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APPROXIMATION OF THE THREE-FIELD STOKES SYSTEM VIA OPTIMIZED QUADRILATERAL FINITE ELEMENTS (*)

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Abstract. — *In a recent paper the first two authors showed that a convenient choice of « bubble tensors » leads to a significant reduction of the number of degrees of freedom needed to define finite element spaces for the extra stresses, suitable for the solution of viscoelastic flow problems. In this context specialists' usual guiding criterion is a stable and accurate approximation of the underlying linear problem : the three-field Stokes system. Keeping this in view, a new element for the approximation of this problem following the same ideas is presented. Some computer tests illustrate the potentialities of the new methods.*

Résumé. — *Dans un article récent, les deux premiers auteurs ont démontré qu'un choix convenable de « tenseurs bulle » permet une réduction très importante du nombre de degrés de liberté nécessaire pour définir des espaces d'approximation de type éléments finis, du tenseur d'extracontraintes, en vue de la simulation d'écoulements de fluides viscoélastiques. Dans ce cadre, il est généralement admis qu'une méthode stable et précise pour l'approximation du problème linéaire sous-jacent est aussi efficace pour les systèmes non linéaires dont il est question. Ainsi on présente et on étudie ici une nouvelle méthode construite selon des principes semblables, pour le cas linéaire, à savoir, le système de Stokes à trois champs. De plus, on met en évidence le potentiel des nouvelles méthodes au travers de quelques exemples numériques.*

1. INTRODUCTION

The Stokes system expressed in terms of three fields, namely, velocity, pressure and extra stress tensor, is generally acknowledged as a basic problem associated with the system describing the motion of a viscoelastic

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liquid. In particular, in the framework of a finite element simulation of this kind of flows, many specialists so far searched for classes of methods that are able to produce fine approximations of the three variables in the case of the Stokes system. In so doing they either hoped or showed experimentally that this is still so in the case of true viscoelastic models such as Maxwell's or Oldroyd's (see e.g. [2]).

Incidentally, at least for the latter case Baranger & Sandri [3] proved that second order convergent finite element methods for the solution of the three-field Stokes system are convergent as well, when applied to the solution of the corresponding viscoelasticity system.

As a significant example of efficient finite element methods for this kind of problem derived in the past few years we should quote the so-called Marchal & Crochet element [9]. Indeed, this second order quadrilateral element allowed a considerable advance in the numerical simulation of viscoelastic flow. In particular they showed through many examples given in [9] and in other papers that followed, that flows involving a rather high degree of elasticity could be represented satisfactorily by a computer solution using such a method.

A little later Fortin & Pierre [8] rigorously justified the performance anticipated by computations, by proving that the Marchal & Crochet method was optimally convergent, precisely in the limiting case of the three-field stationary Stokes system.

In a recent paper, the first two authors presented a method for solving this system, having comparable convergence properties, though at a considerably lower cost. They are based on the same velocity interpolation as that of Marchal & Crochet's, whereas the pressure interpolation is the discontinuous piecewise linear one used by Fortin & Fortin [7]. The main difference with respect to previous work lies in the new type of extra stress interpolation using a well-chosen set of twelve bubble tensors, which enrich the standard continuous finite element space, thereby ensuring the necessary stability.

In this paper we introduce another element based on the same ideas. However in order to allow the use of other types of pressure interpolation, such as the popular continuous bilinear one, we modify the set of bubble tensors, whose number is increased by three.

Notice that in the case of the first element this technique is equivalent to adding four nodes in the interior of the quadrilaterals to the standard bilinear element, while in the case of the new element it corresponds to adding five inner nodes.

After having reviewed in Section 2 the functional background for the three-field Stokes system proposed in [10], we briefly recall in Section 3 its application to the study of one of the elements introduced by the first two authors in [12]. In Section 4 we give a similar analysis adapted to the case of the new method. Incidentally we should mention that the latter method was briefly studied in [5], where a detailed computer study using both elements is given. As we should also add, the convergence results derived in this work are restricted to the case of rectangular elements. However an analysis recently performed by the first author [11] showed that such convergence properties remain valid, at least in the case where the quadrilaterals are not too distorted.

Computer tests for the three-field Stokes system, described in Section 5, using both methods and a classical biquadratic interpolation of both velocity and extra stresses [1], allow a fair comparison of their performances. It is found out in particular that they seem to be superior to the Marchal & Crochet element, from the accuracy point of view too, at least in the case of rectangular meshes.

Before entering into the subject, let us first recall the three-field Stokes system, assuming for simplicity that the velocity vanishes on the boundary Γ of the bounded flow domain of $\mathcal{R}^2$ represented by Ω:

$$\left.\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} + \operatorname{grad} p &= \vec{f}\,, \\ \boldsymbol{\sigma} &= 2\,\eta\,\boldsymbol{\varepsilon}(\vec{u})\,, \\ \operatorname{div} \vec{u} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega\,, \tag{1}$$

where $\boldsymbol{\sigma}$ is the extra stress tensor, $\vec{u}$ is the velocity field, p is the pressure, $\vec{f}$ is a given body forces, η is the viscosity of the fluid and $\boldsymbol{\varepsilon}(\vec{u})$ is the strain rate tensor given by

$$\boldsymbol{\varepsilon}(\vec{u}) = \frac{1}{2}\,[\operatorname{grad} \vec{u} + (\operatorname{grad} \vec{u})^T]\,.$$

2. SUMMARY OF THE FUNCTIONAL BACKGROUND

The natural variational formulation of system (1) is

$$\left.\begin{aligned} &\text{Find } (\vec{u}, p, \boldsymbol{\sigma}) \in V \times Q \times \Sigma \quad \text{such that} \\ &-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\vec{v})\,dx + \int_\Omega p \operatorname{div} \vec{v}\,dx = -\int_\Omega \vec{f}\,.\,\vec{v}\,dx\,, \quad \forall \vec{v} \in V\,, \\ &\frac{1}{2\,\eta}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\,dx - \int_\Omega \boldsymbol{\varepsilon}(\vec{u}) : \boldsymbol{\tau}\,dx = 0\,, \quad \forall \boldsymbol{\tau} \in \Sigma\,, \\ &\int_\Omega q \operatorname{div} \vec{u}\,dx = 0\,, \quad \forall q \in Q\,, \end{aligned}\right\} \tag{2}$$

[1] While keeping the pressure discontinuous linear in all the cases.

where referring to [1] for example, for the notation, we set :

$$V = (H_0^1(\Omega))^2 ,$$

normed by

$$|\vec{v}|_{1,\Omega} = \left(\int_\Omega |\text{grad}\,\vec{v}|^2 \, dx \right)^{1/2} ;$$

$$Q = L_0^2(\Omega) = \left\{ q \mid q \in L^2(\Omega), \int_\Omega q \, dx = 0 \right\} ,$$

normed by

$$\|q\|_{0,\Omega} = \left(\int_\Omega |q|^2 \, dx \right)^{1/2} ;$$

$$\Sigma = \left\{ \boldsymbol{\tau} \mid \boldsymbol{\tau} \in (L^2(\Omega))^{2\times 2}, \ \tau_{ij} = \tau_{ji}, \ 1 \leqslant i, j \leqslant 2 \right\} ,$$

normed by

$$\|\boldsymbol{\tau}\|_{0,\Omega} = \left(\sum_{i,j=1}^{2} \|\tau_{ij}\|_{0,\Omega}^2 \right)^{1/2} .$$

