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F. A. MILNER
M. SURI

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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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MIXED FINITE ELEMENT METHODS FOR QUASILINEAR SECOND ORDER ELLIPTIC PROBLEMS : THE p-VERSION (*)

by F. A. MILNER (1) and M. SURI (2)

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Abstract — *The p-version of the finite element method is analyzed for quasilinear second order elliptic problems in mixed weak form Approximation properties of the Raviart-Thomas projection are demonstrated and L^2-error bounds for the three relevant variables in the mixed method are derived*

Résumé — *Nous analysons la version-p de la méthode d'éléments finis mixtes pour des problèmes quasilinéaires elliptiques du second ordre en forme faible mixte Nous démontrons des propriétés d'approximation de la projection de Raviart-Thomas et on dérive des bornes de l'erreur dans $L^2(\Omega)$ pour les trois variables d'intérêt dans la méthode mixte*

I. INTRODUCTION

We consider here the numerical solution of the following boundary-value problem :

$$\begin{cases} \mathscr{D}(u) = -\underset{\sim}{\nabla} \cdot (a(u)\underset{\sim}{\nabla} u + \underset{\sim}{b}(u)) + c(u) = 0 & \text{in} \quad \Omega , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad u = -g & \text{on} \quad \partial\Omega , \end{cases} \tag{1.1}$$

where Ω is a convex polygon with boundary $\partial\Omega$, $\underset{\sim}{\nabla} w$ denotes the gradient of the scalar function w and $\underset{\sim}{\nabla} \cdot \underset{\sim}{v}$ and div $\underset{\sim}{v}$ denote the divergence of the vector

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(1) Department of Mathematics, Purdue University, West Lafayette, IN 47907, and Dipartimento di Matematica, IIa Università di Roma, 00133 Rome, Italy The first author was supported in part by National Science Foundation Grant DMS-8813258

(2) Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228 The second author was supported in part by Air Force Office of Scientific Research Grant AFOSR-89-0252

function $\underset{\sim}{v}$. We shall assume that for $r \geq 2$ and for each $g \in H^{r-1/2}(\partial\Omega)$ there exists a unique isolated solution $u \in H^r(\Omega)$ of (1.1) (that is, a solution not situated at a bifurcation point). Note that Sobolev's embedding theorem implies then that $u \in W^{r-1-\varepsilon,\infty}(\Omega)$, $\varepsilon > 0$, $\varepsilon \ll 1$, which will be needed throughout the paper.

We shall also assume that the coefficients $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $\underset{\sim}{b} : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ and $c : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with bounded derivatives through second order, and that $a(\underset{\sim}{x}, q) \geq a_1 > 0$. The variable $\underset{\sim}{x}$ will be omitted as explicit argument of all functions, except when necessary to avoid ambiguity.

For $1 \leq s \leq \infty$ and k any nonnegative integer, we let

$$W^{k,s}(\Omega) = \{f \in L^s(\Omega) : D^\alpha f \in L^s(\Omega),\ |\alpha| \leq k\}$$

denote the Sobolev space endowed with its standard norm

$$\|f\|_{k,s,\Omega} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^s_{L^s(\Omega)} \right)^{1/s}, \quad s \leq \infty,$$

$$\|f\|_{k,\infty,\Omega} = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

The subscript Ω in the norms will be omitted. Let $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\|\cdot\|_k = \|\ \|_{k,2}$. In particular, the notation $\|\cdot\|_0$ will mean $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$ For $0 \leq r < \infty$ let $W^{r,s}(\Omega)$, $W^{r,s}(\partial\Omega)$, $H^r(\Omega)$, and $H^r(\partial\Omega)$ denote the fractional order Sobolev spaces with norms $\|\cdot\|_{r,s,\Omega}$, $\|\cdot\|_{r,s,\partial\Omega}$, $\|\cdot\|_{r,\Omega}$ and $\|\cdot\|_{r,\partial\Omega}$, respectively, defined by interpolation [7].

We shall denote by $(\cdot,\cdot)$ the Hilbert inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$ and by $\langle \cdot,\cdot \rangle$ the L^2-inner product on the boundary of Ω. The same notation will be used to indicate the dualities between $W^{r,s}(\Omega)$ and $W^{r,s}(\Omega)'$ and $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$, respectively. Throughout the paper, C, Q, and K will denote generic positive constants which need not have the same value in all their occurrences.

Let

$$\underset{\sim}{V} = \underset{\sim}{H}(\mathrm{div}\,; \Omega) = \left\{\underset{\sim}{v} \in L^2(\Omega)^2 : \mathrm{div}\,\underset{\sim}{v} \in L^2(\Omega)\right\},$$

normed by

$$\|\underset{\sim}{v}\|^2_{H(\mathrm{div},\Omega)} = \|\underset{\sim}{v}\|^2_0 + \|\mathrm{div}\,\underset{\sim}{v}\|^2_0,$$

and

$$W = L^2(\Omega).$$

The mixed finite element method we shall consider seeks simultaneous approximations of the solution of (1.1), u, and of the flux

$$\underset{\sim}{z} = - a(u) \underset{\sim}{\nabla} u - \underset{\sim}{b}(u) . \tag{1.2}$$

The mixed weak formulation of (1.1) consists of finding $(\underset{\sim}{z}, u) \in \underset{\sim}{V} \times W$ such that

$$\begin{cases} (\alpha(u)\, \underset{\sim}{z}, \underset{\sim}{v}) - (u, \operatorname{div} \underset{\sim}{v}) + (\underset{\sim}{\beta}(u), \underset{\sim}{v}) = \langle g, \underset{\sim}{v} \cdot \underset{\sim}{\nu} \rangle , & \underset{\sim}{v} \in \underset{\sim}{V} , \\ (\operatorname{div} \underset{\sim}{z}, w) + (c(u), w) = 0 , & w \in W , \end{cases} \tag{1.3}$$

where we have set

$$\alpha(u) = 1/a(u) , \quad \underset{\sim}{\beta}(u) = \alpha(u)\, \underset{\sim}{b}(u) , \tag{1.4}$$

and $\underset{\sim}{\nu}$ is the outward unit normal vector on $\partial\Omega$. Our mixed finite element method is a discrete form of (1.3).

Let $\mathscr{T}$ be a decomposition of Ω by parallelograms which will be the « elements » E and let $\mathscr{P}_{p,q}(E) = \{$polynomials $f(x, y)$ on E, of degree $\leq p$ in x and degree $\leq q$ in $y\}$, $\mathscr{Q}_p(E) = \{$polynomials of degree $\leq p$ on $E\}$; next define, for each element E,

$$\underset{\sim}{V}^p(E) = \mathscr{P}_{p+1,p}(E) \times \mathscr{P}_{p,p+1}(E) ,$$

and let

$$\underset{\sim}{V}^p \times W^p \subset \underset{\sim}{V} \times W$$

be the Raviart-Thomas-Nedelec space of index $p \geq 0$ associated with this decomposition [3, 5], given by

$$\begin{cases} \underset{\sim}{V}^p = \left(\prod_{E \in \mathscr{T}} \underset{\sim}{V}^p(E) \right) \cap \{ \underset{\sim}{f} : \Omega^2 \to \mathbb{R} \,|\, \underset{\sim}{f} \cdot \nu_E \\ \qquad = \underset{\sim}{f} \cdot \nu_{E'} \text{ on } E \cap E', E, E' \in \mathscr{T} \} \\ W^p = \prod_{E \in \mathscr{T}} \mathscr{Q}_p(E) , \end{cases}$$

where $\underset{\sim}{\nu}_E$ denotes the outward unit normal vector along ∂E, $E \in \mathscr{T}$. It is known [3, 5] that div $\underset{\sim}{V}^p \subset W^p$, a property we shall exploit later.

