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ADDENDUM TO THE PAPER « NUMERICAL SOLUTION OF SECOND-ORDER EQUATIONS ON PLANE DOMAINS (*) »

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In the paper [1] we applied a so-called upwind discretization scheme to the problem

$$\begin{cases} -\Delta u + b \cdot \nabla u + cu = f & \text{in} \quad D \\ u = 0 & \text{on} \quad B , \end{cases} \tag{1}$$

where D is a bounded plane domain with polygonal boundary. We could demonstrate some important properties of the resulting scheme. Moreover, we proved a convergence result. Unfortunately, for that proof we had to require the underlying triangulations to be uniformly acute, i.e. in addition to the quite usual assumption on the existence of a uniform *lower* bound for all interior angles of the triangles (Zlámal's angle condition), we needed a uniform *upper* bound being less than $\frac{\pi}{2}$ strongly. In addition, we had to suppose that the family of triangulations satisfies the so-called inverse assumption (quasiuniformity condition).

In this note, we intend to show that the last two assumptions are not related with the discretization principle itself, but rather with the approximation quality of the coefficients which can be achieved.

Namely, if we redefine the quantities N_{ij}, c_i and f_i from [1] as follows :

$$N_{ij} = \frac{1}{m_{ij}} \int_{B_{ij}} (n \cdot b)\, ds\,, \quad c_i = \frac{1}{m_i} \int_{D_i} c\, dx\,, \quad f_i = \frac{1}{m_i} \int_{D_i} f\, dx\,,$$

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then instead of the assumption (A2) from [1] it is sufficient to suppose that obtuse triangles do not appear in the triangulations. Such triangulations are called *weakly acute*.

It is not difficult to verify that the statements of Lemmata 1-3 in [1] remain valid. However, the proof of the convergence theorem must be modified essentially. Therefore we completely bring the formulation and the proof of this theorem.

THEOREM 1 : *Let the weak solution u of* (1) *belong to* $H_0^1(D) \cap H^2(D)$. *Then the estimate*

$$\|u - u_h ; H^1(D)\| \leq Ch$$

holds, where C is a positive constant independent of h. ∎

Proof : For arbitrary $w_h \in X_h$, we have

$$C\,\|u_h - w_h ; H^1(D)\|^2 \leq a(u_h - w_h, u_h - w_h)\,. \tag{2}$$

Setting $v_h = u_h - w_h$, the right-hand side of this inequality can be treated as follows :

$$\begin{aligned} a(v_h, v_h) &= a(u - w_h, v_h) + a(u_h - u, v_h) \\ &= a(u - w_h, v_h) + a(u_h, v_h) - a_l(u_h, v_h) + (f, v_h)_l - (f, v_h) \\ &= a(u - w_h, v_h) + \delta a \end{aligned}$$

Using the definition of f_i, the term δa admits the following representation, which is different from [1] :

$$\begin{aligned} \delta a = &\sum_{i \in V} \sum_{j \in V_i} (1 - r_{ij}) N_{ij}(u_{hi} - u_{hj})\, v_{hi}\, m_{ij} + \int_D (b \,.\, \nabla u_h)\, v_h\, dx \\ &- \sum_{i \in V} \int_{D_i} [c_i\, u_{hi} v_{hi} - c u_h\, v_h]\, dx - \sum_{i \in V} \int_{D_i} [f v_h - f_i\, v_{hi}]\, dx \\ = &\sum_{i \in V} \left\{ \sum_{j \in V_i} (1 - r_{ij}) N_{ij}(u_{hi} - u_{hj})\, v_{hi}\, m_{ij} + \int_{D_i} (b \,.\, \nabla u_h)\, v_{hi}\, dx \right\} \\ &+ \sum_{i \in V} \int_{D_i} (b \,.\, \nabla u_h)(v_h - v_{hi})\, dx - \sum_{i \in V} \int_{D_i} [c_i\, u_{hi} - c u_h]\, v_{hi}\, dx \\ &+ \sum_{i \in V} \int_{D_i} c u_h (v_h - v_{hi})\, dx - \sum_{i \in V} \int_{D_i} f\,[v_h - v_{hi}]\, dx \\ = &\ \delta_1 + \delta_2 + \delta_3\,, \end{aligned}$$