Now if we are given a quadrilateral family $\{\mathcal{T}_h\}_h$ of finite element meshes of Ω, respecting the usual compatibility and regularity rules, where the parameter h denotes the maximum diameter of the elements of $\mathcal{T}_h$, we associate with $\mathcal{T}_h$ three subspaces V_h, Q_h and Σ_h of V, Q and Σ, respectively. The corresponding sequence of approximate problems is thus defined by :

$$\left.\begin{aligned}
&\text{Find } (\vec{u}_h, p_h, \boldsymbol{\sigma}_h) \in V_h \times Q_h \times \Sigma_h \quad \text{such that} \\
&-\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}(\vec{v}) \, dx + \int_\Omega p_h \,\text{div}\, \vec{v} \, dx = -\int_\Omega \vec{f} \cdot \vec{v} \, dx , \quad \forall \vec{v} \in V_h , \\
&\frac{1}{2\eta} \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau} \, dx - \int_\Omega \boldsymbol{\varepsilon}(\vec{u}_h) : \boldsymbol{\tau} \, dx = 0 , \quad \forall \boldsymbol{\tau} \in \Sigma_h , \\
&\int_\Omega q \,\text{div}\, \vec{u}_h \, dx = 0 , \quad \forall q \in Q_h .
\end{aligned}\right\} \quad (3)$$

We will work with spaces V_h and Q_h that satisfy the well-known stability condition, necessary for well-posedness of problem (3), namely,

$$\exists \beta_1 > 0 \quad \text{such that} \quad \inf_{\substack{q \in Q_h \\ q \neq 0}} \sup_{\substack{\vec{v} \in V_h \\ \vec{v} \neq 0}} \frac{\int_\Omega q \,\text{div}\, \vec{v} \, dx}{\|q\|_{0,\Omega} |\vec{v}|_{1,\Omega}} \geqslant \beta_1 \quad (4)$$

we assume in addition that β_1 is independent of h.

According to [10], the error bound

$$|\vec{u}-\vec{u}_h|_{1,\Omega}^2+\|p-p_h\|_{0,\Omega}^2+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}^2 \leq$$

$$\leq C\left(\inf_{\vec{v}\in V_h}|\vec{u}-\vec{v}|_{1,\Omega}^2+\inf_{q\in Q_h}\|p-q\|_{0,\Omega}^2+\inf_{\boldsymbol{\tau}\in\Sigma_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}\|_{0,\Omega}^2\right) \quad (5)$$

holds for a constant C independent of h, if the following compatibility condition between spaces V_h and Σ_h is fulfilled :

$$\exists\beta_2>0 \quad \text{such that} \quad \inf_{\substack{\vec{v}\in\tilde{U}_h\\ \vec{v}\neq 0}}\sup_{\substack{\boldsymbol{\tau}\in\Sigma_h\\ \boldsymbol{\tau}\neq 0}}\frac{\int_\Omega \boldsymbol{\tau}:\boldsymbol{\varepsilon}(\vec{v})\,dx}{|\vec{v}|_{1,\Omega}\|\boldsymbol{\tau}\|_{0,\Omega}}\geq\beta_2 \quad (6)$$

and that in addition β_2 is independent of h, where $\tilde{U}_h$ is any space containing U_h defined by,

$$U_h=\left\{\vec{v}\mid\vec{v}\in V_h,\ \int_\Omega q\operatorname{div}\vec{v}\,dx=0,\ \forall q\in Q_h\right\}.$$

Likewise in [10], we will use the following result adapted from Fortin's quoted in [8], namely :

PROPOSITION 1 : *If, for each $\boldsymbol{\tau}\in\Sigma$, there exists a $\boldsymbol{\tau}_h\in\Sigma_h$ such that*

$$\int_\Omega \boldsymbol{\tau}_h:\boldsymbol{\varepsilon}(\vec{v})\,dx=\int_\Omega \boldsymbol{\tau}:\boldsymbol{\varepsilon}(\vec{v})\,dx,\ \forall\vec{v}\in\tilde{U}_h, \quad (7)$$

and $\boldsymbol{\tau}_h$ satisfies the inequality

$$\|\boldsymbol{\tau}_h\|_{0,\Omega}\leq C\,\|\boldsymbol{\tau}\|_{0,\Omega}, \quad (8)$$

for a constant C independent of h, then (6) *is fulfilled.*

3. EXTRA STRESSES FOR RECTANGLES WITH A DISCONTINUOUS PRESSURE

Let $\mathcal{T}_h$ consist of rectangles. Let also a and b denote the lengths of the two edges defining a rectangle $R\in\mathcal{T}_h$, assumed to be parallel to the two cartesian axes, say Ox_1 and Ox_2, respectively. Set $r=a/b$ and let $\mathcal{F}_R$ be the invertible affine mapping from the reference square $\hat{R}=[-1,+1]\times[-1,+1]$ of the plane $\hat{O}\hat{x}_1\hat{x}_2$ onto R.

We recall that $\hat{Q}_2$ denotes the space of polynomials, of degree less than or equal to two in each variable $\hat{x}_i$, $i = 1, 2$, over $\hat{R}$. In so doing, we introduce the following auxiliary space :

$$W_h = \left\{ \vec{v} = (v_1, v_2) \middle| \, \vec{v}\,|_R = \vec{\hat{v}} \circ \mathcal{F}_R^{-1} \text{ with } \vec{\hat{v}} \in (\hat{Q}_2)^2, \ \forall R \in \mathcal{T}_h \right\} .$$

Like in [9], we will take $V_h = W_h \cap (C^0(\bar{\Omega}))^2 \cap (H_0^1(\Omega))^2$. However in order to enable the best possible improvement of the space for the extra stress tensor will work with the space

$$Q_h = \tilde{Q}_h \cap L_0^2(\Omega)\,,$$

where

$$\tilde{Q}_h = \{ q \mid q|_R \in P_1, \ \forall R \in \mathcal{T}_h \} \,,$$

P_1 being the space of affine functions over R. Such a pair (V_h, Q_h) is known to satisfy condition (4) (see e.g. [7]).

We will define space Σ_h to be the direct sum of the space Σ_h^1, of continuous symmetric tensors whose restriction to each element of $\mathcal{T}_h$ belongs to $(Q_1)^{2\times 2}$ (that is, linear with respect to x_1 and x_2 separately), with another space $\tilde{\Sigma}_h$ of symmetric tensors that vanish on the boundary of every element $R \in \mathcal{T}_h$, to be specified hereafter.

Next we define our space $\tilde{U}_h$ to be :

$$\tilde{U}_h = \left\{ \vec{v} \mid \vec{v} \in W_h \text{ and } \int_R q \operatorname{div} \vec{v}\, dx = 0, \quad \forall q \in P_1, \quad \forall R \in \mathcal{T}_h \right\} .$$

The inclusion $U_h \subset \tilde{U}_h$ follows directly from the fact that V_h is a subspace of W_h.