We seek $(\underset{\sim}{z}^p, u^p) \in \underset{\sim}{V}^p \times W^p$ so that

$$\begin{cases} (\alpha(u^p)\, \underset{\sim}{z}^p, \underset{\sim}{v}) - (u^p, \operatorname{div} \underset{\sim}{v}) + (\underset{\sim}{\beta}(u^p), \underset{\sim}{v}) = \langle g, \underset{\sim}{v} \cdot \underset{\sim}{\nu} \rangle , & \underset{\sim}{v} \in \underset{\sim}{V}^p , \\ (\operatorname{div} \underset{\sim}{z}^p, w) + (c(u^p), w) = 0 , & w \in W^p . \end{cases} \tag{1.5}$$

Equations (1.5) define the p-version of the mixed finite element approximation for (1.3). This version is based on using a fixed mesh and increasing the degree of the finite elements (as opposed to the h-version that keeps the degree fixed and refines the mesh). The p-version has been analyzed for the linearized version of (1.1) in terms of the standard variational form in [1] and in terms of the mixed variational form in [6]. In this paper, we extend the results obtained in [6] for the linear problem to the quasilinear case. We also obtain an improved version of lemma 3.1 of [6] by reducing the regularity assumed there. We restrict our attention to the mixed method, the corresponding generalization for the standard finite element method is more straightforward.

Milner [3] described the h-version of this method for the same problem, demonstrated the unique solvability (for small h) of the nonlinear algebraic system (1.5) and derived error estimates in $L^s(\Omega)$, $2 \leqslant s \leqslant +\infty$, for the error in u, and in $\underset{\sim}{H}(\text{div}\,;\Omega)$ for the error in $\underset{\sim}{z}$. The assumption there was that the solution of (1.1) was in the space $H^{2+\varepsilon}(\Omega)$. In contrast, for the present paper we shall need an extra half derivative, that is, $u \in H^{5/2+\varepsilon}(\Omega)$.

We shall follow very closely the analysis of [3]. In order to do so we shall use the L^2-projection onto W^p, $P^p : L^2 \to W^p$, given by

$$(P^p w - w, \chi) = 0\,, \quad \chi \in W^p\,, \quad w \in W\,, \tag{1.6}$$

for which the following approximation properties follow by repeating the arguments of [4] in two dimensions and using interpolation from the cases $s = 2$ and $s = \infty$:

$$\| P^p w - w \|_{0,s} \leqslant \underset{\sim}{Q} p^{-m+3/2-3/s} \| w \|_m\,, \quad s \geqslant 2\,, \quad 3/2 - 3/s \leqslant m\,, \tag{1.7}$$

if $w \in H^m(\Omega)$. We shall also use the Raviart-Thomas projection of $\underset{\sim}{V}$ onto $\underset{\sim}{V}^p$, $\pi^p : \underset{\sim}{V} \to \underset{\sim}{V}^p$, [5] for which we shall demonstrate in Section 2 the following approximation property :

$$\| \pi^p \underset{\sim}{v} - \underset{\sim}{v} \|_0 \leqslant \underset{\sim}{Q} p^{1/2-r} \| \underset{\sim}{v} \|_r\,, \quad r > 1/2\,, \quad \underset{\sim}{v} \in H^r(\Omega)^2 \cap \underset{\sim}{V}\,. \tag{1.8}$$

Our proof of (1.8) improves upon the one presented in [6], which imposed extra regularity on $\underset{\sim}{v}$ by requiring that $r > 1$. In contrast, the condition $r > 1/2$ is optimal (see remark 2.1). We also obtain estimates for the approximation properties of π^p in the $W^{0,s}(\Omega)$-norm.

We shall find very useful the following inverse-type inequalities, the two dimensional form of the ones found in [4] :

$$\| \chi \|_{0,s} \leqslant \underset{\sim}{Q} p^{4/r-4/s} \| \chi \|_{0,r}\,, \quad 1 \leqslant r \leqslant s \leqslant \infty\,,$$
$$\chi \in L^s(\Omega) \cap W^p \ (\text{or } \chi \in L^s(\Omega)^2 \cap \underset{\sim}{V}^p)\,. \tag{1.9}$$

The plan of the paper is as follows : in Section 2 we demonstrate (1.8), in Section 3 we prove that, for p sufficiently large, (1.5) is uniquely solvable and its solution $(\underset{\sim}{z}^p, u^p)$ converges to $(\underset{\sim}{z}, u)$ in $\underset{\sim}{V} \cap L^{2+\varepsilon}(\Omega)^2 \times L^{(2+4\varepsilon)/\varepsilon}(\Omega)$ for any fixed ε, $0 \ll \varepsilon \ll 1$, and in Section 4 we establish the rate of convergence of the approximation to the exact solution.

II. THE APPROXIMATION PROPERTIES OF π^p

We recall that $\pi^p \underset{\sim}{v}$ is given locally (on every element E) by the following relations (2.1) and (2.2) (see [5]) :

$$\left\langle [\pi^p \underset{\sim}{v} - \underset{\sim}{v}] \cdot \underset{\sim}{\nu}_E, \varphi \right\rangle_{S_i} = 0 , \quad \varphi \in \mathscr{P}_p , \tag{2.1}$$

where $\langle \cdot , \cdot \rangle_{S_i}$, $1 \leq i \leq 4$, denotes the line integral along each side S_i of the element E and $\mathscr{P}_p$ is the set of all polynomials in one variable of degree less than or equal to p,

$$(\pi^p \underset{\sim}{v} - \underset{\sim}{v}, \underset{\sim}{\psi})_E = 0 , \quad \underset{\sim}{\psi} \in \underset{\sim}{V}^p(E) , \tag{2.2}$$

where $(\cdot , \cdot)_E$ denotes the standard $L^2(E)$-inner product.

Now, let $R = [-1, 1] \times [-1, 1]$ and let $\{P_i\}_{i \geq 0}$ denote the $L^2([-1, 1])$-complete orthogonal Legendre polynomials. For any $\underset{\sim}{v} \in \underset{\sim}{H}(\text{div} ; R)$, let

$$\underset{\sim}{v}(x, y) = \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(x) P_j(y), \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(x) P_j(y) \right] , \tag{2.3}$$

and let

$$\pi^p \underset{\sim}{v}(x, y) = \left[\sum_{i=0}^{p+1} \sum_{j=0}^{p} \tilde{a}_{i,j} P_i(x) P_j(y), \sum_{i=0}^{p} \sum_{j=0}^{p+1} \tilde{b}_{i,j} P_i(x) P_j(y) \right] . \tag{2.4}$$

It follows from (2.2)-(2.4) that

$$\begin{cases} a_{i,j} = \tilde{a}_{i,j} , & 0 \leq i \leq p-1 , \quad 0 \leq j \leq p , \\ b_{kl} = \tilde{b}_{kl} , & 0 \leq k \leq p , \quad 0 \leq l \leq p-1 . \end{cases} \tag{2.5}$$

