^0$ and $W^n = (r(k\mathcal{L}_h))^n P U^0$. Therefore, it follows that

$$((U^h(nk) - W^n, \Psi))_h = ((PU^0, (e^{-\mathcal{L}_h nk} - (r(-k\mathcal{L}_h))^n) P\Psi))_h . \tag{3.5}$$

From Theorem 3.1 of Baker and Bramble [1], we have that

$$\|(e^{-\mathcal{L}_h nk} - (r(-k\mathcal{L}_h))^n) P\Psi\|_h \leq$$
$$\leq C\{h^r[\|\Psi_1\|_{r+1} + \|\Psi_2\|_r] + k^\nu[\|\Psi_1\|_{\nu+1} + \|\Psi_2\|_\nu]\} \tag{3.6}$$

since

$$(e^{-\mathcal{L}_h nk} - (r(-k\mathcal{L}_h))^n) P\Psi =$$
$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (e^{\mathcal{L}_h nk} - (r(k\mathcal{L}_h))^n) P \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix}.$$

It follows from (3.5), (3.6) and the Cauchy-Schwarz inequality that

$$|((U^h(nk) - W^n, \Psi))_h| \leq$$
$$\leq C\|PU^0\|_h\{h^r[\|\Psi_1\|_{r+1} + \|\Psi_2\|_r] + k^\nu[\|\Psi_1\|_{\nu+1} + \|\Psi_2\|_\nu]\} . \tag{3.7}$$

Choosing $\Psi = \begin{pmatrix} \Psi_1 \\ 0 \end{pmatrix}$ in (3.7) and using (2.3) gives (3.4) since

$$\sup_{\Psi_1 \in \dot{H}^\mu(\Omega)} \frac{|(u^h(nk) - W_1^n, \Psi_1)|}{\|\Psi\|_\mu} = \|u^h(nk) - W_1^n\|_{-\mu}$$

where $\mu = \max\{r+1, \nu+1\}$. ■

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