Following again [8], we note that in order to satisfy (7) and (8) it suffices to be able to define a tensor $\tilde{\tau}_h$ belonging to space $\tilde{\Sigma}_h$, such that for every tensor $\tilde{\tau} \in \Sigma$, we have

$$\int_R \tilde{\tau}_h : \varepsilon(\vec{v})\, dx = \int_R \tilde{\tau} : \varepsilon(\vec{v})\, dx\,, \quad \forall \vec{v} \in \tilde{U}_h\,, \quad \forall R \in \mathcal{T}_h \tag{9}$$

and

$$\|\tilde{\tau}_h\|_{0,R} \leq C\, \|\tilde{\tau}\|_{0,R}\,, \tag{10}$$

where C is a constant independent of h.

By changing variables in the above integrals, (9) is seen to be equivalent to stating that with every symmetric tensor $\hat{\tau}$, whose components belong to

$L^2(\hat{R})$, it is possible to associate another tensor $\hat{\boldsymbol{\tau}}_h$ belonging to a certain space $\hat{\Sigma}$ of tensors defined in the reference square, and that vanish on the boundary of $\hat{R}$, such that

$$\int_{\hat{R}} \hat{\boldsymbol{\tau}}_h : \hat{\boldsymbol{\varepsilon}}\, dx = \int_{\hat{R}} \hat{\boldsymbol{\tau}} : \hat{\boldsymbol{\varepsilon}}\, dx\,, \quad \forall \hat{\boldsymbol{\varepsilon}} \in \hat{E}\,, \tag{11}$$

where $\hat{E}$ is the space spanned by the tensors of the form $\boldsymbol{\varepsilon}(\vec{v}) \circ \mathcal{F}_R^{-1}$ for $\vec{v} \in \tilde{U}_h$.

Moreover using the same arguments as in [10] the fact that (11) has a unique solution implies that (9) holds, provided the determinant of the matrix associated with the corresponding system is independent of h.

In the case under study, using the monomials

$$\{1,\, x_1,\, x_2,\, x_1 x_2,\, x_1^2,\, x_2^2,\, x_1^2 x_2,\, x_1 x_2^2,\, x_1^2 x_2^2\}$$

spanning the space Q_2 (i.e., the space analogous to $\hat{Q}_2$ for the variables x_i, $i = 1, 2$) $\hat{E}$ is found to be spanned by the following set of twelve linearly independent tensors (see [5]) :

$$\hat{\boldsymbol{\varepsilon}}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \hat{\boldsymbol{\varepsilon}}_2 = \begin{pmatrix} \hat{x}_1 & 0 \\ 0 & -\hat{x}_1 \end{pmatrix};$$

$$\hat{\boldsymbol{\varepsilon}}_3 = \begin{pmatrix} \hat{x}_2 & 0 \\ 0 & -\hat{x}_2 \end{pmatrix}; \qquad \hat{\boldsymbol{\varepsilon}}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\varepsilon}}_5 = \begin{pmatrix} 0 & \hat{x}_1 \\ \hat{x}_1 & 0 \end{pmatrix}; \qquad \hat{\boldsymbol{\varepsilon}}_6 = \begin{pmatrix} 0 & \hat{x}_2 \\ \hat{x}_2 & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\varepsilon}}_7 = \begin{pmatrix} 2\,\hat{x}_1\,\hat{x}_2 & r\hat{x}_1^2/2 \\ r\hat{x}_1^2/2 & 0 \end{pmatrix}; \qquad \hat{\boldsymbol{\varepsilon}}_8 = \begin{pmatrix} x_2^2 - 1/3 & 2\,r\bar{x}_1\,\bar{x}_2 \\ 2\,r\hat{x}_1\,\hat{x}_2 & r^2(\hat{x}_1^2 - 1/3) \end{pmatrix};$$

$$\hat{\boldsymbol{\varepsilon}}_9 = \begin{pmatrix} \hat{x}_1(1 - 3\,r\hat{x}_2^2) & -3\,r\hat{x}_1^2\,\hat{x}_2/2 \\ -3\,\hat{x}_1^2\,\hat{x}_2/2 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\varepsilon}}_{10} = \begin{pmatrix} 0 & \hat{x}_2^2/2 \\ \hat{x}_2^2/2 & 2\,r\hat{x}_1\,\hat{x}_2 \end{pmatrix};$$

$$\hat{\boldsymbol{\varepsilon}}_{11} = \begin{pmatrix} \hat{x}_2^2 & 0 \\ 0 & [r^2(1 - \hat{x}_1^2) - 1]/3 \end{pmatrix}; \quad \hat{\boldsymbol{\varepsilon}}_{12} = \begin{pmatrix} 0 & -3\,\hat{x}_1\,\hat{x}_2^2/2 \\ -3\,\hat{x}_1\,\hat{x}_2^2/2 & r(\hat{x}_2 - 3\,\hat{x}_1^2\,\hat{x}_2) \end{pmatrix}.$$

Just to clarify the calculations leading to the above result, observe the following. Whenever $\hat{v} \in (Q_2)^2$, for a given rectangle R, $\boldsymbol{\varepsilon}(\vec{v}) \circ \mathcal{F}_R$ may be viewed as a linear combination of the fifteen tensors, $\{\zeta_i\}$, given in the next section. However, if the projection of div $\vec{v}$ onto the space of linear functions in the rectangle R vanishes, the corresponding set of tensors is reduced to the twelve tensors above. This is because three relations among the fifteen original tensors result from this projection condition. Note that this is the reason why we may work with twelve bubble tensors instead of fifteen, whenever the pressure space consists of discontinuous piecewise linear functions.

Let now $\hat{\varphi}$ denote the « *bubble function* » of $\hat{R}$, that is,

$$\hat{\varphi}(\hat{x}_1, \hat{x}_2) = \frac{1}{4}(1 - \hat{x}_1^2)(1 - \hat{x}_2^2).$$

We define $\hat{\Sigma}$ to be the space spanned by the twelve following tensors :

$$\hat{\boldsymbol{\sigma}}_1 = \begin{pmatrix} \hat{\varphi} & 0 \\ 0 & -\hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_2 = \begin{pmatrix} \hat{x}_1\hat{\varphi} & 0 \\ 0 & -\hat{x}_1\hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_3 = \begin{pmatrix} \hat{x}_2\hat{\varphi} & 0 \\ 0 & -\hat{x}_2\hat{\varphi} \end{pmatrix};$$

$$\hat{\boldsymbol{\sigma}}_4 = \begin{pmatrix} 0 & \hat{\varphi} \\ \hat{\varphi} & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_5 = \begin{pmatrix} 0 & \hat{x}_1\hat{\varphi} \\ \hat{x}_1\hat{\varphi} & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_6 = \begin{pmatrix} 0 & \hat{x}_2\hat{\varphi} \\ \hat{x}_2\hat{\varphi} & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\sigma}}_7 = \begin{pmatrix} \hat{x}_1\hat{x}_2\hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_8 = \begin{pmatrix} 0 & \hat{x}_1\hat{x}_2\hat{\varphi} \\ \hat{x}_1\hat{x}_2\hat{\varphi} & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_9 = \begin{pmatrix} \hat{x}_1\hat{\varphi} & 0 \\ 0 & \hat{x}_1\hat{\varphi} \end{pmatrix};$$

$$\hat{\boldsymbol{\sigma}}_{10} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_1\hat{x}_2\hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_{11} = \begin{pmatrix} \hat{x}_2^2\hat{\varphi} & 0 \\ 0 & -\hat{x}_1^2\hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\sigma}}_{12} = \begin{pmatrix} \hat{x}_2\hat{\varphi} & 0 \\ 0 & \hat{x}_2\hat{\varphi} \end{pmatrix}.$$