Next, we see that (2.1), (2.3)-(2.5) imply that

$$\begin{cases} \sum_{i=p}^{p+1} \tilde{a}_{i,j} P_i(\pm 1) = \sum_{i=p}^{\infty} a_{i,j} P_i(\pm 1) , & 0 \leq j \leq p , \\ \sum_{j=p}^{p+1} \tilde{b}_{i,j} P_j(\pm 1) = \sum_{j=p}^{\infty} b_{i,j} P_i(\pm 1) , & 0 \leq i \leq p . \end{cases} \tag{2.6}$$

Since $P_i(-1) = (-1)^i$ and $P_i(1) = 1$, (2.6) implies that

$$\begin{cases} \tilde{a}_{pj} = \sum_{i=0}^{\infty} a_{2i+p,j}\,, & \tilde{a}_{p+1,j} = \sum_{i=0}^{\infty} a_{2i+p+1,j}\,, \quad 0 \leq j \leq p\,, \\ \tilde{b}_{ip} = \sum_{j=0}^{\infty} b_{i,p+2j}\,, & \tilde{b}_{i,p+1} = \sum_{j=0}^{\infty} b_{i,p+1+2j}\,, \quad 0 \leq i \leq p\,. \end{cases} \tag{2.7}$$

PROPOSITION 2.1 : *Let $\underset{\sim}{v} \in \underset{\sim}{V}$ and let $\pi^p \underset{\sim}{v}$ be its Raviart-Thomas projection in $\underset{\sim}{V}^p$ given by* (2.1)-(2.2). *Then, if $\underset{\sim}{v} \in H^r(\Omega)^2$, we have*

$$\| \underset{\sim}{v} - \pi^p \underset{\sim}{v} \|_0 \leq Q p^{1/2 - r} \| \underset{\sim}{v} \|_r\,, \quad r > 1/2\,,$$

where $Q > 0$ is a constant independent of p and $\underset{\sim}{v}$ but depending on r.

Proof : We first assume that $\Omega = R$ and that the decomposition consists of just one element. Then, $\underset{\sim}{v} \in \underset{\sim}{V}$ and $\pi^p \underset{\sim}{v} \in \underset{\sim}{V}^p$ can be given, respectively, by (2.3) and (2.4).

The following relation is a trivial consequence of well known properties of the Legendre polynomials,

$$\begin{aligned} \| \underset{\sim}{v} - \pi^p \underset{\sim}{v} \|_0^2 = & \sum_{i=p}^{p+1} \sum_{j=0}^{p} \frac{4(a_{i,j} - \tilde{a}_{i,j})^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p} \sum_{j=p}^{p+1} \frac{4(b_{i,j} - \tilde{b}_{i,j})^2}{(2i+1)(2j+1)} + \\ & + \sum_{i=0}^{p+1} \sum_{j=p+1}^{\infty} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=p+1}^{\infty} \sum_{j=0}^{p+1} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \\ & + \sum_{i=p+2}^{\infty} \sum_{j=0}^{p} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p} \sum_{j=p+2}^{\infty} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \\ & + \sum_{i=p+2}^{\infty} \sum_{j=p+1}^{\infty} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=p+1}^{\infty} \sum_{j=p+2}^{\infty} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \end{aligned}$$

$= \mathrm{I} + \mathrm{II} + \cdots + \mathrm{VIII}$.

Note that III-VIII can be bounded as follows :

$$\mathrm{III} + \mathrm{V} + \mathrm{VII} \leq Q p^{-2r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{i,j}^2 (1 + i^2 + j^2)^r}{(2i+1)(2j+1)}\,, \quad r \geq 0\,,$$

while

$$\mathrm{IV} + \mathrm{VI} + \mathrm{VIII} \leq Q p^{-2r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{b_{i,j}^2 (1 + i^2 + j^2)^r}{(2i+1)(2j+1)}\,, \quad r \geq 0\,,$$

which implies that (see [6])

$$\mathrm{III} + \cdots + \mathrm{VIII} \leq Q p^{-2r} \| \underset{\sim}{v} \|_r^2\,, \quad r \geq 0\,. \tag{2.9}$$

On the other hand, it follows from (2.7) that

$$\mathrm{I} = \frac{4}{2p+1} \sum_{j=0}^{p} \frac{\left(a_{p,j} - \sum_{i=0}^{\infty} a_{2i+p,j}\right)^2}{2j+1} + \frac{4}{2p+3} \sum_{j=0}^{p} \frac{\left(a_{p+1,j} - \sum_{i=0}^{\infty} a_{2i+p+1,j}\right)^2}{2j+1}$$

$$= \frac{4}{2p+1} \sum_{j=0}^{p} \frac{\left(\sum_{i=1}^{\infty} a_{2i+p,j}\right)^2}{2j+1} + \frac{4}{2p+3} \sum_{j=0}^{p} \frac{\left(\sum_{i=1}^{\infty} a_{2i+p+1,j}\right)^2}{2j+1} \tag{2.10}$$

and

$$\mathrm{II} = \frac{4}{2p+1} \sum_{i=0}^{p} \frac{\left(b_{ip} - \sum_{j=0}^{\infty} b_{i,2j+p}\right)^2}{2i+1}$$

$$+ \frac{4}{2p+3} \sum_{i=0}^{p} \frac{\left(b_{i,p+1} - \sum_{j=0}^{\infty} b_{i,2j+p+1}\right)^2}{2i+1}$$

$$= \frac{4}{2p+1} \sum_{i=0}^{p} \frac{\left(\sum_{j=1}^{\infty} b_{i,2j+p}\right)^2}{2i+1} + \frac{4}{2p+3} \sum_{i=0}^{\infty} \frac{\left(\sum_{j=1}^{\infty} b_{i,2j+p+1}\right)^2}{2i+1}\,. \tag{2.11}$$

Next observe that bounding the series $\sum_{k=p+1}^{\infty} (c+k^2)^{-s}\,(1+2k)$ using the integral method for $\int_p^{\infty} (c+t^2)^{-s}\,(1+2t)\,dt \leqslant \frac{K}{s-1} p^{2-2s}$ $(s>1)$, and using the Cauchy-Schwarz inequality, we see that, for s bounded away from 1,

$$\left(\sum_{i=1}^{\infty} a_{2i+p,j}\right)^2 \leqslant$$

$$\leqslant \sum_{i=1}^{\infty} \frac{a_{2i+p,j}^2}{4i+2p+1}\,[1+(2i+p)^2+j^2]^s \sum_{i=1}^{\infty} [1+(2i+p)^2+j^2]^{-s} \times$$

$$\times\,(1+4i+2p)$$

$$\leqslant \sum_{i=0}^{\infty} \frac{a_{i,j}^2 (1+i^2+j^2)^s}{2i+1} \sum_{i=p+2}^{\infty} (1+i^2+j^2)^{-s}\,(1+2i)$$

$$\leqslant Qp^{2-2s} \sum_{i=0}^{\infty} \frac{a_{i,j}^2 (1+i^2+j^2)^s}{2i+1}\,, \tag{2.12}$$

with exactly the same final bound holding for $\left(\sum_{i=1}^{\infty} a_{2i+p+1,j}\right)^2$. It follows from (2.10) and (2.12) that, for s bounded away from 1,

$$\mathrm{I} \leqslant Qp^{1-2s} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{a_{i,j}^2 (1+i^2+j^2)^s}{(2i+1)(2j+1)}. \tag{2.13}$$