where

$$\delta_1 = \sum_{i \in V} \int_{D_i} [b \,.\, \nabla u_h + c u_h - f\,] (v_h - v_{hi}) \, dx \,,$$

$$\delta_2 = \sum_{i \in V} v_{hi} \left\{ \sum_{j \in V_i} u_{hi} \, N_{ij} \, m_{ij} - \int_{D_i} [(\nabla \,.\, b - c) \, u_h + c_i \, u_{hi}] \, dx \right\} ,$$

$$\delta_3 = \sum_{i \in V} \sum_{j \in V_i} \int_{B_{ij}} \left\{ (n \,.\, b) \, u_h - [r_{ij} \, u_{hi} + (1 - r_{ij}) \, u_{hj}] \, N_{ij} \right\} v_{hi} \, ds \,.$$

By Cauchy's inequality, for δ_1 we obtain

$$\delta_1 \leqslant \| b \,.\, \nabla u_h + c u_h - f \,; L_2(D) \| \left\{ \sum_{i \in V} \int_{D_i} | v_h - v_{hi} | \, dx \right\}^{\frac{1}{2}} .$$

The last term can be estimated by means of Lemma 2.1 from [3], that is

$$\delta_1 \leqslant C h \| b \,.\, \nabla u_h + c u_h - f \,; L_2(D) \| \, | v_h \,; H^1(D) | \,. \tag{3}$$

Now we consider δ_2. Using the definition of c_i, we have

$$\delta_2 =$$

$$\sum_{i \in V} v_{hi} \left\{ \sum_{j \in V_i} u_{hi} \, N_{ij} \, m_{ij} - \int_{D_i} (\nabla \,.\, b) \, u_{hi} \, dx - \int_{D_i} (\nabla \,.\, b - c)(u_h - u_{hi}) \, dx \right\} .$$

Integrating the first integral by parts and using the definition of N_{ij}, we obtain

$$\begin{aligned} \delta_2 &= \sum_{i \in V} \int_{D_i} (c - \nabla \,.\, b)(u_h - u_{hi}) \, v_{hi} \, dx \\ &\leqslant \| c - \nabla \,.\, b \,; L_\infty(D) \| \sum_{i \in V} \int_{D_i} | u_h - u_{hi} | \, | v_{hi} | \, dx \,. \end{aligned}$$

Applying Cauchy's inequality and Lemma 2.1 from [3], it follows

$$\delta_2 \leqslant C h \| c - \nabla \,.\, b \,; L_\infty(D) \| \, | u_h \,; H^1(D) | \, \| v_h \|_l \,.$$

In view of the equivalence of the $\| \,.\, \|_l$-norm with the L_2-norm on X_h (see Lemma 2.2 in [3]), we get

$$\delta_2 \leqslant C h \| c - \nabla \,.\, b \,; L_\infty(D) \| \, | u_h \,; H^1(D) | \, \| v_h \,; L_2(D) \| \,. \tag{4}$$

Thus it remains to consider the third term. By a symmetry argument, we see that

$$\delta_3 = \frac{1}{2} \sum_{i \in V} \sum_{j \in V_i} \int_{B_{ij}} \{(n \,.\, b) u_h - [r_{ij} u_{hi} + (1 - r_{ij}) u_{hj}] N_{ij}\} (v_{hi} - v_{hj}) \, ds \,.$$

Obviously, by means of B_{ij} and x_i there can be defined some triangle T_{ij} in such a way that B_{ij} is one edge and x_i the opposite vertex. On the other hand, it is possible to represent this triangle as the union of two triangles $T_{ij}^{(k)}$ $(k = 1, 2)$ having in common the vertex x_i and the straight-line segment connecting the node x_i with the point $\frac{1}{2}(x_i + x_j)$.