The construction of $\hat{\boldsymbol{\tau}}_h \in \hat{\Sigma}$ satisfying (11) reduces to the solution of a linear system,

$$A\vec{t}_h = \vec{t}$$

where $\vec{t}_h$ is the vector of $\mathscr{R}^{12}$ consisting of the components of $\hat{\boldsymbol{\tau}}_h$ with respect to the basis $\{\hat{\boldsymbol{\sigma}}_j\}_{j=1}^{12}$, $\vec{t}$ is the vector of $\mathscr{R}^{12}$ whose i-th component is given by $\int_{\hat{R}} \hat{\boldsymbol{\tau}} : \hat{\boldsymbol{\varepsilon}}_i \, dx$, and $A = a_{ij}$ is the 12×12 matrix given by $a_{ij} = \int_{\hat{R}} \hat{\boldsymbol{\sigma}}_j : \hat{\boldsymbol{\varepsilon}}_i \, dx$.

Since

$$\int_{-1}^{1} x_1^m (1 - x_i^2) \, dx_i = 0$$

if m is odd, $i = 1$ ou 2, matrix A is sparse. Actually the only non zero entries of A are found to be

$$a_{1,1} = \frac{8}{9}; \qquad a_{1,11} = \frac{8}{45};$$

$$a_{2,2} = \frac{8}{45};$$

$$a_{3,3} = \frac{8}{45};$$

$$a_{4,4} = \frac{8}{9};$$

$$a_{5,5} = \frac{8}{45};$$

$$a_{6,6} = \frac{8}{45};$$

$$a_{7,4} = \frac{4\,r}{45}; \quad a_{7,7} = \frac{8}{225};$$

$$a_{8,1} = \frac{8\,(r^2 - 1)}{135}; \quad a_{8,8} = \frac{16\,r}{225}; \quad a_{8,11} = \frac{8(1 - r^2)}{945};$$

$$a_{9,2} = \frac{8}{225}; \quad a_{9,6} = \frac{-4\,r}{75}; \quad a_{9,9} = \frac{8}{225};$$

$$a_{10,4} = \frac{4}{45}; \quad a_{10,10} = \frac{8\,r}{225};$$

$$a_{11,1} = \frac{8\,(4 - r^2)}{135}; \quad a_{11,11} = \frac{8\,(r^2 - 8)}{945};$$

$$a_{12,3} = \frac{-8\,r}{225}; \quad a_{12,5} = \frac{-4}{75}; \quad a_{12,12} = \frac{8\,r}{225}.$$

After performing some simple Gaussian eliminations, we obtain

$$\det A = \frac{2^{39}}{3^{24} \times 5^{16} \times 7}\, r^3 (r^2 + 1)\,,$$

thus A is an invertible matrix for every $r \neq 0$. This implies not only that system (11) has a unique solution, but that this is also the case of system (9), provided $\tilde{\Sigma}$ is the space of symmetric tensors, whose restriction to every $R \in \mathcal{T}_h$ belongs to the space spanned by $\{\boldsymbol{\sigma}_j^R\}_{j=1}^{12}$ where $\boldsymbol{\sigma}_j^R = \hat{\boldsymbol{\sigma}}_j \circ \mathcal{F}_R^{-1}$.

Like in [8], we choose $\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} - \pi_h^1 \boldsymbol{\tau}$, where $\boldsymbol{\tau} \in \Sigma$ is any tensor that satisfies an inequality obtained from (6), by replacing Σ_h with Σ, and $\pi_h^1 \boldsymbol{\tau}$ is the L^2-projection of $\boldsymbol{\tau}$ onto Σ_h^1. In so doing, using the same kind of arguments as in [10], the fact that the determinant of A is bounded below by a constant independent of h for every $R \in \mathcal{T}_h$, implies that both (7) and (8) hold as well with $\boldsymbol{\tau}_h = \pi_h^1 \boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}_h$, where $\tilde{\boldsymbol{\tau}}_h|_R$ is the solution of (9), $\forall R \in \mathcal{T}_h$.

Finally, according to Proposition 1 and taking into account that the best approximations of $\vec{u}$, p and $\boldsymbol{\sigma}$ in V_h, Q_h and Σ_h are bounded above by $\mathcal{O}(h^2)$ terms if $\vec{u} \in (H^3(\Omega))^2$ and $p \in H^2(\Omega)$, recalling (5) we have :

THEOREM 1 : *If* $\vec{u} \in (H^3(\Omega))^2$, $p \in H^2(\Omega)$, $\Sigma_h = \tilde{\Sigma}_h \oplus \Sigma_h^1$ *and* $\{\mathcal{T}_h\}_h$ *is uniformly regular in the usual sense, then* $\vec{u}_h$, p_h, $\boldsymbol{\sigma}_h$ *converge to* $\vec{u}$, p, $\boldsymbol{\sigma}$ *respectively in the corresponding spaces as h goes to zero, at the same rate as a term* Ch^2 *converges to zero.*

4. A VARIANT FOR THE CASE OF A CONTINUOUS PRESSURE

In this Section we consider an alternative choice of extra stress interpolation similar to the one associated with space $\hat{\Sigma}$. Our aim is to define a three-field mixed element based on biquadratic velocity fields and pressure interpolations other than discontinuous piecewise linear. In particular this is the case of the classical space of continuous piecewise isoparametric bilinear functions, which together with the above mentioned velocity space, satisfies the inf-sup condition (4) according to [4].

Let then $\hat{\Sigma}'$ be the space spanned by the ordered basis $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^{15}$ where :

$$\hat{\boldsymbol{\xi}}_1 = \begin{pmatrix} \hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_3 = \begin{pmatrix} 0 & \hat{\varphi} \\ \hat{\varphi} & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\xi}}_4 = \begin{pmatrix} \hat{x}_1 \hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_5 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_1 \hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_6 = \begin{pmatrix} 0 & \hat{x}_1 \hat{\varphi} \\ \hat{x}_1 \hat{\varphi} & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\xi}}_7 = \begin{pmatrix} \hat{x}_2 \hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_8 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_2 \hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_9 = \begin{pmatrix} 0 & \hat{x}_2 \hat{\varphi} \\ \hat{x}_2 \hat{\varphi} & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\xi}}_{10} = \begin{pmatrix} \hat{x}_1 \hat{x}_2 \hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_{11} = \begin{pmatrix} (\hat{x}_2^2 - \hat{x}_1^2)\, \hat{\varphi} & 0 \\ 0 & 0 \end{pmatrix};$$

$$\hat{\boldsymbol{\xi}}_{12} = \begin{pmatrix} 0 & \hat{x}_1^2 \hat{x}_2 \hat{\varphi} \\ \hat{x}_1^2 \hat{x}_2 \hat{\varphi} & 0 \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_{13} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_1 \hat{x}_2 \hat{\varphi} \end{pmatrix};$$

$$\hat{\boldsymbol{\xi}}_{14} = \begin{pmatrix} 0 & 0 \\ 0 & (\hat{x}_1^2 - \hat{x}_2^2)\, \hat{\varphi} \end{pmatrix}; \quad \hat{\boldsymbol{\xi}}_{15} = \begin{pmatrix} 0 & \hat{x}_1 \hat{x}_2^2 \hat{\varphi} \\ \hat{x}_1 \hat{x}_2^2 \hat{\varphi} & 0 \end{pmatrix}.$$

We will let space $\hat{\Sigma}'$ play here the same role as space $\hat{\Sigma}$ played in the previous Section. In so doing we will prove that a condition analogous to (6) is satisfied for the modified finite element method associated with $\hat{\Sigma}'$.