In an entirely analogous way (replacing $a_{i,j}$ by $b_{i,j}$ and reversing the roles of i and j) we deduce from (2.11) that, for s bounded away from 1,

$$\mathrm{II} \leqslant Qp^{1-2s} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{b_{i,j}^2 (1+i^2+j^2)^s}{(2i+1)(2j+1)}. \tag{2.14}$$

Combining (2.13) and (2.14) results, for s bounded away from 1, in

$$\mathrm{I} + \mathrm{II} \leqslant Qp^{1-2s} \|\underset{\sim}{v}\|_s^2. \tag{2.15}$$

Next note that

$$\begin{aligned} &\underset{\sim}{v} \cdot \underset{\sim}{\nu}\big|_{\partial R} = \\ &v_1(\pm 1, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(\pm 1) P_j(y), \quad -1 \leqslant y \leqslant 1, \\ &v_2(x, \pm 1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(y) P_j(\pm 1), \quad -1 \leqslant x \leqslant 1. \end{aligned} \tag{2.16}$$

The trace theorem (Sobolev's embedding theorem) implies that $v_1, v_2 \in L^2(\partial R)$ for $s > 1/2$. Consequently, since $P_i(1) = 1$ and $P_i(-1) = (-1)^i$, we see from (2.16) that

$$\begin{aligned} \|v_1(\pm 1, .)\|_{0,\partial\Omega}^2 &= 2 \sum_{j=0}^{\infty} \frac{\left[\sum_{i=0}^{\infty} (\pm 1)^i a_{i,j}\right]^2}{2j+1} < \infty, \\ \|v_2(., \pm 1)\|_{0,\partial\Omega}^2 &= 2 \sum_{j=0}^{\infty} \frac{\left[\sum_{j=0}^{\infty} (\pm 1)^j b_{i,j}\right]^2}{2i+1} < \infty. \end{aligned} \tag{2.17}$$

Let now $\underset{\sim}{v} \in H^{1/2+\varepsilon}(\Omega)^2$. We shall prove that

$$\|\pi^p \underset{\sim}{v} - \underset{\sim}{v}\|_0 \leqslant Qp^{-\varepsilon} \|\underset{\sim}{v}\|_{1/2+\varepsilon}. \tag{2.18}$$

In view of (2.8) and (2.9) it is sufficient to prove that I, II $< Qp^{-2\varepsilon} \| \underset{\sim}{v} \|^2_{1/2+\varepsilon}$. It follows from (2.10) that

$$\begin{aligned}
\mathrm{I} &\leqslant 4 p^{-1} \sum_{j=0}^{p} (2j+1)^{-1} \left[\left(\sum_{i=1}^{\infty} a_{2i+p,j} \right)^2 + \left(\sum_{i=1}^{\infty} a_{2i+p+1,j} \right)^2 \right] \\
&= 2 p^{-1} \sum_{j=0}^{p} (2j+1)^{-1} \left[\left(\sum_{i=p+2}^{\infty} a_{i,j} \right)^2 + \left(\sum_{i=p+2}^{\infty} (-1)^i a_{i,j} \right)^2 \right] \\
&= 2 p^{-1} \sum_{j=0}^{p} (2j+1)^{-1} \left[\left(\sum_{i=0}^{\infty} a_{i,j} - \sum_{i=0}^{p+1} a_{i,j} \right)^2 + \right. \\
&\quad \left. + \left(\sum_{i=0}^{\infty} (-1)^i a_{i,j} - \sum_{i=0}^{p+1} (-1)^i a_{i,j} \right)^2 \right] \\
&\leqslant 4 p^{-1} \sum_{j=0}^{p} (2j+1)^{-1} \left[\left(\sum_{i=0}^{\infty} a_{i,j} \right)^2 + \left(\sum_{i=0}^{\infty} (-1)^i a_{i,j} \right)^2 + \left(\sum_{i=0}^{p+1} a_{i,j} \right)^2 + \right. \\
&\quad \left. + \left(\sum_{i=0}^{p+1} (-1)^i a_{i,j} \right)^2 \right] \\
&\leqslant 4 p^{-1} (\| v_1(1, .) \|^2_{0,\partial\Omega} + \| v_1(-1, .) \|^2_{0,\partial\Omega}) + \\
&\quad + 4 p^{-1} \sum_{j=0}^{p} (2j+1)^{-1} \left[\left(\sum_{i=0}^{p+1} a_{i,j} \right)^2 + \left(\sum_{i=0}^{p+1} (-1)^i a_{i,j} \right)^2 \right]. \qquad (2.19)
\end{aligned}$$

Note that the next to last term on the right hand side of (2.19) can be bounded, using the integral method for

$$\int_0^{p+1} (2i+1)(1+i^2+j^2)^{-1/2-\varepsilon} = O(p^{1-2\varepsilon})$$

as $p \to \infty$, as follows :

$$\begin{aligned}
\sum_{j=0}^{p} (2j+1)^{-1} \left(\sum_{i=0}^{p+1} a_{i,j} \right)^2 &\leqslant \sum_{j=0}^{p} (2j+1)^{-1} \sum_{i=0}^{p+1} \frac{a_{i,j}^2 (1+i^2+j^2)^{1/2+\varepsilon}}{2i+1} \\
&\quad \times \sum_{i=0}^{p+1} (2i+1)(1+i^2+j^2)^{-1/2-\varepsilon} \\
&\leqslant Q \| \underset{\sim}{v} \|^2_{1/2+\varepsilon} p^{1-2\varepsilon}, \qquad (2.20)
\end{aligned}$$

with an identical bound holding for the last term of (2.19). Combining (2.19) and (2.20) and using Sobolev's embedding theorem yields

$$\mathrm{I} \leqslant Qp^{-2\varepsilon} \| \underset{\sim}{v} \|^2_{1/2+\varepsilon}. \qquad (2.21)$$

In an entirely analogous fashion we can see that

$$\mathrm{II} \leq Qp^{-2\varepsilon} \| \underset{\sim}{v} \|^2_{1/2+\varepsilon},$$

which together with (2.21) yields (2.18). Using interpolation [7], it follows from (2.15) and (2.18) that, for s bounded away from 1/2,

$$\mathrm{I} + \mathrm{II} \leq Qp^{1-2s} \| \underset{\sim}{v} \|^2_s,$$

which together with (2.8) and (2.9) concludes the proof for the case $\Omega = R$. For the case when Ω is a disjoint union of parallelograms the result follows on each element by using affine mappings onto R. The proposition then follows by summing over all the elements (see [6] for details).

Remark 2.1 : This result differs from the one known for the h-version of the finite element method [3, (1.5)],

$$\| \underset{\sim}{v} - \pi^h \underset{\sim}{v} \|_0 \leq Qh^r \| \underset{\sim}{v} \|_r, \quad r > 1/2 . \tag{2.22}$$

The constraint $r > 1/2$ (or $r \geq 1/2 + \varepsilon$) stems from the fact that, according to the trace theorem, this is the minimal requirement to ensure that $\underset{\sim}{v}$ has a trace on the boundary which is an L^2-function (not just a distribution). In [6] the corresponding result required an additional half derivative on $\underset{\sim}{v}\,(r > 1)$. In contrast, proposition 2.1 assumes the minimum regularity necessary. It is possible, however, that the bound still holds with the exponent of p replaced by $-r$, as suggested by (2.22).