Now we can write $\delta_3 = \delta_{31} + \delta_{32}$, where

$$\delta_{31} = \frac{1}{2} \sum_{i \in V} \sum_{j \in V_i} \int_{B_{ij}} (n \,.\, b - N_{ij}) u_h (v_{hi} - v_{hj}) \, ds \,,$$

$$\delta_{32} = \frac{1}{2} \sum_{i \in V} \sum_{j \in V_i} \int_{B_{ij}} [u_h - r_{ij} u_{hi} - (1 - r_{ij}) u_{hj}] N_{ij} (v_{hi} - v_{hj}) \, ds \,.$$

We turn to the estimation of the integral

$$J_{ij}(b) = \int_{B_{ij}} (n \,.\, b - N_{ij}) u_h \, ds \,.$$

To this end we introduce a reference triangle $\tilde{T}$ with vertices $t_1 = (0, 0)$, $t_2 = (1, 0)$, $t_3 = (0, 1)$ and transform T_{ij} onto $\tilde{T}$ in such a way that B_{ij} has the image $\tilde{B} = \overline{t_1 t_2}$. Then we have

$$J_{ij}(b) = m_{ij} J(\tilde{b}) \,,$$

where

$$J(\tilde{b}) = \int_{\tilde{B}} (n \,.\, \tilde{b} - \tilde{N}_{ij}) \tilde{u}_h \, d\tilde{s} \,.$$

It follows

$$\begin{aligned} |J(\tilde{b})| &\leq \|n \,.\, \tilde{b} - \tilde{N}_{ij} \,;\, L_2(\tilde{B})\| \, \|\tilde{u}_h \,;\, L_2(\tilde{B})\| \\ &\leq 2 \|n \,.\, \tilde{b} \,;\, L_2(\tilde{B})\| \, \|\tilde{u}_h \,;\, L_2(\tilde{B})\| \,. \end{aligned}$$

Using the trace theorem [4], we get

$$|J(\tilde{b})| \leq C \|n \,.\, \tilde{b} \,;\, H^1(\tilde{T})\| \, \|\tilde{u}_h \,;\, H^1(\tilde{T})\| \,.$$

Observing that J vanishes for constant $\tilde{b}$, we conclude from Bramble-Hilbert's lemma [2] the estimate

$$\left|J(\tilde{b})\right| \leqslant C\left|n \cdot \tilde{b} ; H^{1}(\tilde{T})\right|\left\|\tilde{u}_{h} ; H^{1}(\tilde{T})\right\| .$$

The back-transformation (see [2], again) yields

$$\begin{aligned}
\left|J_{ij}(b)\right| & \leqslant C m_{ij} h_{T_{ij}}(\text{meas } T_{ij})^{-1}\left|n \cdot b ; H^{1}(T_{ij})\right|\left\|u_{h} ; H^{1}(T_{ij})\right\| \\
& \leqslant C m_{ij} h_{T_{ij}}(\text{meas } T_{ij})^{-\frac{1}{2}}\left|n \cdot b ; W_{\infty}^{1}(T_{ij})\right|\left\|u_{h} ; H^{1}(T_{ij})\right\| \\
& \leqslant C m_{ij} h_{T_{ij}}(\text{meas } T_{ij})^{-\frac{1}{2}}\left|b ; [W_{\infty}^{1}(T_{ij})]^{2}\right|\left\|u_{h} ; H^{1}(T_{ij})\right\| .
\end{aligned}$$

Thus we have

$$\begin{aligned}
\delta_{31} \leqslant C h\left|b ; [W_{\infty}^{1}(D)]^{2}\right| \sum_{i \in V} \sum_{j \in V_{i}} m_{ij}(\text{meas } T_{ij})^{-\frac{1}{2}} \times & \\
\times\left|v_{hi}-v_{hj}\right|\left\|u_{h} ; H^{1}(T_{ij})\right\| .&
\end{aligned}$$

In view of the relations

$$\left|v_{hi}-v_{hj}\right|=d_{ij}\left|n \cdot \nabla v_{h}\right| \quad \text{and} \quad d_{ij} m_{ij}=4 \text{ meas } T_{ij} \tag{5}$$

it follows

$$\delta_{31} \leqslant C h\left|b ; [W_{\infty}^{1}(D)]^{2}\right| \sum_{i \in V} \sum_{j \in V_{i}}(\text{meas } T_{ij})^{\frac{1}{2}}\left|n \cdot \nabla v_{h}\right|\left\|u_{h} ; H^{1}(T_{ij})\right\| .$$

Cauchy's inequality implies

$$\delta_{31} \leqslant C h\left|b ; [W_{\infty}^{1}(D)]^{2}\right|\left|v_{h} ; H^{1}(D)\right|\left\|u_{h} ; H^{1}(D)\right\| . \tag{6}$$