Such a condition is

$$\exists \beta_3 > 0 \quad \text{such that} \quad \inf_{\vec{v} \in V_h} \sup_{\boldsymbol{\tau} \in \Sigma_h'} \frac{\int_\Omega \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\vec{v})\, dx}{|\vec{v}|_{1,\Omega} \, \|\boldsymbol{\tau}\|_{0,\Omega}} \geq \beta_3 , \tag{12}$$

and that in addition β_3 is independent of h, where Σ'_h is the space given by

$$\Sigma'_h = \Sigma_h^1 \oplus \tilde{\Sigma}'_h$$

with

$$\tilde{\Sigma}'_h = \left\{ \boldsymbol{\tau} \mid \boldsymbol{\tau}|_R = \hat{\boldsymbol{\tau}} \circ \mathscr{F}_R^{-1},\ \hat{\boldsymbol{\tau}} \in \hat{\Sigma}',\ \forall R \in \mathcal{C}_h \right\} .$$

The main difference between condition (12) and condition (6) is the fact that in the former the discrete divergence free condition for fields $\vec{v}$ is not used. For this reason (12) is more restrictive and therefore space Σ'_h should be richer than Σ_h in principle. This explains the choice of fifteen tensors to define $\hat{\Sigma}'$ instead of twelve.

Now we have the following results :

PROPOSITION 2 : *Let $\hat{E}'_R$, $R \in \mathcal{C}_h$ be the space of second order tensors defined by $\varepsilon(\vec{v})|_R \circ \mathscr{F}_R$, $\vec{v} \in V_h$, $\forall R \in \mathcal{C}_h$. $\hat{E}'_R$ is a subspace of the space spanned by set $\{\hat{\zeta}_i\}_{i=1}^{15}$ of linearly independent tensors given by*

$$\hat{\zeta}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \qquad \hat{\zeta}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ;$$

$$\hat{\zeta}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \hat{\zeta}_4 = \begin{pmatrix} \hat{x}_1 & 0 \\ 0 & 0 \end{pmatrix} ;$$

$$\hat{\zeta}_5 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_1 \end{pmatrix} ; \qquad \hat{\zeta}_6 = \begin{pmatrix} 0 & \hat{x}_1 \\ \hat{x}_1 & 0 \end{pmatrix} ;$$

$$\hat{\zeta}_7 = \begin{pmatrix} \hat{x}_2 & 0 \\ 0 & 0 \end{pmatrix} ; \qquad \hat{\zeta}_8 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{x}_2 \end{pmatrix} ;$$

$$\hat{\zeta}_9 = \begin{pmatrix} 0 & \hat{x}_2 \\ \hat{x}_2 & 0 \end{pmatrix} ; \qquad \hat{\zeta}_{10} = \begin{pmatrix} 2\,\hat{x}_1\,\hat{x}_2 & r\hat{x}_1^2/2 \\ r x_1^2/2 & 0 \end{pmatrix} ;$$

$$\hat{\zeta}_{11} = \begin{pmatrix} \hat{x}_2^2 & r\hat{x}_1\,\hat{x}_2 \\ r\hat{x}_1\,\hat{x}_2 & 0 \end{pmatrix} ; \quad \hat{\zeta}_{12} = \begin{pmatrix} 2\,\hat{x}_1\,\hat{x}_2^2 & r\hat{x}_1^2\,\hat{x}_2 \\ r\hat{x}_1^2\,\hat{x}_2 & 0 \end{pmatrix} ;$$

$$\hat{\zeta}_{13} = \begin{pmatrix} 0 & \hat{x}_2^2/2 \\ \hat{x}_2^2/2 & 2\,r\hat{x}_1\,\hat{x}_2 \end{pmatrix} ; \quad \hat{\zeta}_{14} = \begin{pmatrix} 0 & \hat{x}_1\,\hat{x}_2 \\ \hat{x}_1\,\hat{x}_2 & r\hat{x}_1^2 \end{pmatrix} ;$$

$$\hat{\zeta}_{15} = \begin{pmatrix} 0 & \hat{x}_1\,\hat{x}_2^2 \\ \hat{x}_1\,\hat{x}_2^2 & 2\,r\hat{x}_1^2\,\hat{x}_2 \end{pmatrix} .$$

Proof : The proposition is established by means of a straightforward calculation given in [5]. ■

PROPOSITION 3 : *Given a tensor* $\hat{\boldsymbol{\tau}}$ *whose components belong to* $L^2(\hat{R})$ *there exists a unique* $\hat{\boldsymbol{\sigma}}_h \in \hat{\Sigma}'$ *of the form*

$$\hat{\boldsymbol{\sigma}}_h = \sum_{j=1}^{15} c_j \, \hat{\boldsymbol{\xi}}_j ,$$

where the c_j*'s are constants for which*

$$\int_{\hat{R}} \hat{\boldsymbol{\sigma}}_h : \hat{\phi}_h \, dx = \int_{\hat{R}} \hat{\boldsymbol{\tau}} : \hat{\phi}_h \, dx , \quad \forall \hat{\phi}_h \in \hat{E}'_{\hat{R}} . \tag{13}$$

Moreover, $\hat{\boldsymbol{\sigma}}_h$ *satisfies*

$$\|\hat{\boldsymbol{\sigma}}_h\|_{0,\hat{R}} \leq \hat{C} \, \|\hat{\boldsymbol{\tau}}\|_{0,\hat{R}} ,$$

where $\hat{C}$ *is a constant independent of* $\hat{\boldsymbol{\tau}}$ and $\| \, . \, \|_{0,\hat{R}}$ *denotes the standard norm of* $L^2(\hat{R})$.