COROLLARY 2.1 : *For* $s \geq 2$, $r > \max\ \{1/2\,;\,3/2 - 3/s\}$,

$$\| \underset{\sim}{v} - \pi^p \underset{\sim}{v} \|_{0,s} \leq Qp^{5/2-r-4/s} \| \underset{\sim}{v} \|_r .$$

Proof : Let $\underset{\sim}{P}^p\, \underset{\sim}{v}$ be the L^2-projection $P^p \times P^p : \underset{\sim}{V} \to \underset{\sim}{V}^p$. Then the following analogue of (1.7) holds :

$$\| \underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v} \|_{0,s} \leq Qp^{-r+3/2-3/s} \| \underset{\sim}{v} \|_r, \quad s \geq 2\,, \quad 3/2 - 3/s \leq r\,. \tag{2.23}$$

Also,

$$\| \pi^p \underset{\sim}{v} - \underset{\sim}{v} \|_{0,s} \leq \| \underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v} \|_{0,s} + \| \pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v} \|_{0,s} . \tag{2.24}$$

The second term in this expression may be bounded using the inverse inequality (1.9) as follows :

$$\begin{aligned} \| \pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v} \|_{0,s} &\leq Qp^{2-4/s} \| \pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v} \|_0 \\ &\leq Qp^{2-4/s} \left(\| \underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v} \|_0 + \| \pi^p \underset{\sim}{v} - \underset{\sim}{v} \|_0 \right) . \end{aligned} \tag{2.25}$$

Combining (2.23)-(2.25) and using proposition 2.1, we obtain the corollary.

III. SOLVABILITY OF THE DISCRETE PROBLEM

Following [3] we introduce, for $\rho \in W^p$, the notation

$$\alpha(\rho) - \alpha(u) = -\tilde{\alpha}_u(\rho)(u-\rho) = -\alpha_u(u)(u-\rho) + \tilde{\alpha}_{uu}(\rho)(u-\rho)^2, \tag{3.1}$$

where

$$\tilde{\alpha}_u(\rho) = \int_0^1 \alpha_u(\rho + t[u-\rho])\, dt,$$

and

$$\tilde{\alpha}_{uu}(\rho) = \int_0^1 (1-t)\, \alpha_{uu}(u + t[\rho - u])\, dt,$$

are bounded functions in $\bar{\Omega}$. Similarly, we write

$$\underset{\sim}{\beta}(\rho) - \underset{\sim}{\beta}(u) = -\tilde{\underset{\sim}{\beta}}_u(\rho)(u-\rho) = -\underset{\sim}{\beta}_u(u)(u-\rho) + \tilde{\underset{\sim}{\beta}}_{uu}(\rho)(u-\rho)^2, \tag{3.2}$$

and

$$c(\rho) - c(u) = -\tilde{c}_u(\rho)(u-\rho) = -c_u(u)(u-\rho) + \tilde{c}_{uu}(\rho)(u-\rho)^2, \tag{3.3}$$

where $\tilde{\underset{\sim}{\beta}}_u(\rho)$, $\tilde{\underset{\sim}{\beta}}_{uu}(\rho)$, $\tilde{c}_u(\rho)$, and $\tilde{c}_{uu}(\rho)$ are bounded functions in $\bar{\Omega}$. Also, let

$$\underset{\sim}{\Gamma} = \alpha_u(u)\, \underset{\sim}{z} + \underset{\sim}{\beta}_u(u), \qquad \gamma = c_u(u). \tag{3.4}$$

With the notation of (3.1)-(3.4), the following error equations follow from (1.3) and (1.5), [3] :

$$\begin{aligned}(\alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{z}^p], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P^p u - u^p) + ([P^p u - u^p]\, \underset{\sim}{\Gamma}, \underset{\sim}{v}) &= \\ = (\underset{\sim}{q}(u^p, \underset{\sim}{z}^p), \underset{\sim}{v}), \quad \underset{\sim}{v} \in \underset{\sim}{V}^p&, \\ (\operatorname{div}[\pi^p \underset{\sim}{z} - \underset{\sim}{z}^p], w) + (\gamma[P^p u - u^p], w) = (\eta(u^p), w), \quad w \in W^p&,\end{aligned} \tag{3.5}$$

where

$$\underset{\sim}{q}(u^p, \underset{\sim}{z}^p) = \alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{z}] + [P^p u - u]\, \underset{\sim}{\Gamma} + $$
$$+ (u-u^p)^2 [\tilde{\alpha}_{uu}(u^p)\, \underset{\sim}{z} + \tilde{\underset{\sim}{\beta}}_{uu}(u^p)] + \tilde{\alpha}_u(u^p)(u-u^p)(\underset{\sim}{z} - \underset{\sim}{z}^p), \tag{3.6}$$

and

$$\eta(u^p) = \gamma[P^p u - u] + \tilde{c}_{uu}(u^p)(u - u^p)^2 . \tag{3.7}$$

Just as in [3], we let

$$\Phi : \underset{\sim}{V}^p \times W^p \to \underset{\sim}{V}^p \times W^p$$

be given by $\Phi((\underset{\sim}{\mu}, \rho)) = (\underset{\sim}{\lambda}, \kappa)$, $(\underset{\sim}{\lambda}, \kappa)$ being the (unique) solution of the system

$$\begin{aligned}(\alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P^p u - \kappa) + ([P^p u - \kappa]\underset{\sim}{\Gamma}, \underset{\sim}{v}) &= \\ = (\underset{\sim}{q}(\rho, \underset{\sim}{\mu}), \underset{\sim}{v}), &\quad \underset{\sim}{v} \in \underset{\sim}{V}^p , \\ (\operatorname{div}[\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}], w) + (\gamma[P^p u - \kappa], w) = (\eta(\rho), w), &\quad w \in W^p ,\end{aligned} \tag{3.8}$$

where $\underset{\sim}{q}(\rho, \mu)$ and $\eta(\rho)$ are given by (3.6) and (3.7), respectively, replacing u^p by ρ and $\underset{\sim}{z}^p$ by $\underset{\sim}{\mu}$. The unique solvability of this (linear) system follows, for p sufficiently large, from [2], since the left hand side of (3.8) corresponds to the mixed method for the operator $M : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ given by

$$Mw = -\underset{\sim}{\nabla} \cdot (a(u)\underset{\sim}{\nabla} w + a(u) w \underset{\sim}{\Gamma}) + \gamma\omega ,$$

which has a bounded inverse. In fact, note that (1.2), (1.4) and (3.4) give

$$\begin{aligned}Mw &= -\underset{\sim}{\nabla} \cdot [a(u)\underset{\sim}{\nabla} w + a(u) w(\alpha_u(u)\underset{\sim}{z} + \underset{\sim}{\beta}_u(u))] + c_u(u) w \\ &= -\underset{\sim}{\nabla} \cdot \left[a(u)\underset{\sim}{\nabla} w + a(u) w\left[-\frac{a_u(u)}{a^2(u)}(-a(u)\nabla u) + \alpha(u)\underset{\sim}{b}_u(u)\right]\right] + \\ &\quad + c_u(u) w \\ &= -\underset{\sim}{\nabla} \cdot [a(u)\underset{\sim}{\nabla} w + (a_u(u)\nabla u + \underset{\sim}{b}_u(u)) w] + c_u(u) w ,\end{aligned}$$

which shows that M is the linearization of the operator $\mathscr{D}$ in (1.1) about the function u, and, thus, it has a bounded inverse since we have assumed that (1.1) admits unique isolated solutions.