Thus it remains to consider the term δ_{32}. Using the notation $B_{ij}^{(k)}=B_{ij} \cap T_{ij}^{(k)}$, we can write

$$\delta_{32}=\frac{1}{2} \sum_{i \in V} \sum_{j \in V_{i}} \sum_{k=1}^{2} \int_{B_{ij}^{(k)}} u_{hij} N_{ij}(v_{hi}-v_{hj})\, ds ,$$

where

$$u_{hij}=u_{h}-r_{ij} u_{hi}-(1-r_{ij}) u_{hj} .$$

Now, on each triangle $T_{ij}^{(k)}$ we have

$$u_{h}(x)=u_{hi}+\nabla u_{h} \cdot(x-x_{i})$$

with ∇u_h being constant on $T_{ij}^{(k)}$. Thus, u_{hij} can be treated as follows :

$$\begin{aligned} u_{hij} &= (1 - r_{ij})(u_{hi} - u_{hj}) + \nabla u_h \cdot (x - x_i) \\ &= (1 - r_{ij})\, d_{ij}(n \cdot \nabla u_h) + \nabla u_h \cdot (x - x_i) \\ &\leqslant C d_{ij} \|\nabla u_h\| . \end{aligned}$$

Therefore, together with (5) this estimate implies

$$\begin{aligned} \delta_{32} &\leqslant C \sum_{i \in V} \sum_{j \in V_i} d_{ij}^2 N_{ij} \sum_{k=1}^{2} \int_{B_{ij}^{(k)}} \|\nabla u_h\| \, \|\nabla v_h\| \, ds \\ &= C \sum_{i \in V} \sum_{j \in V_i} d_{ij}^2 N_{ij} \sum_{k=1}^{2} \frac{m_{ij}^{(k)}}{\text{meas } T_{ij}^{(k)}} \int_{T_{ij}^{(k)}} \|\nabla u_h\| \, \|\nabla v_h\| \, dx , \end{aligned}$$

where $m_{ij}^{(k)}$ is the length of $B_{ij}^{(k)}$.

Observing that meas $T_{ij}^{(k)} = \dfrac{1}{4} m_{ij}^{(k)} d_{ij}$, after the application of various variants of Cauchy's inequality we obtain

$$\begin{aligned} \delta_{32} &\leqslant C \sum_{i \in V} \sum_{j \in V_i} d_{ij} N_{ij} \sum_{k=1}^{2} \int_{T_{ij}^{(k)}} \|\nabla u_h\| \, \|\nabla v_h\| \, ds \\ &\leqslant C h \big\| b \, ; \, [L_\infty(D)]^2 \big\| \sum_{i \in V} \sum_{j \in V_i} |u_h \, ; H^1(T_{ij})| \, |v_h \, ; H^1(T_{ij})| \\ &\leqslant C h \big\| b \, ; \, [L_\infty(D)]^2 \big\| \sum_{i \in V} |u_h \, ; H^1(D_i)| \, |v_h \, ; H^1(D_i)| \\ &\leqslant C h \big\| b \, ; \, [L_\infty(D)]^2 \big\| \, |u_h \, ; H^1(D)| \, |v_h \, ; H^1(D)| . \end{aligned} \tag{7}$$

Summarizing the relations (3), (4), (6), (7) and using Lemma 1 from [1] in order to derive an a priori estimate for $\|u_h \, ; H^1(D)\|$, we get from (2)

$$\|u_h - w_h \, ; H^1(D)\| \leqslant C \, [\, \|u - w_h \, ; H^1(D)\| + h] .$$

The triangle inequality yields

$$\begin{aligned} \|u - u_h \, ; H^1(D)\| &\leqslant \|u - w_h \, ; H^1(D)\| + \|u_h - w_h \, ; H^1(D)\| \\ &\leqslant C \, [\, \|u - w_h \, ; H^1(D)\| + h] . \end{aligned}$$

Choosing for w_h the Lagrangian interpolant in X_h of the exact solution u of (1), the standard interpolation theory gives

$$\|u - w_h \, ; H^1(D)\| \leqslant C h \, |u \, ; H^2(D)| .$$

Now, the desired result follows immediately. ∎

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