Proof : Given $R \in \mathcal{C}_h$ and $\hat{\boldsymbol{\tau}}$ with $\hat{\tau}_{ij} \in L^2(\hat{R})$ let $\vec{s}$ be the vector of $\mathcal{R}^{15}$ defined by

$$s_i = \int_{\hat{R}} \hat{\boldsymbol{\tau}} : \boldsymbol{\zeta}_i \, dx , \quad 1 \leq i \leq 15$$

and A' be the 15×15 matrix a'_{ij} where

$$a'_{ij} = \int_{\hat{R}} \boldsymbol{\xi}_j : \boldsymbol{\zeta}_i \, dx , \quad 1 \leq i, j \leq 15 .$$

The system of equations defined by (13) may be written as

$$A' \, \vec{s}_h = \vec{s}$$

where $\vec{s}_h$ is the vector whose components are the c_j's. After straightforward calculations the only non zero elements of A' are found to be,

$$a'_{1,1} = \frac{4}{9} ;$$

$$a'_{2,2} = \frac{4}{9} ;$$

$$a'_{3,3} = \frac{8}{9} ;$$

$$a'_{4,4} = \frac{4}{45} ;$$

$$a'_{5,5} = \frac{4}{45} ;$$

$$a'_{6,6} = \frac{8}{45}\,; \quad a'_{6,15} = \frac{8}{225}\,;$$

$$a'_{7,7} = \frac{4}{45}\,;$$

$$a'_{8,8} = \frac{4}{45}\,;$$

$$a'_{9,9} = \frac{8}{45}\,; \quad a'_{9,12} = \frac{8}{225}\,;$$

$$a'_{10,3} = \frac{4r}{45}\,; \quad a'_{10,10} = \frac{8}{225}\,;$$

$$a'_{11,1} = \frac{4}{45}\,; \quad a'_{11,11} = \frac{32}{1\,575}\,;$$

$$a'_{12,4} = \frac{8}{225}\,; \quad a'_{12,9} = \frac{8r}{225}\,; \quad a'_{12,12} = \frac{8r}{525}\,;$$

$$a'_{13,3} = \frac{4}{45}\,; \quad a'_{13,13} = \frac{8r}{225}\,;$$

$$a'_{14,2} = \frac{4r}{45}\,; \quad a'_{14,14} = \frac{32r}{1\,575}\,;$$

$$a'_{15,6} = \frac{8}{225}\,; \quad a'_{15,8} = \frac{8r}{225}\,; \quad a'_{15,15} = \frac{4}{525}\,.$$

Next by performing a couple of simple Gaussian eliminations we obtain :

$$\det A' = \frac{2^{45}}{3^{30} \times 5^{20} \times 7^4} r^3\,.$$

Thus A' is an invertible matrix for every $r \neq 0$ and the result follows from the fact that det A' is bounded below by a constant independent of R if $\{\mathcal{C}_h\}_h$ is a regular family of partitions of Ω (see [10]). ■

PROPOSITION 4 : *The result of Proposition 1 holds if* Σ_h *is replaced with* Σ'_h *and* $\tilde{U}_h$ *is replaced with* U_h.

Proof : This result follows directly from Proposition 3. ■

Finally, let Q'_h be any space, that together with space V_h associated with piecewise biquadratic functions, satisfies the compatibility condition

$$\exists \beta_4 > 0 \quad \text{such that} \quad \inf_{\substack{q \in Q'_h \\ q \neq 0}} \sup_{\substack{\vec{v} \in V_h \\ \vec{v} \neq 0}} \frac{\int_\Omega q \operatorname{div} \vec{v}\, dx}{\|q\|_{0,\Omega}\, |\vec{v}|_{1,\Omega}} \geq \beta_4\,,$$

where β_4 is a constant independent of h. Then, we have :

THEOREM 2 : *Let* $(\vec{u}_h, p_h, \sigma'_h) \in V_h \times Q'_h \times \Sigma'_h$ *be the solution of a problem analogous to* (3) *obtained by replacing* Q_h *with* Q'_h *and* Σ_h *with* Σ'_h. *If* $\{\mathcal{T}_h\}_h$ *is a regular family of partitions of* Ω *into rectangles, there exists a constant* C' *independent of* h *such that*

$$\| (\vec{u}, p, \sigma) - (\vec{u}_h, p_h, \sigma'_h) \|_Z \leq C' h^2 (\| \vec{u} \|^2_{3, \Omega} + \| p \|^2_{2, \Omega})^{1/2},$$

assuming that $\vec{u} \in (H^3(\Omega))^2$ *and* $p \in H^2(\Omega)$, *where* $\| \cdot \|_{m, \Omega}$ *denotes the standard norms of Sobolev space* $H^m(\Omega)$ *and*

$$\| (\vec{u}, p, \sigma) \|^2_Z = | \vec{u} |^2_{1, \Omega} + \| p \|^2_{0, \Omega} + \| \sigma \|^2_{0, \Omega} .$$

Proof : As it has been established in previous work (see e.g. [10] and [5] and references therein) such convergence result is a direct consequence of the fact that condition (12) holds. This in turn is immediately derived from Proposition 3. ■

5. NUMERICAL EXPERIMENTS

The aim of this Section is to verify experimentally the convergence results derived in the two previous Sections, and to check the performance of the corresponding methods when solving the classical stick-slip test problem described in terms of a three-field Stokes system.

We showed that whenever the hypotheses of Theorems 1 and 2 are satisfied, there exists a constant C independent of h and depending on the solution $(\vec{u}, p, \sigma)$ of system (1) such that

$$\| (\vec{u}, p, \sigma) - (\vec{u}_h, p_h, \sigma_h) \|_Z \leq C h^2$$

where $(\vec{u}_h, \sigma_h, p_h) \in V_h \times \bar{\Sigma}_h \times Q_h$ is the solution of the associated discrete problem where $\bar{\Sigma}_h$ is either Σ_h or Σ'_h.

In order to verify experimentally the above convergence result, we will make use of the classical $\log \| (\vec{u}, p, \sigma) - (\vec{u}_h, p_h, \sigma_h) \| \times \log h$ plot obtained from the solution of a test problem proposed in [6].

This test problem governed by the Stokes system, deals with the stationary flow of a fluid confined in the square cavity Ω with unit edge length, and subjected to the action of body forces $\vec{f} = (f_x, f_y)$ where

$$\begin{cases} f_x(x, y) = g(x, y) + y - \dfrac{1}{2}, \\ f_y(x, y) = -g(y, x) + x - \dfrac{1}{2}, \end{cases}$$

with

$$g(x, y) = \\ = 256[6x^2(x-1)^2(2y-1) + y(y-1)(2y-1)(12x^2-12x+2)].$$

The solution of this problem, which satisfies homogeneous Dirichlet boundary conditions, is given by

$$\begin{cases} u_x(x, y) = -256x^2(x-1)^2 y(y-1)(2y-1), \\ u_y(x, y) = -u_x(y, x); \end{cases}$$

$$p(x, y) = \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right),$$

$$\sigma = 2\eta\varepsilon(\vec{u}).$$

We solved this test problem with the element considered in Section 3 only. All the numerical quadrature were performed with the ninth order Gauss-Legendre formula, while the resulting linear system was solved by the frontal method without pivoting.

In figure 1 we display the results obtained with four uniform meshes of Ω, namely those obtained by subdividing the edges into 4, 5, 6 and 7 equal segments, respectively. Quadratic convergence is clearly observed as expected.

Let us now switch to the stick-slip problem. This test case has been considered by many authors in order to check the performance of mixed

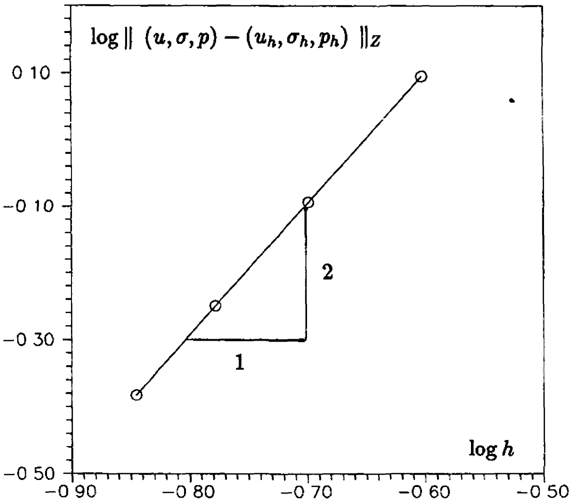

Figure 1. — Log-Log plot of the error for a test problem proposed in [6].

methods for the three-field Stokes system. This is because, whenever an element does not satisfy the extra stress-velocity compatibility condition, severe non physical oscillations of extra stress components appear in the zone where high stress gradients occurs. As it should be noted, in the case of viscoelastic flow those spurious oscillations are amplified, which may spoil completely the accuracy of a finite element simulation of the flow (see e.g. [9]).