The solvability of (1.5) is now equivalent to showing that Φ has a fixed point. This will follow from the Brouwer fixed point theorem if we show that Φ maps a ball of $\underset{\sim}{V}^p \times W^p$ into itself. We shall need the following technical result, a p-version of lemma 2.1 of [3]. Let $\varepsilon > 0$ be fixed for the rest of the paper, $\varepsilon \ll 1$.

LEMMA 3.1 : *Let* $2 \leqslant \theta \leqslant 4 - \varepsilon$. *Let* $\underset{\sim}{\omega} \in \underset{\sim}{V}$, $\underset{\sim}{q} \in L^2(\Omega)^2$, *and* $\eta \in L^2(\Omega)$. *If* $\tau \in W^p$ *satisfies*

$$\begin{cases} (\alpha(u)\,\underset{\sim}{\omega}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) + (\tau \underset{\sim}{\Gamma}, \underset{\sim}{v}) = (\underset{\sim}{q}, \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}^p, \\ (\operatorname{div} \underset{\sim}{\omega}, w) + (\gamma\tau, \omega) = (\eta, w), & w \in W^p, \end{cases}$$

then, there exists a constant $C = C(\theta, u, \alpha, \underset{\sim}{\Gamma}, \gamma, \Omega, \varepsilon)$ *such that, for p sufficiently large, depending upon* ε,

$$\|\tau\|_{0,\theta} \leqslant C\,[p^{1/2 - 2/\theta}\,\|\underset{\sim}{\omega}\|_0 + p^{-1-2/\theta}\,\|\operatorname{div} \underset{\sim}{\omega}\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta\|_0]\,.$$

Proof : We follow the proof of lemma 2.1 of [3]. Let $\theta' = \theta/(\theta - 1)$ be the conjugate exponent of θ. For $\psi \in L^{\theta'}(\Omega)$ let $\phi \in W^{2,\theta'}(\Omega)$ be the (unique) solution of $M^* \phi = \psi$ in Ω, $\psi = 0$ on $\partial\Omega$, where M^* is the formal adjoint of M. It follows that $\|\phi\|_{2,\theta'} \leqslant Q\,\|\psi\|_{0,\theta'}$. We then have [3],

$$\begin{aligned}(\tau, \psi) = {} & (\underset{\sim}{q}, a(u)\,\underset{\sim}{\nabla}\phi) + (\underset{\sim}{q}, \pi^p a(u)\,\underset{\sim}{\nabla}\phi - a(u)\,\underset{\sim}{\nabla}\phi) + \\ & + (\operatorname{div} \underset{\sim}{\omega} + \gamma\tau, \phi - P^p\phi) \\ & + (\alpha(u)\,\underset{\sim}{\omega} + \tau\underset{\sim}{\Gamma}, a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi) + (\eta, \phi) + (\eta, P^p\phi - \phi)\,.\end{aligned} \tag{3.9}$$

Note that Sobolev's embedding theorem implies that

$$(\underset{\sim}{q}, a(u)\,\underset{\sim}{\nabla}\phi) \leqslant C\,\|\underset{\sim}{q}\|_0\,\|\phi\|_1 \leqslant C\,\|\underset{\sim}{q}\|_0\,\|\phi\|_{2,\theta'}\,. \tag{3.10}$$

Next, (1.8) and Sobolev's embedding theorem imply that

$$\begin{aligned}(\underset{\sim}{q} - \alpha(u)\,\underset{\sim}{\omega}, \pi^p a(u)\,\underset{\sim}{\nabla}\phi - a(u)\,\underset{\sim}{\nabla}\phi) & \leqslant \\ & \leqslant C\,(\|\underset{\sim}{q}\|_0 + \|\underset{\sim}{\omega}\|_0)\,p^{1/2 - 2/\theta}\,\|\underset{\sim}{\nabla}\phi\|_{2/\theta} \\ & \leqslant C\,(\|\underset{\sim}{q}\|_0 + \|\underset{\sim}{\omega}\|_0)\,p^{1/2 - 2/\theta}\,\|\phi\|_{2,\theta'}\,,\end{aligned} \tag{3.11}$$

and that

$$\begin{aligned}(\tau\underset{\sim}{\Gamma}, a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi) & \leqslant C\,\|\tau\|_{0,\theta}\,\|a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi\|_0 \\ & \leqslant C\,\|\tau\|_{0,\theta}\,p^{-\varepsilon/8}\,\|\phi\|_{2,\theta'}\,.\end{aligned} \tag{3.12}$$

On the other hand, (1.7) and Sobolev's embedding theorem lead to

$$(\operatorname{div} \underset{\sim}{\omega}, \phi - P^p\phi) \leqslant K\,\|\operatorname{div} \underset{\sim}{\omega}\|_0\,p^{-1-2/\theta}\,\|\phi\|_{2,\theta'}\,, \tag{3.13}$$

$$(\gamma\tau, \phi - P^p\phi) \leqslant K\,\|\tau\|_{0,\theta}\,p^{-1-2/\theta}\,\|\phi\|_{2,\theta'}\,, \tag{3.14}$$

and

$$(\eta, \phi) + (\eta, P^p \phi - \phi) \leq K \|\eta\|_0 \|\phi\|_0 \leq K \|\eta\|_0 \|\phi\|_{2, \theta'} . \quad (3.15)$$

Collecting (3.9)-(3.15) we see that

$$(\tau, \psi) \leq K \|\psi\|_{0, \theta'} [p^{1/2 - 2/\theta} \|\underset{\sim}{\omega}\|_0 + p^{-1 - 2/\theta} \|\mathrm{div}\, \underset{\sim}{\omega}\|_0 + \\ + p^{-\varepsilon/8} \|\tau\|_{0, \theta} + \|\underset{\sim}{q}\|_0 + \|\eta\|_0] ,$$

which, for p sufficiently large, yields the desired estimate.

Now let $\underset{\sim}{\mathcal{V}}^p = \underset{\sim}{V}^p$ with the stronger norm $\|\underset{\sim}{v}\|_{\mathcal{V}^p} = \|\underset{\sim}{v}\|_{0, 2+\varepsilon} + \|\mathrm{div}\, \underset{\sim}{v}\|_0$ and let $\mathcal{W}^p = W^p$ with the stronger norm $\|w\|_{\mathcal{W}^p} = \|w\|_{0, t}$, where $t = \dfrac{4 + 2\varepsilon}{\varepsilon}$. We can prove now the existence of a solution of (1.5).

THEOREM 3.1 : *For $\delta > 0$ sufficiently small (dependent on p) and for p sufficiently large, Φ maps a ball of radius δ centered at $(\pi^p \underset{\sim}{z}, P^p u)$ of $\underset{\sim}{\mathcal{V}}^p \times \mathcal{W}^p$ into itself.*

Proof : Note that $1/t + 1/(2 + \varepsilon) = 1/2$. Let

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\mu}\|_{\mathcal{V}^p} \leq \delta \quad \text{and} \quad \|P^p u - \rho\|_{\mathcal{W}^p} \leq \delta < 1 .$$

Let us use lemma 3.1 on (3.8) with $\tau = P^p u - \kappa$, $\underset{\sim}{w} = \pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}$, $\underset{\sim}{q} = \underset{\sim}{q}(\rho, \underset{\sim}{\mu})$, $\eta = \eta(\rho)$ and $\theta = 4 - \varepsilon$. Observe that (1.7)-(1.9) and corollary 2.1 imply that, for $r > 1/2$, $m = r + 1$,

$$\begin{aligned}
\|\underset{\sim}{q}(\rho, \underset{\sim}{\mu})\|_0 + \|\eta(\rho)\|_0 &\leq \mathcal{Q}[p^{1/2 - r} \|\underset{\sim}{z}\|_r + p^{-m} \|u\|_m + \|u - \rho\|_{0,4}^2 + \\
&\quad + \|u - \rho\|_{0, t} \|\underset{\sim}{z} - \underset{\sim}{\mu}\|_{0, 2+\varepsilon}] \\
&\leq \mathcal{Q}[p^{1/2 - r} \|u\|_{r+1} + (\|u - P^p u\|_{0,4} + \|P^p u - \rho\|_{0,4})^2 + \\
&\quad + (\|u - P^p u\|_{0, t} + \|P^p u - \rho\|_{0, t}) \times \\
&\quad \times (\|\underset{\sim}{z} - \pi^p \underset{\sim}{z}\|_{0, 2+\varepsilon} + \|\pi^p \underset{\sim}{z} - \underset{\sim}{\mu}\|_{0, 2+\varepsilon})] \\
&\leq \mathcal{Q}[p^{1/2 - r} \|u\|_{r+1} + (p^{-m + 3/4} \|u\|_m + \delta)^2 + \\
&\quad + (p^{5/2 - r - 4/(2+\varepsilon)} \|u\|_{r+1} + \delta)(p^{-m + 3/2 - 3/t} \|u\|_m + \delta)] \\
&\leq \mathcal{Q}(\delta^2 + p^{1/2 - r} \|u\|_{r+1}) ,
\end{aligned} \quad (3.16)$$

where $\mathcal{Q}$ depends on $\|u\|_m$. Therefore,

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[p^{-\varepsilon/8}\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 + p^{-1-2/(4-\varepsilon)} \times \times \|\operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})\|_0 + \delta^2 + p^{1/2-r}]. \quad (3.17)$$

On the other hand, taking $\underset{\sim}{v} = \pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}$ and $w = P^p u - \kappa$ in (3.8), we see that

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 \leq \mathcal{Q}[\|P^p u - \kappa\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta\|_0], \quad (3.18)$$

and, taking $w = \operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})$ in the second equation of (3.8) results in

$$\|\operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})\|_0 \leq \mathcal{Q}[\|P^p u - \kappa\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta_0\|]. \quad (3.19)$$

Combining (3.17)-(3.19) yields the relation

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[p^{-\varepsilon/8}\|P^p u - \kappa\|_0 + \delta^2 + p^{1/2-r}],$$

which, for p sufficiently large and $r = 5/2$, implies that

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[\delta^2 + p^{-2}], \quad (3.20)$$

where the constant $\mathcal{Q}$ depends on $\|u\|_{7/2}$. Combining (3.20) with (1.9) we see that

$$\begin{aligned} \|P^p u - \kappa\|_{0,t} &\leq \mathcal{Q}p^{\frac{4}{4-\varepsilon} - \frac{2\varepsilon}{\varepsilon+2}}\|P^p u - \kappa\|_{0,4-\varepsilon} \\ &\leq \mathcal{Q}(p^{1-\varepsilon/4}\delta^2 + p^{-1-\varepsilon/4}), \end{aligned} \quad (3.21)$$

while (1.9), (3.18), (3.16), and (3.20) imply that

$$\begin{aligned} \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{0,2+\varepsilon} &\leq \mathcal{Q}p^{2\varepsilon/(2+\varepsilon)}\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 \\ &\leq \mathcal{Q}(p^{\varepsilon}\delta^2 + p^{-2+\varepsilon}). \end{aligned} \quad (3.22)$$

Combining (3.19) and (3.22) yields

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leq \mathcal{Q}(p^{\varepsilon}\delta^2 + p^{-2+\varepsilon}). \quad (3.23)$$

We can now combine (3.21) and (3.23) in the bound

$$\|P^p u - \kappa\|_{\mathcal{W}^p} + \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leq \mathcal{Q}_1(p^{1-\varepsilon/4}\delta^2 + p^{-1-\varepsilon/4}). \quad (3.24)$$

We want to choose p and δ so that $\mathcal{Q}_1 p^{1-\varepsilon/4}\delta^2 \leq \frac{\delta}{2}$ and $\mathcal{Q}_1 p^{-1-\varepsilon/4} \leq \frac{\delta}{2}$.

Let $p \geqslant (2\,\mathcal{Q}_1)^{4/\varepsilon}$, so that $I = \left[2\,\mathcal{Q}_1\,p^{-1-\varepsilon/4}, \dfrac{p^{\varepsilon/4-1}}{2\,\mathcal{Q}_1}\right]$ is not empty. Then, for $\delta \in I$, (3.24) implies that

$$\|P^p u - \kappa\|_{\mathcal{W}^p} \leqslant \delta \quad \text{and} \quad \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leqslant \delta\,,$$

as we needed.

Remark 3.1 : Note that the choice $\delta = 2\,\mathcal{Q}_1\,p^{-1-\varepsilon/4}$ in theorem 3.1 shows (using (1.7) and (1.8)) not only that (1.5) is solvable but also that, for $p \to \infty$, the solution of (1.5), $(\underset{\sim}{z}^p, u^p)$, differs from $(\underset{\sim}{z}, u)$ in the $\underset{\sim}{\mathcal{V}}^p \times \mathcal{W}^p$ norm by $0(p^{-1-\varepsilon/4})$ at most. We shall need this observation in order to arrive at the correct error estimates.

4. THE L^2-ERROR BOUNDS

Just as in [3], using (3.1)-(3.3) we now rewrite (3.5) in the form

$$\begin{cases} (\alpha(u)\,\underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) + (\tau\tilde{\underset{\sim}{\Gamma}}, \underset{\sim}{v}) = (\underset{\sim}{q}, \underset{\sim}{v})\,, & \underset{\sim}{v} \in \underset{\sim}{V}^p\,, \\ (\operatorname{div} \underset{\sim}{\zeta}, w) + (\tilde{\gamma}\tau, w) = (\eta, w)\,, & w \in W^p\,, \end{cases} \tag{4.1}$$

where $\underset{\sim}{\zeta} = \underset{\sim}{z} - \underset{\sim}{z}^p$, $\tau = P^p u - u^p$, $\tilde{\underset{\sim}{\Gamma}} = \tilde{\alpha}_u(u^p)\,\underset{\sim}{z}^p + \tilde{\underset{\sim}{\beta}}_u(u^p)$, $\tilde{\gamma} = \tilde{c}_u(u^p)$, $\underset{\sim}{q} = (P^p u - u)\,\tilde{\underset{\sim}{\Gamma}}$, and $\eta = (P^p u - u)\,\tilde{\gamma}$. Note that the left hand side of (4.1) corresponds to the mixed method for the operator $N : H^2(\Omega) \to L^2(\Omega)$ given by

$$Nw = -\underset{\sim}{\nabla} \cdot (a(u)\,\underset{\sim}{\nabla} w + a(u)\,w\tilde{\underset{\sim}{\Gamma}}) + \tilde{\gamma} w\,.$$

Therefore, if we show that its formal adjoint, N^*, has a bounded inverse $L^2 \to H^2(\Omega) \cap H_0^1(\Omega)$, then lemma 3.1 would apply to (4.1) without any change in the proof. Since we know that M^* has a bounded inverse, all we need to do is to check that the operator norm of $M^* - N^*$ can be made arbitrarily small by taking p large enough.