Let us recall the problem under consideration referring to figure 2, where the flow domain Ω is illustrated, together with the boundary conditions. Notice that in this problem the most interesting zone is the neighborhood of point (L_1, D) since high stress gradients develop there.

We have further assumed that L_2 is sufficiently large so that we may consider that $u_y = 0$ at $x = L_2$, and that the flow is symmetric with respect to the axis Ox. Moreover we take $\vec{f} = 0$.

In all the cases we used space V_h defined in Section 3 with the usual modification, in order to accommodate non homogeneous velocity boundary conditions. As pointed out before only discontinuous linear pressure fields were used in these computations. On the other hand besides the two extra-stress finite element spaces Σ_h and Σ_h' considered in Sections 3 and 4 respectively, we used the space $\Xi_h = (\Sigma_h^0)^{2 \times 2}_{\text{sym}}$, where

$$\Sigma_h^0 = \left\{ t_h \in C^0(\bar{\Omega}) \,|\, t_h|_R = \hat{t} \circ \mathscr{F}_R^{-1}, \quad \hat{t} \in \hat{Q}_2, \quad \forall R \in \mathcal{T}_h \right\}$$

and $\hat{Q}_2$ is the space of biquadratic functions defined in the reference rectangle $\hat{R}$.

Notice that, though capable of producing reasonable results, this element has not been proved to satisfy the required compatibility condition involving velocity and extra stresses.

We denote in this Section the mixed method associated with extra stress spaces Σ_h, Σ_h' and Ξ_h and the fixed velocity-pressure space specified above by T_{12}, T_{15} and Q_2 respectively.

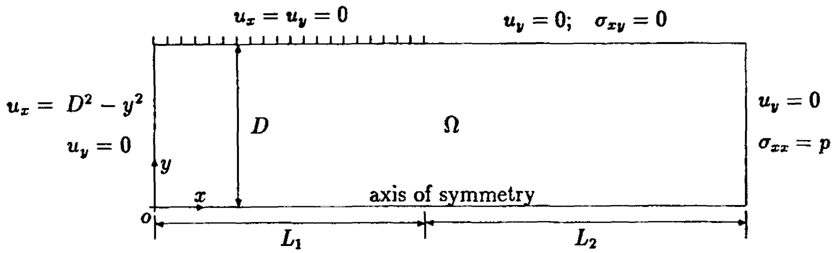

Figure 2. — Geometry and boundary conditions for the stick-slip problem.

The computer results given below were obtained by using the following numerical values of the geometry and equation parameter :

$$\eta = 1.00 \,, \quad D = 1.00 \,, \quad L_1 = 20.00 \,, \quad L_2 = 50.00 \,.$$

Marchal and Crochet gave in [9] results for the Newtonian stick-slip problem in a geometry identical to the one used in this work. They used two different methods : the so-called Marchal and Crochet method denoted by MC_1, and a method identical to Q_2, except for the pressure space that consists of isoparametric continuous bilinear functions. We denote the latter mixed method by MC_2, which is not known to satisfy the compatibility condition of type (6) or (12) either.

In [9] the domain for the stick-slip problem was discretized into 160 rectangles. We did the same but since in that work the value of η is not given and the quadrature formula that has been used is not specified, we could only perform a qualitative comparison with their results.

In figure 3 we illustrate a zoom on the mesh in the neighborhood of point (L_1, D).

The corresponding number of degrees of freedom are as follows : 1 430 for the velocity field, 480 for the pressure, 2 145, 2 514 and 2 994 for the extra stresses in the cases of elements Q_2, T_{12} and T_{15} respectively.

We display in figure 4 the graphs $u_x \times x$ along the line $y = D$ in the neighborhood of point (L_1, D) for elements Q_2, T_{12} and T_{15} as indicated.

Notice that the three curves look basically the same. Moreover by comparing the result obtained with Q_2 to the one presented in [9] for MC_2, one observes that our velocity component u_x does not show the oscillations found by those authors. This seems to indicate that the choice of discontinuous piecewise linear pressures stabilize a little the biquadratic element for extra stresses, which is quite plausible.

Now we illustrate the behavior of the extra stress components σ_{xx}, σ_{xy} and σ_{yy} along the same line in figures 5, 6 and 7, respectively, represented by elements Q_2, T_{12} and T_{15}. Notice that results given by Q_2 show several non physical oscillations near point (L_1, D). This had also been observed for MC_2 [9]. On the other hand when we compare the curves $\sigma_{xx} \times x$ for T_{12} and T_{15} with those given in [9] for MC_1, we observe that they are qualitatively identical. However the former graphs have smaller oscillations around $x = 19.7$ than those observed in their results, although they

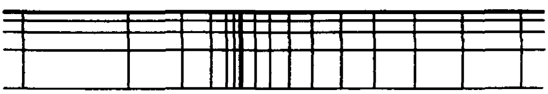

Figure 3. — Zoom on the mesh around point (L_1, D).

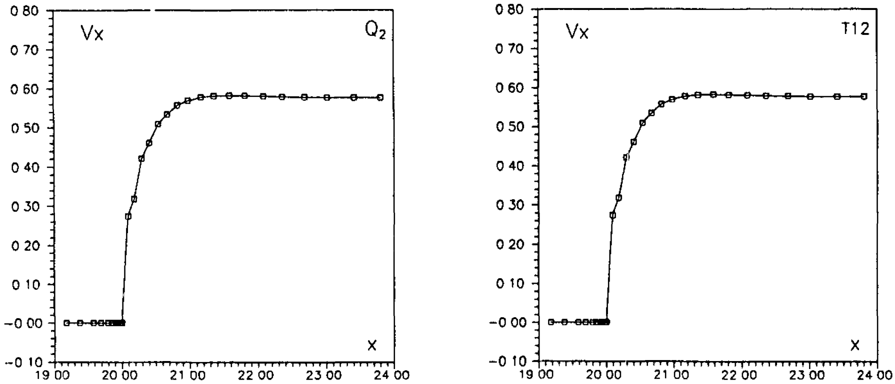

Figure 4. — Curves $u_x \times x$ for the stick-slip problem along line $y = D$ near point (L_1, D) c

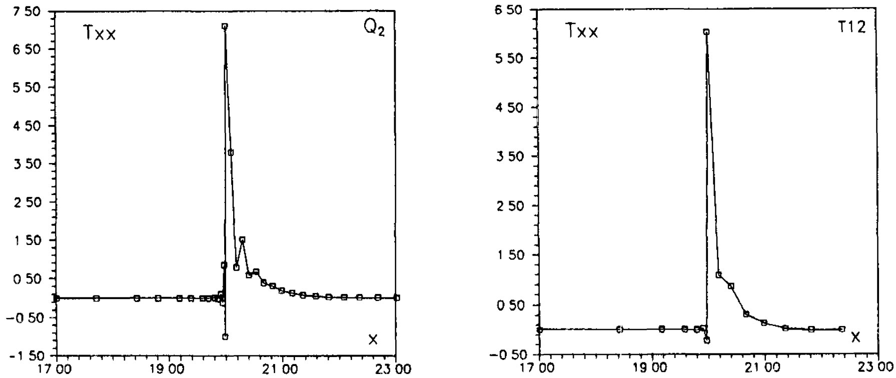

Figure 5. — Curves $\sigma_{xx} \times x$ for the stick-slip problem along line $y = D$ near point (L_1, D) c

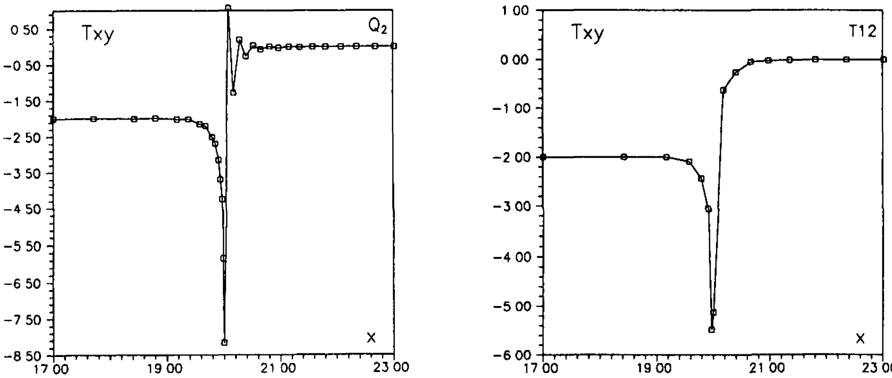

Figure 6. — Curves $\sigma_{xy} \times x$ for the stick-slip problem along line $y = D$ near point (L_1, D)

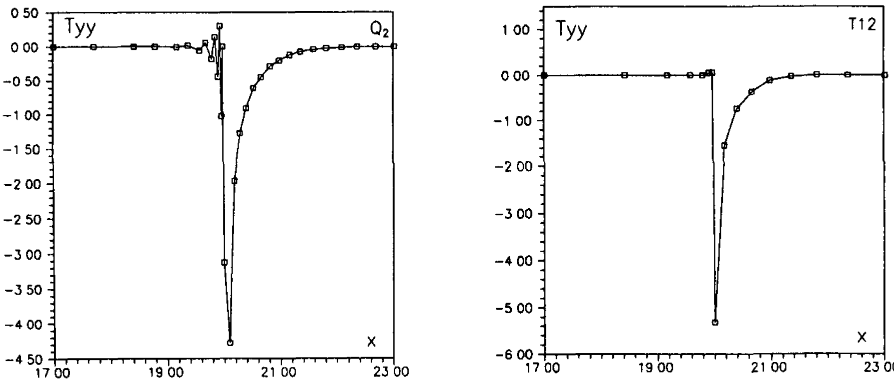

Figure 7. — Curves $\sigma_{yy} \times x$ for the stick-slip problem along line $y = D$ near point (L_1, D)

are practically negligible at the scale used in both works. According to a conjecture in [9] these small oscillations would be due to the relative coarseness of the mesh.

Finally let us add that we rerun the code for a mesh consisting of 315 rectangles. We observed that an attenuation of the oscillations of the extra stresses represented by elements T_{12} and T_{15} effectively occurred.

6. CONCLUDING REMARKS

1. The next point related to the new finite elements for the three-field Stokes system, considered in this work that the authors intend to exploit computationally, is their use in connection with arbitrary quadrilateral meshes. The theoretical results given for the T_{15} element by the first author in [11], indicate that one might expect comparable performance in this case too. Extension of this analysis to the T_{12} element is underway.

2. Another important topic that should be considered soon is the extension of the new elements to the axisymmetric case.

3. Finally let us point out that although we did not present any comparative data concerning CPU time for the different second order methods that were considered in this work, the following a priori argument should be stressed. For a given mesh $\mathcal{T}_h$ the number of degrees of freedom of the space Σ_h for the T_{12} element is roughly 5/16 of the total number of degrees of freedom of the space of piecewise bilinear functions defined on a submesh four times finer than $\mathcal{T}_h$, used to define the extra stresses for the MC_1 element. This ratio increases by 1/16 for the T_{15} element. Moreover, in the case of the T_{3n} element ($n = 4$ or 5) roughly n tensor degrees of freedom out of $n + 1$ can be easily eliminated from the corresponding linear system at a low cost, since they are associated with only one quadrilateral.

REFERENCES

[1] R. A. ADAMS, Sobolev Spaces, *Academic Press*, N.Y., 1968.

[2] R. B. BIRD, R. C. ARMSTRONG and O. HASSAGER, Dynamics of polymeric liquids, Vol. 1, Fluid Mechanics, Second Edition, *John Wiley & Sons*, N.Y., 1987.

[3] J. BARANGER and D. SANDRI, Approximation par élément finis d'écoulements de fluides viscoélastiques. Existence de solutions approchées et majorations d'erreur. I. Contraintes continues, *C.R. Acad. Sci. Paris*, Tome 312, Série I (1991), 541-544.

[4] M. BERCOVIER and O. PIRONNEAU, Error estimates for the finite element method solution of the Stokes problem in the primitive variables, *Numer. Math.*, 33 (1979), 211-224.

[5] J. H. CARNEIRO DE ARAÚJO, Métodos de Elementos Finitos Otimizados para o Sistema de Stokes Associado a Problemas de Viscoelasticidade, *Doctoral dissertation, Pontificia Universidade Católica do Rio de Janeiro,* 1991.

[6] M. S. ENGELMAN, R. L. SANI, P. M. GRESHO and M. BERCOVIER, Consistent *vs*. reduced integration penalty methods for incompressible media using several old and new elements, *Int. J. Num. Methods in Fluids,* 2 (1982), 25-42.

[7] M. FORTIN and A. FORTIN, A new approach for the FEM simulation of viscoelastic flows, *J. Non-Newtonian Fluid Mech.*, 32 (1989) 295-310.

[8] M. FORTIN and R. PIERRE, On the convergence of the mixed method of Crochet and Marchal for viscoelastic flows, *Comput. Methods Appl. Mech. Engrg,* 73 (1989) 341-350.

[9] J. M. MARCHAL and M. CROCHET, A new mixed finite element for calculating viscoelastic flow, *J. Non-Newtonian Fluid Mech.*, 26 (1987) 77-117.

[10] V. RUAS, An optimal three-field finite element approximation of the Stokes system with continuous extra stresses (to appear).

[11] V. RUAS, A convergent three-field quadrilateral finite element method for simulating viscoelastic flow on irregular meshes. Revue Européenne des Eléments Finis, Vol. 1, 4 (1992), 391-406.

[12] V. RUAS and J. H. CARNEIRO DE ARAÚJO, Un método mejorado de segundo orden para la simulación de flujo viscoelástico con elementos finitos quadrilaterales, Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería, Vol. 8, 1 (1992), 77-85.