LEMMA 4.1 : *There exists a positive integer p_0 such that, for all $p \geqslant p_0$, N^* has a bounded inverse $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$. ($N^*$ depends on p through $\tilde{\gamma}$ and $\tilde{\underset{\sim}{\Gamma}}$).*

Proof : Just as in [3], we have

$$(M^* - N^*)\chi = a(u)\left\{[\bar{\alpha}_{uu}\,\underset{\sim}{z} + \bar{\underset{\sim}{\beta}}_{uu}](u - u^p) + \tilde{\alpha}_u(u^p)(\underset{\sim}{z} - \underset{\sim}{z}^p)\right\} \times$$
$$\times\,\underset{\sim}{\nabla}\chi + \bar{c}_{uu}(u - u^p)\,\chi\,, \quad \chi \in L^2(\Omega)\,,$$

where $\bar{\alpha}_{uu} = \dfrac{\alpha_u(u) - \tilde{\alpha}_u(u^p)}{u - u^p}$ and $\bar{\underset{\sim}{\beta}}_{uu}$, and $\bar{c}_{uu}$, defined by analogous relations, are bounded functions in $\bar{\Omega}$. It follows from remark 3 1 and Sobolev's embedding theorem that

$$\begin{aligned}\|(M^* - N^*)\chi\|_0 &\leq K[\|\underset{\sim}{z}\|_{0\ \infty}\ \|u - u^p\|_{0\ t}\ \|\underset{\sim}{\nabla}\chi\|_{0\ 2+\varepsilon} + \\ &\quad + \|\underset{\sim}{z} - \underset{\sim}{z}^p\|_{0\ 2+\varepsilon}\ \|\underset{\sim}{\nabla}\chi\|_{0\ t} + \|u - u^p\|_0\ \|\chi\|_{0\ \infty}] \\ &\leq K(\|\underset{\sim}{\nabla}\chi\|_1 + \|\chi\|_{1+\varepsilon})\,p^{-1-\varepsilon/4} \\ &\leq Kp^{-1}\|\chi\|_2 ,\end{aligned}$$

as needed

To conclude, we establish the rate of convergence of $(\underset{\sim}{z}^p, u^p)$ to (z, u).

THEOREM 4.1 : *Assume that the solution u of* (1.1) *is in* $H^{7/2}(\Omega)$ *There is a positive constant Q, independent of p but dependent on* $\|u\|_{7/2+2\varepsilon}$, *such that, for p sufficiently large and* $m \geq 7/2$,

$$i) \qquad \|u - u^p\|_0 \leq Qp^{1-m}\|u\|_m ,$$

$$ii) \qquad \|\underset{\sim}{z} - \underset{\sim}{z}^p\|_0 \leq Qp^{3/2-m}\|u\|_m ,$$

$$iii) \qquad \|\operatorname{div}(\underset{\sim}{z} - \underset{\sim}{z}^p)\|_0 \leq Qp^{2-m}\|u\|_m$$

Proof In view of remark 3 1 and lemma 4.1, we can use lemma 3.1 on (4 1) with $\theta = 2$. Thus,

$$\|\tau\|_0 \leq C\,[p^{-1/2}\|\underset{\sim}{\zeta}\|_0 + p^{-2}\|\operatorname{div}\underset{\sim}{\zeta}\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta\|_0] \tag{4.2}$$

Note that remark 3.1 together with (1.7) lead to the following estimate for $r \geq 0$, $m > 3/2$,

$$\begin{aligned}\|\underset{\sim}{q}\|_0 + \|\eta\|_0 &= \|(P^p u - u)\,\underset{\sim}{\tilde{\Gamma}}\| + \|(P^p u - u)\,\tilde{\gamma}\| \leq \\ &\leq K(\|P^p u - u\|_0 + \|(P^p u - u)\,\underset{\sim}{z}^p\|_0) \\ &\leq K[p^{-r}\|u\|_r(1 + \|\underset{\sim}{z}\|_{0\ \infty}) + \|\underset{\sim}{z}^p - \underset{\sim}{z}\|_{0,\,2+\varepsilon}\ \|P^p u - u\|_{0,\,t}] \\ &\leq K(p^{-r}\|u\|_r + p^{-1-\varepsilon/4}\,p^{3/2 - 3\varepsilon/[2(2+\varepsilon)] - m}\ \|u\|_m) \\ &\leq Kp^{1-m-\varepsilon}\|u\|_m .\end{aligned} \tag{4 3}$$

Combining (4.2), (4.3), (3.18), (3.19), (1.7) and (1.8) yields,

$$\begin{aligned}\|\tau\|_0 &\leq C\,[p^{-1/2}(\|\underset{\sim}{z}-\pi^p\underset{\sim}{z}\|_0+\|\pi^p\underset{\sim}{z}-\underset{\sim}{z}^p\|_0)+p^{-2}(\|\operatorname{div}\underset{\sim}{z}-P^p\operatorname{div}\underset{\sim}{z}\|_0\\ &\quad+\|\operatorname{div}(\pi^p\underset{\sim}{z}-\underset{\sim}{z}^p)\|_0)+p^{1-m-\varepsilon}\|u\|_m]\\ &\leq C\,[p^{-1/2}\|\tau\|_0+p^{-1/2}p^{1/2-r}\|u\|_{r+1}+p^{-2}p^{-s}\|u\|_{s+2}+\\ &\quad+p^{-1/2}p^{-1-\varepsilon/4}+p^{1-m-\varepsilon}\|u\|_m]\,,\quad r>1/2\,,\quad s\geq 0\,,\quad m>3/2\,,\end{aligned}$$

which, for p sufficiently large, leads to

$$\|\tau\|_0 \leq Cp^{1-m}\|u\|_m\,,\quad m\geq 2\,, \tag{4.4}$$

where the constant C depends on $\|u\|_{7/2}$. The first part of the theorem is an immediate consequence of (1.7) and (4.4). On the other hand, it follows from (1.8), (3.18), (4.3) and (4.4), that

$$\begin{aligned}\|\underset{\sim}{z}-\underset{\sim}{z}^p\|_0 &\leq \|\underset{\sim}{z}-\pi^p\underset{\sim}{z}\|_0+\|\pi^p\underset{\sim}{z}-\underset{\sim}{z}^p\|_0\\ &\leq C\,[p^{3/2-m}\|u\|_m+p^{1-m}\|u\|_m]\,,\end{aligned}$$

which proves the second part of the theorem.

Finally, we deduce from (3.19), (1.7), (4.3) and (4.4) that

$$\begin{aligned}\|\operatorname{div}(\underset{\sim}{z}-\underset{\sim}{z}^p)\|_0 &\leq \|\operatorname{div}\underset{\sim}{z}-P^p\operatorname{div}\underset{\sim}{z}\|_0+\|\operatorname{div}(\pi^p\underset{\sim}{z}-\underset{\sim}{z}^p)\|_0\\ &\leq C\,[p^{2-m}\|u\|_m+p^{1-m}\|u\|_m]\,,\end{aligned}$$

which gives iii).

Remark 4.1 : The estimate for the error in $\underset{\sim}{z}$ is the best we could hope for in view of (1.8). The estimate for the error in div $\underset{\sim}{z}$ is optimal in rate and regularity, while the one for u is probably not sharp in view of (1.7).

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